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THE NONLINEAR TRANSMISSION PROBLEM WITH TIME DEPENDENT COEFFICIENTS

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ABSTRACT.- *In this paper we consider the nonlinear transmission problem for the wave equation with time dependent coefficients and linear internal damping. We prove global existence and exponential decay of solution. The result is achieved by considering energy like - Lyapunov functionals and suitable unique continuation theorem for the wave equation.*

KEY WORDS.- *Transmission problem, wave equation, global existence.*

RESUMEN.- *En este trabajo, consideramos el problema de transmisión no lineal para la ecuación de onda con coeficientes dependientes del tiempo y un damping lineal interno. Probamos la existencia global y el decaimiento exponencial de la solución. Los resultados son obtenidos por la consideración de funcionales tipo Lyapunov y un adecuado teorema de continuación única para la ecuación de onda.*

PALABRAS CLAVE.- *Problema de transmisión, ecuación de la onda, existencia global.*

1. INTRODUCTION

In this work, we consider the transmission problem

$$\rho_1 u_{tt} - b u_{xx} + f_1(u) = 0 \quad \text{in} \quad]0, L_0[\times \mathbb{R}^+ \tag{1.1}$$

$$\rho_2 v_{tt} - (a(x,t) v_x)_x + \alpha v_t + f_2(v) = 0 \quad \text{in} \quad]L_0, L[\times \mathbb{R}^+ \tag{1.2}$$

$$u(0,t) = v(L,t)\ , \quad t > 0 \tag{1.3}$$

$$u(L_0,t) = v(L_0,t)\ , \ b u_x(L_o,t) = a(L_0,t)\, v_x(L_0,t)\ , \ t > 0 \tag{1.4}$$

$$u(x,0) = u^0(x)\ , \ u_t(x,0) = u^1(x)\ , \ x \in]0, L_0[\tag{1.5}$$

$$v(x,0) = v^0(x)\ , \ v_t(x,0) = v^1(x)\ , \ x \in]L_0, L[\tag{1.6}$$

where ρ_1, ρ_2 are differents constants; α, b are positive constants, f, g are nonlinear functions and $a(x,t)$ is a positive Controllability and Stability function. This transmission problem has been studied by many authors (see for example J. L. Lions [7], J. Lagnese [5], W. Liu and G. Williams [8], J. Muñoz Rivera and H. Portillo Oquendo [9], D. Andrade, L. H. Fatori and J. Muñoz Rivera [1]).

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All the authors above mentionated established their results with constant coefficients. In base of our knowledge this is a first publication on transmission problem with time dependent coefficients and nonlinear terms.

The goal of this work is to study the existence and uniqueness of global solutions of (1.1) - (1.6) and the assymptotic behavior of the energy.

2. NOTATIONS AND STATEMENT OF RESULTS

We denote

$$(w, z) = \int_I w(x)\, z(x)\, dx \quad , \quad |z|^2 = \int_I |z(x)|^2\, dx$$

where $I =]0, L_0[$ or $]L_0, L[$ for $u's$ and $v's$ respectively. Now, we state the general hypotheses.

(A.1) The function $f_i \in C^1(\mathbb{R})$, $i = 1, 2$, satisfies

$$f_i(s)\, s \geq 0 \quad , \quad \forall\, s \in \mathbb{R}$$
$$\left|f_i^{(J)}(s)\right| \leq c(1 + |s|)^{\rho - J}, \quad \forall\, s \in \mathbb{R}, \quad j = 0, 1$$

for some $c > 0$ and $\rho \geq 1$.

$$f_1(s) \geq f_2(s)$$
$$F_i(s) = \int_0^s f_i(\xi)\, d\xi \quad , \quad i = 1, 2.$$

(A.2) Assumptions on the coefficient a

$$a \in W^{1,\infty}(0, \infty; C^1([L_0, L])) \cap W^{2,\infty}(0, \infty; L^\infty(L_0, L))$$
$$a_t \in L^1(0, \infty; L^\infty(L_0, L))$$
$$a(x, t) \geq a_0 > 0 \quad , \quad \forall (x, t) \in]L_0, L[\times]0, \infty[$$

By V, we denote the Hilbert space

$$V = \left\{ (w, z) \in H^1(0, L_0) \times H^1(L_0, L) \colon w(0) = z(L) = 0;\ w(L_0) = z(L_0) \right\}$$

By E_1 and E_2, we denote the first order associated energy to each equation,

$$E_1(t, u) = \frac{1}{2}\left\{ \rho_1 |u_t|^2 + b|u_x|^2 + 2\int_0^{L_0} F_1(u)\, dx \right\}$$
$$E_2(t, v) = \frac{1}{2}\left\{ \rho_2 |v_t|^2 + (a, v_x^2) + 2\int_{L_0}^{L} F_2(v)\, dx \right\}$$
$$E(t) = E_1(t, u, v) = E_1(t, u) + E_2(t, v).$$

We conclude this section with the following lemma which will play an essential role when establishing the assymptotic behaviour.

Lema 2.1. *Let* $E\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be a non-increasing function and assume that there exists two constants* $p > 0$ *and* $c > 0$ *such that*

$$\int_s^{+\infty} E^{\frac{p+1}{2}}(t)\, dt \le cE(s) \ , \quad 0 \le s < +\infty$$

then we have

$$E(t) \le cE(0)(1+t)^{-\frac{2}{p-1}} \text{ for all } t \ge 0 \text{ if } p > 1$$
$$E(t) \le cE(0)e^{1-wt} \text{ for all } t \ge 0 \text{ if } p = 1$$

where c and w are positive constants.

Proof.

See reference [[2], Lema 9.1].

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS

First of all, we define the weak solutions of problem (1.1) - (1.6).

Definición 3.1.- *We say that the couple* $\{u, v\}$ *is a solution of (1.1) - (1.6) when*

$$\{u, v\} \in L^\infty(0, T; V) \cap W^{1,\infty}(0, T; L^2(0, L_0) \times L^2(L_0, L))$$

and satisfies

$$-\rho_1 \int_0^{L_0} u^1(x)\, \varphi(x,0)\, dx - \rho_2 \int_{L_0}^{L} v^1(x)\, \psi(x,0)\, dx - \rho_1 \int_0^T \int_0^{L_0} u_t \varphi_t dxdt$$
$$-\rho_2 \int_0^T \int_{L_0}^{L} v_t \psi_t dxdt + b \int_0^T \int_0^{L_0} u_x \varphi_x dxdt + \int_0^T \int_0^{L_0} f_1(u)\, \varphi dxdt$$
$$+ \int_0^T \int_{L_0}^{L} a(x,t)\, v_x \varphi_x dxdt + \int_0^T \int_{L_0}^{L} f_2(v)\, \psi dxdt + \alpha \int_0^T \int_{L_0}^{L} v_t \psi dxdt = 0$$

for any $\{\varphi, \psi\} \in L^\infty(0, T; V) \cap W^{1,\infty}(0, T; L^2(0, L_0) \times L^2(L_0, L))$ *such that* $\varphi(T) = 0,\ \psi(T) = 0.$

In order to show the existence of strong solutions, we need a regularity result for the elliptic system associated to the problem (1.1) - (1.6) whose proof can be obtained, with little modifications, in the book by O.A. Ladyzhenskaya and N.N. Ural'seva ([3], theorem 16.2).

Lema 3.2. *For any given functions* $F \in L^2(0, L_0)$, $G \in L^2(L_0, L)$, *there exists only one solution* $\{u, v\}$ *of*

$$-bu_{xx} = F \text{ in }]0, L_0[$$
$$-(a,(x,t)\, v_x)_x = G \text{ in }]0, L_0[$$
$$u(0) = v(L) = 0$$
$$u(L_0) = v(L_0)\, ,\ bu_x(L_0) = a(L_0, t)\, v_x(L_0)$$

with t a fixed value in $[0, T]$, *satisfying*

$$u \in H^2(0, L_0) \text{ and } v \in H^2(L_0, L).$$

The existence result to the system (1.1) - (1.6) is summarized in the following theorem.

Teorema 3.3. *Suppose that* $\{u^0, v^0\} \in V$, $\{u^1, v^1\} \in L^2(0, L_0) \times L^2(L_0, L)$ *and that assumptions (A.1) - (A.2) holds. Then there exists a unique weak solution of (1.1) - (1.6) satisfying*

$$\{u, v\} \in C(0, T; V) \cap C^1(0, T; L^2(0, L_0) \times L^2(L_0, L)).$$

In addition, if $\{u^0, v^0\} \in H^2(0, L_0) \times H^2(L_0, L)$, $\{u^1, v^1\} \in V$, *verifying the compatibility condition below*

$$b u_x^0(L_0) = a(L_0, 0)\ v_x^0(L_0) \tag{3.1}$$

Then

$$\{u, v\} \in \bigcap_{k=0}^{2} W^{k,\infty}(0, T; H^{2-k}(0, L_0) \times H^{2-k}(L_0, L))$$

Proof. The main idea is to use the Galerkin Method.

Let $\{\{\varphi^i, \psi^i\},\ i = 1, 2, ...\}$ be a basis of V.

Let us consider the Galerkin approximation

$$\{u^m(t), v^m(t)\} = \sum_{i=1}^{m} h_{im}(t)\{\varphi^i, \psi^i\}$$

where u^m and v^m satisfy

$$\begin{aligned} &\rho_1(u_{tt}^m, \varphi^i) + b(u_x^m, \varphi_x^i) + (f_1(u^m), \varphi^1) + \varphi_2(v_{tt}^m, \psi^i) + (a(x,t)v_x^m, \psi_x^i) \\ &+ \alpha(v_t^m, \psi^i) + (f_2(v^m), \psi^i) = 0 \end{aligned} \tag{3.2}$$

where $i = 1, 2, ...$

With initial data

$$\{u^m(0), v^m(0)\} = \{u^0, v^0\};\ \{u_t^m(0), v_t^m(0)\} = \{u^1, v^1\} \tag{3.3}$$

standard results about ordinary differential equations guarantee that there exists only one solution of this system on some interval $[0, T_m[$. The priori estimate that follow imply that in fact $T_m = +\infty$.

Weak solutions. Multiplying (3.2) by $h'_{im}(t)$ integrating by parts and summing over i, we get

$$\frac{d}{dt}E(t,u^m,v^m)+\alpha\left|v_t^m\right|^2\le\frac{|a_t(t)|_{L^\infty}}{a_0}E(t,u^m,v^m) \tag{3.4}$$

Proof. From this inequality, the Gronwall's inequality and taking account the definition of the initial data of $\left\{u^m, v^m\right\}$ we conclude that

$$E(t,u^m,v^m)\le C,\ \forall t\in[0,T],\ \forall m\in\mathbb{N} \tag{3.5}$$

thus we deduce that

$$\left\{u^m, v^m\right\} \text{ is bounded in } L^\infty(0,T;V)$$
$$\left\{u_t^m, v_t^m\right\} \text{ is bounded in } L^\infty(0,T;L^2(0,L_0)\times L^2(L_0,L))$$

wich imply that

$$\left\{u^m, v^m\right\} \to \{u,v\} \text{ weakly } * \text{ in } L^\infty(0,T;V)$$
$$\left\{u_t^m, v_t^m\right\} \to \{u,v_t\} \text{ weakly } * \text{ in } L^\infty(0,T;L^2(0,L_0)\times L^2(L_0,L)).$$

In particular, by application of the Lions - Lemma [, theorem 5.1] we have $\left\{u^m, v^m\right\}\to\{u,v\}$ strongly in $L^2(0,T;L^2(0,L_0)\times L^2(L_0,L))$ and consequently

$$u^m\to u \text{ a. e. in } \left]0,L_0\right[\text{ and } f_1(u^m)\to f_1(u) \text{ a.e in } \left]0,L_0\right[$$
$$v^m\to v \text{ a. e. in } \left]L_0,L\right[\text{ and } f_2(v^m)\to f_2(v) \text{ a.e in } \left]L_0,L\right[.$$

Besides, from the growth condition in (A.1) we have that

$$f_1(u^m) \text{ is bounded in } L^\infty(0,T;L^2(0,L_0))$$
$$f_2(v^m) \text{ is bounded in } L^\infty(0,T;L^2(L_0,L))$$

and therefore.

$$\left\{f_1(u^m), f_2(v^m)\right\}\to\left\{f_1(u), f_2(v)\right\} \text{ in } L^2(0,T;L^2(0,L_0)\times L^2(L_0,L)).$$

The rest of the proof of the existence of a weak solution is matter of routine.

Regularity of solution: To get the regularity, we take a basis $B=\left\{\left\{\varphi^i,\psi^i\right\}, i\in\mathbb{N}\right\}$ such that

$$\left\{u^0, v^0\right\},\ \left\{u^1, v^1\right\}\in \text{span}\left\{\left\{\varphi^0,\psi^0\right\},\left\{\varphi^1,\psi^1\right\}\right\}.$$

Let us differenciate the approximate equation and multiply by $h''_{im}(t)$. Using a similar argument as before we obtain

$$\begin{aligned}\frac{d}{dt}E_2(t,u^m,v^m)+\alpha\left|v_{tt}^m\right|^2 &= -(f_1'(u^m)u_t^m, u_{tt}^m)-(f_2'(v^m)v_t^m, v_{tt}^m)\\ &\quad -(a_t v_x^m, v_{xtt}^m)+\frac{1}{2}(a_t,(v_{xt}^m)^2)\end{aligned} \tag{3.6}$$

where

$$E_2(t,u,v)=\frac{\rho_1}{2}\left|u_{tt}\right|^2+\frac{b}{2}\left|u_{xt}\right|^2+\frac{\rho_2}{2}\left|v_{tt}\right|^2+\frac{1}{2}\left(a,v_{xt}\right)^2.$$

Note that

$$-(a_t v_x^m, v_{xtt}^m)=-(a_t v_x^m, v_{xt}^m)_t+(a_{tt}v_x^m, v_{xt}^m)+(a_t,(v_{xt}^m)^2), \tag{3.7}$$

$E_2(0,u^m,v^m)$ is bounded, because of our choice of the basis.

From the assumption (A.1) and from the Sobolev imbedding we have

$$\int_0^{L_0} f_1'(u^m)\, u_t^m u_{tt}^m dx \le C\left[\int_0^{L_0}\left(1+\left|u_x^m\right|\right)^2 dx\right]^{\frac{p-1}{2}}\left|u_{xt}^m\right|\left|u_{tt}^m\right| \tag{3.8}$$

and similarly

$$\int_{L_0}^{L} f_2'(v^m)\, v_t^m v_{tt}^m dx \le C\left[\int_{L_0}^{L}\left(1+\left|v_x^m\right|\right)^2 dx\right]^{\frac{p-1}{2}}\left|v_{xt}^m\right|\left|v_{tt}^m\right| \tag{3.9}$$

Substituting (3.7), the inequalities (3.8) - (3.9), using the estimative (3.5) in (3.6) and applying Gronwall inequality we conclude that

$$E_2(t,u^m,v^m)\le C \tag{3.10}$$

which imply that

$$\begin{aligned}\left\{u_t^m, v_t^m\right\} &\to \left\{u_t, v_t\right\} \ \text{weakly} * \ \text{in } L^\infty(0,T;H^1(0,L_0)\times H^1(L_0,L))\\ \left\{u_{tt}^m, v_{tt}^m\right\} &\to \left\{u_{tt}, v_{tt}\right\} \ \text{weakly} * \ \text{in } L^\infty(0,T;L^2(0,L_0)\times L^2(L_0,L)).\end{aligned}$$

Therefore we have $\{u,v\}$ satisfies (1.1) - (1.4) and we have

$$\left|\begin{array}{l}-bu_{xx}=-\rho_1 u_{tt}-f_1(u)\in L^2(0,L_0)\\ -(a(x,t)v_x)_x=-\rho_2 v_{tt}-f_2(v)-\alpha v_t\in L^2(L_0,L)\\ u(L_0,t)=v(L_0,t),\ bu_x(L_0,t)=a(L_0,t)\,v_x(L_0,t)\\ u(0,t)=0=v(L,t)\end{array}\right.$$

then using Lemma 3.2 we have the required regularity to $\{u,v\}$.

4. EXPONENTIAL DECAY

In this section we prove that the solution of the system (1.1) - (1.6) decays exponentially as time goes to infinity. In the remainder of this paper we denote by c a positive constant which takes different values in different places.

We shall suppose that $\rho_1\le\rho_2$ and $a(x,t)\le b,\ a_t(x,t)\le 0,\ \forall (x,t)\in\left]L_0,L\right[\times\left]0,\infty\right[$.

Teorema 4.1. *Take* $\left\{u^0,v^0\right\}\in V$ *and* $\left\{u^1,v^1\right\}\in L^2(0,L_0)\times L^2(L_0,L)$ *with*

$$u_x^0(L_0)=0, \tag{4.1}$$

then there exists positive constants γ *and c such that*

$$E(t) \le cE(0)e^{-\gamma^t}, \quad \forall t \ge 0. \tag{4.2}$$

We shall prove this theorem for strong solutions; our conclusion follows by standard density arguments.

The dissipative property of system (1.1) - (1.6) is given by the following lemma.

Lema 4.2. *The first order energy satisfies*

$$\frac{d}{dt} E_1(t, u, v) = -\alpha |v_t|^2 + (a_t, v_x^2) \tag{4.3}$$

Proof. Multiplying equation (1.1) by u_t, equation (1.2) by v_t and performing an integration by parts, we get the result.

Let $\psi \in C_0^\infty(0,L)$ be such that $\psi = 1$ in $]L_0 - \delta, L_0 + \delta[$ for some $\delta > 0$, small constant. Let us introduce the following functional

$$I(t) = \int_0^{L_0} \rho_1 u_t q u_x dx + \int_{L_0}^{L} \rho_2 v_t \psi q v_x dx$$

where $q(x) = x$.

Lema 4.3. *There exists* $c_1 > 0$ *such that*

$$\begin{aligned}\frac{d}{dt} I(t) \le & -\frac{L_0}{2}\left\{(\rho_2 - \rho_1) v_t^2(L_0, t) + a(L_0, t)\left[1 - \frac{a(L_0, t)}{b}\right] v_x^2(L_0, t)\right\} \\ & - L_0(F_1(u(L_0, t)) - F_2(v(L_0, t))) - \frac{1}{2}\int_0^{L_0} (\rho_1 u_t + b u_x^2 + 2F(u))\, dx \\ & -\frac{1}{4}\int_{L_0}^{L_0+\delta} a v_x^2 dx + c_1\left(\int_{L_0+\delta}^{L_0} (v_t^2 + a v_x^2)\, dx + \int_{L_0}^{L} v_t^2 dx + \int_0^{L_0} u^2 dx\right. \\ & \left. + \int_{L_0}^{L} v^2 dx\right) + \varepsilon E(t, u, v)\end{aligned}$$

for any $\varepsilon > 0$.

Proof. Multiplying equation (1.1) by qu_x, equation (1.2) by $\psi q v_x$, integrating by parts and using the corresponding boundary conditions we obtain

$$\begin{aligned}\frac{d}{dt}(\rho, u_t, qu_x) = & \frac{L_0}{2}\left[\rho_1 u_t^2(L_0, t) + b u_x^2(L_0, t)\right] - L_0 F_1(u(L_0, t)) - \\ & \frac{1}{2}\int_0^{L_0} \rho_1 u_t^2 + b u_x^2 + 2F_1(u) dx\end{aligned} \tag{4.4}$$

$$\begin{aligned}\frac{d}{dt}(\rho_2 v_t, \psi q u_x) \le & \frac{L_0}{2}\left[\rho_2 v_t^2(L_0, t) + a(L_0, t)\, v_x^2(L_0, t)\right] + L_0 F_2(v(L_0, t)) \\ & -\frac{1}{4}\int_{L_0}^{L_0+\delta} a v_x^2 dx + + c_1\left[\int_{L_0+\delta}^{L} (v_t^2 + a v_x^2)\, dx + \int_{L_0}^{L} (v_t^2 + F_2(v))\, dx\right]\end{aligned} \tag{4.5}$$

Summing up (4.4) with (4.5), we get

$$\frac{d}{dt}I(t) \le -\frac{L_0}{2}\Big[(\rho_2-\rho_1)\,v_t^2(L_0,t)+a(L_0,t)\,v_x^2(L_0,t)-bu_x^2(L_0,t)\Big]$$
$$-L_0\Big[F_1(u(L_0,t))-F_2(v(L_0,t))\Big]-\frac{1}{2}\int_0^{L_0}(\rho_1u_t^2+bu_x^2+2F_1(u))\,dx$$
$$-\frac{1}{4}\int_{L_0}^{L_0+\delta}av_x^2dx+c_1\bigg(\int_{L_0+\delta}^{L}(v_t^2+av_x^2)\,dx+\int_{L_0}^{L}(v_t^2+F_2(v))\,dx \tag{4.6}$$
$$+\int_0^{L_0}F(u)\,dx\bigg)$$

According to (A.1), we have $f_i(0)=0$ and

$$\left|f_i(s)\right|\le c(|s|+|s|^{\rho}) \tag{4.7}$$

this implies

$$\left|F_i(s)\right|\le c\left(|s|^2+|s|^{\rho+1}\right)\le c\left(|s|^2+|s|^{2\rho}\right). \tag{4.8}$$

From the interpolation inequality

$$|y|_p\le|y|_2^{\alpha}\,|y|_q^{1-\infty},\quad \frac{1}{p}=\frac{\alpha}{2}+\frac{1-\alpha}{q},\quad \alpha\in[0,1]$$

and the immersion $H^1(\Omega)\mapsto L^{2p}(\Omega)$, $\Omega=\left]0,L_0\right[\ or\ \left]L_0,L\right[$, we obtain for all $t\ge 0$

$$|u(t)|_{2\rho}^{2\rho}\le c_\varepsilon\,[E(0)]^{2(\rho-1)}|u(t)|_2^2+\frac{\varepsilon}{[E(0)]^{2(\rho-1)}}|u_x(t)|_2^{\frac{2\rho-1}{2}},$$

considering that

$$|u_x(t)|_2^2\le cE(0,u,v)\equiv c_1E(0)$$

we have

$$|u(t)|_{2\rho}^{2\rho}\le c_\varepsilon\,[E(0)]^{2(\rho-1)}|u(t)|_2^2+\varepsilon E(t,u,v). \tag{4.9}$$

Replacing the inequalities (4.7) - (4.9) in (4.6) our conclusion follows.

Let $\varphi\in C^\infty(\mathbb{R})$ a nonnegative function such that $\varphi=0$ in $I_{\delta/2}=\left]L_0-\frac{\delta}{2},L_0+\frac{\delta}{2}\right[$ and $\varphi=1$ in $\mathbb{R}\setminus I_\delta$ and consider the functional

$$J(t)=\int_{L_0}^{L}\rho_2v_t\varphi v\,dx.$$

We have the following lemma

Lemma 4.4. *Given $\varepsilon>0$, there exists a positive constant c_ε such that*

$$\frac{d}{dt}J(t)\le-\frac{1}{2}\int_{L_0+\delta}^{L}av_x^2\,dx+\varepsilon\int_{L_0}^{L_0+\delta}av_x^2\,dx+c_\varepsilon\int_{L_0}^{L}(v^2+v_t^2)\,dx$$

Proof. Multiplying equation (1.2) by φv and integrating by parts we get

$$\frac{d}{dt}J(t) = -(av_x, \varphi v_x) - (av_x, \varphi_x v) - \alpha(v_t, \varphi v) - (\varphi, f_2(v)v) + (v_t, \varphi v_t).$$

Applying Young's Inequality and hypothesis (A.1) we concludes our assertion.

Let us consider the following functional

$$K(t) = I(t) + (2c_1 + 1)\, J(t)$$

and we take $\varepsilon = \varepsilon_1$ in lemma 4.4, where ε_1 is the solution of the equation

$$(2c_1 + 1)\,\varepsilon_1 = \frac{1}{8},$$

taking in consideration (A.1) in lemma 4.3 we obtain

$$\begin{aligned} \frac{d}{dt}K(t) \le & -E_1(t,u) - \frac{1}{8}\int_{L_0}^{L}(av_x^2 + 2F_2(v))\,dx + \varepsilon E(t,u,v) + \\ & + c_2\left(\int_{L_0}^{L}(v_t^2 + v^2)\,dx + \int_0^{L_0} u^2 dx\right). \end{aligned} \tag{4.10}$$

Now in order to estimate the last two terms of (4.10) we need the following result

Lema 4.5. *Let $\{u, v\}$ be a solution in theorem 3.3. Then there exists $T_0 > 0$ such that if $T \ge T_0$ we have*

$$\begin{aligned} \int_s^T \left(|v|^2 + |u|^2\right) ds \le \varepsilon\left[\int_s^T \left(b|u_x|^2 + |u_t|^2\right) ds + \int_s^T \left|a^{1/2}v_x\right|^2 ds\right] \\ + c_\varepsilon \int_s^t |v_t|^2\, ds \end{aligned} \tag{4.11}$$

for any $\varepsilon > 0$ and c_ε is a constant depending on T and ε, by independent of $\{u, v\}$, for any initial data $\{u^0, v^0\}$, $\{u^1, v^1\}$ satisfying $E(0,u,v) \le R$, where $R > 0$ is fixed and $0 < S < T < +\infty$.

Proof. We use a contradiction method. If (4.11) was false there would exist a sequence of solutions $\{u^\nu, v^\nu\}$ such that

$$\int_S^T \left(|v^\nu|^2 + |u^\nu|^2\right) ds \ge \nu \int_S^t |v_t^\nu|^2\, ds + c_0 \int_S^T \left(b|u_x^\nu|^2 + |u_t|^2 + |a^{1/2}v_x|^2\right) ds$$

and $E(0, u^\nu, v^\nu) \le R$, $\forall \nu$.

Let

$$\lambda_\nu^2 = \int_S^T \left(|v^\nu|^2 + |u^\nu|^2 \right) ds$$

$$w^\nu(x,t) = \frac{u^\nu(x,t)}{\lambda_\nu} \quad , \quad z^\nu(x,t) = \frac{v^\nu(x,t)}{\lambda_\nu} \quad , \quad 0 \le t \le T.$$

Then we have

$$\nu \int_S^T |z_t^\nu|^2 ds + c_0 \int_S^T \left(b|w_x^\nu|^2 + |w_t^\nu|^2 + |a^{1/2} z_x^\nu|^2 \right) ds \le 1$$

and consequently

$$\int_S^T |z_t^\nu|^2 \; ds \to 0 \;\; as \;\; \nu \to \infty \tag{4.12}$$

$$\int_S^T \left(b|w_x^\nu|^2 + |w_t^\nu|^2 + |a^{1/2} z_x^\nu|^2 \right) ds \le c. \tag{4.13}$$

Also we have

$$\int_S^T \left(|z^\nu|^2 + |w^\nu|^2 \right) ds = 1 \tag{4.14}$$

As S is chosen in the interval $[0, T[$, we obtain from (4.12) - (4.13) that, there exists a subsequence $\{w^\nu, z^\nu\}$ which we denote in the same way, such that

$$\begin{aligned} w^\nu &\to w \quad \text{in} \quad L^2(0,T; H^1(0,L_0)) \\ w_t^\nu &\to w_t \quad \text{in} \quad L^2(0,T; L^2(0,L_0)) \\ z^\nu &\to z \quad \text{in} \quad L^2(0,T; H^1(L_0,L)) \\ z_t^\nu &\to 0 \quad \text{in} \quad L^2(0,T; L^2(L_0,L)). \end{aligned}$$

From which

$$\begin{aligned} w^\nu &\to w \quad \text{in} \quad L^2(0,T; L^2(0,L_0)) \\ z^\nu &\to z \quad \text{in} \quad L^2(0,T; L^2(L_0,L)). \end{aligned}$$

This implies

$$\int_0^T \left(|z|^2 + |w|^2 \right) ds = 1. \tag{4.15}$$

Besides, from the uniqueness of the limit we conclude that

$$z_t(x,0) = 0$$

and therefore

$$z(x,t) = \varphi(x) \tag{4.16}$$

Note that $\{w^\nu, z^\nu\}$ satisfies

$$\left|\begin{array}{l}
\rho_1 w_{tt}^\nu - b w_{xx}^\nu + \dfrac{1}{\lambda_\nu} f_1(\lambda_\nu w^\nu) = 0 \text{ in }]0, L_0[\times]0, T[\\
\rho_2 z_{tt}^\nu - (a(x,t) z_x^\nu)_x + \dfrac{1}{\lambda_\nu} f_2(\lambda_\nu z^\nu) + \alpha z_t^\nu = 0 \text{ in }]L_0, L[\times]0, T[\\
w^\nu(0,t) = 0 = z^\nu(L,t) \\
w^\nu(L_0,t) = z^\nu(L_0,t) \\
b w_x^\nu(L_0,t) = a(L_0, L)\, z_x^\nu(L_0,t) \\
w^\nu(x,0) = \dfrac{u^{\nu,0}(x)}{\lambda_\nu}, \quad w_t^\nu(x,0) = \dfrac{1}{\lambda_\nu} u^{\nu,1}(x) \\
z^\nu(x,0) = \dfrac{1}{\lambda_\nu} v^{\nu,0}(x) \,, \quad z_t^\nu(x,0) = \dfrac{1}{\lambda_\nu} v^{\nu,1}(x).
\end{array}\right. \tag{4.17}$$

Now, we observe that $\{\lambda_\nu\}_{\nu \geq 1}$ is a bounded sequence

$$\lambda_\nu = \left[\int_S^T \left(|v^\nu|^2 + |u^\nu|^2 \right) ds \right]^{1/2} \leq c \left[\int_S^T \left(|v_x^\nu|^2 + |u_x^\nu|^2 \right) ds \right]^{1/2}$$
$$\leq cE(0, u, v) \leq cR, \ R \text{ fixed}.$$

because the initial data are in the ball $B(\theta, R)$.

Hence, ther exists a subsequence of $\{\lambda_\nu\}_{\nu \geq 1}$ (still denoted by (λ_ν)) such that

$$\lambda_\nu \to \lambda \in]0, +\infty[.$$

In this case passing to limit in (4.17) when $\nu \to \infty$ we get for $\{w, z\}$

$$\left|\begin{array}{l}
\rho_1 w_{tt} - b w_{xx} + \dfrac{1}{\lambda} f_1(\lambda w) = 0 \text{ in }]0, L_0[\times]0, T[\\
(a(x,t) z_x)_x + \dfrac{1}{\lambda} f_2(\lambda z) = 0 \quad \text{in }]L_0, L[\times]0, T[\\
w(0,t) = 0 = z(L,t) \\
w(L_0,t) = z(L_0,t) \\
b w_x(L_0,t) = a(L_0, L)\, z_x(L_0,t) \\
z_t(x,0) = 0 \qquad\qquad \text{in }]L_0, L[\times]0, T[
\end{array}\right. \tag{4.18}$$

and for $y = w_t$

$$\left|\begin{array}{l}
\rho_1 y_{tt} - b y_{xx} + f'(\lambda w) y = 0 \quad in \ ]0, L_0[\times]0, T[\\
y(0,t) = 0 = y(L_0,t) \\
b y_x(L_0,t) = a_t(L_0,t)\, z_x(L_0,t).
\end{array}\right. \tag{4.19}$$

Here, we observe that

$$\frac{w_{xt}(L_0,t)}{w_x(L_0,t)}=\frac{a_t(L_0,t)}{a(L_0,t)}$$

then we get after an integration

$$w_x(L_0,t)=k\,a(L_0,t),\quad k \text{ is a constant.}$$

But, using the hypotheses we obtain

$$0=\lim_{t\to 0^+} w_x(L_0,t)=k\,a(L_0,0).$$

Consequently $k=0$ and $y_x(L_0,t)=0$.

Thus, the function y satisfies

$$\left|\begin{array}{ll} \rho_1 y_{tt}-by_{xx}+f'(\lambda w)\,y=0 & \text{in }]0,L_0[\times]0,T[\\ y(0,t)=0=y(L_0,t) & \text{on }]0,T[\\ y_x(L_0,t)=0 & \text{on }]0,T[. \end{array}\right. \tag{4.19}$$

Here, we observe that

$$\frac{w_{xt}(L_0,t)}{w_x(L_0,t)}=\frac{a_t(L_0,t)}{a(L_0,t)}$$

then we get after an integration

$$w_x(L_0,t)=k\,a(L_0,t),\quad k \text{ is a constant.}$$

But, using the hypotheses we obtain

$$0=\lim_{t\to 0^+} w_x(L_0,t)=k\,a(L_0,0).$$

Consequently $k=0$ and $y_x(L_0,t)=0$.

Thus, the function y satisfies

$$\left|\begin{array}{ll} \rho_1 y_{tt}-by_{xx}+f'(\lambda w)\,y=0 & \text{in }]0,L_0[\times]0,T[\\ y(0,t)=0=y(L_0,t) & \text{on }]0,T[\\ y_x(L_0,t)=0 & \text{on }]0,T[. \end{array}\right.$$

Then, using the result of [[4]] (based on Ruiz arguments [[10]]) adapted to our case we conclude that $y = 0$, that is $w_t(x,t) = 0$, for T suitable big.

Returning to (4.18) we obtain the following elliptic system

$$\left|\begin{array}{l} -bw_{xx} + \frac{1}{\lambda} f_1(\lambda w) = 0 \\ (a(x,t) z_x)_x + \frac{1}{\lambda} f_2(\lambda z) = 0 \end{array}\right.$$

multiplying by u and v respectively, integrating, and summing up we arrive at

$$b\int_0^{L_0} w_x^2 dx + \int_{L_0}^{L} a(x,t)\, z_x^2 dx + \frac{1}{\lambda}\int_0^{L_0} f_1(\lambda w)\, w dx + \frac{1}{\lambda}\int_{L_0}^{L} f_2(\lambda z)\, z dx = 0$$

So we have $w = 0$ and $z = 0$, wich contradicts (4.15).

If we are not in the above situation and there exists a subsequence satisfying

$$\lambda_\nu \to 0$$

and applying inequality (4.10) to the solutions $\{u^\nu, v^\nu\}$ we have

$$\frac{d}{dt} K^\nu(t) \le -\delta_0 E(t, u^\nu, v^\nu) + c_3\left(\int_{L_0}^{T} ((v_t^\nu)^2 + (v^\nu)^2)\, dx + \int_0^{L_0} (u^\nu)^2\, dx\right),$$

integrating from s to T we get

$$K^\nu(T) + \delta_0 \int_S^T E(t, u^\nu, v^\nu)\, dt \le K(S) + c_3\left(\int_S^T\left(|v_t^\nu|^2 + |v^\nu|^2 + |u^\nu|^2\right)\right) dt.$$

Since K^ν satisfies

$$c_0 E(t, u^\nu, v^\nu) \le K^\nu(T) \le c_1 E(t, u^\nu, v^\nu)$$

and E is a decreasing function we have

$$E(T, u^\nu, v^\nu) + \delta_0' \int_S^T E(t, u^\nu, v^\nu)\, dt \le \frac{c_1'}{T}\int_S^T E(t, u^\nu, v^\nu)\, dt + $$

$$+ c_3 \int_S^T\left(|v_t^\nu|^2 + |v^\nu|^2 + |u^\nu|^2\right) dt;$$

thus, we obtain

$$E(T, w^\nu, z^\nu) + \left(\delta_0' - \frac{c_1'}{T}\right)\int_S^T E(t, w^\nu, z^\nu)\, dt \le c_3 \int_S^T\left(|z_t^\nu|^2 + |z^\nu|^2 + |w^\nu|^2\right) dt.$$

Using (4.12) and (4.14), taking T large enough, we conclude that $E(T, w^\nu, z^\nu)$ is bounded. Now, multiplying equation $(4.17)_1$, $(4.17)_2$ by w_t^ν and z_t^ν respectively, performing an integration by parts we get

$$E(t, w^\nu, z^\nu) \le E(t, w^\nu, z^\nu) + \alpha \int_S^T \left|z_t^\nu\right|^2 dt - \int_S^T \left(a_t, (z_x^\nu)^2\right) dt.$$

From (4.12) and (4.13) we deduce that $E(t, w^\nu, z^\nu)$ is bounded for all $t \in [S, T]$.

Then in particular, on a subsequence we obtain

$$\begin{array}{lll} w^\nu \to w & \text{weak} * \text{in} & L^\infty(0, T; H^1(0, L_0)) \\ w_t^\nu \to w_t & \text{weak} * \text{in} & L^\infty(0, T; L^2(0, L_0)) \\ z^\nu \to z & \text{weak} * \text{in} & L^\infty(0, T; H^1(L_0, L)) \\ z_t^\nu \to z_t & \text{weak} * \text{in} & L^\infty(0, T; L^2(L_0, L)) \\ w^\nu \to w & & \text{in } L^2(0, T; L^2(0, L_0)) \\ z^\nu \to z & & \text{in } L^2(0, T; L^2(L_0, L)) \end{array}$$

Now, the limit funtion $\{w, z\}$ satisfies

$$\left| \begin{array}{ll} \rho_1 w_{tt} - b w_{xx} + f_1'(0)\, w = 0 & \text{in }]0, L_0[\times]0, T[\\ (a(x,t) z_x)_x + f_2'(0)\ z = 0 & \text{in }]L_0, L[\times]0, T[\\ w(0, t) = 0 = z(L, t) & \\ w(L_0, t) = z(L_0, t) & \\ b w_x(L_0, t) = a(L_0, L)\, z_x(L_0, t) & \\ z_t(x, 0) = 0 & \text{in }]L_0, L[\times]0, T[\end{array} \right.$$

Repeating the above procedure, we get $w = 0$ and $z = 0$ which is a contradiction.

The proof of lemma 4.5 is now complete.

Proof of theorem 4.1.

Let us introduce the functional

$$L(t) = N\, E(t) + K(t)$$

with $N > 0$. Using Young's Inequality and taking N large enough we find that

$$\theta_0 E(t) \le L(t) \le \theta_1 E(t) \tag{4.20}$$

for some positive constants θ_0 and θ_1.

Applying the inequalities (4.9) and (4.20), along with the ones in Lemma 4.5 and integrating from S to T where $0 \le S \le T < \infty$ we obtain

$$\int_S^T E(t)\, dt \le c\, E(S).$$

In this condition, lemma 2.1 implies that

$$E(t) \le c\, E(0) e^{-rt},$$

this completes the proof.

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