DOI: http://dx.doi.org/10.21123/bsj.2018.15.3.0344

An Efficient Numerical Method for Solving Volterra-Fredholm Integro-Differential Equations of Fractional Order by Using Shifted Jacobi-Spectral Collocation Method

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Received 24/3/2017, Accepted 1/7/2018, Published 13/9/2018

Abstract:

The aim of this article is to solve the Volterra-Fredholm integro-differential equations of fractional order numerically by using the shifted Jacobi polynomial collocation method. The Jacobi polynomial and collocation method properties are presented. This technique is used to convert the problem into the solution of linear algebraic equations. The fractional derivatives are considered in the Caputo sense. Numerical examples are given to show the accuracy and reliability of the proposed technique.

Keywords: Collocation Method, Fractional derivative, Jacobi polynomial, Volterra-Fredholm integro-differential equations of fractional order.

Introduction:

Fractional Calculus has been pulling in the consideration of researchers and specialists from long time ago, resulting in the improvement of numerous applications. Since the nineties of a century ago fractional calculus is being rediscovered and connected in an expanding number of fields, namely in several areas of Physics, Control Engineering, and Signal Processing, such as electromagnetism, communications, sciences, control, robotics, information and many other physical sciences and also in medical sciences (1,2,3,4,5).

Integral equations can be described as being a functional equation involving the unknown function under one or more integrals.

Differential equations as well as integral equations of fractional order belong to a wider class of equations in which the unknown object is a function (scalar function or vector function). Such kinds of equations are often encountered in mathematics and in various sciences that use the mathematical apparatus, and they are generally called functional equations.

Because of the broad utilizations of differential equations and integral equations of fractional order in engineering and science; research in this area has become essentially all around the globe (6-16).

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An Integro-differential equation is an equation in which the unknown functions appear with derivatives, and either the unknown functions, or their derivatives, or both, appear under the sign of integration. This, however, is a purely formal classification, since we can easily pass from one type to the other. Numerous scientific models of physical wonders contain integro-differential equations; these equations emerge in numerous fields like physics, potential theory, astronomy, biological models, chemical kinetics and fluid dynamics.

Integro-differential equations are normally not being easy to solve analytically; so, it is required to obtain an effective approximate solution (17-21). Fractional integro-differential equation is considered as an important model for various physical wonders in engineering and scientific fields. Some numerical calculation for solving integro-differential equation of fractional order can be summarized as follows: variational iteration method (22, 23), Haar wavelets method (24), Adomian decomposition method (25, 26, 27), Laplace decomposition method (28), differential transform method (29, 30), Legendre wavelets method (31, 32) and Chebyshev wavelets method (33, 34, 35).

As a special form of integro-differential equations of fractional order Volterra-Fredholm integro-differential equations of fractional order (36, 37).

That primary point of the Jacobi-collocation method over other techniques may be that Jacobi-collocation method provides a great finer rate for convergence (38, 39).

In this article, we consider a Volterra-Fredholm integro-differential equation of fractional order as follows:

$$\frac{d^2u(x)}{dx^2} + \frac{du(x)}{dx} + D_x^v u(x) + u(x) =$$
$$f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)dt + \lambda_2 \int_a^b K_2(x,t)u(t)\,dt,$$
$$0 < v \le 1 \qquad \ldots(1)$$

Subject to the homogenous boundary conditions:

$$u(a) = 0,\ u(b) = 0,\ a \le x \le b \qquad \ldots(2)$$

The organization of the rest of this article is as follows: in section 2 we present some essential definitions of the fractional calculus theory, in section 3 the Jacobi polynomial, and its properties are presented. While in section 4 we show how Jacobi polynomial with collocation technique may be used to replace problem (1)–(2) by an explicit system of linear algebraic equations. Moreover in section 5, we introduce some numerical cases to show the adequacy of the proposed method, concluding remarks are given in the last section.

Fractional Derivative and Integration

In this section, we might survey the essential definitions and properties of fractional integral and derivatives, which are utilized further in (3).

Definition (1):- The left-sided and the right-sided Riemann-Liouville fractional integrals $I_{a+}^v f$ and $I_{b-}^v f$ of order $v \in \mathbb{C}$ $(\Re(\alpha) > 0)$ are defined by:-

$$(I_{a+}^v f)(x) = \frac{1}{\Gamma(v)} \int_a^x \frac{f(t)dt}{(x-t)^{1-v}} \quad (x > a;\ \Re(v) > 0) \ldots(3)$$

$$(I_{b-}^v f)(x) = \frac{1}{\Gamma(v)} \int_x^b \frac{f(t)dt}{(t-x)^{1-v}} \quad (x < b;\ \Re(v) > 0), \ldots(4)$$

$$I_x^0 f(x) = f(x) \qquad \ldots(5)$$

Definition (2):- The left and right Riemann-Liouville fractional derivative $D_{a+}^v y$ and $D_{b-}^v y$ of order $v \in \mathbb{C}$ $(\Re(v) \ge 0)$ are defined respectively by :-

$$\begin{cases} (D_{a+}^v y)(x) := \left(\frac{d}{dx}\right)^n (I_{a+}^{n-v} y)(x) \\ = \frac{1}{\Gamma(n-v)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{y(t)dt}{(x-t)^{v-n+1}} \quad (n = [\Re(v)] + 1; x > v) \\ (D_{b-}^v y)(x) := \left(-\frac{d}{dx}\right)^n (I_{b-}^{n-v} y)(x) \\ = \frac{1}{\Gamma(n-v)} \left(-\frac{d}{dx}\right)^n \int_x^b \frac{y(t)dt}{(t-x)^{v-n+1}} \quad (n = [\Re(v)] + 1; x < b) \end{cases} \ldots(6)$$

Where $[\Re(v)]$ means The integral part of $\Re(v)$.

Definition (3):- The Caputo fractional derivative operator of order v for the function $f: [a,b] \to \mathbb{R}$, is given as follows:

$${}^cD_x^v f(x) = \frac{1}{\Gamma(n-v)} \int_0^x (x-t)^{n-v-1} f^{(n)}(t)dt,\ x > 0 \quad \ldots(7)$$

Where $v > 0$, n is an integer and $n - 1 < v \le n$.

The relation between Caputo fractional derivative and Riemann-Liouville:

$$I_x^v\ {}^cD_x^v f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!} \qquad \ldots(8)$$

Where n is an integer and $n - 1 < v \le n$.

Also, for the Caputo fractional derivative we have

$${}^cD_x^v C = 0,\ (\text{C is a constant}) \qquad \ldots(9)$$

$${}^cD_x^v x^\beta = \begin{cases} 0 & \text{for } \beta \in N_0 \text{ and } \beta < \lceil v \rceil \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-v)} x^{\beta-v}, & \text{for } \beta \in N_0 \text{ and } \beta \ge \lceil v \rceil \text{ or } \beta \notin N \text{ and } \beta > \lfloor v \rfloor. \end{cases} \quad \ldots(10)$$

Where $\lceil v \rceil$ and $\lfloor v \rfloor$ be the ceiling and the floor function respectively.

The Jacobi Polynomials

The well-known Jacobi polynomials (40) are defined on the interval [-1,1] and can be generated with the aid of the following recurrence formula:

$$J_i^{(\gamma,\delta)}(t) =$$
$$\frac{(\gamma+\delta+2i-1)\{(\gamma^2-\delta^2+t(\gamma+\delta+2i))(\gamma+\delta+2i-2)\}}{2i(\gamma+\delta+i)(\gamma+\delta+2i-2)} J_{i-1}^{(\gamma,\delta)}(t)$$
$$-\frac{(\gamma+i-1)(\delta+i-1)(\gamma+\delta+2i)}{i(\gamma+\delta+i)(\gamma+\delta+2i-2)} J_{i-2}^{(\gamma,\delta)}(t), i = 1,2, \ldots \quad \ldots(11)$$

where $J_0^{(\gamma,\delta)}(t) = 1$ and $J_1^{(\gamma,\delta)}(t) = \frac{\gamma+\delta+2}{2} t + \frac{\gamma-\delta}{2}$ with $J_i^{(\gamma,\delta)}(-t) = (-1)^i J_i^{(\gamma,\delta)}(t)$,

$$J_i^{(\gamma,\delta)}(-1) = \frac{(-1)^i \Gamma(\gamma+\delta+1)}{i!\Gamma(\delta+1)}, \qquad \ldots(12)$$

Moreover, the nth derivative of $J_i^{(\gamma,\delta)}(t)$, can be obtained from

$$\frac{d^n}{dx^n} J_i^{(\gamma,\delta)}(t) = \frac{\Gamma(i+n+\gamma+\delta+1)}{2^n \Gamma(i+\gamma+\delta+1)} J_{i-n}^{(\gamma+n,\delta+n)}(t), \quad \ldots(13)$$

We also define the so-called shifted Jacobi polynomials of degree $i \in \mathbb{N}$ on the interval $[0, L]$ by using the change of variable $t = \frac{2x}{L} - 1$. So Shifted Jacobi polynomials $J_i^{(\gamma,\delta)}(\frac{2x}{L} - 1)$ are denoted by $J_{L,i}^{(\gamma,\delta)}(x)$. Shifted Jacobi polynomials of x can be determined with the aid of the

$$J_{L,i}^{(\gamma,\delta)}(x) =$$
$$\frac{(\gamma+\delta+2i-1)\{(\gamma^2-\delta^2+(\frac{2x}{L}-1)(\gamma+\delta+2i))(\gamma+\delta+2i-2)\}}{2i(\gamma+\delta+i)(\gamma+\delta+2i-2)} J_{L,i-1}^{(\gamma,\delta)}(x)$$

$-\frac{(\gamma+i-1)(\delta+i-1)(\gamma+\delta+2i)}{i(\gamma+\delta+i)(\gamma+\delta+2i-2)} J_{L,i-2}^{(\gamma,\delta)}(x), i = 1,2,\ldots$ (14)

Where $J_{L,0}^{(\gamma,\delta)}(x) = 1$ and $J_{L,1}^{(\gamma,\delta)}(x) = \frac{\gamma+\delta+2}{2}\left(\frac{2x}{L} - 1\right) + \frac{\gamma-\delta}{2}$.

The analytic form of the i-degree shifted Jacobi polynomials is given by

$$J_{L,i}^{(\gamma,\delta)}(x) = \sum_{k=0}^{i}(-1)^{i-k} \frac{\Gamma(\gamma+\delta)\Gamma(i+K+\gamma+\delta+1)}{\Gamma(K+\delta+1)\Gamma(i+\gamma+\delta+1)(i-K)!K!L^k} x^k, \ i = 1,2,\ldots, \qquad \ldots(15)$$

Where

$$J_{L,i}^{(\gamma,\delta)}(0) = (-1)^i \frac{\Gamma(i+\delta+1)}{i!\Gamma(\delta+1)}$$

and $J_{L,i}^{(\gamma,\delta)}(L) = \frac{\Gamma(i+\gamma+1)}{i!\Gamma(\gamma+1)}$ …(16)

The n[th] order derivative of shifted Jacobi polynomial can be written as(41) :

$$\frac{d^n}{dx^n} J_{L,i}^{(\gamma,\delta)}(x) = b_{i,n}^{(\gamma,\delta)} J_{L,i-n}^{(\gamma+n,\delta+n)}(x), \qquad \ldots(17)$$

Where $b_{i,n}^{(\gamma,\delta)} = \frac{\Gamma(i+n+\gamma+\delta+1)}{L^n\Gamma(i+\gamma+\delta+1)}$

The orthogonality condition of shifted Jacobi polynomials is

$$\int_0^L J_{L,j}^{(\gamma,\delta)}(x) J_{L,k}^{(\gamma,\delta)}(x)\Omega_L^{(\gamma,\delta)}(x)dx = \ell_k, \qquad \ldots(18)$$

Where $\Omega_L^{(\gamma,\delta)}(x) = x^\delta(L-x)^\gamma$

And

$$\ell_k = \begin{cases} \frac{L^{\gamma+\delta+1}\Gamma(k+\gamma+1)\Gamma(k+\delta+1)}{(2k+\gamma+\delta+1)k!\Gamma(k+\gamma+\delta+1)}, & i = j \\ 0 & i \neq j. \end{cases} \qquad \ldots(19)$$

A special case for $\gamma = \delta = -1/2$ and $\gamma = \delta = 0$, the Chebyshev and Legendre of the first and kinds polynomials respectively,

A function u(x), which is square integrable in (0,1) may be expressed in terms of shifted Jacobi polynomials as

$$u(x) = \sum_{j=0}^{\infty} c_j J_{L,j}^{(\gamma,\delta)}(x)$$

Where the coefficients c_j are given by

$$c_j = \frac{1}{h_j}\int_0^1 \Omega^{(\gamma,\delta)}(x)y(x)J_{L,j}^{(\gamma,\delta)}(x)dx \qquad j=0,1,\ldots .$$

Now, by considering the first (m+1)-term of shifted Jacobi polynomials.

Hence u(x) can be written in the form

$$u(x) = \sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(x) \qquad \ldots(20)$$

Theorem (1):- Let u(x) be approximated by the shifted Jacobi polynomials as (20) and also suppose $v > 0$ then:

$$^cD_0^v u(t) = \sum_{j=\lceil v\rceil}^{m}\sum_{n=0}^{j-\lceil v\rceil} c_j\, b_{j,n}^{(v)} t^{j-n-v}, \qquad \ldots(21)$$

Where $b_{j,n}^{(v)}$ is given by:

$$b_{j,n}^{(v)} = (-1)^{j-n}\frac{\Gamma(\gamma+\delta)\Gamma(m+j+\gamma+\delta+1)}{\Gamma(j+\delta+1)\Gamma(m+\gamma+\delta+1)(m-j)!j!L^j} \ ..(22)$$

Proof:- Since the Caputo's fractional differentiation is a linear operation we have:

$$^cD_0^v(u(x)) = \sum_{j=0}^{m} c_j \ ^cD_0^v(J_{L,j}^{(\gamma,\delta)}(x)) \qquad \ldots(23)$$

Now, to calculate $^cD_0^v(u(x))$ by using Eqs. (9), (10) in eq. (15) we have

$$^cD_0^v(J_{L,j}^{(\gamma,\delta)}(x)) = \sum_{n=0}^{j}(-1)^{i-k}\frac{\Gamma(\gamma+\delta)\Gamma(j+K+\gamma+\delta+1)}{\Gamma(n+\delta+1)\Gamma(j+\gamma+\delta+1)(j-K)!n!L^n} \ ^cD_0^v(x^{j-n}), \ j = \lceil v\rceil, \lceil v\rceil+1,\ldots,m \qquad \ldots(24)$$

Since the degree of Jacobi polynomial is n, then:

$$^cD_0^v(J_{L,j}^{(\gamma,\delta)}(x)) = 0, \ \forall\, n = 0,1,2,\ldots,\lceil v\rceil - 1, \ v > 0 \qquad \ldots(25)$$

A combination of (23)-(25), leads to the following form:

$$^cD_0^v u(x) = \sum_{j=\lceil v\rceil}^{m}\sum_{n=0}^{j-\lceil v\rceil} c_j b_{j,n}^{(v)} = (-1)^{j-n}\frac{\Gamma(\gamma+\delta)\Gamma(m+j+\gamma+\delta+1)}{\Gamma(j+\delta+1)\Gamma(m+\gamma+\delta+1)(m-j)!j!L^j} x^{j-n-v},$$

$$= \sum_{j=\lceil v\rceil}^{m}\sum_{n=0}^{j-\lceil v\rceil} c_j\, b_{j,k}^{(v)} x^{j-n-v}, \qquad \ldots(26)$$

This is the end of the proof.

Function approximation

Consider linear Volterra-Fredholm integro-differential equations of fractional order derivative of the form:

$$\frac{d^2u(x)}{dx^2} + \frac{du(x)}{dx} + \ ^cD_0^v u(x) + u(x) = f(x) + \lambda_1\int_a^x K_1(x,t)u(t)dt + \lambda_2\int_a^b K_2(x,t)u(t)\,dt, \quad \ldots(27)$$

subject to the homogenous boundary condition (2) using Eqs.(17),(25) in Eq.(27) we have:

$$\sum_{j=0}^{m} c_j \frac{d^2}{dx^2}J_{L,j}^{(\gamma,\delta)}(x) + \sum_{j=0}^{m} c_j \frac{d}{dx}J_{L,j}^{(\gamma,\delta)}(x) + \sum_{j=\lceil v\rceil}^{m}\sum_{n=0}^{j-\lceil v\rceil} c_j\, b_{j,k}^{(v)} x^{j-n-v} + \sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(x) = f(x) + \lambda_1\int_a^x K_1(x,t)\sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(t)\,dt + \lambda_2\int_a^b K_2(x,t)\sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(t)\,dt, \qquad \ldots(28)$$

$$\sum_{j=0}^{m} c_j\, b_{j,2}^{(\gamma,\delta)} J_{L,j-2}^{(\gamma+2,\delta+2)}(x) + \sum_{j=0}^{m} c_j\, b_{j,1}^{(\gamma,\delta)} J_{L,j-1}^{(\gamma+1,\delta+1)}(x) + \sum_{j=\lceil v\rceil}^{m}\sum_{n=0}^{j-\lceil v\rceil} c_j\, b_{j,k}^{(v)} x^{j-n-v} + \sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(x) = f(x) + \lambda_1\int_a^x K_1(x,t)\sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(t)\,dt + \lambda_2\int_a^b K_2(x,t)\sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(t)\,dt, \qquad \ldots(29)$$

Now we arrange Eq. (29) at $(m + 1 - \lceil v\rceil)$ points x_q as:

$$\sum_{j=0}^{m} c_j\, b_{j,2}^{(\gamma,\delta)} J_{L,j-2}^{(\gamma+2,\delta+2)}(x_q) + \sum_{j=0}^{m} c_j\, b_{j,1}^{(\gamma,\delta)} J_{L,j-1}^{(\gamma+1,\delta+1)}(x_q) +$$

$\sum_{j=\lceil v\rceil}^{m}\sum_{n=0}^{j-\lceil v\rceil} c_j\, b_{j,k}^{(v)} x_q{}^{j-n-v} + \sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(x_q) = f(x_q) + \lambda_1 \int_a^{x_q} K_1(x_q,t)\sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(t)\,dt + \lambda_2 \int_a^{b} K_2(x_q,t)\sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(t)\,dt, \quad q = 0,1,\dots,\lceil v\rceil.$ …(30)

For suitable collocation points we use roots of shifted Jacobi polynomial $J_{L,m+1-\lceil v\rceil}^{(\gamma,\delta)}(x)$.

In order to use the quadrature rule for Eq. (30), we transfer the interval $[a,x_q]$ and the interval $[a,b]$ to fixed interval $[-1,1]$ by means of the transformation $t=\frac{b-a}{2}t+\frac{a+b}{2}$

Eq. (30), for $p = 0,1,\dots,m+1-\lceil v\rceil$, may be restated as:

$$\sum_{j=0}^{m} c_j\, b_{i,2}^{(\gamma,\delta)} J_{L,j-2}^{(\gamma+2,\delta+2)}(x_q) + \sum_{j=0}^{m} c_j\, b_{i,1}^{(\gamma,\delta)} J_{L,j-1}^{(\gamma+1,\delta+1)}(x_q) + \sum_{j=\lceil v\rceil}^{m}\sum_{n=0}^{j-\lceil v\rceil} c_j\, b_{j,k}^{(v)} x_q{}^{j-n-v} + \sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(x_q) = f(x_q) + \frac{\lambda_1(x_q-a)}{2}\int_{-1}^{1} K_1(x_q,\xi)\sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(\xi)\,d\tau + \frac{\lambda_2(b-a)}{2}\int_{-1}^{1} K_2(x_q,\chi)\sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(\chi)\,d\tau, \quad q = 0,1,\dots,\lceil v\rceil. \quad \dots(31)$$

Where $\xi=\frac{x_q-a}{2}t+\frac{x_q+a}{2}$ and $\chi=\frac{(b-a)}{2}t+\frac{(b-a)}{2}$,

By using the Gaussian integration formula, for $q = 0,1,\dots,m-\lceil v\rceil$, we get:

$$\sum_{j=0}^{m} c_j\, b_{i,2}^{(\gamma,\delta)} J_{L,j-2}^{(\gamma+2,\delta+2)}(x_q) + \sum_{j=0}^{m} c_j\, b_{i,1}^{(\gamma,\delta)} J_{L,j-1}^{(\gamma+1,\delta+1)}(x_q) + \sum_{j=\lceil v\rceil}^{m}\sum_{n=0}^{j-\lceil v\rceil} c_j\, b_{j,k}^{(v)} x_q{}^{j-n-v} + \sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(x_q) = f(x_q) + \frac{\lambda_1(x_q-a)}{2}\sum_{d=0}^{m} w_d K_1(x_q,\xi_{t_d})\sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(\xi_{t_d}) + \frac{\lambda_2(b-a)}{2}\sum_{d=0}^{m} w_d K_2(x_q,\chi_{t_d})\sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(\chi_{t_d}), \quad q = 0,1,\dots,\lceil v\rceil \quad \dots(32)$$

Where t_d are $m+1$ roots of Jacobi polynomial $J_{m+1}^{(\alpha,\delta)}(t)$ and w_d are their weights given in (7). The idea of the above approximation is the exactness of the quadrature rule for polynomials of degree does not exceed 2m + 1.

Also, by substituting Eq. (20) in boundary conditions, we can find $\lceil v\rceil$ of equations we obtain

$$u(x) = \sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(0) = 0, \quad \dots(33)$$

$$u(x) = \sum_{j=0}^{m} c_j J_{L,j}^{(\gamma,\delta)}(1) = 0 \quad \dots(34)$$

Next, Eqs. (33-34), after using (15), can be written as

$$\sum_{j=0}^{m} c_j(-1)^i \frac{\Gamma(j+\delta+1)}{i!\Gamma(\delta+1)} = 0, \quad \dots(35)$$

$$\sum_{j=0}^{m} c_j \frac{\Gamma(j+\gamma+1)}{i!\Gamma(\gamma+1)} = 1 \quad \dots(36)$$

So, from using equations (32) with (35) and (36), we get $(m+1)$ linear algebraic equations which can be solved for the unknown coefficients c_j.

Numerical Examples:

In this section, we will examine the accuracy and efficiency of the proposed method by the following two examples:

Example 1(36): Given the following linear singular fractional order Volterra-Fredholm integro-differential equation:

$$\frac{d^2u(x)}{dx^2}+\frac{1}{x}D_x^{v}u(x)+\frac{1}{x^2}u(x) = f(x)+\int_0^x K_1(x,t)u(t)dt+\int_0^1 K_2(x,t)u(t)\,dt, \quad 0< v\le 1 \quad \dots(37)$$

with boundary conditions as follows:

$$u(0) = u(1) = 0$$

Where $f(x) = 5+1.50451x^{0.5}-13x^{1.5}-x^2+x^3-2.0674\cos(x)+5.95385\sin(x)$,

And $k_1(x,t)=\sin(x-t)$, $k_2(x,t)=\cos(x-t)$.

The exact solution of this problem for $v=1$ is $u(x)=x^2(1-x)$.

We apply the proposed method for solving Eq. (37) the absolute error AE between our methods with the method given in (36) for different values of γ , δ with $v=0.9$ and $m=8$ are shown in (Table 1), where x_q are roots of the shifted Jacobi polynomial$J_8^{(\gamma,\delta)}(x)$. The diagrams of the exact and approximate solution for $\gamma=\delta=0$, $\gamma=\delta=-0.5$ and for $m=4,16$ together with $v=0.75$, $v=0.9$ are given in Fig. (1,2).

Table 1. The comparison of AE between approximate solution for m=8 and the solution in method (36) for *m=32 and* $\boldsymbol{v=0.9}$

Our methods				Method (36)		
$\gamma=\delta=0$ x_q	m=8	$\gamma=\delta=-0.5$ x_q	m=8	x	m=32
0.02	2.7061e-4	0.01	3.3954e-05	0.1	7.309e-06
0.10	1.7168e-4	0.08	2.9156e-05	0.2	2.048e-05
0.24	3.0262e-4	0.22	5.9122e-05	0.3	2.606e-05
0.41	2.2260e-4	0.40	6.3878e-05	0.4	2.503e-05
0.59	6.4991e-4	0.60	2.5112e-05	0.6	1.789e-05
0.76	3.8925e-4	0.78	5.1331e-05	0.7	1.202e-05
0.90	1.1815e-4	0.92	1.3613e-05	0.8	7.682e-06
0.98	1.7664e-4	0.99	1.4384e-05	0.9	3.034e-06

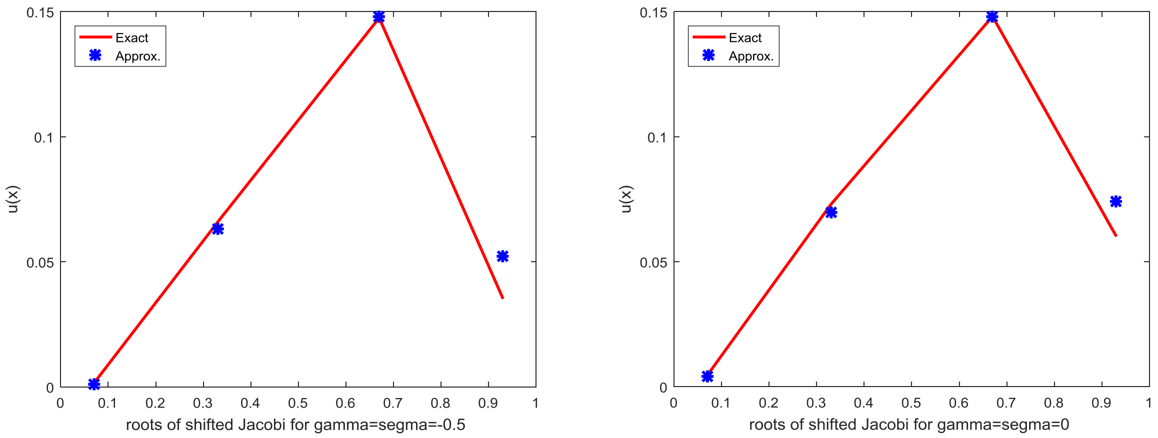

Figure 1. The comparison between the exact and approximate solution for $\mathbf{m=4}$, $\boldsymbol{\gamma=\delta=0}$, left and $\mathbf{m=4}$, $\boldsymbol{\gamma=\delta=-0.5}$ right with $\boldsymbol{v=0.75}$.

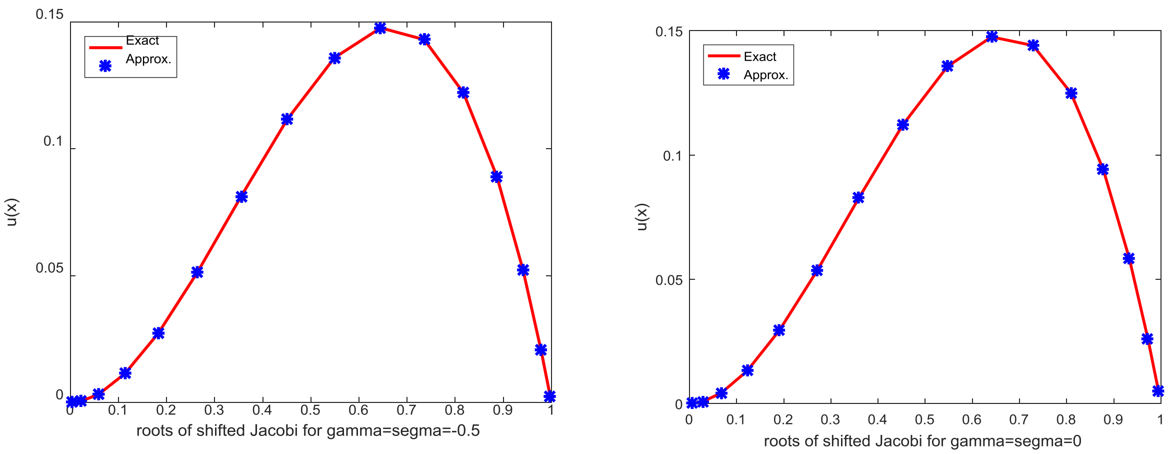

Figure 2. The comparison between the exact and approximate solution for $\mathbf{=16}$, $\boldsymbol{\gamma=\delta=0}$, left and $\boldsymbol{\gamma=\beta=-0.5}$ right with $\boldsymbol{v=0.9}$

Example 2(36):- Given the following linear fractional order Volterra-Fredholm integro-differential equation:

$$\frac{d^2u(x)}{dx^2}+D_x^{v}u(x)+u(x)=f(x)-2\int_0^x K_1(x,t)u(t)dt+\int_0^1 K_2(x,t)u(t)\,dt,\ 0<\alpha\le 1 \quad \ldots(38)$$

With boundary conditions as follows:

$$u(0)=u(1)=0 \quad \ldots(39)$$

Where $f(x)=-\frac{1}{30}-6x+\frac{181x^2}{20}+4x^3-\frac{x^5}{10}+\frac{x^6}{15}$,
And $k_1(x,t)=x-t$, $k_2(x,t)=x^2-t$.

The exact solution of this problem for $v=1$ is $u(x)=x^3(x-1)$.
The approximate solutions which are obtained by using the present method for $j=4$ $q=0,1,\ldots,3$ where x_q are roots of the shifted Jacobi polynomial $J_4^{(\gamma,\delta)}(x)$ and their values for γ and δ are:

x_q	$\gamma=\delta=0$	$\gamma=\delta=-0.5$
x_0	0.9306	0.9619
x_1	0.6700	0.6913
x_2	0.3300	0.3087
x_3	0.0694	0.0381

Also t_d are the roots of Jacobi polynomial $J_5^{(\gamma,\delta)}(t)$ with w_d are the corresponding weights and their values are:

	t_d	w_d
$\gamma=\delta=-0.5$	$t_0=0$	$w_0=0.628319$
	$t_1=-0.587785$	$w_1=0.628319$
	$t_2=0.587785$	$w_2=0.628319$
	$t_3=-0.951057$	$w_3=0.628319$
	$t_4=0.951057$	$w_4=0.628319$
$\gamma=\delta=0$	$t_0=-0.9061798459$	$w_0=0.2369268850$
	$t_1=0.9061798459$	$w_1=0.2369268850$
	$t_2=-0.5384693101$	$w_2=0.4786286704$
	$t_3=0.5384693101$	$w_3=0.4786286704$
	$t_4=0$	$w_4=0.5688888888$

Using Eq.(27) we have:

$$\sum_{j=0}^{4} c_j\, b_{j,2}^{(\gamma,\delta)} J_{L,j-2}^{(\gamma+2,\delta+2)}(x_q) + \sum_{j=1}^{4}\sum_{n=0}^{j-1} c_j\, b_{j,n}^{(0.75)} x_q^{\,j-n-0.75} + \sum_{j=0}^{4} c_j J_{L,j}^{(\gamma,\delta)}(x_q) = f(x_q) - \lambda_1 x_q \sum_{d=0}^{4} w_d\left(x_q-\xi_{t_d}\right) \sum_{j=0}^{4} c_j J_{L,j}^{(\gamma,\delta)}\left(\xi_{t_d}\right) + \frac{\lambda_2(b-a)}{2}\sum_{d=0}^{m} w_d\left(x_q^2-\chi_{t_d}\right)\sum_{j=0}^{4} c_j\, J_{L,j}^{(\gamma,\delta)}\left(\chi_{t_d}\right)$$

Next, after using (15), in Eq. (39), we can get

$$\sum_{j=0}^{4} c_j (-1)^i \frac{\Gamma(i+\delta+1)}{i!\,\Gamma(\delta+1)} = 0,$$

$$\sum_{j=0}^{4} c_j \frac{\Gamma(i+\gamma+1)}{i!\,\Gamma(\gamma+1)} = 1$$

By applying the suggested method for solving Eq. (38), the diagrams of the exact and approximate solution when $\boldsymbol{v}=0.9$ and $\mathrm{m}=16$ are presented in Fig.3. Also the AE for different values of γ, δ with $\boldsymbol{v}=0.5$ and $\mathrm{m}=4$, $\mathrm{m}=8$ between our methods with the method given in (36) are shown in Tables 2 and 3.

Table 2. The comparison of AE between approximate solution and the solution in method (36) for $\mathbf{m}=4$, with $\boldsymbol{v}=\mathbf{0.5}$

Our method for $\mathbf{m}=4, \boldsymbol{v}=\mathbf{0.5}$				method (36)	
$\mathbf{x_q}$	$\gamma=\delta=-\mathbf{0.5}$	$\mathbf{x_q}$	$\gamma=\delta=\mathbf{0}$	**x**	$\mathbf{v}=\mathbf{0.5}$
0.0381	5.1756e-04	0.0694	5.3877e-04	0.1	1.945e-03
0.3087	2.6469e-03	0.3300	3.3310e-03	0.3	4.551e-03
0.6913	4.3335e-4	0.6700	2.6200e-04	0.7	2.394e-03
0.9619	1.7069e-02	0.9306	1.4028e-02	0.9	`4.607-03

Table 3. The comparison of AE between approximate solution and the solution in method (36) for $\mathbf{m}=8$, with $\boldsymbol{v}=\mathbf{0.5}$

Our method for $\boldsymbol{m}=\mathbf{8}$ $\boldsymbol{v}=\mathbf{0.75}$				method (36)	
$\boldsymbol{x_q}$	$\gamma=\delta=-\mathbf{0.5}$	$\boldsymbol{x_q}$	$\gamma=\delta=\mathbf{0}$	**x**	$\mathbf{v}=\mathbf{0.5}$
0.0096	3.3954e-05	0.0199	2.7061e-04	0.1	1.945e-03
0.0843	2.9156e-04	0.1017	1.7168e-04	0.2	3.602 e-03
0.2222	5.9122e-04	0.2372	3.0262e-03	0.3	4.551 e-03
0.4025	6.3878e-04	0.4083	2.2260e-03	0.4	4.411 e-03
0.5975	2.5112e-04	0.5917	6.4991e-03	0.6	5.421 e-04
0.7778	5.1331e-04	0.7628	3.8925e-04	0.7	2.394 e-03
0.9157	1.3613e-03	0.8983	1.1815e-04	0.8	4.665 e-03
0.9904	1.4384e-03	0.9801	1.7664e-02	0.9	4.607 e-03

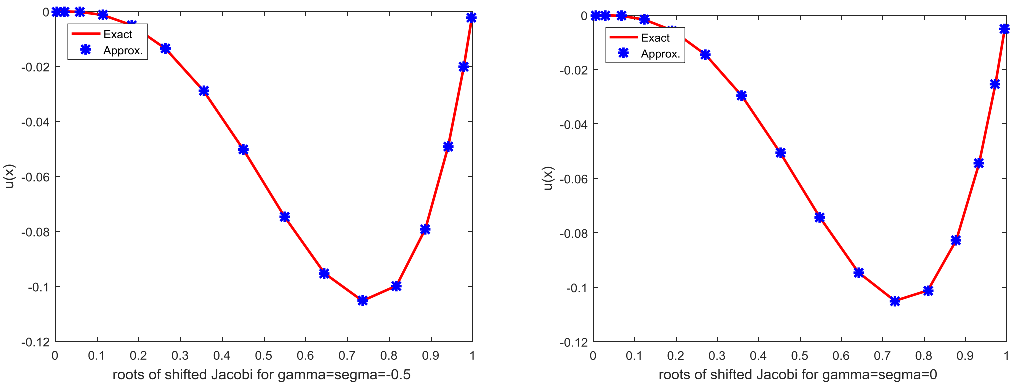

Figure 3. The comparison between the exact and approximate solution for $\mathbf{m}=\mathbf{16}$ and $\boldsymbol{\gamma}=\boldsymbol{\delta}=\mathbf{0}$ left $\boldsymbol{\gamma}=\boldsymbol{\delta}=-\mathbf{0.5}$ right with $\boldsymbol{v}=\mathbf{0.9}$

Conclusions:

In this work, Jacobi-collocation method is used to solve fractional order Volterra-Fredholm integro-differential equation. The properties of shifted Jacobi polynomials together with the collocation method are utilized to reduce the fractional order Volterra-Fredholm integro-differential equation to the solution of algebraic equations. For the effectiveness for this method, it is applied to some examples and obtained results are compared between the approximate solution of the proposed method with the solution given in (36) and they were presented.

Conflicts of Interest: None.

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طريقة عددية كفوءة لحل المعادلة التفاضلية التكاملية من نوع فولتيرا-فريدهولم ذات الرتبة الكسرية بأستخدام متعددة حدود جاكوبي وطريقة الحشد

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الخلاصة:
ان الهدف من هذا البحث هو ايجاد الحل العددي للمعادلة التفاضلية التكاملية من نوع فريدهولم-فولتيرا ذات الرتب الكسرية بأستخدام متعددة حدود جاكوبي المتحولة وطريقة الحشد حيث تم التطرق الى خواص متعددة حدود جاكوبي وطريقة الحشد. يتم تحويل المسألة بهذه الطريقة المقترحة الى مسألة لحل نظام من المعادلات الجبرية الخطية، المشتقة الكسرية هنا من نوع كابوتو. تم اعتماد مجموعة من الامثلة لأثبات دقة وموثوقية الطريقة المقترحة.

الكلمات المفتاحية: طريقة الحشد، المشتقة الكسرية، متعددة حدود جاكوبي، المعادلات تفاضلية تكاملية من نوع فولتيرا-فريدهولم ذات الرتبة الكسرية .