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ESTIMATION OF DYNAMIC CUMULATIVE PAST ENTROPY FOR POWER FUNCTION DISTRIBUTION

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1. Introduction

The power function distribution (PFD) is a flexible model, which is used for analysing different types of data in the area of lifetime, reliability and income distributions etc. The probability density function of PFD is given by

$$f(x,\beta,\alpha)=\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{-(\alpha-1)}; \quad 0<x<\beta, \quad \alpha,\beta>0, \tag{1}$$

and cumulative distribution function is given by

$$F(x)=\left(\frac{x}{\beta}\right)^{\alpha}; \quad 0<x<\beta, \quad \alpha,\beta>0, \tag{2}$$

where α is the scale parameter and β is the shape parameter. Many authors have discussed the problem of estimation of parameters of PFD distribution using Bayesian techniques. Bagchi and Sarkar (1986) discussed Bayes interval estimation for the shape parameter and the reliability function of the PFD, Meniconi and Barry (1996) discussed the electrical component reliability using the PFD, Omar and Low (2012) have discussed Bayesian estimation of generalized PFD under non-informative and informative priors, Rahman *et al.* (2012) discussed the Bayesian method to estimate the parameters of PFD, Zaka and Akhter (2014) discussed Bayesian analysis of PFD using different loss functions. Abdul-Sathar *et al.* (2015a) have discussed Bayes estimation of Lorenz curve and Gini-index for PFD and Abdul-Sathar *et al.* (2015b) have discussed quasi-Bayesian estimation of Lorenz curve and Gini-index in the Power model. However not much work

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is carried out on estimation of entropy functions of PFD using different loss functions. Hence in this paper we proposed the estimation of parameters and dynamic cumulative past entropy of two parameter PFD.

A fundamental uncertainty measure of a random variable is known as entropy and was introduced by Shannon (1948). Let X be a non-negative absolutely continuous random variable with probability density function $f(x)$, then the Shannon entropy is defined as

$$H(f) = -\int_0^\infty f(x)\ln[f(x)]\,dx. \tag{3}$$

Frequently, in survival analysis and in life testing one has information about the current age of the component under consideration. In such cases, the age must be taken into account when measuring uncertainty. Obviously, the measure $H(f)$ defined in (3) is unsuitable in such situations and must be modified to take the age into account. Given that a component has survived up to time t, then residual entropy is defined as

$$H(f,t) = -\int_t^\infty \frac{f(x)}{\overline{F}(t)}\ln\left[\frac{f(x)}{\overline{F}(t)}\right]dx. \tag{4}$$

It is reasonable to presume that in many realistic situations uncertainty is not necessarily related to the future but can also refer to the past. For instance, if at time t, a system which is observed only at certain preassigned inspection times, is found to be down; then the uncertainty of the system life relies on the past, that is, on which instant in (0, t) it has failed. Based on this idea Di Crescenzo and Longobardi (2002) have studied the past entropy over (0, t). They have discussed the necessity of the past entropy, its relation with residual entropy and many interesting results. Di Crescenzo and Longobardi (2004) proposed a measure of discrimination based on past entropy. If X is a random variable having an absolutely continuous distribution function F with probability density function f, then dynamic past entropy of the random variable X is defined as

$$\overline{H}(t) = \int_0^t \frac{f(x)}{F(t)}\ln\left[\frac{f(x)}{F(t)}\right]dx. \tag{5}$$

However, some inefficiencies inherited by (3) motivated various authors to introduce other suitable measures of information. Di Crescenzo and Longobardi (2009) introduced cumulative past entropy (CPE), which is given by

$$\mathcal{E}(t) = -\int_0^\infty F(x)\ln[F(x)]\,dx. \tag{6}$$

Although CPE defined above have wide range of applications, this measure is not applicable for a system which have survived up to some unit of time or failed before some specified time instant. When the duration of phenomenon is considered, dynamic cumulative past entropy (DCPE) measure, which is useful to measure information on the inactivity time $[t - X | X \leq t]$, is defined by Di Crescenzo and Longobardi (2009) and is given by

$$\overline{\mathscr{E}}(t) = -\int_0^t \frac{F(x)}{F(t)} \ln\left[\frac{F(x)}{F(t)}\right] dx. \tag{7}$$

For practical purpose we need to develop some inference techniques using (7). Hence in this paper proposed estimators of DCPE of PFD using MLE and Bayesian methods.

The rest of the article is organized as follows. MLE, asymptotic and bootstrap confidence intervals for DCPE of the PFD are discussed in Section 2. In Section 3, we discussed the Bayes estimators of the DCPE using the loss functions SELF, ELF and LLF. We used important sampling procedure and Lindley approximation methods to simplify the ratio of integrals involved in the proposed Bayes estimators of $\overline{\mathscr{E}}(t)$. We also proposed HPD credible intervals of $\overline{\mathscr{E}}(t)$ in the same section. In Section 4, Monte Carlo simulation study is discussed for studying the performance of the estimators. In Section 5, a real life data set is used to illustrate the estimation procedure and a conclusion is given in Section 6.

2. MLE, ASYMPTOTIC AND BOOTSTRAP CONFIDENCE INTERVALS

In this section, we discuss the MLE, asymptotic and bootstrap confidence intervals of the parameters and DCPE for the PFD. Belzunce *et al.* (1998) obtained the ML estimates of the parameters of PFD and is given respectively by

$$\hat{\alpha} = \frac{n}{\sum_{j=1}^{n} (X_j - X_{(1)})} \quad \text{and} \quad \hat{\beta} = X_{(n)}, \tag{8}$$

where $X_{(n)} = \text{Max}(X_1, X_2, ..., X_n)$ and $X_{(1)} = \text{Min}(X_1, X_2, ..., X_n)$.
The DCPE for (1) is simplified as

$$\overline{\mathscr{E}}(t) = \frac{\alpha t}{(\alpha + 1)^2}. \tag{9}$$

Using the invariance property of MLE, the MLE of the DCPE is given by

$$\hat{\overline{\mathscr{E}}}(t) = \frac{\hat{\alpha} t}{(\hat{\alpha} + 1)^2},$$

where $\hat{\alpha}$ is given by (8).

2.1. Asymptotic confidence interval

In this section, we derived the asymptotic confidence interval and coverage probability of $\overline{\mathscr{E}}(t)$. In practice, the observed information matrix is used as a consistent estimator of the Fisher information matrix. The Fisher information matrix of $\phi = (\alpha, \beta)$ is given by

$$I(\phi) = E \begin{bmatrix} -\frac{\partial^2 \ln[L(\alpha,\beta)]}{\partial \alpha^2} & -\frac{\partial^2 \ln[L(\alpha,\beta)]}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \ln[L(\alpha,\beta)]}{\partial \beta \partial \alpha} & -\frac{\partial^2 \ln[L(\alpha,\beta)]}{\partial \beta^2} \end{bmatrix}.$$

The likelihood function using (1) can be derived as

$$L(\alpha, \beta) = \frac{\alpha^n}{\beta^{n\alpha}} \prod_{i=1}^{n} x_i^{(\alpha-1)}. \tag{10}$$

Differentiating twice the logarithm of (10) with respect to α and β respectively, we get

$$\frac{\partial^2 \ln[L(\alpha, \beta)]}{\partial \alpha^2} = -n/\alpha^2,$$

$$\frac{\partial^2 \ln[L(\alpha, \beta)]}{\partial \beta^2} = -n\alpha/\beta^2$$

and

$$\frac{\partial^2 \ln[L(\alpha, \beta)]}{\partial \alpha \partial \beta} = \frac{\partial^2 \ln[L(\alpha, \beta)]}{\partial \beta \partial \alpha} = -n/\beta.$$

Using the delta method, we derive the asymptotic distribution of $\hat{\overline{\mathscr{E}}}(t)_{mle}$. For that, we have

$$v\hat{a}r(\hat{\overline{\mathscr{E}}}(t)_{mle}) = v\hat{a}r(\overline{\mathscr{E}}(\hat{\alpha})) \approx \omega^{'}((\hat{\alpha})[I(\hat{\alpha})]^{-1}\omega((\hat{\alpha})),$$

where

$$\omega(\hat{\alpha}) = \left[\frac{\partial(\overline{\mathscr{E}}(\hat{\alpha}))}{\partial(\alpha)} \right]_{(\alpha)=(\hat{\alpha})} = [\omega_\alpha],$$

with

$$\omega_\alpha = \frac{t(1-\hat{\alpha})}{(\hat{\alpha}+1)^3}.$$

Thus $\frac{\hat{\overline{\mathscr{E}}}(t)_{mle} - \overline{\mathscr{E}}(t)}{\sqrt{v\hat{a}r(\hat{\overline{\mathscr{E}}}(t))}}$ is asymptotically distributed as $N(0, 1)$. Hence $100(1-\xi)\%$ confidence interval for $\hat{\overline{\mathscr{E}}}(t)$ is given as $\hat{\overline{\mathscr{E}}}(t)_{mle} \pm z_{\xi/2}\sqrt{\hat{V}ar\left(\hat{\overline{\mathscr{E}}}(t)_{mle}\right)}$. Also, the coverage probability for $\overline{\mathscr{E}}(t)_{mle}$ is defined as

$$CP_{\overline{\mathscr{E}}(t)_{mle}} = P\left[\left|\frac{\left(\hat{\overline{\mathscr{E}}}(t)_{mle} - \overline{\mathscr{E}}(t)_{mle}\right)}{\sqrt{\hat{V}ar\left(\hat{\overline{\mathscr{E}}}(t)_{mle}\right)}}\right| \leq z_{\xi/2}\right],$$

where $z_{\xi/2}$ is the sample $(\xi/2)^{th}$ percentile of standard normal distribution.

2.2. *Bootstrap confidence interval*

In this section, we derive the confidence intervals for $\overline{\mathscr{E}}(t)_{mle}$ based on the percentile bootstrap method discussed by Davison and Hinkley (1997). The percentile bootstrap confidence interval can be derived using the follows steps.

1. Compute the MLE $\hat{\alpha}^{(0)}$ of α using the sample.
2. Generate a bootstrap sample using $\hat{\alpha}^{(\omega-1)}$ and obtain the MLE $\hat{\alpha}^{(\omega)}$ using the bootstrap sample.
3. Obtain the MLE of DCPE $\hat{\overline{\mathscr{E}}}(t)_{\omega} = \overline{\mathscr{E}}(\hat{\alpha}^{\omega})$
4. Put $\omega = \omega + 1$.
5. Repeat 2-4, N times to have $\hat{\overline{\mathscr{E}}}(t)_{w}$ for $w = 1, 2, ..., N$.
6. Arrange $\hat{\overline{\mathscr{E}}}(t)_{\omega}$ for $\omega = 1, 2, ..., N$ in ascending order as $\hat{\overline{\mathscr{E}}}(t)_{(1)} \leq \hat{\overline{\mathscr{E}}}(t)_{(2)} .... \leq \hat{\overline{\mathscr{E}}}(t)_{(N)}$ respectively.

Then the $100(1-\xi)$ percentile bootstrap CI for DCPE is given by

$$(\hat{\overline{\mathscr{E}}}(t)_{(N(\xi/2)}, \hat{\overline{\mathscr{E}}}(t)_{(N(1-\xi/2))}).$$

3. BAYES ESTIMATION

In this section, we discuss the Bayesian estimation of $\overline{\mathscr{E}}(t)$ for PFD using different loss functions. In Bayes estimation, the unknown parameter is treated as a random variable and assumes a prior distribution. Here we use the independent gamma priors for the parameters α and β of PFD and is given respectively as

$$g_1(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)} \alpha^{a_1 - 1} e^{-b_1 \alpha}, \alpha > 0, a_1, b_1 > 0 \tag{11}$$

and

$$g_2(\beta) = \frac{{b_2}^{a_2}}{\Gamma(a_2)} \beta^{a_2-1} e^{-b_2\beta}, \beta > 0, a_2, b_2 > 0. \tag{12}$$

It is noted that the non-informative priors on the shape and scale parameters are the special cases of independent gamma priors. Then the joint prior of α and β is obtained as

$$g(\alpha,\beta) \propto \alpha^{a_1-1}\beta^{a_2-1}e^{-(b_1\alpha+b_2\beta)}. \tag{13}$$

Using (10) and (13), the joint posterior distribution can be written as

$$\Pi(\alpha,\beta) \;=\; K\alpha^{\lambda_1-1}e^{-\gamma_1\alpha}\beta^{\lambda_2-1}e^{-\gamma_2\beta}e^{-\sum_{i=1}^{n}\ln(x_i)}, \tag{14}$$

where $\lambda_1 = n + a_1$, $\gamma_1 = b_1 - \sum_{i=1}^{n} \ln(x_i)$, $\lambda_2 = a_2 - n\alpha$, $\gamma_2 = b_2$ and K is the normalizing constant and is given by

$$K^{-1} = \int_0^\infty \int_0^\infty \alpha^{\lambda_1-1}e^{-\gamma_1\alpha}\beta^{\lambda_2-1}e^{-\gamma_2\beta}e^{-\sum_{i=1}^{n}\ln(x_i)}d\alpha d\beta.$$

In Bayesian approach, to arrive at the best estimator, one has to choose a loss function corresponding to each of the possible estimates. We consider symmetric as well as asymmetric loss functions. Here we use the squared error loss function (SELF), entropy loss function (ELF) and Linex loss function (LLF) for estimating the parameters. Symmetric loss function is the squared error loss (SEL) function which is defined as

$$L_1[d(\phi), d(\hat{\phi})] = [d(\phi) - d(\hat{\phi})]^2$$

with $d(\hat{\phi})$ being an estimate of $d(\phi)$. Here $d(\phi)$ denotes some parametric function of ϕ. For this situation, the Bayesian estimate, say $d(\hat{\phi})$, is given by the posterior mean of $d(\phi)$.

One of the most commonly used asymmetric loss function is the Linex loss (LL) function which is defined by

$$L_2[d(\phi), d(\hat{\phi})] = \exp[h(d(\phi) - d(\hat{\phi}))] - h(d(\phi) - d(\hat{\phi})) - 1, h \neq 0.$$

Bayes estimate of $d(\phi)$ for LL function is obtained as

$$d(\hat{\phi}) = -\frac{1}{h}\log(E_\phi(\exp(-h\phi)/X)),$$

provided the above expectation exists.

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $d(\hat{\phi})/d(\phi)$. In this case, a useful asymmetric loss function is the general entropy loss (EL) function proposed by Calabria and Pulcini (1996) and is given by

$$L_3[d(\phi), d(\hat{\phi})] \propto \left[\frac{[d(\hat{\phi})]}{d(\phi)}\right]^c - c\left(\log\left(\frac{[d(\hat{\phi})]}{d(\phi)}\right)\right) - 1, c \neq 0.$$

Bayes estimate of $d(\phi)$ using EL function is obtained as

$$\hat{d}_E(\phi) = [E_\phi(\phi^{-c})/X]^{-1/c},$$

provided the above expectation exists.

The Bayes estimators of $\overline{\mathscr{E}}(t)$, under SEL, LL, and EL are the posterior expectation of $\overline{\mathscr{E}}(t)$ and is given respectively as

$$\overline{\mathscr{E}}(t)_{self} = \frac{\int_0^\infty \int_0^\infty \overline{\mathscr{E}}(t) \Pi(\alpha,\beta) g(\alpha,\beta) d\alpha d\beta}{\int_0^\infty \int_0^\infty \Pi(\alpha,\beta) g(\alpha,\beta) d\alpha d\beta}, \tag{15}$$

$$\overline{\mathscr{E}}(t)_{elf} = \frac{\int_0^\infty \int_0^\infty \overline{\mathscr{E}}(t)^{-c} \Pi(\alpha,\beta) g(\alpha,\beta) d\alpha d\beta}{\int_0^\infty \int_0^\infty \Pi(\alpha,\beta) g(\alpha,\beta) d\alpha d\beta} \tag{16}$$

and

$$\overline{\mathscr{E}}(t)_{llf} = -\frac{1}{h}\log\left[\frac{\int_0^\infty \int_0^\infty e^{-h\overline{\mathscr{E}}(t)} \Pi(\alpha,\beta) g(\alpha,\beta) d\alpha d\beta}{\int_0^\infty \int_0^\infty \Pi(\alpha,\beta) g(\alpha,\beta) d\alpha d\beta}\right], \tag{17}$$

where $\Pi(\alpha,\beta)$ is given by (14) and $g(\alpha,\beta)$ is given by (13). It can be see that Bayes estimators are in the form of ratio of integrals, which cannot be simplified to closed forms. So, we use two approximation methods, namely, the Lindley approximation and importance sampling methods to solve the above ratio of integrals.

3.1. Lindley's approximation method

A number of approximate methods are available to solve the ratio of integrals. One of the simplest method is Lindley's approximation method proposed by Lindley (1980).

The Bayes estimates under SEL, EL and LL given in (15) to (17) using Lindley approximation can be written respectively as

$$\hat{\overline{\mathscr{E}}}(t)_{self} = \overline{\mathscr{E}}(t) + \frac{1}{2}[H + R_{30}H_{12} + R_{21}G_{12} + R_{12}G_{21} + R_{03}H_{21}], \quad (18)$$

$$\hat{\overline{\mathscr{E}}}(t)_{elf} = [\overline{\mathscr{E}}(t)]^{-c} + \frac{1}{2}[H + R_{30}H_{12} + R_{21}G_{12} + R_{12}G_{21} + R_{03}H_{21}] \quad (19)$$

and

$$\hat{\overline{\mathscr{E}}}(t)_{llf} = e^{-h[\overline{\mathscr{E}}(t)]} + \frac{1}{2}[H + R_{30}H_{12} + R_{21}G_{12} + R_{12}G_{21} + R_{03}H_{21}], \quad (20)$$

where $H = \sum_{i=1}^{2}\sum_{j=1}^{2}\delta_{ij}\nu_{ij}$, $R_{ij} = \frac{\partial^{i+j}\ln[\Pi(\alpha,\beta)]}{\partial(\alpha)^i\partial(\beta)^j}, i,j = 0,1,2,3$ and $i + j = 3$, $\delta_i = \frac{\partial\overline{\mathscr{E}}(t)}{\partial\alpha_i}$, $\delta_j = \frac{\partial\overline{\mathscr{E}}(t)}{\partial\beta_j}$, $\delta_{ij} = \frac{\partial\overline{\mathscr{E}}(t)}{\partial\alpha_i\partial\beta_j}$ and $i \neq j$, $H_{ij} = (\delta_i\nu_{ii} + \delta_j\nu_{ij})\nu_{ii}$, $G_{ij} = 3\delta_i\nu_{ii}\nu_{ij} + \delta_j(\nu_{ii}\nu_{jj} + 2\nu_{ij}^2)$ and ν_{ij} is the $(i,j)^{\text{th}}$ element in the inverse of matrix $\overline{\mathscr{E}}(t) = \{-\overline{\mathscr{E}}(t)_{ij}\}$ with $\overline{\mathscr{E}}(t)_{ij} = \frac{\partial^2\overline{\mathscr{E}}(t)}{\partial\alpha_i\partial\beta_j}$.

3.2. *Importance sampling procedure*

In this section, we discuss the importance sampling procedure to derive the ratio of integrals for finding the Bayes estimator of $\overline{\mathscr{E}}(t)$. We also derive the HPD credible intervals of $\overline{\mathscr{E}}(t)$. The joint posterior distribution given in (14) can be written as

$$\begin{aligned} \Pi(\alpha,\beta) &\propto \alpha^{\lambda_1-1}e^{-\alpha\gamma_1}\beta^{\lambda_2-1}e^{-\beta\gamma_2}e^{-[(a2-n\alpha)\ln b2-\sum_{i=1}^{n}\ln(x_i)]} \\ &\propto f(\alpha;\lambda_1,\gamma_1)f(\beta|\alpha;\lambda_2,\gamma_2)h(\alpha,\beta), \end{aligned} \quad (21)$$

where

$$h(\alpha,\beta) = \Gamma(a_2 - n\alpha) * e^{-[(a_2-n\alpha)\ln b_2-\sum_{i=1}^{n}\ln(x_i)]} \quad (22)$$

and

$$f(\alpha;\lambda_1,\gamma_1) \propto \alpha^{\lambda_1-1}e^{-\alpha\gamma_1}. \quad (23)$$

$$f(\beta;\lambda_2,\gamma_2) \propto \beta^{\lambda_2-1}e^{-\beta\gamma_2} \quad (24)$$

The following steps are used in the important sampling procedure.

1. Generate β_1 from $f(\beta;\lambda_2,\gamma_2)$
2. For the generated value of β_1, generate α_1 from $f(\beta|\alpha;\lambda_1,\gamma_1)$
3. Repeat 1–2, n times, to obtain the importance sample $(\alpha_1,\beta_1),(\alpha_2,\beta_2),...,(\alpha_n,\beta_n)$.

Using important sampling procedure, the Bayes estimators of the $\overline{\mathscr{E}}(t)$ given in (15) to (17) are given respectively by

$$\hat{\overline{\mathscr{E}}}(t)_{self} = \frac{\sum_{j=1}^{n} \overline{\mathscr{E}}(t) h\left(\alpha_j, \beta_j\right)}{\sum_{j=1}^{n} h\left(\alpha_j, \beta_j\right)}, \tag{25}$$

$$\hat{\overline{\mathscr{E}}}(t)_{llf} = -\frac{1}{h} \log\left[\frac{\sum_{j=1}^{n} e^{-h[\overline{\mathscr{E}}_X(t)]} h\left(\alpha_j, \beta_j\right)}{\sum_{j=1}^{n} h\left(\alpha_j, \beta_j\right)}\right] \tag{26}$$

and

$$\hat{\overline{\mathscr{E}}}(t)_{elf} = \frac{\sum_{j=1}^{n} \overline{\mathscr{E}}(t)^{-c} h\left(\alpha_j, \beta_j\right)}{\sum_{j=1}^{n} h\left(\alpha_j, \beta_j\right)}, \tag{27}$$

where $h\left(\alpha_j, \beta_j\right)$ is given by (22).

3.3. HPD credible interval estimation

In this subsection, we construct the HPD intervals of $\overline{\mathscr{E}}(t)$ using the procedure discussed by Chen and Shao (1999). Define $\overline{\mathscr{E}}(t)_{(s)} = \overline{\mathscr{E}}(\alpha^{(s)}, \beta^{(s)})$ where $\alpha^{(s)}$ and $\beta^{(s)}$ for $s = 1, 2, ..., M$ are posterior samples generated respectively using (23) and (24) for α and β. Let $\overline{\mathscr{E}}(t)_{(s)}$ be the ordered values of $\overline{\mathscr{E}}(t)$. Define

$$\delta_i = \frac{h(\alpha^{(s)}, \beta^{(s)})}{\sum_{i=1}^{M} h(\alpha^{(s)}, \beta^{(s)})}.$$

Therefore, when the q^{th} rate of $\overline{\mathscr{E}}(t)$ can be estimated as

$$\hat{\overline{\mathscr{E}}}(t) = \begin{cases} \overline{\mathscr{E}}(t)_{(1)} & \text{if } q = 0 \\ \overline{\mathscr{E}}(t)_{(i)} & \text{if } \sum_{j=1}^{i-1} \delta_j < q < \sum_{j=1}^{i} \delta_j \end{cases}$$

The $100 \times (1-\xi)\%$ where $0 < \xi < 1$, confidence interval for $\overline{\mathscr{E}}(t)$ is given by $(\hat{\overline{\mathscr{E}}}(t)^{j/M}, \hat{\overline{\mathscr{E}}}(t)^{(j+[(1-\xi)M])/M})$, $j = 1, 2, ..., M$, where $[.]$ is the greatest integer function. Then the desired HPD interval for $\overline{\mathscr{E}}(t)$ is the interval with smallest width.

4. Simulation study

In this section, we conduct a simulation to study the performance of the estimators developed in the previous sections. We have generated sample of sizes: $n = 20$, 40, 70, 100, 150 and 200 from (1) with $\alpha = (1.5, 7, \text{by} = 0.5)$. We fix $(t, c, h) = (3, 1, 2)$. The results of the simulation study are presented in Tables 1-5. The MSE of $\overline{\mathscr{E}}(t)$ using Lindley approximation method and important sampling procedures are summarized in Table 2-5. In Figures 1-4, we plot the MSE of Bayes estimates of $\overline{\mathscr{E}}(t)$ against t.

From Tables 1-5, the following conclusions are made: 1) In most of the cases, the bias and MSE of all estimators decreases as n increases; 2) Bayes estimators perform better in terms of MSE compared to the MLE; 3) The perform of the estimators are more and less same for different loss functions.

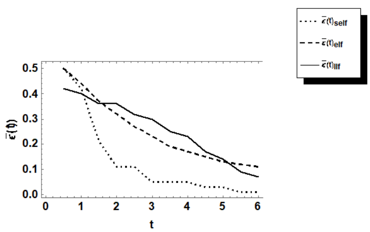

Figure 1 – When $c = 1$ and $h = 1$, the MSE of $\overline{\mathscr{E}}(t)$ using Lindley approximation method.

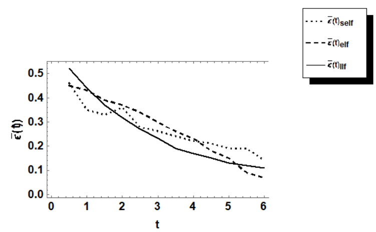

Figure 2 – When $c = 1$ and $h = 1$, the MSE of $\overline{\mathscr{E}}(t)$ using important sampling procedure.

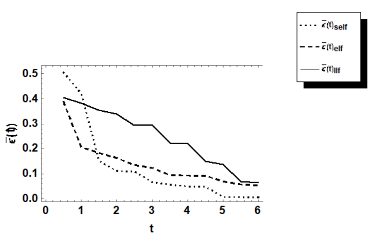

Figure 3 – When $c = 2$ and $h = 2$, the MSE of $\overline{\mathscr{E}}(t)$ using Lindley approximation method.

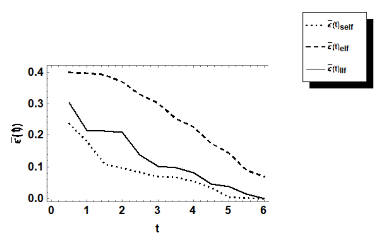

Figure 4 – When $c = 2$ and $h = 2$, the MSE of $\overline{\mathscr{E}}(t)$ using important sampling procedure.

TABLE 1
The bias and MSE for MLE and AIL and CP for CI for $\overline{\mathscr{E}}(t)$.

n	α	$\overline{\mathscr{E}}(t)$	$\overline{\mathscr{E}}(t)_{mle}$		Confidence interval				HPD	
					Bootstrap		Asymptotic			
			Bias	MSE	AIL	CP	AIL	CP	AIL	CP
20	1.5	0.72	0.58	0.43	0.49	0.99	0.14	0.99	0.63	0.99
20	2	0.67	0.29	0.08	0.33	0.96	0.12	0.94	0.59	0.99
40	2.5	0.61	0.27	0.07	0.60	0.99	0.14	0.99	0.56	0.99
40	3	0.56	0.23	0.06	0.55	0.99	0.19	0.96	0.48	0.98
70	3.5	0.52	0.11	0.01	0.62	0.99	0.18	0.96	0.34	0.91
70	4	0.48	0.07	0.00	0.48	0.94	0.12	0.94	0.33	0.90
100	4.5	0.45	0.02	0.00	0.10	0.97	0.19	0.96	0.32	0.91
100	5	0.42	0.02	0.00	0.75	0.99	0.16	0.95	0.31	0.97
150	5.5	0.39	0.01	0.00	0.43	0.95	0.16	0.96	0.30	0.92
150	6	0.37	0.02	0.00	0.37	0.95	0.17	0.96	0.30	0.91
200	6.5	0.35	-0.02	0.00	0.57	0.94	0.11	0.94	0.29	0.90
200	7	0.33	-0.01	0.00	0.48	0.96	0.19	0.96	0.26	0.93

TABLE 2
The bias and MSE of Lindley approximation method for Bayes estimators of $\overline{\mathscr{E}}(t)$ using $c = 1$ and $h = 1$.

n	α	$\overline{\mathscr{E}}(t)$	$\overline{\mathscr{E}}(t)_{self}$		$\overline{\mathscr{E}}(t)_{elf}$		$\overline{\mathscr{E}}(t)_{llf}$	
			Bias	MSE	Bias	MSE	Bias	MSE
20	1.5	0.72	-0.71	0.51	-0.71	0.50	0.64	0.42
20	2	0.67	-0.65	0.42	-0.66	0.44	0.64	0.40
40	2.5	0.61	-0.41	0.21	-0.61	0.37	0.60	0.36
40	3	0.56	-0.33	0.11	-0.56	0.32	0.59	0.36
70	3.5	0.52	-0.33	0.11	-0.52	0.27	0.56	0.32
70	4	0.48	-0.22	0.05	-0.48	0.23	0.55	0.30
100	4.5	0.44	-0.21	0.05	-0.46	0.19	0.49	0.25
100	5	0.42	-0.21	0.06	-0.47	0.17	0.48	0.23
150	5.5	0.39	-0.13	0.03	-0.39	0.15	0.41	0.18
150	6	0.37	-0.13	0.03	-0.37	0.13	0.38	0.14
200	6.5	0.35	-0.08	0.00	-0.35	0.12	0.29	0.09
200	7	0.33	-0.02	0.00	-0.33	0.12	0.27	0.07

TABLE 3
The bias and MSE of important sampling procedure for Bayes estimators of $\overline{\mathscr{E}}(t)$ using $c=1$ and $h=1$.

n	α	$\overline{\mathscr{E}}(t)$	$\overline{\mathscr{E}}(t)_{self}$		$\overline{\mathscr{E}}(t)_{elf}$		$\overline{\mathscr{E}}(t)_{llf}$	
			Bias	MSE	Bias	MSE	Bias	MSE
20	1.5	0.72	0.21	0.46	0.67	0.45	0.72	0.52
20	2	0.67	0.13	0.35	0.65	0.43	0.66	0.44
40	2.5	0.61	0.11	0.33	0.63	0.39	0.61	0.37
40	3	0.56	0.09	0.36	0.61	0.37	0.56	0.32
70	3.5	0.52	0.08	0.28	0.58	0.33	0.52	0.27
70	4	0.48	0.07	0.26	0.55	0.30	0.48	0.23
100	4.5	0.45	0.06	0.24	0.51	0.26	0.45	0.19
100	5	0.42	0.04	0.22	0.48	0.23	0.42	0.17
150	5.5	0.39	0.04	0.21	0.43	0.18	0.39	0.15
150	6	0.37	0.05	0.19	0.38	0.15	0.37	0.13
200	5.5	0.35	0.04	0.19	0.31	0.09	0.34	0.11
200	7	0.33	0.01	0.14	0.27	0.07	0.33	0.11

TABLE 4
The bias and MSE Lindley approximation method for Bayes estimators of $\overline{\mathscr{E}}(t)$ using $c=2$ and $h=2$.

n	α	$\overline{\mathscr{E}}(t)$	$\overline{\mathscr{E}}(t)_{self}$		$\overline{\mathscr{E}}(t)_{elf}$		$\overline{\mathscr{E}}(t)_{llf}$	
			Bias	MSE	Bias	MSE	Bias	MSE
20	1.5	0.72	-0.71	0.50	0.49	0.39	0.64	0.41
20	2	0.67	-0.65	0.42	0.45	0.21	0.62	0.38
40	2.5	0.61	-0.34	0.15	0.42	0.18	0.59	0.35
40	3	0.56	-0.33	0.11	0.37	0.16	0.58	0.34
70	3.5	0.52	-0.33	0.11	-0.35	0.13	0.54	0.29
70	4	0.48	-0.26	0.07	-0.34	0.12	0.54	0.29
100	4.5	0.45	-0.23	0.06	-0.31	0.09	0.47	0.22
100	5	0.42	-0.17	0.05	-0.29	0.09	0.47	0.22
150	5.5	0.39	-0.22	0.05	-0.28	0.09	0.39	0.15
150	6	0.37	-0.05	0.00	-0.26	0.07	0.37	0.14
200	6.5	0.35	-0.07	0.00	-0.22	0.06	0.26	0.07
200	7	0.33	-0.01	0.00	-0.19	0.05	0.25	0.06

TABLE 5
The bias and MSE of important sampling procedure for Bayes estimators of $\overline{\mathscr{E}}(t)$ using $c = 2$ and $h = 2$.

n	α	$\overline{\mathscr{E}}(t)$	$\overline{\mathscr{E}}(t)_{self}$		$\overline{\mathscr{E}}(t)_{elf}$		$\overline{\mathscr{E}}(t)_{llf}$	
			Bias	MSE	Bias	MSE	Bias	MSE
20	1.5	0.72	0.49	0.24	0.67	0.39	0.55	0.30
20	2	0.67	0.44	0.18	0.65	0.39	0.34	0.22
40	2.5	0.61	0.33	0.11	0.63	0.39	0.49	0.22
40	3	0.56	0.32	0.09	0.61	0.37	0.46	0.21
70	3.5	0.52	0.29	0.08	0.57	0.33	0.37	0.14
70	4	0.48	0.26	0.07	0.55	0.30	0.32	0.10
100	4.5	0.44	0.26	0.07	0.50	0.25	0.19	0.09
100	5	0.42	0.24	0.06	0.48	0.23	0.19	0.08
150	6.5	0.39	0.21	0.03	0.42	0.17	0.16	0.05
150	6	0.37	0.07	0.00	0.38	0.14	0.14	0.04
200	6.5	0.35	0.04	0.00	0.29	0.09	0.14	0.02
200	7	0.33	0.07	0.00	0.26	0.07	0.02	0.00

TABLE 6
The test of the fitted PFD models to data sets.

	Statistic	P-value
Anderson-Darling	1.27	0.24
Cramer- Von Mises	0.08	0.71
Kolmogrov-Smirnov	0.15	0.60
Watson U^2	0.06	0.58

5. Real data analysis

In this section, we considered the real data-set obtained from Huss and Holme (2007) representing the degrees of metabolites in the metabolic network of the bacterium Escherichia Coli. For this model, using MLE, the estimated parameters are $\alpha = 1.4258$, $\beta = 3.37$, and $\overline{\mathscr{E}}(t)$ =1.45378. Table 6 shows the test values by fitting PFD to the data set. Table 7 shows the value of $\overline{\mathscr{E}}(t)$ under different loss functions using Lindley approximation and important sampling procedure based on real data set. Table 8 provided the CIs and HPD intervals for $\overline{\mathscr{E}}(t)$ based on real data set.

TABLE 7
The Lindley approximation and importance sampling procedure for $\overline{\mathscr{E}}(t)$.

Bayes estimate	$\overline{\mathscr{E}}(t)_{self}$		$\overline{\mathscr{E}}(t)_{elf}$		$\overline{\mathscr{E}}(t)_{llf}$	
	Bias	MSE	Bias	MSE	Bias	MSE
LAM	-0.18	0.14	0.65	0.14	0.53	0.22
ISP	-0.22	0.03	0.26	0.20	0.67	0.41

TABLE 8
The bias and MSE for MLE and AIL and CP for CI for $\overline{\mathscr{E}}(t)$.

Estimate		Confidence interval				HPD	
		Bootstrap		Asymptotic			
Bias	MSE	AIL	CP	AIL	CP	AIL	CP
-0.13	0.02	1.16	0.99	0.19	0.97	0.14	0.95

Figure 5 shows the MLE and Bayes estimates of $\overline{\mathscr{E}}(t)$ under different loss functions with varying values of t. From Figure (5), we can conclude that in general the uncertainty values of the degree of metabolites increases as t increases.

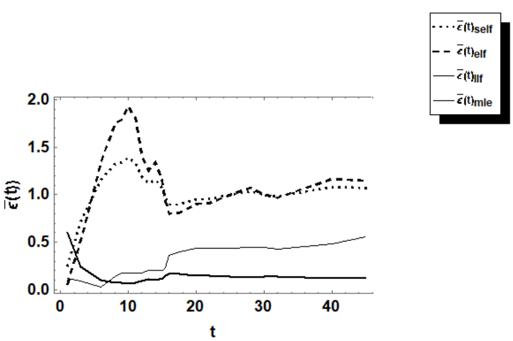

Figure 5 – MLE and Bayes estimates of $\overline{\mathscr{E}}(t)$ under different loss functions for the real data.

6. Conclusion

In this paper, we proposed the estimation of $\overline{\mathscr{E}}(t)$ for a two parameter power function distribution using MLE and Bayesian estimation techniques. The Bayes estimates are simplified using Lindley approximation method and the important sampling procedures. Monte Carlo simulation is used to compare the Bayes estimates of $\overline{\mathscr{E}}(t)$ under different loss functions. A real data set is also used to illustrate the estimation procedures.

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SUMMARY

In this paper, we proposed MLE and Bayes estimators of parameters and DCPE for the two parameter power function distribution. Bayes estimators under different loss functions are obtained using Lindley approximation method and important sampling procedures. A real life data set and a Monte Carlo simulation are used to study the performance of the estimators derived in the article.

Keywords: Power function distribution; Bayes estimators; DCPE; Lindley approximation; Importance sampling; HPD credible interval.