Application of the Nullcline Method to a Certain Model of Competitive Species

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***Abstract* – In this paper we will observe the model of competitive types and it will be analyzed using the nullcline method. It will be shown that this model has four points of equilibria, which are stable or unstable depending on the parameters a and b. The local stability of these points was investigated and global dynamics was determined using nullcline methods, that is, the bases of attraction of these points were shown.**

***Keywords* – Competitive model, Equilibrium Point, Growth rate, Nullcline, Stability.**

1. Introduction

One of the basic methods in studying the systems of ordinary differential equations in the plane is the nullclines method. If we consider the system of differential equations

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y), \end{cases} \tag{1}$$

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then the x-nullclines are obtained by solving the equation $f(x,y) = 0$. Similarly, y-nullclines are obtained by solving the equation $g(x,y) = 0$. The vector field point on x-nullcline is directed up or down, and on the y-nullcline is directed left or right. The intersection points of the x-nullclines and y-nullclines are the equilibrium points. Thus, x-nullclines and y-nullclines separate the plane in the area so that in each of these areas the vector field is one of four directions: southwest, southeast, northwest or northeast. Using a vector field in each of these areas, it is possible to sketch the phase portrait of the system (1) and this greatly helps us to qualitatively analyze the system of equations (1). In this paper we observed a competitive model

$$\begin{cases} \dot{x} = x(a - x - ay) \\ \dot{y} = y(b - bx - y), \end{cases} \tag{2}$$

where $x \geq 0$, $y \geq 0$, $a > 0$ and $b > 0$, which is a special case of a general competitive model

$$\begin{cases} \dot{x} = xf(x,y) \\ \dot{y} = yg(x,y). \end{cases} \tag{3}$$

The variables x and y denote two species, and the functions f and g depend on both variables and represent a growth rate of species. Competitive models have great applications in biology. In medicine we have applied this model to infectious diseases (see [1]). The Lotka-Volterra model is also a special case of a competitive model (see [2],[3]). A better insight into the relationship between predators and prey to the Lotka-Volterra model with infected prey is given in the delayed equation (see [4]). One of the basic theorems for the model of competitive species is:

Theorem 1: *The flow ϕ_t of the competitive species system has the following property: For all points (x,y), with $x \geq 0$ and $y \geq 0$, the limit*

$lim_{t\to\infty} \phi_t(x, y)$ exists and is one of a finite number of equilibria (see [5]).

2. Model of competitive species

Now we will examine model (2), where $x \geq 0$ and $y \geq 0$ are two species, and parameters $a > 0$ and $b > 0$. The equilibrium points of the model (2) are the solutions of the system:

$$\begin{cases} x(a - x - ay) = 0 \\ y(b - bx - y) = 0. \end{cases} \quad (4)$$

These are actually intersect of nullclines (see[7],[9]). The system (4) has four solutions: $O(0,0), A(a, 0), B(0, b)$ and $C\left(\frac{ab-a}{ab-1}, \frac{ab-b}{ab-1}\right)$.

Lemma 1: *Equilibrium $C\left(\frac{ab-a}{ab-1}, \frac{ab-b}{ab-1}\right)$ is a positive if $0 < a < 1$ and $0 < b < 1$, or $a > 1$ and $b > 1$.*

Proof: Let

$$\frac{ab-a}{ab-1} > 0 \quad (5)$$

and

$$\frac{ab-b}{ab-1} > 0. \quad (6)$$

If $ab - 1 > 0$, i.e. $ab > 1$, then from inequality (5) we get that $ab - a > 0$, i.e. $a(b - 1) > 0$. Since $a > 0$, it implies that $b > 1$. Similarly, from inequality (6) we get that $a > 1$. So, if $ab > 1$, then $a > 1$ and $b > 1$.

If $ab - 1 < 0$, i.e. $ab < 1$, then from (5) we get that $ab - a < 0$, i.e. $a(b - 1) < 0$. The last inequality implies that $b < 1$. Similary, using inequality (6) we get that $a < 1$. So, if $ab < 1$, then $a < 1$ and $b < 1$. Hence, the Lemma 1 is proved.

The system (2) can be written with

$$\dot{x} = \begin{pmatrix} x(a - x - ay) \\ y(b - bx - y) \end{pmatrix}.$$

The Jacobian matrix is given with

$$D_f = \begin{pmatrix} a - 2x - ay & -ax \\ -by & b - bx - 2y \end{pmatrix}. \quad (7)$$

Theorem 2: *If $0 < a < 1$ and $0 < b < 1$, then for system (2) it means that:*

- *equilibrium point $O(0,0)$ is a source;*
- *equilibrium $A(a, 0)$ is a saddle point;*
- *equilibrium $B(0, b)$ is a saddle point;*
- *equilibrium $C\left(\frac{ab-a}{ab-1}, \frac{ab-b}{ab-1}\right)$ is a real sink.*

Proof: The Jacobian matrix (7) at the point $O(0,0)$ is

$D_f(O) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Since both the eigenvalues of the matrix $D_f(O)$ are positive, then the equilibrium point $O(0,0)$ is a source (see [6]).

The Jacobian matrix (7) at the point $A(a, 0)$ has a form

$$D_f(A) = \begin{pmatrix} -a & -a^2 \\ 0 & b - ab \end{pmatrix}.$$

Since

$$\det(D_f(A)) = -a(b - ab) = -ab(1 - a) < 0,$$

then equilibrium point $A(a, 0)$ is a saddle point (see [6]).

The Jacobian matrix (7) at the point $B(0, b)$ has a form

$$D_f(B) = \begin{pmatrix} a - ab & 0 \\ -b^2 & -b \end{pmatrix}.$$

Similarly, for the point $B(0, b)$ we get that $\det(D_f(B)) = -b(a - ab) = -ab(1 - b) < 0,$

from which we conclude that equilibrium point $B(0, b)$ is a saddle point (see [6]).

The Jacobian matrix (7) evaluated at the point $C\left(\frac{ab-a}{ab-1}, \frac{ab-b}{ab-1}\right)$ is

$D_f(C) = \begin{pmatrix} \frac{a-ab}{ab-1} & \frac{-a(ab-a)}{ab-1} \\ \frac{-b(ab-b)}{ab-1} & \frac{b-ab}{ab-1} \end{pmatrix}$. Determinant of this matrix is

$$\det(D_f(C)) = \frac{(ab-a)(ab-b)}{1-ab} = \frac{ab(b-1)(a-1)}{1-ab} > 0.$$

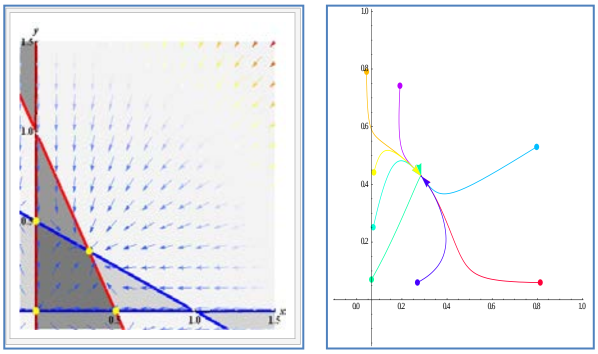
Trace of matrix $D_f(C)$ is

$$tr\left(D_f(C)\right) = \frac{a + b - 2ab}{ab - 1} < 0,$$

since $ab - 1 < 0$ and $a + b - 2ab > a^2 + b^2 - 2ab = (a - b)^2 \geq 0$. Besides, we have that

$$\left(tr\left(D_f(C)\right)\right)^2 - 4\,det\left(D_f(C)\right)$$
$$= \left(\frac{a+b-2ab}{ab-1}\right)^2 - 4\frac{ab(b-1)(a-1)}{1-ab}$$
$$= \frac{-2ab+b^2+4a^3b^3-4a^3b^2+a^{2-}4a^2b^3+4a^2b^2}{(ab-1)^2}$$
$$= \frac{(a-b)^2+4a^2b^2(1-a)(1-b)}{(ab-1)^2} > 0.$$

Based on the above, we conclude that equilibrium point C $\left(\frac{\mathrm{ab-a}}{\mathrm{ab-1}}, \frac{\mathrm{ab-b}}{\mathrm{ab-1}}\right)$ is a real sink (see [5]).



(a) (b)

Figure 1. (a) The nullclines of the system (2) in the case when $\boldsymbol{a = 0,5}$ ***and*** $\boldsymbol{b = 0,6}$ *, and (b) trajectories in the case* $\boldsymbol{a = 0,5}$ ***and*** $\boldsymbol{b = 0,5}$.

Theorem 3: *If* $a > 1$ *and* $b > 1$, *then for system (2) it means that:*

- *equilibrium point* $O(0,0)$ *is a source;*
- *equilibrium* $A(a,0)$ *is a real sink;*
- *equilibrium* $B(0,b)$ *is a real sink;*
- *equilibrium* $C\left(\frac{ab-a}{ab-1}, \frac{ab-b}{ab-1}\right)$ *is a saddle point.*

Proof: Same as in the case when it is $0 < a < 1$ and $0 < b < 1$, the Jacobian matrix $D_f(O)$ has both positive eigenvalues and equilibrium point $O(0,0)$ is a source.

For the point $A(a,0)$ we have that

$$D_f(A) = \begin{pmatrix} -a & -a^2 \\ 0 & b-ab \end{pmatrix},$$

$$\det(D_f(A)) = -a(b-ab) = -ab(1-a) > 0,$$

$$\left(tr\left(D_f(A)\right)\right)^2 - 4\,det\left(D_f(A)\right) = (a+b-ab)^2 > 0$$

and

$$tr\left(D_f(A)\right) = -a+b-ab = -a+b(1-a) < 0.$$

It implies that equilibrium point $A(a,0)$ is a sink (see [5],[8]).

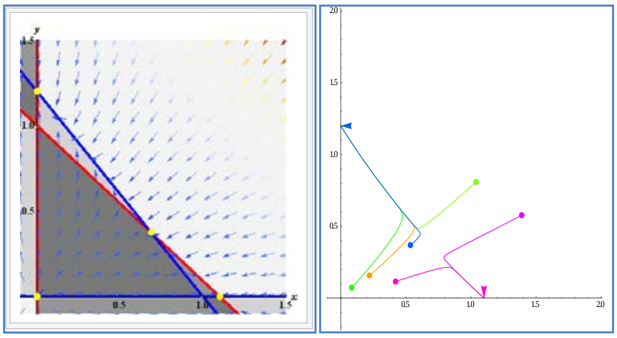
On the basis of the symmetry of the system (2) and equilibrium points $A(a,0)$ and $B(0,b)$ we conclude in the same way that the equilibrium point $B(0,b)$ is a sink.

For the point $C\left(\frac{ab-a}{ab-1}, \frac{ab-b}{ab-1}\right)$ we have that

$$\det(D_f(C)) = \frac{(ab-a)(ab-b)}{1-ab} = \frac{ab(b-1)(a-1)}{1-ab} < 0$$

and it implies that equilibrium point $C\left(\frac{ab-a}{ab-1}, \frac{ab-b}{ab-1}\right)$ is a saddle point (see [5]).

In all other cases, the system (2) has exactly 3 equilibrium points. In addition, point $O(0,0)$ is in all of the following cases, as well as the earlier is a source. For this reason, we will only examine the behavior of points $A(a,0)$ and $B(0,b)$.



(a) (b)

Figure 2. (a) The nullclines of the system (2), and (b) trajectories, both in the case $\boldsymbol{a = 1,1}$ ***and*** $\boldsymbol{b = 1,2}$.

Theorem 4: *If* $0 < a < 1$ *and* $b > 1$, *then equilibrium* $A(a,0)$ *is a saddle point, and equilibrium point* $B(0,b)$ *is a real sink.*

Proof: Now we have that

$$det(D_f(A)) = -ab(1-a) < 0.$$

and it means that equilibrium point $A(a,0)$ is a saddle. For the point $B(0,b)$ follows

$$\det(D_f(B)) = -ab(1-b) > 0,$$

$$\left(tr\left(D_f(B)\right)\right)^2 - 4\,det\left(D_f(B)\right) = (a+b-ab)^2 > 0$$

and

$$tr\left(D_f(B)\right) = -b+a-ab < 0.$$

So, equilibrium point $B(0,b)$ is a real sink.

Theorem 5: *If* $0 < b < 1$ *and* $a > 1$, *then equilibrium* $A(a, 0)$ *is a real sink, and equilibrium point* $B(0, b)$ *is a saddle.*

Proof: Now we have that

$$\det(D_f(B)) = -ab(1-b) < 0.$$

and it means that equilibrium point $B(0, b)$ is a saddle. For the point $A(a, 0)$ we have that

$$\det(D_f(A)) = -ab(1-a) > 0,$$

$$\left(tr\left(D_f(A)\right)\right)^2 - 4\,det\left(D_f(A)\right) = (a+b-ab)^2 > 0$$

and

$$tr\left(D_f(A)\right) = -a + b - ab < 0.$$

From the last three inequalities, equilibrium point $A(a, 0)$ is a real sink.

Theorem 6: If $a = 1$ and $0 < b < 1$, *then equilibrium* $A(a, 0)$ *is a center, and equilibrium point* $B(0, b)$ *is a saddle.*

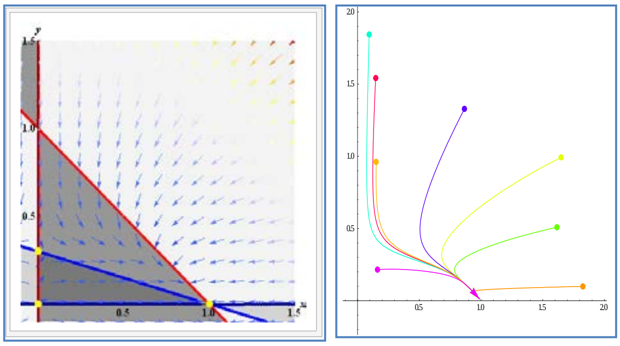
Proof: We have

$$\det(D_f(A)) = -ab(1-a) = 0.$$

It implies that equilibrium point $A(a, 0)$ is a center. Since,

$$\det(D_f(B)) = -ab(1-b) < 0,$$

then equilibrium point $B(0, b)$ is a saddle.



(a) (b)

Figure 3. (a) The nullclines of the system (2), and (b) trajectories, both in the case $\boldsymbol{a = 1}$ ***and*** $\boldsymbol{b = 0,3}$.

Theorem 7: If $b = 1$ and $0 < a < 1$, *then equilibrium* $A(a, 0)$ *is a saddle, and equilibrium point* $B(0, b)$ *is a centre.*

Proof: Proof of Theorem 7 is completely similar to the proof of Theorem 6.

Theorem 8: If $a = 1$ and $b > 1$, *then equilibrium* $A(a, 0)$ *is a center, and equilibrium point* $B(0, b)$ *is a sink.*

Proof: For equilibrium point $A(a, 0)$ we have

$$\det(D_f(A)) = -ab(1-a) = 0,$$

and it implies that $A(a, 0)$ is a center.

For point $B(0, b)$ it means that:

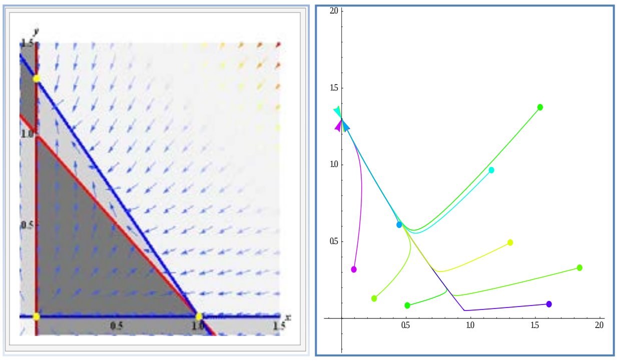
$$\det(D_f(B)) = -ab(1-b) > 0,$$

$$\left(tr\left(D_f(B)\right)\right)^2 - 4\,det\left(D_f(B)\right) = (a+b-ab)^2 > 0$$

and

$$tr\left(D_f(B)\right) = a - b - ab = 1 - 2b < 0.$$

From there, equilibrium point $B(0, b)$ is a real sink.



(a) (b)

Figure 4. (a) The nullclines of the system (2), and (b) trajectories, both in the case $\boldsymbol{a = 1}$ ***and*** $\boldsymbol{b = 1,3}$.

Theorem 9: If $b = 1$ and $a > 1$, *then equilibrium* $A(a, 0)$ *is a sink, and equilibrium point* $B(0, b)$ *is a center.*

Proof: Proof of Theorem 9 is similar to the proof of Theorem 8.

Theorem 10: *If* $a = b = 1$ *then system (2) has equilibrium point* $O(0,0)$ *which is a source, and infinitely many equilibrium points in the form* $(d, 1-d)$ *where* $(0 \le d \le 1)$, which are the centers. *The solution of the system (2) was given with* $y = c_1 x$, $\frac{x}{(c_1+1)x-1} = c_2 e^t$.

Proof: *If* $a = b = 1$, *then system (2) has the form*

$$\begin{cases} \dot{x} = x(1-x-y) \\ \dot{y} = y(1-x-y). \end{cases} \qquad (8)$$

If we solve the system

$$x(1-x-y) = 0$$
$$y(1-x-y) = 0,$$

we have that $x = y = 0$, or $x + y = 1$.

Since, $\det(D_f(d, 1-d)) = 0$, equilibrium points $(d, 1-d)$ are the centers.

The system (8) can be written in the form

$$\frac{dx}{x(1-x-y)} = \frac{dy}{y(1-x-y)} = dt.$$

From

$$\frac{dx}{x(1-x-y)} = \frac{dy}{y(1-x-y)},$$

we have that

$$\frac{dx}{x} = \frac{dy}{y}.$$

The solution of the last equation is $y = c_1 x$.

Now it is

$$\frac{dx}{x(1-x-c_1 x)} = dt.$$

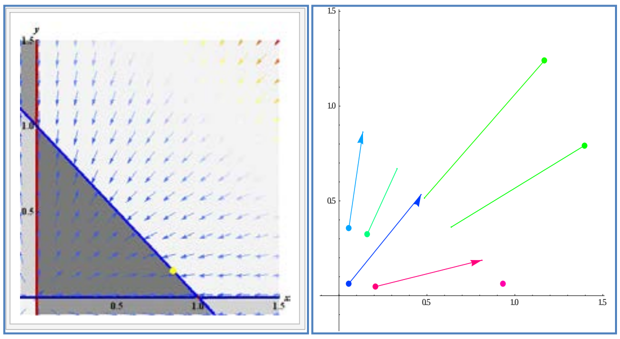
If we integrate the last equation, then we get

$$\int \frac{dx}{x(1-x-c_1 x)} = \int dt,$$

and from there

$$\frac{x}{(c_1+1)x-1} = c_2 e^t.$$

Remark: *Every point that belongs to the part of line $x + y = 1$ which lies in the first quadrant $(x \geq 0, y \geq 0)$ is the equilibrium point. In addition, each solution tends to the one of these points, in the line $x + y = 1$, because the point $O(0,0)$ is the source.*



(a) *(b)*

Figure 5. (a) The nullclines of the system (2), and (b) trajectories, both in the case $\boldsymbol{a = 1}$ ***and*** $\boldsymbol{b = 1}$.

3. Conclusion

According to all of the above, we can see that every solution of system (2) will tend to some of the equilibrium points. For all parameter values $a > 0$ and $b > 0$ point $O(0,0)$ will be the source. The other three equilibrium points will be stable or unstable depending on the parameter value $a > 0$ and $b > 0$. If $a = b = 1$, then every solution of system (2) tends to some of the equilibrium points in form $(d, 1-d)$, where $0 \leq d \leq 1$. If $a < 1$ and $b < 1$, then every solution of system (2) tends to equilibrium point $\left(\frac{ab-a}{ab-1}, \frac{ab-b}{ab-1}\right)$. If $a < 1$ and $b \geq 1$, then every solution of system (2) tends to the equilibrium point $B(0, b)$. If $a \geq 1$ and $b < 1$, then due to symmetry of system (2) every solution of system (2) tends to the equilibrium point $A(a, 0)$. If $a > 1$ and $b > 1$, then every solution of system (2) tends to one of the equilibrium points $A(a, 0)$ and $B(0, b)$.

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