Research Article

Characterization of the Solvability of Generalized Constrained Variational Equations

Manuel Ruiz Galán

*Departamento de Matemática Aplicada, E.T.S. de Ingeniería de Edificación, Universidad de Granada,
c/Severo Ochoa s/n, 18071 Granada, Spain*

Correspondence should be addressed to Manuel Ruiz Galán, mruizg@ugr.es

In a general context, that of the locally convex spaces, we characterize the existence of a solution
for certain variational equations with constraints. For the normed case and in the presence of some
kind of compactness of the closed unit ball, more specifically, when we deal with reflexive spaces
or, in a more general way, with dual spaces, we deduce results implying the existence of a unique
weak solution for a wide class of linear elliptic boundary value problems that do not admit a
classical treatment. Finally, we apply our statements to the study of linear impulsive differential
equations, extending previously stated results.

1. Introduction

It is common knowledge that in studying differential problems, variational methods have
come to be essential. For instance, in [1], for a certain impulsive differential equation, its
variational structure as well as the existence and uniqueness of a weak solution is shown.
Specifically, given $T > 0$, $f \in L^2(0, T)$, $\lambda > -\pi^2/T^2$, $t_0 = 0 < t_1 < \cdots < t_k < t_{k+1} = T$, and
$d_1, \ldots, d_k \in \mathbb{R}$, the impulsive linear problem

$$-x''(t) + \lambda x(t) = f(t), \quad t \in (0, T),$$

$$x(0) = x(T) = 0, \tag{1.1}$$

$$\Delta x'(t_j) = d_j, \quad j = 1, \ldots, k,$$

where $\Delta x'(t_j) := x'(t_j^+) - x'(t_j^-)$, is considered, and as a direct application of the classical Lax-Milgram theorem ([2, Corollary 5.8]), possibly the most popular variational tool, it is proven that there exists a unique $x_0 \in H_0^1(0, T)$ such that

$$y \in H_0^1(0, T) \implies \int_0^T (x_0'y' + \lambda x_0 y) = \int_0^T fy - \sum_{j=1}^k d_j y(t_j). \tag{1.2}$$

In this paper we replace the Lax-Milgram theorem with a characterization of the unique solvability of a certain type of variational equation with constraints. Such a constrained variational equation arises naturally; for instance, when in the variational formulation of an elliptic partial differential equation, its essential boundary constraints are treated as constraints. This result allows us to consider problems without data functions in a Hilbert space, which is beyond the control of the classical theory ([3, section II.1 Proposition 1.1], [4, Lemma 4.67]). In particular, it extends the class of the said impulsive linear problems admitting a weak solution, since we prove that for any $f \in L^p(0, T)$, where $1 < p < \infty$ and not necessarily $p = 2$, and for all $\lambda > -\lambda_{p,T}$, for some $\lambda_{p,T} > 0$ only depending on p and T, the impulsive equation has one and only one solution.

To help understand specifically the sort of constrained variational inequality under consideration, we have selected a simple but illustrative model problem, which will adequately serve our purposes related to linear impulsive problems. For $T > 0$, $v_0, v_T \in \mathbb{R}$, and $f \in L^2(0, T)$, let us consider the corresponding Poisson' equation with nonhomogeneous Dirichlet boundary conditions

$$\begin{aligned} -x'' &= f \quad \text{in } (0, T), \\ x(0) &= v_0, \qquad x(T) = v_T, \end{aligned} \tag{1.3}$$

whose usual weak formulation is

$$\text{find } x_0 \in H^1(0, T) \text{ such that } \begin{cases} x \in H_0^1(0, T) \implies \int_0^T x_0'x' = \int_0^T fx, \\ x_0(0) = v_0, \qquad x_0(T) = v_T. \end{cases} \tag{1.4}$$

By imposing essential boundary conditions weakly, we can equivalently write that variational formulation as a constrained variational equation. To be more concrete, let X be the Sobolev space $H^1(0, T)$, let $Y := \mathbb{R}^2$, and let $a : X \times X \to \mathbb{R}$ and $b : X \times Y \to \mathbb{R}$ be the continuous bilinear forms given by

$$\begin{aligned} a(x, \tilde{x}) &:= \int_0^T x'\tilde{x}', \quad (x, \tilde{x} \in X), \\ b(x, y) &:= (x(0), x(T)) \cdot y, \quad (x \in X, \ y \in Y) \end{aligned} \tag{1.5}$$

("$\cdot$" stands for the Euclidean inner product in $\mathbb{R}^2$), and let $x_0^* : X \to \mathbb{R}$ and $y_0^* : Y \to \mathbb{R}$ be the continuous linear functionals defined by

$$x_0^*(x) := \int_\Omega fx, \quad (x \in X),$$

$$y_0^*(y) := (v_0, v_T) \cdot y, \quad (y \in Y). \tag{1.6}$$

Since

$$K := \{x \in X : y \in Y \Longrightarrow b(x, y) = 0\} = H_0^1(\Omega), \tag{1.7}$$

then the weak formulation above leads us to consider the following variational equation with constraints: find $x_0 \in X$ such that

$$x \in K \Longrightarrow a(x_0, x) = x_0^*(x),$$

$$y \in Y \Longrightarrow b(x_0, y) = y_0^*(y). \tag{1.8}$$

With regard to this problem, a more abstract approach has been adopted: let X and Y be the Hilbert spaces, let $x_0^* : X \to \mathbb{R}$ and $y_0^* : Y \to \mathbb{R}$ be continuous linear functionals, let $a : X \times X \to \mathbb{R}$ and $b : X \times Y \to \mathbb{R}$ be continuous bilinear forms, and let $K := \{x \in X : y \in Y \Longrightarrow b(x, y) = 0\}$. Under these assumptions,

$$\text{find } x_0 \in X \text{ such that } \begin{cases} x \in K \Longrightarrow a(x_0, x) = x_0^*(x) \\ y \in Y \Longrightarrow b(x_0, y) = y_0^*(y). \end{cases} \tag{1.9}$$

The known classical results are nothing more than sufficient conditions guaranteeing that such a variational equation with constraints has a solution. However, when the function data do not belong to Hilbert spaces, these results do not apply. For this reason we study a more general type of variational equation with constraints, whose most important particular case relies on this construction: given a reflexive Banach space X, normed spaces Y, Z, and W, continuous bilinear forms $a : X \times Y \to \mathbb{R}$, $b : Y \times Z \to \mathbb{R}$, and $c : X \times W \to \mathbb{R}$, and continuous linear functionals $y_0^* : Y \to \mathbb{R}$ and $w_0^* : W \to \mathbb{R}$, denoting

$$K_b := \{y \in Y : b(y, \cdot) = 0\}, \tag{1.10}$$

find, if possible, $x_0 \in X$ such that

$$y \in K_b \Longrightarrow a(x_0, y) = y_0^*(y),$$

$$w \in W \Longrightarrow c(x_0, w) = w_0^*(w). \tag{1.11}$$

In Theorem 2.2 of Section 2 we characterize when this constrained variational equation, in effect a more general variational inequality with constraints in the framework of locally

convex spaces, admits a solution. The particular normed case is discussed in Sections 3 and 4. In the first one, the version for normed spaces of Theorem 2.2 leads to an easier statement and, under hypotheses of uniqueness, it is possible to obtain a stability estimation for the solution. The reflexive case, which immediately follows and extends the classical known results, is illustrated with a non-Hilbertian data example. Section 4 completes the normed space setting, and as an application of Theorem 2.2 we also obtain analogous results for dual normed spaces. Finally, Section 5 is concerned with solving weakly the aforementioned kind of impulsive differential equation, generalizing the linear results in [1] to the reflexive context.

From now on, we assume that all the spaces are real, although our results are equally valid and easily adapted to the complex case.

2. Variational Inequalities with Constraints in Lcs

We first discuss a characterization of the existence of solutions to some constrained variational inequalities in the general setting of locally convex spaces. In order to state our main result, Theorem 2.2, the Hahn-Banach theorem, is required. Although there is a long history of using the Hahn-Banach theorem, recently a fine reformulation of this fundamental result has been developed in [5, 6] (see Proposition 2.1 below). It is known as the *Hahn-Banach-Lagrange theorem* and has encountered numerous applications in different branches of the mathematical analysis (see [5–8]). Let us recall that if X is a real vector space, a function $S : X \to \mathbb{R}$ is *sublinear* provided that it is subadditive and positively homogeneous. For such an S, if C is a nonempty convex subset of a vector space, then $j : C \to X$ is said to be *S-convex* if

$$x, y \in C, \quad 0 < t < 1 \implies S\big(j(tx + (1-t)y) - tj(x) - (1-t)j(y)\big) \leq 0. \tag{2.1}$$

Finally, a convex function $k : C \to \mathbb{R} \cup \{\infty\}$ is *proper* when there exists $x \in C$ with $k(x) < \infty$.

Proposition 2.1 ([6, Theorem 2.9]). *Let X be a nontrivial vector space, and let $S : X \to \mathbb{R}$ be a sublinear function. Assume in addition that C is a nonempty convex subset of a vector space, $k : C \to \mathbb{R} \cup \{\infty\}$ is a proper convex function, and $j : C \to X$ is S-convex. Then there exists a linear functional $L : X \to \mathbb{R}$ such that*

$$L \leq S,$$
$$\inf_C (L \circ j + k) = \inf_C (S \circ j + k). \tag{2.2}$$

Now we state the main result of this section, along the lines of [5–7]. To this end, some notations are required. For two real vector spaces X and Y, a bilinear form $a : X \times Y \to \mathbb{R}$, and $x \in X$, $y \in Y$, and $a(\cdot, y)$ stands for the linear functional on X

$$x \in X \longmapsto a(x, y) \in \mathbb{R} \tag{2.3}$$

and $a(x, \cdot)$ for the analogous linear functional on Y. In addition, given a real Hausdorff locally convex space X, we will write X^* to denote its dual space (continuous linear functionals on X).

The characterization is stated as follows.

Theorem 2.2. *Let X be a real Hausdorff locally convex space such that its dual space X^* is also a real Hausdorff locally convex space. Suppose that Y and W are real vector spaces, C and D are convex subsets of Y and W, respectively, with $(0,0) \in C \times D$, $\Phi : C \to \mathbb{R} \cup \{-\infty\}$ and $\Psi : D \to \mathbb{R} \cup \{-\infty\}$ are concave functions such that $\Phi(0) \geq 0$ and $\Psi(0) \geq 0$, and $a : X \times Y \to \mathbb{R}$ and $c : X \times W \to \mathbb{R}$ are bilinear forms satisfying that*

$$(y, w) \in C \times D \Longrightarrow a(\cdot, y), \ c(\cdot, w) \in X^*. \tag{2.4}$$

Then,

$$\text{there exists } x_0^{**} \in X^{**} \text{ such that} \begin{cases} y \in C \Longrightarrow \Phi(y) \leq x_0^{**}(a(\cdot, y)) \\ w \in D \Longrightarrow \Psi(w) \leq x_0^{**}(c(\cdot, w)) \end{cases} \tag{2.5}$$

if, and only if, there exists a continuous seminorm $p : X^* \to \mathbb{R}$ so that

$$(y, w) \in C \times D \implies \Phi(y) + \Psi(w) \leq p(a(\cdot, y) + c(\cdot, w)). \tag{2.6}$$

Furthermore, if one of these equivalent conditions holds, then it is possible to choose x_0^{**} and p with $x_0^{**} \leq p$.

Proof. We can assume without loss of generality that X is nontrivial, which is exactly the same as X^* being nontrivial, thanks to the Hahn-Banach theorem.

Let us first assume that (2.6) is true for some continuous seminorm $p : X^* \to \mathbb{R}$. The Hahn-Banach-Lagrange theorem (Proposition 2.1) applies, with the sublinear function $S = p$, the S-convex mapping $j : C \times D \to X^*$ defined as

$$j(y, w) := a(\cdot, y) + c(\cdot, w), \quad ((y, w) \in C \times D), \tag{2.7}$$

and the proper convex function $k : C \times D \to \mathbb{R} \cup \{\infty\}$ given by

$$k(y, w) := -\Phi(y) - \Psi(w), \quad ((y, w) \in C \times D), \tag{2.8}$$

obtaining thus that there exists a linear functional $L : X^* \to \mathbb{R}$ such that, on the one hand,

$$L \leq p, \tag{2.9}$$

and therefore $L = x_0^{**} \in X^{**}$ for some $x_0^{**} \in X^{**}$, and on the other hand, satisfyies

$$\inf_{(y,w) \in C \times D} \left(x_0^{**}(a(\cdot, y) + c(\cdot, w)) - \Phi(y) - \Psi(w) \right)$$
$$= \inf_{(y,w) \in C \times D} \left(p(a(\cdot, y) + c(\cdot, w)) - \Phi(y) - \Psi(w) \right). \tag{2.10}$$

But we are assuming that

$$\inf_{(y,w)\in C\times D}\left(p(a(\cdot,y)+c(\cdot,w))-\Phi(y)-\Psi(w)\right)\geq 0. \qquad (2.11)$$

Hence

$$\inf_{(y,w)\in C\times D}\left(x_0^{**}(a(\cdot,y)+c(\cdot,w))-\Phi(y)-\Psi(w)\right)\geq 0, \qquad (2.12)$$

that is,

$$(y,w)\in C\times D\Longrightarrow\Phi(z)+\Psi(w)\leq x_0^{**}(a(\cdot,y)+c(\cdot,w)). \qquad (2.13)$$

Since $(0,0)\in C\times D$ and $\Phi(0)\geq 0$ and $\Psi(0)\geq 0$, taking in this last inequality $(y,0)\in C\times D$ yields

$$y\in C\Longrightarrow\Phi(y)\leq x_0^{**}(a(\cdot,y)), \qquad (2.14)$$

while for $(0,w)\in C\times D$ it implies

$$w\in D\Longrightarrow\Psi(w)\leq x_0^{**}(c(\cdot,w)), \qquad (2.15)$$

and we have proven (2.5) as we wish.

And conversely, if $x_0^{**}\in X^{**}$ satisfies (2.5), then

$$(y,w)\in C\times D\Longrightarrow\Phi(y)+\Psi(w)\leq x_0^{**}(a(\cdot,y)+c(\cdot,w)), \qquad (2.16)$$

so for the continuous seminorm p on X^*

$$p(x^*):=\left|x_0^{**}(x^*)\right|,\quad(x^*\in X^*), \qquad (2.17)$$

we have that

$$\Phi(y)+\Psi(w)\leq p(a(\cdot,y)+c(\cdot,w)). \qquad (2.18)$$

In any case we have stated the inequality $x_0^{**}\leq p$.   □

As we can see, all the topological assumptions fall on X. Thus, C and D are nothing more than convex sets, and there is no topological assumption on them, not even that they are closed. In particular, no continuity is supposed for Φ or Ψ.

Let us also note that the condition $x_0^{**}\leq p$, which seems to be irrelevant, will entail in the normed case a control of the norm $\|x_0^{**}\|$.

Let us also emphasize that Theorem 2.2 captures the essence of the Hahn-Banach theorem from a variational standpoint. Theorem 2.2 extends the Lax-Milgram-type result

given in [8, Theorem 1.2]. But the latter in turn is an equivalent reformulation of the Hahn-Banach theorem (Proposition 2.1), as shown in [8, Theorem 3.1]. Because the Hahn-Banach theorem and the Hahn-Banach-Lagrange theorem are equivalent results, Theorem 2.2 is nothing more than an equivalent version of the Hahn-Banach theorem.

3. Constrained Variational Equations in Reflexive Banach Spaces

Both in this section and the next one we turn to the study of constrained variational equations, only in the case of X being a normed space and considering a kind of locally convex topology that, in some sense, satisfies a compactness property, which actually ensures that the solutions $x_0^{**} \in X^{**}$ of the constrained variational inequality (2.5) belong to X. To be more precise, in this section we fix the norm topology in X and deduce that $x_0^{**} \in X$ in an obvious way when X is reflexive; that is, its closed unit ball B_X is weakly compact ([2, Theorem 3.17]), which suffices for the applications in Section 5. In the next section we assume that X is a dual Banach space, that is, $X = E^*$ for some normed space E, endowed with its weak-star topology $w(E^*, E)$, in which, by the way, its closed unit ball is closed ([2, Theorem 3.16]). Continuing with the contents of this section, we provide an estimation of the norm of the solution only in terms of the data. We also generalize to the reflexive framework the classical Hilbertian characterization [3, 4] of those constrained problems (1.9) that admit a solution, indeed obtaining a proper extension of a result in the reflexive context that follows from [9, Theorem 2.1], developed for mixed variational formulations of elliptic boundary value problems.

Thus it is that we consider a real normed space X, equipped with its norm topology, and the topology in X^* is taken to be that associated with the canonical norm of X^*. In this way we obtain the said estimation of the norm of a solution to the variational inequality with constraints (2.5). Moreover, we replace the existence of the continuous seminorm p in Theorem 2.2 with that of a nonnegative constant.

Corollary 3.1. *Suppose that X is a real normed space, Y and W are real vector spaces, C and D are convex subsets of Y and W, respectively, $(0,0) \in C \times D$, $\Phi : C \to \mathbb{R} \cup \{-\infty\}$ and $\Psi : D \to \mathbb{R} \cup \{-\infty\}$ are concave functions such that $\Phi(0) \geq 0$ and $\Psi(0) \geq 0$. If $a : X \times Y \to \mathbb{R}$ and $c : X \times W \to \mathbb{R}$ are bilinear forms so that*

$$(y, w) \in C \times D \Longrightarrow a(\cdot, y), \ c(\cdot, w) \in X^*, \tag{3.1}$$

then the constrained variational problem

$$\text{find } x_0^{**} \in X^{**} \text{ such that } \begin{cases} y \in C \Longrightarrow \ \Phi(z) \leq x_0^{**}(a(\cdot, y)) \\ w \in D \Longrightarrow \ \Psi(w) \leq x_0^{**}(c(\cdot, w)) \end{cases} \tag{3.2}$$

is solvable if, and only if, for some $\rho \geq 0$

$$(y, w) \in C \times D \ \Longrightarrow \Phi(y) + \Psi(w) \leq \rho \| a(\cdot, y) + c(\cdot, w) \|. \tag{3.3}$$

Moreover, when these statements hold and there exists $(y, w) \in C \times D$ *satisfying* $a(\cdot, y) + c(\cdot, w) \neq 0$, *then*

$$\min\left\{ \|x_0^{**}\| : x_0^{**} \in X^{**}, \ \begin{cases} y \in C \Longrightarrow \Phi(y) \le x_0^{**}(a(\cdot, y)) \\ w \in D \Longrightarrow \Psi(w) \le x_0^{**}(c(\cdot, w)) \end{cases} \right\}$$
$$= \left(\sup_{\substack{(y,w)\in C\times D, \\ a(\cdot,y)+c(\cdot,w)\neq 0}} \frac{\Phi(y) + \Psi(w)}{\|a(\cdot, y) + c(\cdot, w)\|} \right)_+ . \tag{3.4}$$

Proof. The equivalence between (3.2) and (3.3) clearly follows from **Theorem 2.2** for the real Hausdorff locally convex space X endowed with its norm topology and considering in its dual space X^* the dual norm topology, and from the fact that if $p : X^* \to \mathbb{R}$ is a continuous seminorm, then it is bounded above by a suitable positive multiple of the norm.

Let us finally suppose that (3.2) or equivalently (3.3) holds. In order to prove (3.4) provided that there exists $(y, w) \in C \times D$ such that $a(\cdot, y) + c(\cdot, w) \neq 0$, let us start by fixing an arbitrary element x_0^{**} in X^{**} for which (3.2) is valid. Then,

$$(y, w) \in C \times D \Longrightarrow \Phi(y) + \Psi(w) \le \|x_0^{**}\| \|a(\cdot, y) + c(\cdot, w)\|. \tag{3.5}$$

Hence, if

$$\alpha := \left(\sup_{\substack{(y,w)\in C\times D, \\ a(\cdot,y)+c(\cdot,w)\neq 0}} \frac{\Phi(y) + \Psi(w)}{\|a(\cdot, y) + c(\cdot, w)\|} \right)_+ \tag{3.6}$$

then

$$\|x_0^{**}\| \ge \alpha, \tag{3.7}$$

and so

$$\alpha < \infty. \tag{3.8}$$

But in addition we have that

$$(y, w) \in C \times D \Longrightarrow \Phi(y) + \Psi(w) \le \alpha \|a(\cdot, y) + c(\cdot, w)\|, \tag{3.9}$$

since for $(y, w) \in C \times D$ with $a(\cdot, y) + c(\cdot, w) \neq 0$ it clearly holds, and when $a(\cdot, y) + c(\cdot, w) = 0$, it suffices to make use of the fact that (3.2) is valid to arrive at the same conclusion. In summary, the continuous seminorm $p : X^* \to \mathbb{R}$ given for each $x^* \in X^*$ by

$$p(x^*) := \alpha \|x^*\| \tag{3.10}$$

satisfies (2.6); therefore, Theorem 2.2 implies the existence of $x_0^{**} \in X^{**}$ for which (3.2) is valid and $x_0^{**} \leq p$, that is,

$$\left\| x_0^{**} \right\| \leq \alpha, \tag{3.11}$$

which together with (3.7) finally yields (3.4). □

Of course, when in Corollary 3.1 we additionally assume that X is reflexive, then the existence of $\rho \geq 0$ such that

$$(y, w) \in C \times D \Longrightarrow \Phi(y) + \Psi(w) \leq \rho \left\| a(\cdot, y) + c(\cdot, w) \right\| \tag{3.12}$$

is equivalent to the existence of a solution in X to the constrained variational inequality, that is,

$$\text{there exists } x_0 \in X \text{ such that } \begin{cases} y \in C \Longrightarrow \Phi(y) \leq a(x_0, y), \\ w \in D \Longrightarrow \Psi(w) \leq c(x_0, w). \end{cases} \tag{3.13}$$

Next we focus our effort on proving that Corollary 3.1, with the additional hypothesis of the reflexivity of X, provides a result, Theorem 3.8, generalizing the classical Hilbertian theory.

Lemma 3.2. *Assume that X is a real normed space, Y and W are real vector spaces, and $a : X \times Y \to \mathbb{R}$, $b : Y \times Z \to \mathbb{R}$, and $c : X \times W \to \mathbb{R}$ are bilinear forms. If one writes*

$$K_b := \{ y \in Y : b(y, \cdot) = 0 \}, \qquad K_c := \{ x \in X : c(x, \cdot) = 0 \} \tag{3.14}$$

and supposes that

$$y \in K_b, \quad w \in W \Longrightarrow a(\cdot, z), \ c(\cdot, w) \in X^*, \tag{3.15}$$

then

$$y \in K_b, \quad w \in W \Longrightarrow \left\| a(\cdot, y)_{|K_c} \right\| \leq \left\| a(\cdot, y) + c(\cdot, w) \right\|. \tag{3.16}$$

Proof. Let $y \in K_b$ and $\varepsilon > 0$. Since $a(\cdot, y)_{|K_c} \in X^*$, choose $x_0 \in K_c$ so that

$$\| x_0 \| = 1,$$
$$a(x_0, y) > \left\| a(\cdot, y)_{|K_c} \right\| - \varepsilon. \tag{3.17}$$

Then, given $w \in W$, we have that

$$
\begin{aligned}
\left\| a(\cdot, y)_{|K_c} \right\| &< a(x_0, y) + \varepsilon \\
&= a(x_0, y) + c(x_0, w) + \varepsilon \quad (x_0 \in K_c) \\
&\leq \|x_0\| \left\| a(\cdot, y) + c(\cdot, w) \right\| + \varepsilon \\
&= \left\| a(\cdot, y) + c(\cdot, w) \right\| + \varepsilon,
\end{aligned}
\tag{3.18}
$$

and the announced inequality follows from the arbitrariness of $\varepsilon > 0$.   □

In the next result we establish the first characterization of the solvability of a variational equation with constraints.

Theorem 3.3. *Let X be a real reflexive Banach space, let Y, Z, and W be real normed spaces, and let $a : X \times Y \to \mathbb{R}$, $b : Y \times Z \to \mathbb{R}$, and $c : X \times W \to \mathbb{R}$ be bilinear forms with a and c being continuous. Let $y_0^* \in Y^*$ and $w_0^* \in W^*$, and write*

$$
K_b := \{ y \in Y : b(y, \cdot) = 0 \}, \qquad K_c := \{ x \in X : c(x, \cdot) = 0 \},
$$
$$
R_{w_0^*} := \{ x \in X : c(x, \cdot) = w_0^* \}.
\tag{3.19}
$$

Then, the corresponding constrained variational equation admits a solution; that is,

$$
\textit{there exists } x_0 \in X \textit{ such that } \begin{cases} y \in K_b \Longrightarrow a(x_0, y) = y_0^*(y), \\ w \in W \Longrightarrow c(x_0, w) = w_0^*(w), \end{cases}
\tag{3.20}
$$

if, and only if,

$$
R_{w_0^*} \neq \emptyset,
\tag{3.21}
$$

$$
\forall x \in R_{w_0^*} \textit{ there exists } \delta \geq 0 \textit{ such that}
$$
$$
y \in K_b \Longrightarrow y_0^*(y) - a(x, y) \leq \delta \left\| a(\cdot, y)_{|K_c} \right\|.
\tag{3.22}
$$

In addition, if one of these equivalent conditions is valid and there exists $y \in K_b$ with $a(\cdot, y)_{|K_c} \neq 0$, then one can take $x_0 \in X$ in (3.20) with

$$
\|x_0\| = \min_{x \in R_{w_0^*}} \left(\sup_{\substack{y \in K_b, \\ a(\cdot, y)_{|K_c} \neq 0}} \frac{y_0^*(y) - a(x, y)}{\left\| a(\cdot, y)_{|K_c} \right\|} + \|x\| \right).
\tag{3.23}
$$

Proof. Let us begin by stating (3.20)⟹(3.22). Let x_0 be a solution to the variational equation with constraints (3.20). Then, in particular,

$$R_{w_0^*} = x_0 + K_c. \tag{3.24}$$

Thus, given $x \in R_{w_0^*}$ there exists $x_1 \in K_c$ such that $x = x_0 + x_1$, so for all $y \in K_b$

$$
\begin{aligned}
y_0^*(y) - a(x,y) &= -a(x_1,y) \\
&\leq \|x_1\| \big\| a(\cdot,y)_{|K_c} \big\|,
\end{aligned}
\tag{3.25}
$$

and we have shown (3.22).

To conclude, we prove the converse (3.22)⟹(3.20). Let $y \in K_b$ and $w \in W$, and let $x \in R_{w_0^*}$, whose existence guarantees (3.22). Then

$$
\begin{aligned}
y_0^*(y) + w_0^*(w) &= y_0^*(y) - a(x,y) + a(x,y) + c(x,w) \quad \left(x \in R_{w_0^*} \right) \\
&\leq y_0^*(y) - a(x,y) + \|x\| \|a(\cdot,y) + c(\cdot,w)\| \\
&\leq \delta \big\| a(\cdot,y)_{|K_c} \big\| + \|x\| \|a(\cdot,y) + c(\cdot,w)\| \quad \text{(by (3.22))} \\
&\leq (\delta + \|x\|) \|a(\cdot,y) + c(\cdot,w)\| \quad \text{(by Lemma 3.2).}
\end{aligned}
\tag{3.26}
$$

Therefore, there exists $\rho := \delta + \|x\| \geq 0$ such that

$$y \in K_b, \ w \in W \implies y_0^*(y) + w_0^*(w) \leq \rho \|a(\cdot,y) + c(\cdot,w)\|, \tag{3.27}$$

so Corollary 3.1 (in combination with the reflexivity of X) for $C := K_b$, $D := W$, $\Phi := y_0^*$, and $\Psi := w_0^*$ ensures, on the one hand, that the variational system (3.20) has a solution, hence stating the equivalence between (3.20) and (3.22), and, on the other hand, that if one of these conditions holds, then

$$
\min\left\{ \|x_0\| : x_0 \in R_{w_0^*}, y_{0|K_b}^* = a(x_0,\cdot)_{|K_b} \right\} = \left(\sup_{\substack{(y,w) \in K_b \times W, \\ a(\cdot,y)+c(\cdot,w) \neq 0}} \frac{y_0^*(y) + w_0^*(w)}{\|a(\cdot,y) + c(\cdot,w)\|} \right)_+
$$
$$
= \sup_{\substack{(y,w) \in K_b \times W, \\ a(\cdot,y)+c(\cdot,w) \neq 0}} \frac{y_0^*(y) + w_0^*(w)}{\|a(\cdot,y) + c(\cdot,w)\|} \, . \tag{3.28}
$$

So, for establishing (3.23) and concluding the proof, it suffices to state the equality

$$
\sup_{\substack{(y,w) \in K_b \times W, \\ a(\cdot,y)+c(\cdot,w) \neq 0}} \frac{y_0^*(y) + w_0^*(w)}{\|a(\cdot,y) + c(\cdot,w)\|} = \min_{x \in R_{w_0^*}} \left(\sup_{\substack{y \in K_b \\ a(\cdot,y)_{|K_c} \neq 0}} \frac{y_0^*(y) - a(x,y)}{\big\| a(\cdot,y)_{|K_c} \big\|} + \|x\| \right). \tag{3.29}
$$

Suppose, on the one hand, that $x \in R_{w_0^*}$, $y \in K_b$, and $w \in W$, with $a(\cdot, y)_{|K_c} \neq 0$, which in view of Lemma 3.2 implies that $a(\cdot, y) + c(\cdot, w) \neq 0$. Then

$$
\begin{aligned}
\frac{y_0^*(y) + w_0^*(w)}{\|a(\cdot, y) + c(\cdot, w)\|} &= \frac{y_0^*(y) - a(x, y)}{\|a(\cdot, y) + c(\cdot, w)\|} + \frac{a(x, y) + c(x, w)}{\|a(\cdot, y) + c(\cdot, w)\|} \quad \left(x \in R_{w_0^*} \right) \\
&\leq \frac{|y_0^*(y) - a(x, y)|}{\|a(\cdot, y) + c(\cdot, w)\|} + \|x\| \\
&\leq \frac{|y_0^*(y) - a(x, y)|}{\left\|a(\cdot, y)_{|K_c}\right\|} + \|x\| \quad \text{(by Lemma 3.2)}.
\end{aligned}
\tag{3.30}
$$

Thus

$$
\begin{aligned}
\sup_{\substack{(y,w)\in K_b \times W \\ a(\cdot,y)_{|K_c} \neq 0}} \frac{y_0^*(y) + w_0^*(w)}{\|a(\cdot, y) + c(\cdot, w)\|} &\leq \sup_{\substack{y\in K_b \\ a(\cdot,y)_{|K_c} \neq 0}} \frac{|y_0^*(y) + w_0^*(w)|}{\left\|a(\cdot, y)_{|K_c}\right\|} + \|x\| \\
&= \sup_{\substack{y\in K_b \\ a(\cdot,y)_{|K_c} \neq 0}} \frac{y_0^*(y) + w_0^*(w)}{\left\|a(\cdot, y)_{|K_c}\right\|} + \|x\|,
\end{aligned}
\tag{3.31}
$$

and therefore, as $x \in R_{w_0^*}$ is arbitrary,

$$
\sup_{\substack{(y,w)\in K_b \times W, \\ a(\cdot,y)_{|K_c} \neq 0}} \frac{y_0^*(y) + w_0^*(w)}{\|a(\cdot, y) + c(\cdot, w)\|} \leq \inf_{x\in R_{w_0^*}} \left(\sup_{\substack{y\in K_b \\ a(\cdot,y)_{|K_c} \neq 0}} \frac{y_0^*(y) - a(x, y)}{\left\|a(\cdot, y)_{|K_c}\right\|} + \|x\| \right).
\tag{3.32}
$$

And, on the other hand, if $x \in R_{w_0^*}$, $y \in K_b$, and $w \in W$ satisfy $a(\cdot, y) + c(\cdot, w) \neq 0$ but $a(\cdot, y)_{|K_c} = 0$,

$$
\begin{aligned}
\frac{y_0^*(y) + w_0^*(w)}{\|a(\cdot, y) + c(\cdot, w)\|} &= \frac{y_0^*(y) - a(x, y)}{\|a(\cdot, y) + c(\cdot, w)\|} + \frac{a(x, y) + c(x, w)}{\|a(\cdot, y) + c(\cdot, w)\|} \quad \left(x \in R_{w_0^*} \right) \\
&= \frac{a(x, y) + c(x, w)}{\|a(\cdot, y) + c(\cdot, w)\|} \\
&\leq \|x\|.
\end{aligned}
\tag{3.33}
$$

Hence

$$
\sup_{\substack{(y,w)\in K_b \times W, \\ a(\cdot,y)+c(\cdot,w) \neq 0, a(\cdot,y)_{|K_c} = 0}} \frac{y_0^*(y) + w_0^*(w)}{\|a(\cdot, y) + c(\cdot, w)\|} \leq \|x\|,
\tag{3.34}
$$

and thus

$$\sup_{\substack{(y,w)\in K_b\times W,\\ a(\cdot,y)+c(\cdot,w)\neq 0, a(\cdot,y)_{|K_c}=0}} \frac{y_0^*(y)+w_0^*(w)}{\|a(\cdot,y)+c(\cdot,w)\|} \leq \inf_{x\in R_{w_0^*}}\|x\|$$

$$\leq \inf_{x\in R_{w_0^*}}\left(\sup_{\substack{y\in K_b\\ a(\cdot,y)_{|K_c}\neq 0}} \frac{y_0^*(y)-a(x,y)}{\left\|a(\cdot,y)_{|K_c}\right\|}+\|x\|\right). \tag{3.35}$$

Therefore, it follows from (3.32), Lemma 3.2, and (3.35) that

$$\sup_{\substack{(y,w)\in K_b\times W,\\ a(\cdot,y)+c(\cdot,w)\neq 0}} \frac{y_0^*(y)+w_0^*(w)}{\|a(\cdot,y)+c(\cdot,w)\|} \leq \inf_{x\in R_{w_0^*}}\left(\sup_{\substack{y\in K_b\\ a(\cdot,y)_{|K_c}\neq 0}} \frac{y_0^*(y)-a(x,y)}{\left\|a(\cdot,y)_{|K_c}\right\|}+\|x\|\right). \tag{3.36}$$

But thanks to (3.28) we can choose $x_0\in R_{w_0^*}$ with $y_{0|K_b}^* = a(x_0,\cdot)_{|K_b}$ in such a way that

$$\|x_0\| = \sup_{\substack{(y,w)\in K_b\times W,\\ a(\cdot,y)+c(\cdot,w)\neq 0}} \frac{y_0^*(y)+w_0^*(w)}{\|a(\cdot,y)+c(\cdot,w)\|}, \tag{3.37}$$

so finally

$$\|x_0\| = \min_{x\in R_{w_0^*}}\left(\sup_{\substack{y\in K_b\\ a(\cdot,y)_{|K_c}\neq 0}} \frac{y_0^*(y)-a(x,y)}{\left\|a(\cdot,y)_{|K_c}\right\|}+\|x\|\right)$$

$$= \sup_{\substack{(y,w)\in K_b\times W,\\ a(\cdot,y)+c(\cdot,w)\neq 0}} \frac{y_0^*(y)+w_0^*(w)}{\|a(\cdot,y)+c(\cdot,w)\|}, \tag{3.38}$$

and (3.29) is proven. □

Remark 3.4. In Theorem 3.3 we do not need to suppose that a and c are continuous: it suffices to impose

$$(y,w)\in K_b\times W \implies a(\cdot,y),\ c(\cdot,w)\in X^*, \tag{3.39}$$

as seen in its proof. Since for our applications the bilinear forms a and c are continuous, for the sake of simplicity, we have assumed this hypothesis. However, in Theorem 4.2 we impose these less restrictive assumptions, in a more general setting.

Let us note that according to Corollary 3.1 (or [8, Corollary 1.3]) we have, with X being reflexive,

$$R_{w_0^*} \neq \emptyset \tag{3.40}$$

if, and only if, there exists $\mu \geq 0$ such that

$$w \in W \implies w_0^*(w) \leq \mu \|c(\cdot, w)\|. \tag{3.41}$$

In connection with uniqueness we have the following elementary characterization, which establishes the equivalence of such uniqueness with that of the corresponding homogeneous variational equation with constraints.

Lemma 3.5. *Let one make the same assumptions and use the same notations as in Theorem 3.3. If the variational equation with constraints* (3.20) *has a solution, then it is unique if, and only if,*

$$x \in K_c, \qquad a(x, \cdot)_{|K_b} = 0 \implies x = 0. \tag{3.42}$$

Proof. If a satisfies the nondegeneracy condition (3.42), then for any $x \in R_{w_0^*}$ we have $R_{w_0^*} = x + K_c$, so if (3.20) has two solutions x_1 and $\tilde{x}_1$, then for some x_2, $\tilde{x}_2 \in K_c$, $x_1 = x + x_2$ and $\tilde{x}_1 = x + \tilde{x}_2$; hence

$$y \in K_b \implies a(x_2 - \tilde{x}_2, y) = 0. \tag{3.43}$$

By hypothesis it follows that $x_2 = \tilde{x}_2$; that is, $x_1 = \tilde{x}_1$.

And conversely, if there exists $x_0 \in K_c$, $x_0 \neq 0$, such that

$$y \in K_b \implies a(x_0, y) = 0, \tag{3.44}$$

then given a solution x of (3.20), $x + x_0$ is also a solution, which is different than x. $\square$

Hypotheses more restrictive than those of Theorem 3.3 imply uniqueness of the solution and simplify the control of the norm of the solution.

Corollary 3.6. *Let X be a real reflexive Banach space, let Y, Z, and W be real normed spaces, let $w_0^* \in W^*$, and let $a : X \times Y \to \mathbb{R}$, $b : Y \times Z \to \mathbb{R}$, and $c : X \times W \to \mathbb{R}$ be bilinear forms such that a and c are continuous. Let one take*

$$K_b := \{y \in Y : b(y, \cdot) = 0\}, \qquad K_c := \{x \in X : c(x, \cdot) = 0\} \tag{3.45}$$

and suppose that

$$x \in K_c, \quad a(x, \cdot)_{|K_b} = 0 \implies x = 0 \tag{3.46}$$

and that there exist constants α, $\mu > 0$ with

$$y \in K_b \Longrightarrow \alpha\|y\| \leq \left\|a(\cdot, y)_{|K_c}\right\|, \tag{3.47}$$

$$w \in W \Longrightarrow w_0^*(w) \leq \mu\|c(\cdot, w)\|. \tag{3.48}$$

Then, for each $y_0^* \in Y^*$, the corresponding variational equation with constraints admits one and only one solution; that is,

$$\text{there exists a unique } x_0 \in X \text{ such that } \begin{cases} y \in K_b \Longrightarrow a(x_0, y) = y_0^*(y), \\ w \in W \Longrightarrow c(x_0, w) = w_0^*(w). \end{cases} \tag{3.49}$$

Furthermore, if one defines

$$R_{w_0^*} := \left\{x \in X : c(x, \cdot) = w_0^*\right\}, \tag{3.50}$$

then the a priori estimate

$$\begin{aligned} \|x_0\| &\leq \frac{\|y_0^*\|}{\alpha} + \left(1 + \frac{\|a\|}{\alpha}\right) \min_{x \in R_{w_0^*}} \|x\| \\ &\leq \frac{\|y_0^*\|}{\alpha} + \left(1 + \frac{\|a\|}{\alpha}\right)\mu \end{aligned} \tag{3.51}$$

is valid for the norm of the solution x_0.

Proof. Corollary 3.1 (or [8, Corollary 1.3]) and Theorem 3.3, together with conditions (3.47) and (3.48), imply the existence of a solution whose uniqueness follows from Lemma 3.5. Besides, we deduce from Theorem 3.3 that for the solution x_0, the identity

$$\|x_0\| = \min_{x \in R_{w_0^*}} \left(\sup_{\substack{y \in K_b, \\ a(\cdot, y)_{|K_c} \neq 0}} \frac{y_0^*(y) - a(x, y)}{\left\|a(\cdot, y)_{|K_c}\right\|} + \|x\| \right) \tag{3.52}$$

holds. Finally, we have by condition (3.47) that

$$
\min_{x \in R_{w_0^*}} \left(\sup_{\substack{y \in K_b, \\ a(\cdot,y)_{|K_c} \neq 0}} \frac{y_0^*(y) - a(x,y)}{\left\| a(\cdot,y)_{|K_c} \right\|} + \|x\| \right) = \min_{x \in R_{w_0^*}} \left(\sup_{\substack{y \in K_b, \\ y \neq 0}} \frac{y_0^*(y) - a(x,y)}{\left\| a(\cdot,y)_{|K_c} \right\|} + \|x\| \right)
$$

$$
\leq \inf_{x \in R_{w_0^*}} \left(\sup_{\substack{y \in K_b, \\ y \neq 0}} \frac{y_0^*(y) + \|a\|\|x\|\|y\|}{\alpha \|y\|} + \|x\| \right)
$$

$$
= \frac{\|y_0^*\|}{\alpha} + \left(1 + \frac{\|a\|}{\alpha} \right) \inf_{x \in R_{w_0^*}} \|x\|,
$$

$$(3.53)$$

and by Corollary 3.1 (or [8, Corollary 1.3]) and (3.48) that

$$
\inf_{x \in R_{w_0^*}} \|x\| = \min_{x \in R} \|x\|
$$

$$
= \sup_{\substack{w \in W, \\ c(\cdot,w) \neq 0}} \frac{w_0^*(w)}{\|c(\cdot,w)\|}
$$

$$(3.54)$$

$$
\leq \mu,
$$

and thus we have the announced bound.   □

If we assume a condition on c stronger than (3.48), the so-called inf-sup or *Babuška-Brezzi condition* (see [10–14] for some recent developments); that is, there exists $\gamma > 0$ such that

$$
w \in W \implies \gamma \|w\| \leq \|c(\cdot,w)\|,
$$

$$(3.55)$$

then we have the stability estimate

$$
\|x_0\| \leq \frac{\|y_0^*\|}{\alpha} + \left(1 + \frac{\|a\|}{\alpha} \right) \frac{\|w_0^*\|}{\gamma}
$$

$$(3.56)$$

for the solution.

Taking into account that when $X = Y$, $Z = W$, and $b = c$ conditions (3.42) and (3.47) are satisfied if a is coercive on $K_b \times K_b$, we deduce the following immediate consequence, which is well known (see [3, section II.1 Proposition 1.1] and [4, Lemma 4.67]) in the particular case of X and Z being Hilbert spaces.

Corollary 3.7. *Suppose that X and Z are reflexive real Banach spaces, $x_0^* \in X^*$, $z_0^* \in Z^*$, and $a : X \times X \to \mathbb{R}$ and $b : X \times Z \to \mathbb{R}$ are continuous bilinear forms. Suppose in addition that, taking*

$$K_b := \{x \in X : b(x, \cdot) = 0\}, \qquad R_{z_0^*} := \{x \in X : b(x, \cdot) = z_0^*\}, \tag{3.57}$$

there exists $\alpha > 0$ such that

$$x \in K_b \Longrightarrow \alpha \|x\|^2 \leq a(x, x) \tag{3.58}$$

and that $R_{z_0^*} \neq \emptyset$. Then

$$\text{there exists a unique } x_0 \in X \text{ such that } \begin{cases} x \in K_b \Longrightarrow a(x_0, x) = x_0^*(x), \\ z \in Z \Longrightarrow b(x_0, z) = z_0^*(z). \end{cases} \tag{3.59}$$

Besides, the solution x_0 satisfies the a priori estimate:

$$\|x_0\| \leq \frac{\|x_0^*\|}{\alpha} + \left(1 + \frac{\|a\|}{\alpha}\right) \min_{x \in R_{z_0^*}} \|x\|. \tag{3.60}$$

In particular, if there exists $\gamma > 0$ such that

$$z \in Z \Longrightarrow \gamma \|z\| \leq \|b(\cdot, z)\|, \tag{3.61}$$

then for the norm of x_0 the following estimation:

$$\|x_0\| \leq \frac{\|x_0^*\|}{\alpha} + \left(1 + \frac{\|a\|}{\alpha}\right) \frac{\|z_0^*\|}{\gamma} \tag{3.62}$$

holds.

In Theorem 3.3 and Lemma 3.5 we drive a characterization when the variational equation with constraints (3.20) admits a unique solution, for two fixed functionals $y_0^* \in Y^*$ and $w_0^* \in W^*$. The known particular cases in the literature are stated for arbitrary functionals $y_0^* \in Y^*$ and $w_0^* \in W^*$. Now, we derive a characterization along those lines. The particular case $X = Y$, $Z = W$, and $b = c$ was stated in [15, Corollary 2.7].

Theorem 3.8. *Let X be a real reflexive Banach space, let Y, Z, and W be real normed spaces, and let $a : X \times Y \to \mathbb{R}$, $b : Y \times Z \to \mathbb{R}$, and $c : X \times W \to \mathbb{R}$ be bilinear forms with a and c being continuous. Let*

$$K_b := \{y \in Y : b(y, \cdot) = 0\}, \qquad K_c := \{x \in X : c(x, \cdot) = 0\}. \tag{3.63}$$

Then, for all $y_0^* \in Y^*$ and $w_0^* \in W^*$,

$$\text{there exists one and only one } x_0 \in X \text{ such that } \begin{cases} y \in K_b \Longrightarrow a(x_0, y) = y_0^*(y), \\ w \in W \Longrightarrow c(x_0, w) = w_0^*(w) \end{cases} \tag{3.64}$$

if, and only if,

$$x \in K_c, \quad a(x, \cdot)_{|K_b} = 0 \Longrightarrow x = 0 \tag{3.65}$$

and there exist $\alpha,\ \gamma > 0$ so that

$$y \in K_b \Longrightarrow \alpha \|y\| \le \left\| a(\cdot, y)_{|K_c} \right\|, \tag{3.66}$$

$$w \in W \Longrightarrow \gamma \|w\| \le \|c(\cdot, w)\|. \tag{3.67}$$

In addition, if one of these equivalent conditions is satisfied, one has the following stability estimate:

$$\|x_0\| \le \frac{\|y_0^*\|}{\alpha} + \left(1 + \frac{\|a\|}{\alpha}\right) \frac{\|w_0^*\|}{\gamma}. \tag{3.68}$$

Proof. In view of **Corollary 3.6** and **Lemma 3.5** we deduce, provided that (3.65), (3.66), and (3.67) hold, that for all $y_0^* \in Y^*$ and $w_0^* \in W^*$ the constrained variational problem (3.64) admits a unique solution, whose norm satisfies estimation (3.68).

And conversely, suppose that for arbitrary $y_0^* \in Y^*$ and $w_0^* \in W^*$ there exists a unique solution of (3.64). Then obviously we have that (3.65) holds, from **Lemma 3.5**. Moreover, since, in particular, for all $w_0^* \in W^*$ there exists $x_0 \in X$ with $w_0^* = c(x_0, \cdot)$; then, the uniform boundedness theorem and **Corollary 3.1** (or [8, Corollary 1.3]) imply (3.67). In a similar way we can arrive at (3.66): taking $w_0^* = 0 \in W^*$ we have that for all $y_0^* \in Y^*$ there exists $x_0 \in K_c$ such that $a(x_0, \cdot)_{|K_b} = y_{0|K_b}^*$, which according to the Hahn-Banach theorem, the uniform boundedness theorem and **Corollary 3.1** (or [8, Corollary 1.3]) are exactly (3.66). □

Remark 3.9. We emphasize, in view of **Remark 3.4**, that in this proper extension of the Lax-Milgram theorem we just need to assume that

$$(y, w) \in K_b \times W \Longrightarrow a(\cdot, y),\ c(\cdot, w) \in X^*, \tag{3.69}$$

and not necessarily that a and c are continuous. **Theorem 4.3** is stated in these terms, and in a more general framework.

Let us again take up the elliptic boundary value problem considered in Introduction, in this case a more general one with non-Hilbertian data. We make use of our results with that elliptic boundary value problem for which the classical theory in the Hilbert framework does not apply. Thus we show how **Theorem 3.8** (or **Theorem 3.3**) increases the class of elliptic boundary value problems for which it is known that the corresponding constrained

variational equation derived from weakly imposing boundary conditions has a unique solution. But before doing so, we give a technical result, interesting in itself, and recall some common notations. For $T > 0$ and $1 < p < \infty$, $\| \cdot \|_p$ stands for the usual norm in the Lebesgue space $L^p(0, T)$. In addition, the standard norm in the Sobolev space $W^{1,p}(0, T)$ is given by

$$\|x\|_{1,p} := \left(\|x\|_p^p + \|x'\|_p^p \right)^{1/p}, \quad \left(x \in W^{1,p}(0, T) \right), \tag{3.70}$$

and the inherited norm on the subspace $W_0^{1,p}(0, T)$ is equivalent to the norm $| \cdot |_{1,p}$ defined as

$$|x|_{1,p} := \|x'\|_p, \quad \left(x \in W_0^{1,p}(0, T) \right), \tag{3.71}$$

thanks to the well-known Poincaré inequality ([16, Theorem 6.30]), which asserts that there exists a constant $c_{p,T} > 0$, depending only on T and p, in such a way that

$$x \in W_0^{1,p}(0, T) \Longrightarrow \|x\|_p \le c_{p,T} |x|_{1,p}. \tag{3.72}$$

In fact, it is easy to check that we can take

$$c_{p,T} = \frac{T}{p^{1/p}}, \tag{3.73}$$

and for $p = 2$ the optimal $c_{2,T}$ is given by

$$c_{2,T} = \frac{T}{\pi} \tag{3.74}$$

(see [17, Section 1.1.3]). As usual, we denote by $W^{-1,p'}(0, T)$ the dual space of $W_0^{1,p}(0, T)$, where p' is the conjugate exponent of p defined through the relation $1/p + 1/p' = 1$. The dual norm of $\| \cdot \|_{1,p}$, when restricted to $W_0^{1,p}(0, T)$, will be denoted by $\| \cdot \|_{-1,p'}$ and that of $| \cdot |_{1,p}$ by $| \cdot |_{-1,p'}$.

The following result is a particular case of [18, Theorem 4], although we include it because we provide an explicit inf-sup constant, which will allow us to improve such a result in the concrete case to be used in Section 5.

Proposition 3.10. *Let $T > 0$ and $1 < p < \infty$, with conjugate exponent p', and consider the continuous bilinear form $a_0 : W_0^{1,p}(0, T) \times W_0^{1,p'}(0, T) \to \mathbb{R}$ given for each $x \in W_0^{1,p}(0, T)$ and $y \in W_0^{1,p'}(0, T)$ by*

$$a_0(x, y) := \int_0^T x' y'. \tag{3.75}$$

Then a_0 satisfies the inf-sup *condition. More specifically,*

$$y \in W_0^{1,p'}(0,T) \implies \frac{1}{1+T^{-1/p'}}|y|_{1,p'} \le |a_0(\cdot,y)|_{-1,p'}. \tag{3.76}$$

Proof. It is sufficient to deal with $y \in C_0^\infty(0,T)$. The description of the dual space $W^{-1,p'}(0,T)$ of $W_0^{1,p}(0,1)$ (see for instance [2, Proposition 8.14]) guarantees that for some $y_0 \in L^{p'}(0,T)$,

$$x \in W_0^{1,p}(0,T) \implies a_0(x,y) = \int_0^T x' y_0 \tag{3.77}$$

$$|a_0(\cdot,y)|_{-1,p'} = \|y_0\|_{p'}.$$

Then

$$x \in W_0^{1,p}(0,T) \implies \int_0^T x'(y'-y_0) = 0, \tag{3.78}$$

and so

$$x \in C_0^\infty(0,T) \implies \int_0^T x'(y'-y_0) = 0. \tag{3.79}$$

But taking into account that

$$\{x' : x \in C_0^\infty(0,T)\} = \left\{ z \in C_0^\infty(0,T) : \int_0^T z = 0 \right\}, \tag{3.80}$$

if $x_0 \in C_0^\infty(0,T)$ with $\int_0^T x_0 = 1$, then

$$\{x' : x \in C_0^\infty(0,T)\} = \left\{ w - \left(\int_0^T w \right) x_0 : w \in C_0^\infty(0,T) \right\}, \tag{3.81}$$

so (3.79) is equivalent to

$$w \in C_0^\infty(0,T) \implies \int_0^T \left(w - \left(\int_0^T x \right) x_0 \right) (y'-y_0) = 0, \tag{3.82}$$

or in other words,

$$w \in C_0^\infty(0,T) \implies \int_0^T w(y'-y_0) = \int_0^T w \int_0^T x_0(y'-y_0), \tag{3.83}$$

that is,

$$y' - y_0 = \int_0^T x_0 (y' - y_0). \tag{3.84}$$

Therefore, we have that for some $\lambda \in \mathbb{R}$

$$y' - y_0 = \lambda. \tag{3.85}$$

Hence, integrating and noting that $y \in C_0^\infty (0, T)$,

$$\lambda = -\frac{1}{T} \int_0^T y_0, \tag{3.86}$$

and thus, as a consequence of the triangular and the Holdër inequalities,

$$\begin{aligned}
|y|_{1,p'} &\leq \|y_0\|_{p'} + \frac{|\lambda|}{T} \\
&\leq \left(1 + T^{-1/p'} \right) \|y_0\|_{p'},
\end{aligned} \tag{3.87}$$

so

$$\begin{aligned}
\frac{1}{1 + T^{-1/p'}} |y|_{1,p'} &\leq \|y_0\|_{p'} \\
&= |a_0 (\cdot, y)|_{-1,p'}. \qquad \Box
\end{aligned} \tag{3.88}$$

The corresponding bilinear form $a : W_0^{1,p}(\Omega) \times W_0^{1,p'}(\Omega) \to \mathbb{R}$ defined for each (x, y) in $W_0^{1,p}(\Omega) \times W_0^{1,p'}(\Omega)$ as

$$a(x, y) := \int_\Omega \nabla x \cdot \nabla y, \tag{3.89}$$

when Ω is a convex bounded plane polygon domain also satisfies the inf-sup condition (see [19, Theorem 2.1]). However we are just interested in the 1D case, since it is the one to be used in the applications of Section 5.

Now we are in a position to return to the mentioned example.

Example 3.11. Assume that $T > 0$, $v_0, v_T \in \mathbb{R}$, $1 < p < \infty$, $f \in L^p(\Omega)$, and consider the elliptic boundary value problem

$$\begin{aligned}
-x'' &= f \quad \text{in } (0, T), \\
x(0) &= v_0, \qquad x(T) = v_T,
\end{aligned} \tag{3.90}$$

which does not admit the classical treatment ([3, section II.1 Proposition 1.1], [4, Lemma 4.67]), since the involved spaces are not Hilbert, except for $p = 2$. However, Theorems 3.3 and 3.8 apply. Indeed, if we multiply equation $-x'' = f$ in $(0, T)$ by a test function $y \in W_0^{1,p'}(0, T)$ $(1/p + 1/p' = 1)$, and integrate by parts, then we obtain the weak formulation of problem (3.90), that in the particular case $p = 2$ coincides with the classical one:

$$\text{find } x_0 \in W^{1,p}(0, T) \text{ such that } \begin{cases} y \in W_0^{1,p'}(0, T) \Longrightarrow \int_0^T x_0' y' = \int_0^T f y \\ x_0(0) = v_0, \qquad x_0(T) = v_T. \end{cases} \tag{3.91}$$

Let us consider the real reflexive Banach space

$$X := W^{1,p}(0, T) \tag{3.92}$$

and the normed spaces

$$Y := W^{1,p'}(0, T), \qquad Z := W := \mathbb{R}^2, \tag{3.93}$$

with $\mathbb{R}^2$ endowed with its $\| \cdot \|_1$ norm, the continuous linear functionals $y_0^* \in Y^*$ and $w_0^* \in W^*$ defined for each $y \in Y$ and $w \in W$, respectively, as

$$y_0^*(y) := \int_0^T f y, \tag{3.94}$$

$$w_0^*(w_1, w_2) := (v_0, v_T) \cdot w, \tag{3.95}$$

and the continuous bilinear forms $a : X \times Y \to \mathbb{R}$, $b : Y \times Z \to \mathbb{R}$, and $c : X \times W \to \mathbb{R}$ given by

$$a(x, y) := \int_0^T x' y', \quad (x \in X, \ y \in Y),$$

$$b(y, (z_1, z_2)) := (y(0), y(T)) \cdot z, \quad (y \in Y, \ z \in Z), \tag{3.96}$$

$$c(x, (w_1, w_2)) := (x(0), x(T)) \cdot w, \quad (x \in X, \ w \in W).$$

Now $K_b = W_0^{1,p'}(0, T)$, so the variational formulation above is nothing more than the variational equation with the following constraints:

$$\text{find } x_0 \in X \text{ such that } \begin{cases} y \in K_b \Longrightarrow y_0^*(y) = a(x_0, y), \\ w \in W \Longrightarrow w_0^*(w) = c(x_0, w). \end{cases} \tag{3.97}$$

In order to prove that this problem has a unique solution, let us check (3.65), (3.66), and (3.67). The first of these conditions follows from the duality $1/p + 1/p' = 1$ and Proposition 3.10, which imply

$$x \in W_0^{1,p}(0,T) \implies \frac{1}{1 + T^{-1/p}} |x|_{1,p} \le |a_0(x, \cdot)|_{-1,p'}, \tag{3.98}$$

which, in view of the fact that $K_c = W_0^{1,p}(0,T)$, is equivalent to

$$x \in K_c \implies \frac{1}{1 + T^{-1/p}} |x|_{1,p} \le \left| a(x, \cdot)_{|K_b} \right|_{-1,p'}, \tag{3.99}$$

and the equivalence of the norm $|\cdot|_{1,p}$ and the usual one in $W_0^{1,p}(0,T)$ yields (3.65).

To prove that condition (3.66) is satisfied, we fix $y \in K_b = W_0^{1,p'}(0,T)$ and apply Proposition 3.10 using equivalence of the norm $|\cdot|_{1,p'}$ and the usual one in $W_0^{1,p'}(0,T)$.

In order to conclude, let us deduce the inf-sup condition (3.67), more specifically, that there exists $\gamma_{p,T} > 0$, depending only on p and T, such that

$$w \in \mathbb{R}^2 \implies \gamma_{p,T} \|w\|_1 \le \|c(\cdot, w)\|_{W^{1,p}(0,T)^*}, \tag{3.100}$$

where $\| \cdot \|_{W^{1,p}(0,T)^*}$ denotes the dual norm of $\| \cdot \|_{1,p}$ in $W^{1,p}(0,T)^*$, equivalently,

$$\inf_{\substack{w \in \mathbb{R}^2, \\ \|w\|_1 = 1}} \|c(\cdot, w)\|_{W^{1,p}(0,T)^*} \ge \gamma_{p,T}. \tag{3.101}$$

Thus, let $w = (w_1, w_2) \in \mathbb{R}^2$ with $\|w\|_1 = 1$. On the one hand, let us assume that $w_1, w_2 \ge 0$. Then we define the function $x_0 \in W^{1,p}(0,T)$

$$x_0(t) := 1, \quad (t \in [0,T]), \tag{3.102}$$

for which

$$\|x_0\|_{1,p} = \frac{1}{T^{1/p}}. \tag{3.103}$$

Then

$$\begin{aligned} \|c(\cdot, w)\|_{W^{1,p}(0,T)^*} &\ge \frac{c(x_0, w)}{T^{1/p}} \\ &= \frac{w_1 + w_2}{T^{1/p}} \\ &= \frac{1}{T^{1/p}}. \end{aligned} \tag{3.104}$$

On the other hand, if $w_1 < 0 < w_2$, for

$$x_0(t) := -1 + \frac{2}{T}t, \quad (t \in [0, T]), \tag{3.105}$$

we have that $x_0 \in W^{1,p}(0, T)$ and

$$\|x_0\|_{1,p} \le T^{1/p}\left(3^p + \left(\frac{2}{T}\right)^p\right)^{1/p}, \tag{3.106}$$

and so,

$$
\begin{aligned}
\|c(\cdot, w)\|_{W^{1,p}(0,T)^*} &\ge \frac{c(x_0, w)}{\|x_0\|_{1,p}} \\
&\ge \frac{-w_1 + w_2}{T^{1/p}(3^p + (2/T)^p)^{1/p}} \\
&= \frac{1}{T^{1/p}(3^p + (2/T)^p)^{1/p}}.
\end{aligned}
\tag{3.107}
$$

As the cases $w_1, w_2 \le 0$ and $w_2 < 0 < w_1$ follow from the preceding ones, we arrive at the inf-sup condition for c (3.101) with

$$\gamma_{p,T} := \min\left\{\frac{1}{T^{1/p}}, T^{-1/p}\left(3^p + \left(\frac{2}{T}\right)^p\right)^{-1/p}\right\} > 0. \tag{3.108}$$

Therefore the variational equation with constraints (3.97), that is, the weak formulation obtained by imposing weakly the boundary conditions of (3.90), admits a unique solution $x_0 \in W^{1,p}(0, T)$, for whose norm the stability estimation

$$\|x_0\| \le \theta_{p,T}\left(\|f\|_p + \|(v_0, v_T)\|_\infty\right) \tag{3.109}$$

is valid, where $\theta_{p,T} > 0$ depends only on p and T.

With the applications of Section 5 in mind, we are just interested in the 1D case, yet as we commented previously, in [19, Theorem 2.1] an analogue of Proposition 3.10 is established for a convex bounded polygon $\Omega \subset \mathbb{R}^2$. In such a case, it is possible to prove that the boundary value problem

$$
\begin{aligned}
-\Delta x &= f \quad \text{in } \Omega, \\
x &= g \quad \text{on } \Gamma
\end{aligned}
\tag{3.110}
$$

admits a constrained variational formulation similar to that of Example 3.11, where Γ is the topological boundary of Ω, $1 < p < \infty$, $f \in L^p(\Omega)$, and $g \in W^{1/p',p}(\Gamma)$, the space of traces on Γ of functions in the Sobolev space $W^{1,p}(\Omega)$.

Next we show that the reflexivity of X is essential in Theorem 3.8.

Proposition 3.12. *If X is a nonreflexive real normed space, then there exist real normed spaces Y, Z, and W, continuous bilinear forms $a : X \times Y \to \mathbb{R}$, $b : Y \times Z \to \mathbb{R}$, and $c : X \times W \to \mathbb{R}$, and $y_0^* \in Y^*$ and $w_0^* \in W^*$ satisfying the assumptions in Theorem 3.8, except obviously the reflexivity of X, but for any $x_0 \in X$ the constrained variational equation*

$$
\begin{aligned}
y \in K_b &\implies a(x_0, y) = y_0^*(y), \\
w \in W &\implies c(x_0, w) = w_0^*(w)
\end{aligned}
\tag{3.111}
$$

does not hold.

Proof. It suffices to take $Z := X$, $Y := W := X^*$, a, b and c the corresponding duality pairings, $y_0^* \in Y^*$ any continuous linear functional on Y and $w_0^* := x_0^{**} \in X^{**} \setminus X$. Then, since $K_b = 0$ and $K_c = 0$, the first condition in the characterization holds. The second one is also satisfied, because

$$
w \in W \implies \|w\| = \|c(\cdot, w)\|.
\tag{3.112}
$$

However, although for all $x_0 \in X$

$$
y \in K_b \implies a(x_0, y) = y_0^*(y),
\tag{3.113}
$$

for each $x_0 \in X$,

$$
c(x_0, \cdot) \neq w_0^*,
\tag{3.114}
$$

since $c(\cdot, x_0) = x_0 \in X$ but $w_0^* = x_0^{**} \notin X$. $\qquad\square$

The following result, stated in [9, Theorem 2.1], has a certain relation with Theorem 3.8 and ensures that the mixed variational formulation of some boundary value problems is uniquely solvable, when the data functions belong to reflexive spaces, unlike the classical Babuška-Brezzi theory ([3, 11, 20, 21]), developed only for the Hilbert framework: if X, Y, Z, and W are real reflexive Banach spaces and $a : X \times Y \to \mathbb{R}$, $b : Y \times Z \to \mathbb{R}$, and $c : X \times W \to \mathbb{R}$ are continuous bilinear forms, and we write

$$
\begin{aligned}
K_b &:= \{y \in Y : b(y, \cdot) = 0\}, \\
K_c &:= \{x \in X : c(x, \cdot) = 0\}.
\end{aligned}
\tag{3.115}
$$

Then, for all $y_0^* \in Y^*$, $w_0^* \in W^*$ there exists a unique $(x_0, z_0) \in X \times Z$ such that

$$
\begin{aligned}
y \in Y &\implies y_0^*(y) = a(x_0, y) + b(y, z_0), \\
w \in W &\implies w_0^*(w) = c(x_0, w)
\end{aligned}
\tag{3.116}
$$

if, and only if,

$$x \in K_c, \qquad a(x, \cdot)_{|K_b} = 0 \Longrightarrow x = 0, \tag{3.117}$$

and there exist $\alpha, \beta, \gamma > 0$ such that

$$y \in K_b \Longrightarrow \alpha \|y\| \le \left\| a(\cdot, y)_{|K_c} \right\|,$$

$$z \in Z \Longrightarrow \beta \|z\| \le \|b(\cdot, z)\|, \tag{3.118}$$

$$w \in W \Longrightarrow \gamma \|w\| \le \|c(\cdot, w)\|.$$

This result establishes the existence of one and only one solution of the constrained variational equation (3.64) under consideration in Theorem 3.8: if the required assumptions in [9, Theorem 2.1] hold, then in particular we deduce the existence of a unique solution for the variational equation with constraints (3.64), since if $(x_0, z_0) \in X \times Z$ is the unique solution of the mixed problem (3.116), then x_0 is the unique solution of the constrained variational equation (3.64). However, [9, Theorem 2.1] has some additional hypotheses (continuity of b and its inf-sup condition), in addition to the reflexivity of Y, Z, and W, so they are independent statements. Indeed, let X and Y be the sequential space ℓ_2, endowed with its usual Hilbertian norm, let $\{e_n\}_{n \ge 1}$ be the usual basis of ℓ_2, and let $Z := \mathrm{span}\{e_{2n} : n \ge 1\}$, the subspace of ℓ_2 endowed with its inherited norm, and $W := 0$. We define the bilinear forms $a : X \times Y \to \mathbb{R}, b : Y \times Z \to \mathbb{R}$, and $c : X \times W \to \mathbb{R}$ as follows: given $(x, y) \in X \times Y$,

$$a(x, y) := \sum_{n=1}^{\infty} x(n) y(2n - 1), \tag{3.119}$$

so a is continuous and $\|a\| = 1$; for $y \in Y$ and $n \ge 1$,

$$b(y, e_{2n}) := n y(2n) \tag{3.120}$$

and it is extended to $Y \times Z$ by linearity, so b is not continuous; obviously, since $W = 0$, then $c = 0$. In particular (3.67) holds. Now

$$K_b = \{y \in \ell_2 : n \ge 1 \Longrightarrow y(2n) = 0\},$$

$$K_c = \ell_2. \tag{3.121}$$

Let us check condition (3.65). If $x \in K_c$ and $a(x, \cdot)_{|K_b} = 0$, then for $n \ge 1$, $e_{2n-1} \in K_b$, so

$$x(n) = a(x, e_{2n-1}) = 0, \tag{3.122}$$

and therefore $x = 0$. Finally, condition (3.66) is also satisfied, since for each $y \in K_b$,

$$y = \sum_{n=1}^{\infty} y(2n - 1) e_{2n-1}, \tag{3.123}$$

and thus defining

$$\widetilde{y} := \sum_{n=1}^{\infty} y(2n-1)e_n, \tag{3.124}$$

then

$$\|y\| = \|\widetilde{y}\|,$$
$$a(\widetilde{y}, y) = \|y\|^2, \tag{3.125}$$

and since $\|a\| = 1$, we arrive at

$$\|y\| = \|a(\cdot, y)\|. \tag{3.126}$$

Thus Theorem 3.8 guarantees that the corresponding constrained variational equation (3.64) has a unique solution, unlike [9, Theorem 2.1], which does not apply.

Let us point out another difference between [9, Theorem 2.1] and our results: that theorem gives a characterization for all $y_0^* \in Y^*$ and $w_0^* \in W^*$, whereas in Theorem 3.3 we have the advantage that we drive a characterization when the constrained variational problem (3.64) is uniquely solvable, but for two fixed functionals $y_0^* \in Y^*$ and $w_0^* \in W^*$.

4. A Word on Dual Banach Spaces

As we commented in Section 3, we now consider a real dual normed space, endowed with its weak-star topology, and its dual space with its norm topology. The main results, Theorems 4.2 and 4.3, extend the corresponding ones in the reflexive case, Theorems 3.3 and 3.8. Yet these results allow us to obtain our applications in the next section, so we have decided to introduce them first. Furthermore, for the sake of the simplicity in the exposition, we consider it appropriate.

Since the results are analogously stated as those in Section 3, we merely enunciate them.

We begin with the analogue to Corollary 3.1, which also follows from Theorem 2.2, but with a different locally convex space.

Corollary 4.1. *Assume that X is a real normed space, Y and W are real vector spaces, C and D are convex subsets of Y and W, respectively, with $(0,0) \in C \times D$, and $\Phi : C \to \mathbb{R} \cup \{-\infty\}$ and $\Psi : D \to \mathbb{R} \cup \{-\infty\}$ are concave functions such that $\Phi(0) \geq 0$ and $\Psi(0) \geq 0$. If in addition $a : X^* \times Y \to \mathbb{R}$ and $c : X^* \times W \to \mathbb{R}$ are bilinear forms satisfying*

$$(y, w) \in C \times D \Longrightarrow a(\cdot, y), \ c(\cdot, w) \in X, \tag{4.1}$$

then

$$
\text{there exists } x_0^* \in X^* \text{ such that} \begin{cases} y \in C \Longrightarrow \Phi(z) \le a(x_0^*, y), \\ w \in D \Longrightarrow \Psi(w) \le c(x_0^*, w), \end{cases} \tag{4.2}
$$

if, and only if, there exists $\rho \ge 0$ such that

$$
(y, w) \in C \times D \Longrightarrow \Phi(y) + \Psi(w) \le \rho \|a(\cdot, y) + c(\cdot, w)\|. \tag{4.3}
$$

Furthermore, if one of these equivalent conditions holds and there exists $(y, w) \in C \times D$ so that $a(\cdot, y) + c(\cdot, w) \ne 0$, then

$$
\min \left\{ \|x_0^*\| : x_0^* \in X^*, \begin{cases} y \in C \Longrightarrow \Phi(y) \le a(x_0^*, y) \\ w \in D \Longrightarrow \Psi(w) \le c(x_0^*, w) \end{cases} \right\}
$$
$$
= \left(\sup_{\substack{(y,w) \in C \times D, \\ a(\cdot, y) + c(\cdot, w) \ne 0}} \frac{\Phi(y) + \Psi(w)}{\|a(\cdot, y) + c(\cdot, w)\|} \right)_+ . \tag{4.4}
$$

Corollaries 3.1 and 4.1 are different results, although the latter is more general than the particular case of the former when X is reflexive. For this reason, the next two statements are extensions of Theorems 3.3 and 3.8.

Theorem 4.2. *Let X, Y, Z, and W be real normed spaces, and let $a : X^* \times Y \to \mathbb{R}$, $b : Y \times Z \to \mathbb{R}$, and $c : X^* \times W \to \mathbb{R}$ be bilinear forms. Let one write*

$$
K_b := \{y \in Y : b(y, \cdot) = 0\}, \qquad K_c := \{x^* \in X^* : c(x^*, \cdot) = 0\}, \tag{4.5}
$$

and assume that

$$
(y, w) \in K_b \times W \Longrightarrow a(\cdot, y), \ c(\cdot, w) \in X. \tag{4.6}
$$

If in addition $y_0^* \in Y^*$ and $w_0^* \in W^*$ and

$$
R_{w_0^*} := \{x^* \in X^* : c(x^*, \cdot) = w_0^*\}, \tag{4.7}
$$

then

$$
\text{there exists } x_0^* \in X^* \text{ such that} \begin{cases} y \in K_b \Longrightarrow a(x_0^*, y) = y_0^*(y), \\ w \in W \Longrightarrow c(x_0^*, w) = w_0^*(w), \end{cases} \tag{4.8}
$$

if, and only if,

$$R_{w_0^*} \neq \emptyset$$

$$\forall x^* \in R_{w_0^*} \text{ there exists } \delta \geq 0 \text{ such that} \tag{4.9}$$

$$y \in K_b \Longrightarrow y_0^*(y) - a(x^*, y) \leq \delta \left\| a(\cdot, y)_{|K_c} \right\|.$$

Moreover, when these conditions are satisfied and there exists $y \in K_b$ such that $a(\cdot, y)_{|K_c} \neq 0$, then one can take a solution $x_0^* \in X^*$ with

$$\|x_0^*\| = \min_{x^* \in R_{w_0^*}} \left(\sup_{\substack{y \in K_b, \\ a(\cdot, y)_{|K_c} \neq 0}} \frac{y_0^*(y) - a(x^*, y)}{\left\| a(\cdot, y)_{|K_c} \right\|} + \|x^*\| \right). \tag{4.10}$$

The version for any $y_0^* \in Y^*$ and $w_0^* \in W^*$, with the ingredient of uniqueness, is stated in these terms.

Theorem 4.3. *Let X, Y, Z, and W be real normed spaces, let $a : X^* \times Y \to \mathbb{R}$, $b : Y \times Z \to \mathbb{R}$, and $c : X^* \times W \to \mathbb{R}$ be bilinear forms, and let*

$$K_b := \{ y \in Y : b(y, \cdot) = 0 \}, \qquad K_c := \{ x^* \in X^* : c(x^*, \cdot) = 0 \}, \tag{4.11}$$

and assume that

$$y \in K_b, \ w \in W \Longrightarrow a(\cdot, y), \ c(\cdot, w) \in X. \tag{4.12}$$

Then, for all $y_0^* \in Y^*$ and $w_0^* \in W^*$, the corresponding constrained variational equation is uniquely solvable, that is,

$$\text{there exists one and only one } x_0^* \in X^* \text{ such that } \begin{cases} y \in K_b \Longrightarrow a(x_0^*, y) = y_0^*(y), \\ w \in W \Longrightarrow c(x_0^*, w) = w_0^*(w) \end{cases} \tag{4.13}$$

if, and only if,

$$x^* \in K_c, \quad a(x^*, \cdot)_{|K_b} = 0 \Longrightarrow x^* = 0 \tag{4.14}$$

and there exist $\alpha, \gamma > 0$ such that

$$y \in K_b \Longrightarrow \alpha \|y\| \leq \left\| a(\cdot, y)_{|K_c} \right\|,$$

$$w \in W \Longrightarrow \gamma \|w\| \leq \|c(\cdot, w)\|. \tag{4.15}$$

In addition, if one of these equivalent conditions holds, then

$$\|x_0^*\| \le \frac{\|y_0^*\|}{\alpha} + \left(1 + \frac{\|a\|}{\alpha}\right)\frac{\|w_0^*\|}{\gamma}. \tag{4.16}$$

As in the reflexive case, [22, Theorem 2.2] provides conditions that imply that the constrained variational equation (4.13) is uniquely solvable, but that result is more restrictive than Theorem 4.3, which is shown similarly as in Section 3 with [9, Theorem 2.1] and Theorem 3.8.

5. Application to Linear Impulsive Differential Equations

We now apply the same technique that motivated our results, as in Example 3.11; namely, the boundary conditions are weakly imposed. To be more concrete, we consider the impulsive differential problem in Section 1, previously studied in [1], but with nonhomogenous Dirichlet conditions and a non-Hilbertian data function: given $f \in L^p(0, T)$, with $1 < p < \infty$, $\lambda \in \mathbb{R}$, $t_0 = 0 < t_1 < \cdots < t_k < t_{k+1} = T$, and $v_0, v_T, d_1, \ldots, d_k \in \mathbb{R}$, the impulsive linear problem in question is

$$
\begin{aligned}
-x''(t) + \lambda x(t) &= f(t), \quad t \in (0, T), \\
x(0) &= v_0, \qquad x(T) = v_T, \\
\Delta x'(t_j) &= d_j, \quad j = 1, \ldots, k,
\end{aligned}
\tag{5.1}
$$

where $\Delta x'(t_j) = x'(t_j^+) - x'(t_j^-)$. Now the notion of a weak solution, along the lines of [1] (multiply by a test function $y \in W_0^{1,p'}(0, T)$, with $1/p + 1/p' = 1$, integrate by parts and take into account the impulsive conditions $\Delta x'(t_j) = d_j$), is defined as

$$
x_0 \in W^{1,p}(0, T) \text{ such that } \begin{cases} y \in W_0^{1,p'}(0, T) \implies \int_0^T (x_0'y' + \lambda x_0 y) = \int_0^T fy - \sum_{j=1}^{k} d_j y(t_j), \\ x_0(0) = v_0, \qquad x_0(T) = v_T. \end{cases}
\tag{5.2}
$$

The classical Lax-Milgram theorem obviously does not apply in this context. For this very reason we express equivalently this variational formulation as a variational equation with constraints. To this end, two technical results are required. The first of them generalizes Proposition 3.10. Moreover, for the bilinear form under consideration we obtain something better than [18, Theorem 4], since λ admits certain negative values.

Proposition 5.1. *Let $T > 0$, $1 < p < \infty$ with conjugate exponent p' and $\lambda \in \mathbb{R}$, and consider the continuous bilinear form $a_\lambda : W_0^{1,p}(0,T) \times W_0^{1,p'}(0,T) \to \mathbb{R}$ given for each $x \in W_0^{1,p}(0,T)$ and $y \in W_0^{1,p'}(0,T)$ by*

$$a_\lambda(x,y) := \int_0^T (x'y' + \lambda x y). \tag{5.3}$$

Then there exists $\delta_{p,T} > 0$ such that if $\lambda > -\delta_{p,T}$, then a_λ satisfies the inf-sup condition. More precisely, for all $\lambda > -\delta_{p,T}$, where

$$\delta_{p,T} := \frac{p^{1/p} p'^{1/p'}}{T^2 (1 + T^{-1/p'})}, \tag{5.4}$$

there exists $\alpha_{p,T,\lambda} > 0$, depending only on p, T, and λ, such that

$$y \in W_0^{1,p'}(0,T) \implies \alpha_{p,T,\lambda} |y|_{1,p'} \leq |a_\lambda(\cdot,y)|_{-1,p}. \tag{5.5}$$

Proof. The mentioned result in [18] gives us proof for the case $\lambda \geq 0$. On the other hand, in Proposition 3.10 we have shown that a_0 (the bilinear form a_λ with $\lambda = 0$) satisfies the inf-sup condition, obtaining a concise inf-sup constant. In fact we are going to make use of that result. To do this, let us note that $a_\lambda = a_0 + \lambda \tilde{a}$, where $\tilde{a} : W_0^{1,p}(0,T) \times W_0^{1,p'}(0,T) \to \mathbb{R}$ is the continuous bilinear form defined for each $(x,y) \in W_0^{1,p}(0,T) \times W_0^{1,p'}(0,T)$ as

$$\tilde{a}(x,y) := \int_0^T xy. \tag{5.6}$$

But for all $(x,y) \in W_0^{1,p}(0,T) \times W_0^{1,p'}(0,T)$ we have, in view of (3.72), that

$$\begin{aligned}
\tilde{a}(x,y) &\leq \|x\|_p \|y\|_{p'} \\
&\leq c_{p,T} c_{q,T} |x|_{1,p} |y|_{1,p'} \\
&\leq \frac{T^2}{p^{1/p} p'^{1/p'}} |x|_{1,p} |y|_{1,p'},
\end{aligned} \tag{5.7}$$

so

$$|\tilde{a}(\cdot,y)|_{-1,p} \leq \frac{T^2}{p^{1/p} p'^{1/p'}} |y|_{1,p'}. \tag{5.8}$$

Taking into account this last inequality, Proposition 3.10, and the fact that $a_\lambda = a_0 + \lambda\tilde{a}$, we conclude that for

$$0 > \lambda > -\frac{p^{1/p}p'^{1/p'}}{T^2(1 + T^{-1/p'})} \tag{5.9}$$

it follows that

$$y \in W_0^{1,p'}(0,T) \implies \left(\frac{1}{1 + T^{-1/p'}} + \lambda\frac{T^2}{p^{1/p}p'^{1/p'}}\right)|y|_{1,p'} \le |a_0(\cdot,y)|_{-1,p'}, \tag{5.10}$$

and the proof is complete.   □

The second technical result is very simple but will be useful to establish the stability of the solution.

Lemma 5.2. *Let $\lambda \in \mathbb{R}$, $T > 0$, $d_1,\dots,d_k \in \mathbb{R}$, $1 < p < \infty$, $1/p + 1/p' = 1$, and suppose that $f \in L^p(0,T)$. Then the linear functional $y_0^* : W^{1,p'}(0,T) \to \mathbb{R}$ given for each $y \in W^{1,p'}(0,T)$ by*

$$y_0^*(y) := \int_0^T fy - \sum_{j=1}^k d_j y(t_j) \tag{5.11}$$

is continuous, and in addition

$$|y_0^*|_{-1,p'} \le \left(\frac{T}{p'^{1/p'}}\|f\|_p + T^{1/p}\sum_{j=1}^k |d_j|\right). \tag{5.12}$$

Proof. Assume without loss of generality that $y \in C_0^\infty(0,T)$. Given $j = 1,\dots,k$ it follows from the Hölder inequality that

$$\begin{aligned}
|y(t_j)| &= \left|\int_0^{t_j} y'\right| \\
&\le |y|_{1,1} \\
&\le T^{1/p}|y|_{1,p'},
\end{aligned} \tag{5.13}$$

from which the announced inequality clearly follows.   □

Now we can show that the impulsive linear problem (5.1) admits a unique weak solution.

Theorem 5.3. *Let $T > 0$, $1 < p < \infty$, whose conjugate exponent is p', $f \in L^p(0,T)$, $t_0 = 0 < t_1 < \cdots < t_k < t_{k+1} = T$, and $v_0, v_T, d_1, \ldots, d_k \in \mathbb{R}$. Then there exists $\lambda_{p,T} > 0$, specifically,*

$$\lambda_{p,T} := \frac{p^{1/p} p'^{1/p'}}{T^2} \min\left\{\frac{1}{1 + T^{-1/p}}, \frac{1}{1 + T^{-1/p'}}\right\}, \tag{5.14}$$

in such a way that if $\lambda > -\lambda_{p,T}$, then the corresponding impulsive linear problem

$$\begin{aligned}
-x''(t) + \lambda x(t) &= f(t), \quad t \in (0,T), \\
x(0) &= v_0, \qquad x(T) = v_T, \\
\Delta x'(t_j) &= d_j, \quad j = 1, \ldots, k,
\end{aligned} \tag{5.15}$$

admits a unique weak solution. Furthermore, there exists $\chi_{p,T,\lambda} > 0$, depending only on p, T, and λ, such that

$$\|x_0\|_{W_0^{1,p}(0,T)} \leq \chi_{p,T,\lambda}\left(\|f\|_p + \sum_{j=1}^{k}|d_j| + \|(v_0, v_T)\|_\infty\right). \tag{5.16}$$

Proof. Our aim is to prove the existence of one and only one $x_0 \in W^{1,p}(0,T)$ such that

$$\begin{aligned}
y \in W_0^{1,p'}(0,T) &\Longrightarrow \int_0^T (x_0'y' + \lambda x_0 y) = \int_0^T fy - \sum_{j=1}^{k} d_j y(t_j), \\
x_0(0) &= v_0, \qquad x_0(T) = v_T
\end{aligned} \tag{5.17}$$

which moreover depends continuously on the initial data. Yet this variational problem admits an equivalent reformulation as the variational problem with constraints

$$\text{find } x_0 \in X \text{ such that } \begin{cases} y \in K_b \Longrightarrow y_0^*(y) = a(x_0, y), \\ w \in W \Longrightarrow w_0^*(w) = c(x_0, w), \end{cases} \tag{5.18}$$

where X is now the real reflexive Banach space

$$X := W^{1,p}(0,T) \tag{5.19}$$

and Y, Z, and W are the real normed spaces

$$Y := W^{1,p'}(0,T), \qquad Z := W := \mathbb{R}^2, \tag{5.20}$$

$\mathbb{R}^2$ is equipped with its $\| \cdot \|_1$ norm, $a : X \times Y \to \mathbb{R}$, $b : Y \times Z \to \mathbb{R}$, and $c : X \times W \to \mathbb{R}$ are the continuous bilinear forms defined by

$$
\begin{aligned}
a(x,y) &:= \int_0^T (x'y' + \lambda xy), \quad (x \in X, \ y \in Y), \\
b(y,z) &:= (y(0), y(T)) \cdot z, \quad (y \in Y, \ z \in Z), \\
c(x,w) &:= (x(0), x(T)) \cdot w, \quad (x \in X, \ w \in W),
\end{aligned}
\tag{5.21}
$$

and $y_0^* \in Y^*$ and $w_0^* \in W^*$ are the continuous linear functionals given as

$$
\begin{aligned}
y_0^*(y) &:= \int_0^T fy - \sum_{j=1}^k d_j y(t_j), \quad (y \in Y), \\
w_0^*(w) &:= (v_0, v_T) \cdot w, \quad (w \in W).
\end{aligned}
\tag{5.22}
$$

The fact that the variational problem coincides with the constrained variational equation follows from the fact that $K_b = W_0^{1,p'}(0,T)$.

Let us finally check (3.65), (3.66), and (3.67) in order to apply Theorem 3.8 and conclude the proof.

Since $K_b = W_0^{1,p'}(0,T)$ and $K_c = W_0^{1,p}(0,T)$, (3.65) follows from Proposition 5.1 and (3.66) from this equivalent reformulation of such a result: there exists $\tilde{\alpha}_{p,T,\lambda} > 0$, depending only on p, T, and λ, such that

$$
x \in W_0^{1,p}(0,T) \Longrightarrow \tilde{\alpha}_{p,T,\lambda} |x|_{1,p} \le |a_\lambda(x,\cdot)|_{-1,p'},
\tag{5.23}
$$

provided that

$$
\lambda > -\frac{p^{1/p} p'^{1/p'}}{T^2 (1 + T^{-1/p'})}.
\tag{5.24}
$$

On the other hand, (3.67) was proven in Example 3.11.

Finally, the continuous dependence property follows from (3.68) and Lemma 5.2. $\square$

Theorem 5.3 generalizes [1, Theorem 3.3] to the reflexive framework, although a constant better than

$$
\lambda_{2,T} = \frac{2}{T^2} \frac{1}{1 + T^{-1/2}}
\tag{5.25}
$$

is obtained, to be more precise,

$$
\frac{\pi^2}{T^2}.
\tag{5.26}
$$

In [23, 24] the damped case is also considered, but only in the Hilbertian framework. For some recent developments of the theory of impulsive differential equations with a variational approach, we also refer to [25–27] and their extension [28], and to [29]. Sea also [30] for impulsive differential equations in abstract Banach spaces.
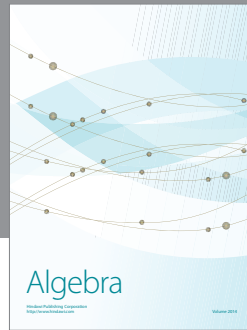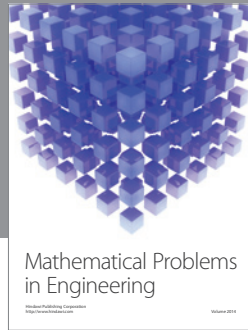
Acknowledgment

References

[1] J. J. Nieto and D. O'Regan, "Variational approach to impulsive differential equations," *Nonlinear Analysis*, vol. 10, no. 2, pp. 680–690, 2009.

[2] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, NY, USA, 2011.

[3] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, vol. 15 of *Springer Series in Computational Mathematics*, Springer, New York, NY, USA, 1991.

[4] C. Grossmann, H.-G. Roos, and M. Stynes, *Numerical Treatment of Partial Differential Equations*, Universitext, Springer, Berlin, Germany, 2007.

[5] S. Simons, "A new version of the Hahn-Banach theorem," *Archiv der Mathematik*, vol. 80, no. 6, pp. 630–646, 2003.

[6] S. Simons, "The Hahn-Banach-Lagrange theorem," *Optimization*, vol. 56, no. 1-2, pp. 149–169, 2007.

[7] S. Simons, *From Hahn-Banach to Monotonicity*, vol. 1693 of *Lecture Notes in Mathematics*, Springer, New York, NY, USA, 2nd edition, 2008.

[8] M. Ruiz Galán, "A version of the Lax-Milgram theorem for locally convex spaces," *Journal of Convex Analysis*, vol. 16, no. 3-4, pp. 993–1002, 2009.

[9] C. Bernardi, C. Canuto, and Y. Maday, "Generalized inf-sup conditions for Chebyshev spectral approximation of the Stokes problem," *SIAM Journal on Numerical Analysis*, vol. 25, no. 6, pp. 1237–1271, 1988.

[10] I. Babuška, "Error-bounds for finite element method," *Numerische Mathematik*, vol. 16, pp. 322–333, 1971.

[11] F. Brezzi, "On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers," *RAIRO Modélisation Mathématique et Analyse Numérique*, vol. 8, no. R-2, pp. 129–151, 1974.

[12] G. N. Gatica and F.-J. Sayas, "Characterizing the inf-sup condition on product spaces," *Numerische Mathematik*, vol. 109, no. 2, pp. 209–231, 2008.

[13] V. J. Ervin, E. W. Jenkins, and S. Sun, "Coupled generalized nonlinear Stokes flow with flow through a porous medium," *SIAM Journal on Numerical Analysis*, vol. 47, no. 2, pp. 929–952, 2009.

[14] J. S. Howell and N. J. Walkington, "Inf-sup conditions for twofold saddle point problems," *Numerische Mathematik*, vol. 118, no. 4, pp. 663–693, 2011.

[15] M. Ruiz Galán, "Variational equations with constraints," *Applied Mathematics Letters*, vol. 23, no. 7, pp. 801–806, 2010.

[16] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, vol. 140, Elsevier, Amsterdam, The Netherlands, 2nd edition, 2003.

[17] I. Babuška and J. Osborn, "Eigenvalue problems," in *Handbook of Numerical Analysis, Vol. II*, pp. 641–787, North-Holland, Amsterdam, The Netherlands, 1991.

[18] I. Babuška and J. Osborn, "Analysis of finite element methods for second order boundary value problems using mesh dependent norms," *Numerische Mathematik*, vol. 34, no. 1, pp. 41–62, 1980.

[19] J. Pousin, "Inf-sup conditions for an elliptic operator in the spaces $W_0^{1,p} W_0^{1,q}$ approximated with Lagrange finite element," in *Partial Differential Equation Methods in Control and Shape Analysis (Pisa)*, vol. 188 of *Lecture Notes in Pure and Appl. Math.*, pp. 259–273, Dekker, New York, NY, USA, 1997.

[20] J. E. Roberts and J.-M. Thomas, "Mixed and hybrid methods," in *Handbook of Numerical Analysis, Vol. II*, Finite Element Methods (Part 1), pp. 523–639, North-Holland, Amsterdam, The Netherlands, 1991.

[21] D. Boffi, F. Brezzi, L. F. Demkowicz, R. G. Durán, R. S. Falk, and M. Fortin, *Mixed Finite Elements, Compatibility Conditions, and Applications*, vol. 1939 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2008.

[22] C. Amrouche and R. Ratsimahalo, "Conditions "inf sup" dans les espaces de Banach non réflexifs," *Comptes Rendus de l'Académie des Sciences I*, vol. 330, no. 12, pp. 1069–1072, 2000.

[23] J. J. Nieto, "Variational formulation of a damped Dirichlet impulsive problem," *Applied Mathematics Letters*, vol. 23, no. 8, pp. 940–942, 2010.

[24] J. Zhou and Y. Li, "Existence and multiplicity of solutions for some Dirichlet problems with impulsive effects," *Nonlinear Analysis*, vol. 71, no. 7-8, pp. 2856–2865, 2009.

[25] J. Sun and H. Chen, "Multiplicity of solutions for a class of impulsive differential equations with Dirichlet boundary conditions via variant fountain theorems," *Nonlinear Analysis*, vol. 11, no. 5, pp. 4062–4071, 2010.

[26] J. Xiao and J. J. Nieto, "Variational approach to some damped Dirichlet nonlinear impulsive differential equations," *Journal of the Franklin Institute*, vol. 348, no. 2, pp. 369–377, 2011.

[27] Y. Tian and W. Ge, "Multiple solutions of Sturm-Liouville boundary value problem via lower and upper solutions and variational methods," *Journal of Mathematical Analysis and Applications*, vol. 387, pp. 475–489, 2012.

[28] J. Xiao, J. J. Nieto, and Z. Luo, "Multiplicity of solutions for nonlinear second order impulsive differential equations with linear derivative dependence via variational methods," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 1, pp. 426–432, 2012.

[29] W. Ding and D. Qian, "Periodic solutions for sublinear systems via variational approach," *Nonlinear Analysis*, vol. 11, no. 4, pp. 2603–2609, 2010.

[30] H. Fan and Y. Li, "Multiple positive solutions for singular periodic boundary value problems of impulsive differential equations in Banach spaces," *Abstract and Applied Analysis*, vol. 2011, Article ID 493639, 12 pages, 2011.