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Convergence and quantale-enriched categories

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Abstract. Generalising Nachbin's theory of "topology and order", in this paper we continue the study of quantale-enriched categories equipped with a compact Hausdorff topology. We compare these $\mathcal{V}$-categorical compact Hausdorff spaces with ultrafilter-quantale-enriched categories, and show that the presence of a compact Hausdorff topology guarantees Cauchy completeness and (suitably defined) codirected completeness of the underlying quantale enriched category.

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1 Introduction

1.1 Motivation This paper continuous a line of research initiated in [72] which combines Nachbin's theory of "topology and order" [59] with the setting of monad-quantale enriched categories [40]. Over the past century, combining order with compact Hausdorff topologies has proven to be very fruitful in various parts of mathematics: in the form of spectral spaces, these structures appear in Stone duality for distributive lattices [70] and Hochster's characterisation of prime spectra of commutative rings [34], the connection between spectral spaces and certain partially ordered compact spaces was made explicit in [61, 62] (see also [18, 22]), and was further extended to an equivalence between all partially ordered compact spaces and stably compact topological spaces in the 1970s (see [25]). Subsequently, stably compact spaces have also played a central role in the development of domain theory, see [49] for details. In a more general context, compact Hausdorff spaces combined with the structure of a quantale-enriched category have been essential in the study of topological structures as categories: they appear in the definition of "dual space", still implicitly in [15] and more explicitly in [29, 37, 38]. This notion turned out to be an essential ingredient in the investigation of (co)completeness properties of monad-quantale enriched categories. In [12] we also explain the connection of Nachbin's work with the theory of multicategories [32, 33].

Motivated by this development, we focus here on the ultrafilter monad and study quantale-enriched categories equipped with a compact Hausdorff topology; our examples include ordered, metric, and probabilistic metric compact Hausdorff spaces. We show that the presence of a compact Hausdorff topology guarantees Cauchy completeness and (suitably defined) codirected completeness of the underlying quantale enriched category. Our investigation relies on a connection between these $\mathcal{V}$-categorical compact Hausdorff spaces and monad-quantale enriched categories which generalises the equivalence between partially ordered compact spaces and stably compact topological spaces (see [40, Section III.5]). Another important ingredient is the concept of Cauchy completeness *à la* Lawvere for monad-quantale enriched categories as introduced in [15]. In order to include probabilistic metric spaces in our study, our setting is slightly weaker than the one considered in [15]. Due to these weaker assumptions, we have to overcome some technical difficulties which force us to revise and extend some notions and results of [15].

In order to explain our motivation more in detail, we find it useful to place it in a historical context.

1.2 Historical background Right from its origins at the beginning of the 20th century, one major concern of set-theoretic topology was the development of a satisfactory notion of convergence. This in turn was motivated by the increasing use of abstract objects in mathematics: besides numbers, mathematical theories deal with sequences of functions, curves, surfaces,.... To the best of our knowledge, a first attempt to treat convergence abstractly is presented in [24]. Whereby the main contribution of [24] is the concept of a (nowadays called) metric space, the starting point of [24] is actually an abstract theory of sequential convergence. Fréchet considers a function associating to every sequence of a set X a point of X, its convergence point, subject to the following axioms:

(A) Every constant sequence $(x, x, \dots)$ converges to x;

(B) If a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x, then also every subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to x.

Under these conditions, Fréchet gave indeed a generalisation of Weierstraß's theorem [23]; however, these constraints seem to be too weak in general *since the limit axioms* (A), (B) *are not very meaningful ...* ([31, page 266], original in German; our translation). In [30], Hausdorff introduces the notion of topological space via neighbourhood systems and compares the notions of distance, topology and sequential convergence as *... the theory of distances seems to be the most specific, the limes theory the most general...* ([31, page 211], original in German; our translation). In the introduction to Chapter 7 "Punktmengen in allgemeinen Räumen", Hausdorff affirms that *the greatest triumph of set theory lies in its application to the point sets of the space, in the clarification and sharpening of the geometric notions...* ([31, page 209], original in German; our translation). According to Hausdorff, these geometric notions not only involve approximation and distance, but also the theory of (partially) ordered sets to which he dedicates a substantial part of his book. Thinking of an order relation on a set M as a function

$$f\colon M \times M \longrightarrow \{<, >, =\},$$

Hausdorff also foresees that ([31, page 210], original in German; our translation)

> *Now there stands nothing in the way of a generalisation of this idea, and we can think of an arbitrary function of pairs of points which associates to each pair (a, b) of elements of a set M a specific element $n = f(a, b)$ of a second set N. Generalising further, we can consider a function of triples, sequences, complexes, subsets, etc.*

In particular, Hausdorff already presents metric spaces as a direct generalisation of ordered sets where now f associates to each pair (a, b) the distance between a and b. This point of view was taken much further in [50]: not only the structure but also the axioms of an ordered set and of a metric space are very similar and, moreover, can be seen as instances of the definition of a category. Furthermore, Hausdorff sees also the definition of a topological space as a generalisation of the concept of a partially ordered set: instead of a relation between points, sequential convergence relates sequences with their convergence points, and a neighbourhood system relies on a relation between points and subsets. Surprisingly, also here the relevant axioms on such relations can be formulated so that they resemble the ones of a partially ordered set. We refer the reader to the monograph [40] for an extensive presentation of this theory and for further pointers to the literature.

Clearly, Hausdorff considers topologies as generalised partial orders; however, a more direct relation between the two concepts was only given more than twenty years later. In [2], Alexandroff observes that every partial order on a set X defines a topology, and from this topology one can reconstruct the given order relation via

$$x \le y \iff y \text{ belongs to every neighbourhood of } x. \tag{1.2.1}$$

Furthermore, Alexandroff characterises the topological spaces obtained this way as the so called “diskrete Räume”, namely as those T0 spaces where the intersection of open subsets is open. These spaces, without assuming the T0 separation axiom, are nowadays called *Alexandroff spaces.* In this paper we depart from Hausdorff’s nomenclature since partial orders seem to be more frequent than total ones. Therefore we call a binary relation $\le$ on a set X an *order relation* whenever $\le$ is reflexive and transitive, and speak of a *total order* whenever all elements are comparable. Furthermore, we think of the anti-symmetry condition as a(n often unnecessary) separation axiom. We write Ord for the category of ordered sets and monotone maps and, with Top denoting the category of topological spaces and continuous maps, Alexandroff's construction extends to a functor

$$\mathsf{Top} \longrightarrow \mathsf{Ord}$$

which commutes with the underlying forgetful functors to the category Set of sets and functions. The order relation defined by (1.2.1) is now known as the *specialisation order* of the space X. This order looses most of the topological information of a space X and does not seem to be very useful for the study of topological properties. Nevertheless, there are some properties of a space X which are reflected in the specialisation order, in particular the lower separation axioms:

- X is T0 if and only if the specialisation order of X is separated (=anti-symmetric); and
- X is T1 if and only if the specialisation order of X is discrete.

The latter equivalence might be the reason why this order relation does not play a dominant role in general topology. More interesting seems to be the reverse question: which order properties are guaranteed by certain topological properties? For instance, the following observation is very relevant for our paper:

- if X is sober, then the specialisation order of X is directed complete (see [44, Lemma II.1.9]).

The specialisation order plays also a role in Hochster's study of ring spectra: [34] characterises the prime spectra of commutative rings as precisely Stone's spectral spaces [70]. Here, for a commutative ring R, the order of the topology on $\Sigma(R)$ should match the inclusion order of prime ideals; by that reason Hochster considers the dual of the specialisation order. Motivated by the convergence theoretic approach described below, in this paper we will also consider this *underlying* order of a topological space instead of the specialisation order. A deep connection between topological properties and order properties is made in [68] where injective topological T0 spaces are characterised in terms of their underlying partial order.

Whereby in the considerations above the order relation is the one induced by a given topology, a different road was taken in [59] in his study of ordered topological spaces where topology and order are two independent structures, subject to a mild compatibility condition. This combination allows for a substantial extension of the scope of various important notions and results in topology, we mention here the concept of order-normality and the Urysohn Lemma. Of special interest to us is a particular class of separated ordered topological spaces, namely the compact ones, which are described in [45] as "precisely the T0 analogues of compact Hausdorff spaces". These spaces can be equivalently described in purely topological terms: firstly, there is a comparison functor

$$K\colon \mathsf{PosComp} \longrightarrow \mathsf{Top}$$

between the category $\mathsf{PosComp}$ of separated ordered compact spaces and monotone continuous maps and Top; secondly, this functor restricts to an equivalence $\mathsf{PosComp} \simeq \mathsf{StablyComp}$ where $\mathsf{StablyComp}$ denotes the category of stably compact spaces and spectral maps. These facts are known since the beginning of the 1970's and were first published in [25]. To explain this connection better, we find it useful to return to the story of convergence.

After Hausdorff's fundamental book [31], the notion of convergence does not seem to have played a prominent role in the development of topology. The notion of sequence proved to be insufficient, and only in the 1930s [6] appeared a characterisation of topological T1 spaces in terms of an abstract concept of convergence based on the notion of *Moore-Smith sequence* [57, 58]. At the same time, Cartan introduced the concept of filter convergence [9, 10], and this idea was met with enthusiasm within the Bourbaki group [8]. However, it seems to us that this enthusiasm was not shared by most treatments of topology as convergence plays often only a secondary role. We refer to [13] for more information on convergence and its history.

Using either filters or nets (as Moore-Smith sequence are typically called nowadays), convergence finally conquered its appropriate place in topology. This also led to the consideration of abstract (ultra)filter convergence structures, we mention here the papers [17, 27, 28, 79] where topological convergence structures are characterised among more general ones. In our opinion, the most useful descriptions were obtained around 1970: firstly, Manes characterises compact Hausdorff spaces as precisely the Eilenberg–Moore algebras for the ultrafilter monad $\mathbb{U} = (U, m, e)$ on Set [55], and

Barr characterises topological spaces as the *lax* algebras for the ultrafilter monad [4]. More in detail, a compact Hausdorff space is given by a set X together with a *map* $\alpha\colon UX \to X$ so that the diagrams

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "1_X"'] & UX \arrow[d, "\alpha"] \\ & X \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} UUX \arrow[r, "m_X"] \arrow[d, "U\alpha"'] & UX \arrow[d, "\alpha"] \\ UX \arrow[r, "\alpha"'] & X \end{tikzcd}$$

commute in Set; whereby a general topological space is given by a set X together with a *relation* $a\colon UX \relbar\joinrel\mapstochar\joinrel\rightarrow X$ so that the inequalities

$$\begin{tikzcd} X \arrow[r, "e_X", "|" marking] \arrow[dr, "1_X"', "|" marking] \arrow[dr, phantom, shift left=3, "\leq"] & UX \arrow[d, "a", "|" marking] \\ & X \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} UUX \arrow[r, "m_X", "|" marking] \arrow[d, "Ua"', "|" marking] \arrow[dr, phantom, "\leq"] & UX \arrow[d, "a", "|" marking] \\ UX \arrow[r, "a"', "|" marking] & X \end{tikzcd}$$

hold in the ordered category Rel of sets and relations, ordered by inclusion. Elementwise, the latter axioms read as

$$e_X(x) \to x \qquad \text{and} \qquad (\mathfrak{X} \to \mathfrak{x} \;\&\; \mathfrak{x} \to x) \implies m_X(\mathfrak{X}) \to x,$$

for all $x \in X$, $\mathfrak{x} \in UX$ and $\mathfrak{X} \in UUX$. Note that the second condition talks about the convergence of an ultrafilter of ultrafilters $\mathfrak{X}$ to an ultrafilter $\mathfrak{x}$, which comes from applying the ultrafilter functor U to the *relation* $a : UX \relbar\joinrel\mapstochar\joinrel\rightarrow X$. Hence, this description involves the additional difficulty of extending the functor $U : \mathsf{Set} \to \mathsf{Set}$ in a suitable way to a locally monotone endofunctor on Rel; but it is extremely useful since it does not only provide axioms but also a calculus to deal with these axioms since they are formulated within the structure of the ordered category Rel. Barr's characterisation gives also new evidence to Hausdorff's intuition that topological spaces are generalised orders, as the two axioms are clearly reminiscent to the reflexivity and the transitivity condition defining an order relation. We also note that the underlying order of a topology $a\colon UX \relbar\joinrel\mapstochar\joinrel\rightarrow X$ is simply the composite $a \cdot e_X : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$.

Using this language, Tholen [72] shows that an ordered compact Hausdorff space can be equivalently described as a set X equipped with an order relation $\leq : X \nrightarrow X$ and a compact Hausdorff topology $\alpha \colon UX \to X$ which must be compatible in the sense that

$$\alpha \colon (UX, U{\leq}) \longrightarrow (X, \leq)$$

is monotone. Moreover, the object part of the functor $K \colon \mathsf{PosComp} \to \mathsf{Top}$ mentioned above can now be simply described by relational composition

$$(X, \leq, \alpha) \longmapsto (X, \leq \cdot \alpha);$$

a simple calculation shows that $\leq \cdot \alpha \colon UX \nrightarrow X$ satisfies indeed the two axioms of a topology. More importantly, as already initiated in [72], this approach paves the way to mix topology with metric structures or other "generalised orders" in the spirit of Hausdorff; or better: enriched categories in the spirit of Lawvere [50]. Undoubtedly, topology is already omnipresent in the study of metric spaces; however, there does not seem to exist a systematic account in the literature thinking of metric and topology as a generalisation of Nachbin's ordered topological spaces. This motivation brings us to the following considerations.

- Instead of analysing a metric space (X, d) using the topology *induced* by d, we ask what properties of d are ensured by a compact Hausdorff topology *compatible* with d.
- To answer this question, we look back and ask the same question for the ordered case. Surprisingly, there is a quick answer: since every separated ordered compact space corresponds to a stably compact space which is in particular sober, every separated ordered compact space has codirected infima and, by duality, also directed suprema.
- To transport this argumentation back to the metric case, we need a metric variant of sober topological spaces, which is provided by the notion of sober approach space [3, 53, 73].
- we also consider the notion of codirected completeness for metric spaces which implies Cauchy completeness. We compare this notion to other concepts of (co)directedness in the literature.

The principal aim of this paper is to present a theory which encompasses the steps above, for quantale-enriched categories equipped with a compact

Hausdorff topology; our examples include ordered, metric, and probabilistic metric compact Hausdorff spaces. We place this study in the general framework of topological theories [36] and monad-quantale-enriched categories (see [40]), for the ultrafilter monad $\mathbb{U}$ on Set.

2 Basic notions

In this section, we recall various aspects of the theory of quantale-enriched categories. All results presented here are well-known, for more information about enriched category theory we refer to the classic [46] and to [71]. In our examples we focus on quantales based on the lattices $\mathbf{2} = \{0,1\}$, $[0,1]$, $[0,\infty]$ and the lattice $\mathcal{D}$ of distribution functions. The book [40] presents and uses quantale-enriched categories in the context of topology, and in this section we follow the notation of [40].

2.1 Quantale A *quantale* $\mathcal{V} = (\mathcal{V}, \otimes, k)$ is a complete lattice $\mathcal{V}$, with the order relation denoted by $\leq$, equipped with a monoidal structure given by a commutative and associative binary operation $\otimes$, with identity k, which distributes over joins:

$$u \otimes \left(\bigvee_{i\in I} u_i\right) = \bigvee_{i\in I}(u \otimes u_i).$$

Thus, by Freyd's Adjoint Functor Theorem, for each $u \in \mathcal{V}$, the monotone map $u \otimes - \colon \mathcal{V} \to \mathcal{V}$ has a right adjoint $\hom(u,-)$ characterised by

$$u \otimes v \leq w \iff v \leq \hom(u,w),$$

for all $v, w \in \mathcal{V}$. Our principal examples include the following.

Example 2.1. (1) The two element chain $\mathbf{2} = \{0,1\}$ of truth values with $0 \leq 1$ is a quantale for $\otimes = \&$ being the logical operation "and"; in this case $\hom(u,v)$ is just the implication $u \implies v$. More general, every Heyting algebra with $\otimes = \wedge$ being infimum and the identity given by the top element $\top$ is a quantale.

(2) The extended real half line $\overleftarrow{[0,\infty]}$ ordered by the "greater or equal" relation $\geqslant$ becomes a quantale with the tensor product given by the usual

addition $+$, denoted by $\overleftarrow{[0,\infty]}_+$. In this case, $\hom(u,v) = v \ominus u = \max\{v - u, 0\}$, for all $u, v \in [0,\infty]$. According to (1), one can also equip $\overleftarrow{[0,\infty]}$ with the infimum $\otimes = \max$ of this lattice, we denote the resulting quantale as $\overleftarrow{[0,\infty]}_\wedge$.

(3) Similar to (2), we consider the unit interval $[0,1]$ with the "greater or equal" relation $\geqslant$ and the tensor

$$u \oplus v = \min\{1, u + v\},$$

for all $u, v \in [0,1]$. This quantale will be denoted by $\overleftarrow{[0,1]}_\oplus$.

(4) The quantales introduced in (2) and (3) can be more uniformly described using the unit interval $[0,1]$ equipped with the usual order $\leqslant$. In fact, $[0,1]$ admits several interesting quantale structures, the most important ones to us are the minimum $\wedge$, the usual multiplication $*$, and the Lukasiewicz sum defined by $u \odot v = \max\{0, u + v - 1\}$, for all $u, v \in [0,1]$. The corresponding operation hom is given, respectively, by

$$\hom(u,v) = \begin{cases} 1, & \text{if } u \leq v \\ v, & \text{else} \end{cases}, \quad \hom(u,v) = \begin{cases} \min\{\frac{v}{u}, 1\}, & \text{if } u \neq 0 \\ 1, & u = 0 \end{cases},$$

$$\hom(u,v) = \min\{1, 1 - u + v\} = 1 - \max\{0, u - v\},$$

for $u, v \in [0,1]$. We will denote these quantales by $[0,1]_\wedge$, $[0,1]_*$, and $[0,1]_\odot$, respectively. Then, through the map

$$[0,\infty] \longrightarrow [0,1],\ u \longmapsto e^{-u}$$

with $e^{-\infty} = 0$, the quantale $\overleftarrow{[0,\infty]}_+$ is isomorphic to $[0,1]_*$ and $\overleftarrow{[0,\infty]}_\wedge$ is isomorphic to $[0,1]_\wedge$. Finally, the quantale $\overleftarrow{[0,1]}_\oplus$ of (3) is isomorphic to the quantale $[0,1]_\odot$, via the lattice isomorphism $u \mapsto 1 - u$.

(5) Another way to equip the unit interval $[0,1]$ with a quantale structure is to consider the usual order and to give $\otimes$ by the *nilpotent minimum*

$$u \otimes v = \begin{cases} \min\{u,v\} & \text{if } u + v > 1, \\ 0 & \text{else} \end{cases}$$

for $u, v \in [0,1]$, for which $\hom(u,v) = \max\{1 - u, v\}$. This is a classical example of a tensor in $[0,1]$ that is left continuous but not continuous.

(6) The set

$$\mathcal{D} = \{f\colon [0,\infty] \longrightarrow [0,1] \mid f(\alpha) = \bigvee_{\beta<\alpha} f(\beta) \text{ for all } \alpha \in [0,\infty]\}$$

of left continuous distribution functions, ordered pointwise, is a complete lattice. Here the supremum of a family $(h_i)_{i\in I}$ of elements of $\mathcal{D}$ can be calculated pointwise as $h(\alpha) = \bigvee_{i\in I} h_i(\alpha)$, for all $\alpha \in [0,\infty]$. The infimum of an arbitrary collection of elements of $\mathcal{D}$ cannot be obtained by an analogous process since the pointwise infimum of a family of left continuous maps need not be left continuous. However, the infimum of a family $(f_i)_{i\in I}$ in $\mathcal{D}$ is given by

$$\bigwedge_{i\in I} f_i(\alpha) = \sup_{\beta<\alpha} \inf_{i\in I} f_i(\beta),$$

for every $\alpha \in [0,\infty]$, due to the adjunction $i \dashv c$, where i is the embedding $\mathcal{D} \to \mathsf{Ord}([0,\infty],[0,1])$ and $c\colon \mathsf{Ord}([0,\infty],[0,1]) \to \mathcal{D}$, such that $c(f)(\alpha) = \sup_{\beta<\alpha} f(\beta)$.

For each of the tensor products $\otimes$ on $[0,1]$ defined in (4), $\mathcal{D}$ becomes a quantale with

$$f \otimes g(\gamma) = \bigvee_{\alpha+\beta\leqslant\gamma} f(\alpha) \otimes g(\beta),$$

for all $\gamma \in [0,\infty]$; the identity is given by

$$\sigma_{0,1}(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0, \\ 1 & \text{else.} \end{cases}$$

For more information about this quantale we refer to [16, 20, 39].

2.2 Completely distributive lattices In this subsection we recall some properties of complete lattices and quantales which will be useful in the sequel. First of all, we call a quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$ *non-trivial* whenever $k > \bot$. More generally:

Definition 2.2. The neutral element k of a quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$ is called $\vee$-*irreducible* whenever $k > \bot$ and, for all $u, v \in \mathcal{V}$, $k \leq u \vee v$ implies $k \leq u$ or $k \leq v$.

For an ordered set X, we denote by $\mathrm{P}_{\downarrow}X$ the complete lattice of down sets of X ordered by inclusion. The ordered set X can be embedded into $\mathrm{P}_{\downarrow}X$ by

$$\downarrow_X : X \longrightarrow \mathrm{P}_{\downarrow}X,\ x \longmapsto \downarrow x = \{y \in X \mid y \leq x\};$$

and X is complete if and only if $\downarrow_X : X \to \mathrm{P}_{\downarrow}X$ has a left adjoint $\bigvee\colon \mathrm{P}_{\downarrow}X \to X$. In this paper we will often require that the complete lattice $\mathcal{V}$ is completely distributive, a notion introduced in [64]. Therefore we recall succinctly the notions and results needed in this paper and refer for all details to [78].

Definition 2.3. A complete ordered set X is called *completely distributive* whenever the map $\bigvee\colon \mathrm{P}_{\downarrow}X \to X$ preserves all infima.

Hence, since $\mathrm{P}_{\downarrow}X$ is complete, the lattice X is completely distributive if and only if $\bigvee$ has a left adjoint $\Downarrow_X\colon X \to \mathrm{P}_{\downarrow}X$. We recall that

$$\Downarrow_X \dashv \bigvee \qquad \iff \qquad \forall x \in X\, \forall A \in \mathrm{P}_{\downarrow}X\,.\, (\Downarrow_X x \subseteq A \iff x \leq \bigvee A).$$

Definition 2.4. Let X be a complete ordered set X. For all $x, y \in X$, *x is totally below y ($x \ll y$)* whenever, for all $A \in \mathrm{P}_{\downarrow}X$,

$$y \leq \bigvee A \implies x \in A.$$

Proposition 2.5. *Let $\ll$ be the totally below relation in a complete ordered set X with order relation $\leq$. Then, for all $x, y, z \in X$:*
(1) $x \ll y \implies x \leq y$;
(2) $x \leq y \ll z \implies x \ll z$;
(3) $x \ll y \leq z \implies x \ll z$;
(4) $x \ll y \implies \exists z \in X\,.\, x \ll z \ll y$.

If X is a completely distributive lattice, then, for every $y \in X$,

$$\Downarrow y = \bigcap \{A \in \mathrm{P}_{\downarrow}X \mid y \leq \bigvee A\};$$

therefore $x \in \Downarrow y$ if and only if $x \ll y$.

Theorem 2.6. *A complete lattice X is completely distributive if and only if every $y \in X$ can be expressed as $y = \bigvee\{x \in X \mid x \ll y\}$.*

Remark 2.7. A complete ordered set X is completely distributive if and only if X^{op} is so.

Example 2.8. (1) The complete lattice $\mathbf{2}$ is completely distributive where $x \ll y$ if and only if $y = 1$.

(2) The lattices $[0,1]$ and $[0,\infty]$, ordered by $\leqslant$, are completely distributive with $\ll$ being the usual smaller relation $<$. Similarly, $\overleftarrow{[0,1]}$ and $\overleftarrow{[0,\infty]}$, with the "greater or equal relation" $\geqslant$, are completely distributive where the totally below relation is the larger relation $>$.

(3) In order to show that the complete lattice $\mathcal{D}$ is completely distributive, it is useful to introduce some special elements that will allow a more simplified description of $\mathcal{D}$ and of its properties. The step functions $\sigma_{n,\varepsilon}$, with $n \in [0,\infty]$ and $\varepsilon \in [0,1]$, are elements of $\mathcal{D}$ defined by

$$\sigma_{n,\varepsilon}(\alpha) = \begin{cases} 0 & \text{if } \alpha \leqslant n, \\ \varepsilon & \text{if } \alpha > n; \end{cases}$$

for all $\alpha \in [0,\infty]$. It is shown in [20] that, for all $f, \sigma_{n,\varepsilon} \in \mathcal{D}$, $\sigma_{n,\varepsilon} \ll f$ if and only if $\varepsilon < f(n)$. This observation allows to write every element $f \in \mathcal{D}$ as the supremum of those step functions totally bellow f: $f = \bigvee\{\sigma_{n,\varepsilon} \in \mathcal{D} \mid \sigma_{n,\varepsilon} \ll f\}$. A complete description of the totally below relation on $\mathcal{D}$ can be found in [16].

Definition 2.9. For a quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$, we say that k is *approximated* whenever the set

$$\Downarrow\! k = \{u \in \mathcal{V} \mid u \ll k\}$$

is directed and $k = \bigvee \Downarrow\! k$.

We note that in each of the quantales of Examples 2.8 the neutral element is approximated.

Proposition 2.10. *Let $\mathcal{V} = (\mathcal{V}, \otimes, k)$ be a quantale where k is approximated. Then k is $\vee$-irreducible and $k = \bigvee_{u \ll k} u \otimes u$.*

Proof. Assume that k is approximated. First note that $k > \bot$ since, being directed, $\Downarrow\! k$ is non-empty. Furthermore, k is $\vee$-irreducible by [39, Remark 4.21], and the second assertion follows from [19, Theorem 1.12]. $\square$

2.3 $\mathcal{V}$-relations For a quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$, a *$\mathcal{V}$-relation* $r\colon X \relbar\!\!\mapsto Y$ is a map $X \times Y \to \mathcal{V}$. Given $\mathcal{V}$-relations $r\colon X \relbar\!\!\mapsto Y$ and $s\colon Y \relbar\!\!\mapsto Z$, their composite $s \cdot r\colon X \relbar\!\!\mapsto Z$ is defined by

$$s \cdot r(x,z) = \bigvee_{y\in Y} r(x,y) \otimes s(y,z),$$

for all $x \in X$ and $z \in Z$. Every map $f\colon X \to Y$ can be seen as a $\mathcal{V}$-relation $f\colon X \relbar\!\!\mapsto Y$ with

$$f(x,y) = \begin{cases} k & \text{if } f(x) = y,\\ \bot & \text{else;}\end{cases}$$

and the identity map $1_X\colon X \to X$ induces the identity $\mathcal{V}$-relation $1_X\colon X \relbar\!\!\mapsto X$. The resulting category of sets and $\mathcal{V}$-relations is denoted by $\mathcal{V}$-Rel. We note that the construction above defines a functor $\mathsf{Set} \to \mathcal{V}$-$\mathsf{Rel}$ which is faithful if and only if $\mathcal{V}$ is non-trivial.

The set $\mathcal{V}\text{-}\mathsf{Rel}(X,Y)$ of $\mathcal{V}$-relations from X to Y is actually a complete ordered set where the supremum of a family $(\varphi_i\colon X \relbar\!\!\mapsto Y)_{i\in I}$ is calculated pointwise. Since the tensor product of $\mathcal{V}$ preserves suprema, for every $\mathcal{V}$-relation $r\colon X \relbar\!\!\mapsto Y$, the maps $(-)\cdot r\colon \mathcal{V}\text{-}\mathsf{Rel}(Y,Z) \to \mathcal{V}\text{-}\mathsf{Rel}(X,Z)$ and $r \cdot (-)\colon \mathcal{V}\text{-}\mathsf{Rel}(Z,X) \to \mathcal{V}\text{-}\mathsf{Rel}(Z,Y)$ preserve suprema as well; which tells us that $\mathcal{V}$-Rel is actually a quantaloid (see [65]). In particular, both maps have right adjoints in Ord.

Explicitly, the right adjoint $- \bullet\!\!- r$ of $(-)\cdot r$ is given by

$$t \bullet\!\!- r(y,z) = \bigwedge_{x\in X} \hom(r(x,y), t(x,z)),$$

for each $t\colon X \relbar\!\!\mapsto Z$; we call $t \bullet\!\!- r$ the *extension of t along r*. Similarly, we denote the right adjoint of $r\cdot(-)$ by $r \multimap -$; for each $t\colon Z \relbar\!\!\mapsto Y$, the $\mathcal{V}$-relation $r \multimap t$ is called the *lifting of t along r* and can be calculated as

$$r \multimap t(z,x) = \bigwedge_{y\in Y} \hom(r(x,y), t(z,y)).$$

Another important feature which comes from the fact that $\mathcal{V}$-Rel is locally ordered, is the possibility to define adjoint $\mathcal{V}$-relations: $r\colon X \relbar\!\!\mapsto Y$ is left adjoint to $s\colon Y \relbar\!\!\mapsto X$ if and only if $1_X \le s \cdot r$ and $r \cdot s \le 1_Y$.

For each $\mathcal{V}$-relation $r\colon X \to\!\!\!\!\!\!+\, Y$ one can consider its opposite $r^\circ\colon Y \to\!\!\!\!\!\!+\, X$ given by $r^\circ(x,y) = r(y,x)$, for all $x \in X$ and all $y \in Y$. This construction defines a locally monotone functor $(-)^{\mathrm{op}}\colon \mathcal{V}\text{-}\mathsf{Rel}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathsf{Rel}$. We also note that $f \dashv f^\circ$ in $\mathcal{V}$-Rel, for every function $f\colon X \to Y$.

2.4 $\mathcal{V}$-categories We introduce now categories enriched in a quantale $\mathcal{V}$.

Definition 2.11. Let $\mathcal{V} = (\mathcal{V}, \otimes, k)$ be a quantale. A $\mathcal{V}$-*category* is a pair (X,a) consisting of a set X and a $\mathcal{V}$-relation $a\colon X \to\!\!\!\!\!\!+\, X$ satisfying $1_X \le a$ and $a \cdot a \le a$; in pointwise notation:

$$k \le a(x,x) \qquad \text{and} \qquad a(x,y) \otimes a(y,z) \le a(x,z),$$

for all $x,y,z \in X$. A $\mathcal{V}$-*functor* $f\colon (X,a) \to (Y,b)$ between $\mathcal{V}$-categories is a map $f\colon X \to Y$ such that $f \cdot a \le b \cdot f$; equivalently, for all $x, x' \in X$, $a(x,x') \le b(f(x), f(x'))$.

With the usual composition of maps and the identity maps, $\mathcal{V}$-categories and $\mathcal{V}$-functors provide the category $\mathcal{V}$-Cat. Note that $1_X \le a$ implies $a \le a \cdot a$; therefore $a \cdot a = a$, for every $\mathcal{V}$-category (X,a). The quantale $\mathcal{V}$ is itself a $\mathcal{V}$-category with structure given by $\hom\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$. To every $\mathcal{V}$-category (X,a) one can associate its dual $\mathcal{V}$-category $X^{\mathrm{op}} = (X, a^\circ)$, and this construction defines a functor

$$(-)^{\mathrm{op}}\colon \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$$

commuting with the canonical forgetful functor $O_{\mathcal{V}}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Set}$.

Definition 2.12. A $\mathcal{V}$-category X is called *symmetric* whenever $X = X^{\mathrm{op}}$.

Due to the fact that the forgetful functor $O_{\mathcal{V}}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Set}$ is topological (see [1, 14]), the category $\mathcal{V}$-Cat admits all limits and colimits. Moreover, $O_{\mathcal{V}}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Set}$ has a left adjoint and the free $\mathcal{V}$-category over the one-element set $1 = \{\star\}$ is given by $G = (1,k)$, where $k(\star,\star) = k$. For every set X, we have the X-fold power $\mathcal{V}^X$ of the $\mathcal{V}$-category $\mathcal{V}$ whose elements are maps $\varphi\colon X \to \mathcal{V}$ and, for maps $\varphi_1, \varphi_2\colon X \to \mathcal{V}$,

$$[\varphi_1, \varphi_2] := \varphi_2 \multimapdotinv \varphi_1 = \bigwedge_{x \in X} \hom(\varphi_1(x), \varphi_2(x))$$

describes the $\mathcal{V}$-categorical structure of $\mathcal{V}^X$. Another example is the product of $\mathcal{V}$-categories (X,a) and (Y,b), which is the $\mathcal{V}$-category $X\times Y=(X\times Y,d)$, where, for $(x,y),(x',y')\in X\times Y$, $d((x,y),(x',y'))=a(x,x')\wedge b(y,y')$. However one can also consider the structure $a\otimes b$ on $X\times Y$: $a\otimes b((x,y),(x',y'))=a(x,x')\otimes b(y,y')$. Both operations are commutative and associative but the neutral objects differ in general: $(1,\top)$ is the neutral object for the first product while $G=(1,k)$ is the neutral object for the second.

We consider now the quantales of Examples 2.1.

Example 2.13. (1) The objects of $\mathbf{2}$-Cat are ordered sets (that is, sets equipped with a reflexive and transitive binary relation) and the morphisms are monotone maps; thus, $\mathbf{2}\text{-}\mathsf{Cat}\simeq\mathsf{Ord}$.

(2) A $\overleftarrow{[0,\infty]}_+$-category is a generalised metric space in the sense of [50] and a $\overleftarrow{[0,\infty]}_+$-functor is a non-expansive map. We write Met for the resulting category, that is, $\overleftarrow{[0,\infty]}_+\text{-}\mathsf{Cat}\simeq\mathsf{Met}$. Due to the lattice isomorphism $\overleftarrow{[0,\infty]}_+\simeq[0,1]_*$, also $[0,1]_*\text{-}\mathsf{Cat}\simeq\mathsf{Met}$. Similarly, for $\mathcal{V}=\overleftarrow{[0,\infty]}_\wedge$, a $\mathcal{V}$-category is a (generalised) ultrametric space and, since $\overleftarrow{[0,\infty]}_\wedge\simeq[0,1]_\wedge$, we have $\overleftarrow{[0,\infty]}_\wedge\text{-}\mathsf{Cat}\simeq[0,1]_\wedge\text{-}\mathsf{Cat}\simeq\mathsf{UMet}$. Finally, we can interpret $\overleftarrow{[0,1]}_\oplus$-categories and $[0,1]_\odot$-categories as bounded-by-1 metric spaces and $[0,1]_\odot$-functors as non-expansive maps, so that $\overleftarrow{[0,1]}_\oplus\text{-}\mathsf{Cat}\simeq[0,1]_\odot\text{-}\mathsf{Cat}\simeq\mathsf{BMet}$.

(3) A $\mathcal{D}$-category consists of a set equipped with a structure $a\colon X\times X\to\mathcal{D}$ such that, for all $x,y,z\in X$ and $t\in[0,\infty]$:

$$1\leqslant a(x,y)(t)\quad\text{and}\quad\bigvee_{q+r\leqslant t}a(x,y)(q)\otimes a(y,z)(r)\leqslant a(x,z)(t),$$

and a $\mathcal{D}$-functor $f\colon(X,a)\to(Y,b)$ satisfies $a(x,y)(t)\leqslant b(f(x),f(y))(t)$, for $x,y\in X$ and $t\in[0,\infty]$. Therefore the category $\mathcal{D}$-Cat is isomorphic to the category of (generalised) probabilistic metric spaces $\mathsf{ProbMet}$. The classical definition of probabilistic metric space (see [56, 67]) demands that (X,a) is separated (see Definition 2.16), symmetric and finitary ($a(x,y)\in\mathcal{D}$ should be finite for all $x,y\in X$). A detailed study of probabilistic metric spaces as enriched categories can be found in [39].

Definition 2.14. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be quantales, we write $\otimes$ for the multiplication in both $\mathcal{V}_1$ and $\mathcal{V}_2$, and k_1 denotes the neutral element of $\mathcal{V}_1$ and

k_2 the neutral element of $\mathcal{V}_2$. A *lax quantale morphism* $\varphi\colon \mathcal{V}_1 \to \mathcal{V}_2$ is a monotone map between the underlying ordered sets satisfying

$$k_2 \le \varphi(k_1) \quad \text{and} \quad \varphi(u) \otimes \varphi(v) \le \varphi(u \otimes v),$$

for all $u, v \in \mathcal{V}_2$.

These properties ensure that the mapping $(X, a) \mapsto (X, \varphi a)$ sends $\mathcal{V}_1$-categories to $\mathcal{V}_2$-categories; hence, this construction defines a functor

$$B_\varphi\colon \mathcal{V}_1\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}_2\text{-}\mathsf{Cat}$$

which commutes with the forgetful functors to Set.

Example 2.15. (1) The identity map on $[0,\infty]$ defines a lax quantale morphism

$$\overleftarrow{[0,\infty]}_\wedge \longrightarrow \overleftarrow{[0,\infty]}_+,$$

and the map $[0,\infty] \to [0,1]$, $u \mapsto \min\{u, 1\}$ gives a lax quantale morphism

$$\overleftarrow{[0,\infty]}_+ \longrightarrow \overleftarrow{[0,1]}_\oplus.$$

The corresponding functors produce the canonical chain of functors

$$\mathsf{UMet} \longrightarrow \mathsf{Met} \longrightarrow \mathsf{BMet}.$$

(2) The quantale $\overleftarrow{[0,\infty]}_+$ embeds canonically into $\mathcal{D}$ via $I_\infty\colon \overleftarrow{[0,\infty]}_+ \to \mathcal{D}$, taking an element $n \in [0,\infty]$ to $\sigma_{n,1} \in \mathcal{D}$. This map is a lax quantale morphism and it admits a right and a left adjoint

$$\begin{tikzcd} \overleftarrow{[0,\infty]}_+ \arrow[r, "I_\infty" description] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{D} \arrow[l, bend right=45, "O_\infty"'] \arrow[l, bend left=45, "P_\infty"] \end{tikzcd}$$

with $P_\infty(f) = \inf\{n \in [0,\infty] \mid f(n) = 1\}$ and $O_\infty(f) = \sup\{n \in [0,\infty] \mid f(n) = 0\}$, for all $f \in \mathcal{D}$ with P_∞ and O_∞ being lax quantale morphisms. These lax morphisms induce adjoint functors

$$\begin{tikzcd} \mathsf{Met} \arrow[r, "I_\infty" description] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{ProbMet}. \arrow[l, bend right=45, "O_\infty"'] \arrow[l, bend left=45, "P_\infty"] \end{tikzcd}$$

between the categories Met and $\mathsf{ProbMet}$.

For every quantale $\mathcal{V}$, the canonical map

$$i\colon \mathbf{2} \longrightarrow \mathcal{V},\ 0 \longmapsto \bot,\ 1 \longmapsto k$$

is a lax quantale morphism, which induces the functor $B_i\colon \mathsf{Ord} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$. The monotone map $i\colon \mathbf{2} \to \mathcal{V}$ has a right adjoint

$$p\colon \mathcal{V} \longrightarrow \mathbf{2},\ v \longmapsto \begin{cases} 1 & \text{if } v \geq k, \\ 0 & \text{else} \end{cases}$$

which is a lax morphism of quantales too and induces the functor $B_p\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Ord}$; explicitly,

$$x \leq y \iff k \leq a(x, y),$$

for all elements x, y of a $\mathcal{V}$-category X.

Definition 2.16. A $\mathcal{V}$-category $X = (X, a)$ is called *separated* whenever $B_p X$ is separated; that is, for all $x, y \in X$, $k \leq a(x, y)$ and $k \leq a(y, x)$ imply $x = y$.

2.5 $\mathcal{V}$-distributors Besides $\mathcal{V}$-functors, there is another important type of morphisms between categories, called $\mathcal{V}$-distributors. The notion of distributor was introduced by Bénabou in the 1960s and provides "a generalisation of relations between sets to 'relations between (small) categories' " (see [5]).

Definition 2.17. For $\mathcal{V}$-categories $X = (X, a)$ and $Y = (Y, b)$, a $\mathcal{V}$-*distributor* $\varphi\colon X \circ\!\!\!\rightarrow Y$ is a $\mathcal{V}$-relation $\varphi\colon X \nrightarrow Y$ compatible with both structures, meaning that $\varphi \cdot a \leq \varphi$ and that $b \cdot \varphi \leq \varphi$.

In fact, these inequalities are equalities due to the reflexivity of a and b. Thus the identity distributor on (X, a) is actually a and, considering the composition of $\mathcal{V}$-relations, we obtain the category $\mathcal{V}\text{-}\mathsf{Dist}$. We also note that a $\mathcal{V}$-relation $\varphi\colon X \nrightarrow Y$ is a $\mathcal{V}$-distributor precisely when $\varphi\colon X^{\mathrm{op}} \otimes Y \to \mathcal{V}$ is a $\mathcal{V}$-functor (see [50]).

For $\mathcal{V}$-categories X and Y, the subset

$$\mathcal{V}\text{-}\mathsf{Dist}(X, Y) \hookrightarrow \mathcal{V}\text{-}\mathsf{Rel}(X, Y)$$

is closed under suprema; hence, the supremum of a family $(\varphi_i\colon X \mathrel{\circ\!\!\to} Y)_{i\in I}$ of $\mathcal{V}$-distributors can be calculated pointwise. As in Subsection 2.3, for a $\mathcal{V}$-distributor $\varphi\colon X \mathrel{\circ\!\!\to} Y$, both maps $(-)\cdot\varphi$ and $\varphi\cdot(-)$ have right adjoint given, respectively, by the extension and lifting along φ.

Every $\mathcal{V}$-functor $f\colon (X,a) \to (Y,b)$ induces a pair of $\mathcal{V}$-distributors $f_*\colon (X,a) \mathrel{\circ\!\!\to} (Y,b)$ and $f^*\colon (Y,b) \mathrel{\circ\!\!\to} (X,a)$ given by $f_* = b\cdot f$ and $f^* = f^\circ\cdot b$; in pointwise notation, for $x\in X$ and $y\in Y$,

$$f_*(x,y) = b(f(x),y) \quad \text{and} \quad f^*(y,x) = b(y,f(x)),$$

which characterise the functors $(-)_*\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Dist}$ and $(-)^*\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Dist}^{\mathrm{op}}$. An important fact about these induced $\mathcal{V}$-distributors is that they form an adjunction $f_* \dashv f^*$ in $\mathcal{V}\text{-}\mathsf{Dist}$. For the particular case of a $\mathcal{V}$-functor of the form $x\colon G \to X$ we obtain $x_* = a(x,-)$ and $x^* = a(-,x)$.

Definition 2.18. A $\mathcal{V}$-functor $f\colon (X,a) \to (Y,b)$ is called *fully faithful* whenever $f^*\cdot f_* = a$, and f is called *fully dense* whenever $f_*\cdot f^* = b$.

The underlying order of $\mathcal{V}$-categories extends point-wise to an order relation between $\mathcal{V}$-functors. This order relation can be equivalently described using $\mathcal{V}$-distributors: for $\mathcal{V}$-functors $f,g\colon (X,a)\to(Y,b)$,

$$f\le g \iff f^*\le g^* \iff g_*\le f_*.$$

Furthermore, the composition from either side preserves this order, and therefore $\mathcal{V}\text{-}\mathsf{Cat}$ is actually an ordered category. An important consequence is the possibility to define adjoint $\mathcal{V}$-functors: a pair of $\mathcal{V}$-functors $f\colon (X,a)\to (Y,b)$ and $g\colon (Y,b)\to(X,a)$ forms an adjunction, $f\dashv g$, whenever, $1_X\le g\cdot f$ and $\cdot fg \le 1_Y$. Since

$$f\dashv g \iff g_*\dashv f_* \iff f_* = g^*,$$

$f\dashv g$ if and only if, for all $x\in X$ and $y\in Y$, $a(x,g(y)) = b(f(x),y)$.

2.6 Cauchy complete $\mathcal{V}$-categories In 1973, Lawvere [50] proved that a metric space X is Cauchy complete if and only if every adjunction $\varphi\dashv\psi\colon Y \mathrel{\circ\!\!\to} X$ of $\overleftarrow{[0,\infty]}_+$-distributors is of the form $f_*\dashv f^*$, for some non-expansive map $f\colon X\to Y$. This observation motivates the following nomenclature.

Definition 2.19. Let $\mathcal{V} = (\mathcal{V}, \otimes, k)$ be a quantale. A $\mathcal{V}$-category (X, a) is *Cauchy complete* if every adjunction of $\mathcal{V}$-distributors $(\varphi \colon X \mathrel{-\!\!\circ\!\!\rightarrow} Y) \dashv (\psi \colon Y \mathrel{-\!\!\circ\!\!\rightarrow} X)$ is representable, meaning that there is a $\mathcal{V}$-functor $f \colon Y \to X$ such that $\varphi = f_*$ and $\psi = f^*$.

Although the definition requires the representability of every adjunction, it is enough to consider the case $Y = G$. Thus, a $\mathcal{V}$-category X is Cauchy complete if and only if every adjunction $(\varphi \colon G \mathrel{-\!\!\circ\!\!\rightarrow} X) \dashv (\psi \colon X \mathrel{-\!\!\circ\!\!\rightarrow} G)$ is representable by some $x \in X$. Subsequent developments established conditions under which results relating Cauchy sequences, convergence of sequences, adjunctions of distributors and representability can be generalised to $\mathcal{V}$-Cat (see [11, 15, 19, 39, 42, 75]).

Finally, we recall some topological notions for $\mathcal{V}$-categories from [42]. For a quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$, a $\mathcal{V}$-category (X, a) and a subset $M \subseteq X$, the *L-closure* $\overline{M}$ of M is given by the collection of all $x \in X$ which *represent adjoint distributors on* M. More precisely, $x \in \overline{M}$ whenever $i^* \cdot x_* \dashv x^* \cdot i_*$, where $i \colon M \hookrightarrow X$ is the inclusion $\mathcal{V}$-functor. In more elementary terms, we have (see [42]):

Proposition 2.20. *Let (X, a) be a $\mathcal{V}$-category, $M \subseteq X$ and $x \in X$. Then $x \in \overline{M}$ if and only if $k \leq \bigvee_{z \in M} a(x, z) \otimes a(z, x)$.*

The above proposition also shows that (X, a) and $(X, a)^{\mathrm{op}}$ induce the same closure operator on the set X. The following two results can be found in [42] and describe fundamental properties of the L-closure.

Proposition 2.21. *Let $\mathcal{V} = (\mathcal{V}, \otimes, k)$ be a quantale. For a $\mathcal{V}$-functor $f \colon X \to Y$ and $M, M' \subseteq X$, $N \subseteq Y$, one has:*

(1) *$M \subseteq \overline{M}$ and $M \subseteq M'$ implies $\overline{M} \subseteq \overline{M'}$.*

(2) *$\overline{\overline{M}} = \overline{M}$.*

(3) *$f(\overline{M}) \subseteq \overline{f(M)}$ and $f^{-1}(\overline{N}) \supseteq \overline{f^{-1}(N)}$.*

(4) *If k is $\vee$-irreducible, then $\overline{M \cup M'} = \overline{M} \cup \overline{M'}$ and $\overline{\varnothing} = \varnothing$.*

Corollary 2.22. *If k is $\vee$-irreducible in $\mathcal{V}$, then the L-closure operator defines a topology on X such that every $\mathcal{V}$-functor becomes continuous. Hence, in this case the L-closure defines a functor $L_{\mathcal{V}} \colon \mathcal{V}$-$\mathsf{Cat} \to \mathsf{Top}$.*

3 Combining convergence and $\mathcal{V}$-categories

In this section, we study $\mathcal{V}$-categories equipped with a compatible convergence structure. As we explained in Section 1, this study has its roots in Nachbin's "Topology and Order" [59] as presented in [72]. We recall the notion of topological theory $\mathcal{U}$ [36], which provides enough structure to extend the ultrafilter monad $\mathbb{U}$ to a monad on $\mathcal{V}$-Cat; the algebras for this monad we designate as *$\mathcal{V}$-categorical compact Hausdorff spaces.* We also recall the notions of $\mathcal{U}$-category and $\mathcal{U}$-functor and the comparison between $\mathcal{V}$-categorical compact Hausdorff spaces and $\mathcal{U}$-categories, which can already be found in [72]. In Subsection 3.6 we use the closure operator on $\mathcal{V}$-Cat (see Subsection 2.6) to define *compact* $\mathcal{V}$-categories, and show, under some conditions on $\mathcal{V}$, that compact separated $\mathcal{V}$-categories provide examples of $\mathcal{V}$-categorical compact Hausdorff spaces.

3.1 Ultrafilter theories Of particular interest to us is the ultrafilter monad $\mathbb{U} = (U, m, e)$ which is induced by the adjunction

$$\begin{tikzcd} \mathsf{Boole}^{\mathrm{op}} \arrow[r, bend left, "\mathsf{Boole}(-{,}\mathbf{2})"] \arrow[r, phantom, "\top"] & \mathsf{Set}; \arrow[l, bend left, "\mathsf{Set}(-{,}\mathbf{2})"] \end{tikzcd}$$

for more information on monads we refer to [54]. Here the functor $U\colon \mathsf{Set} \to \mathsf{Set}$ takes a set X to the set UX of ultrafilters on X and, for a map $f\colon X \to Y$ and $\mathfrak{x} \in UX$, $Uf(\mathfrak{x}) = \{A \subseteq Y \mid f^{-1}(A) \in \mathfrak{x}\}$. The unit $e_X\colon X \to UX$ on X sends $x \in X$ to the principal ultrafilter $\dot{x}$ on X, and the multiplication $m_X\colon U^2X \to UX$ is characterised, for every $\mathfrak{X} \in U^2X$, by $m_X(\mathfrak{X}) = \{A \in UX \mid A^{\#} \in \mathfrak{X}\}$, where $A^{\#} = \{\mathfrak{x} \in UX \mid A \in \mathfrak{x}\}$. The Eilenberg-Moore category of $\mathbb{U}$, $\mathsf{Set}^{\mathbb{U}}$, is equivalent to the category of compact Hausdorff topological spaces and continuous maps (see [55]).

The following result (see [70]) ensures the existence of certain ultrafilters and will be very important for later usage.

Lemma 3.1. *Let $\mathfrak{f}$ be a filter and $\mathfrak{j}$ be an ideal on a set X such that $\mathfrak{f} \cap \mathfrak{j} = \varnothing$. Then there is an ultrafilter $\mathfrak{x}$ that extends $\mathfrak{f}$ and excludes $\mathfrak{j}$; that is, $\mathfrak{f} \subseteq \mathfrak{x}$ and $\mathfrak{j} \cap \mathfrak{x} = \varnothing$.*

In this paper, we consider a particular case $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ of a topological theory (in the sense of [36]) based on the ultrafilter monad $\mathbb{U} = (U, m, e)$, a quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$ and a map $\xi \colon U\mathcal{V} \to \mathcal{V}$. Here we require $(\mathbb{U}, \mathcal{V}, \xi)$ to satisfy all the axioms of the definition of a *strict* topological theory with the exception of the axiom regarding the tensor product $\otimes$ of $\mathcal{V}$, for which it is enough to have lax continuity. We call such a theory an *ultrafilter theory*. More in detail:

- the map $\xi \colon U\mathcal{V} \to \mathcal{V}$ is the structure of an Eilenberg–Moore algebra on $\mathcal{V}$,

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "1_X"'] & UX \arrow[d, "\xi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} UUX \arrow[r, "m_X"] \arrow[d, "U\xi"'] & UX \arrow[d, "\xi"] \\ UX \arrow[r, "\xi"'] & X \end{tikzcd}$$

  that is, $\xi \colon U\mathcal{V} \to \mathcal{V}$ is the convergence of a compact Hausdorff topology on $\mathcal{V}$;

- The tensor product is "laxly continuous":

$$\begin{tikzcd} U(\mathcal{V} \times \mathcal{V}) \arrow[r, "U\otimes"] \arrow[d, "\langle \xi U\pi_1, \xi U\pi_2 \rangle"'] \arrow[dr, phantom, "\leq"] & U\mathcal{V} \arrow[d, "\xi"] \\ \mathcal{V} \times \mathcal{V} \arrow[r, "\otimes"'] & \mathcal{V} \end{tikzcd}$$

- ξ is "compatible with suprema in $\mathcal{V}$" as specified in condition $(\mathrm{Q}_{\bigvee})$ in [36].

We call a theory $\mathcal{U}$ satisfying even equality in the axiom involving the tensor product a *strict ultrafilter theory*. We note that every ultrafilter theory $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ based on a frame $\mathcal{V}$ with $\otimes = \wedge$ is strict. Furthermore, $\mathcal{U}$ is called *compatible with finite suprema* whenever the diagram

$$\begin{tikzcd} U(\mathcal{V} \times \mathcal{V}) \arrow[r, "U\vee"] \arrow[d, "\langle \xi U\pi_1, \xi U\pi_2 \rangle"'] & U\mathcal{V} \arrow[d, "\xi"] \\ \mathcal{V} \times \mathcal{V} \arrow[r, "\vee"'] & \mathcal{V} \end{tikzcd}$$

commutes. Note that we do not need to impose a condition on the empty supremum since, for every ultrafilter theory, the diagram

$$\begin{tikzcd} U1 \arrow[r, "U\perp"] \arrow[d] & U\mathcal{V} \arrow[d, "\xi"] \\ 1 \arrow[r, "\perp"'] & \mathcal{V} \end{tikzcd}$$

commutes. For $u \in \mathcal{V}$, we consider the map

$$t_u : \mathcal{V} \longrightarrow \mathcal{V},\, v \longmapsto u \otimes v.$$

An ultrafilter theory $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ is called *pointwise strict* whenever, for all $u \in \mathcal{V}$, the diagram

$$\begin{tikzcd} U\mathcal{V} \arrow[r, "Ut_u"] \arrow[d, "\xi"'] & U\mathcal{V} \arrow[d, "\xi"] \\ \mathcal{V} \arrow[r, "t_u"'] & \mathcal{V} \end{tikzcd}$$

commutes. Clearly, every strict ultrafilter theory is pointwise strict.

The following result (see [36, Theorem 3.3]) provides examples of ultrafilter theories.

Theorem 3.2. *For every completely distributive quantale $\mathcal{V}$, the map*

$$\xi \colon U\mathcal{V} \longrightarrow \mathcal{V},\, \mathfrak{v} \longmapsto \bigwedge_{A \in \mathfrak{v}} \bigvee_{u \in A} u$$

defines an ultrafilter theory $(\mathbb{U}, \mathcal{V}, \xi)$.

Somehow surprisingly, the formula above depends only on the lattice structure of $\mathcal{V}$; moreover, it is *self-dual* in the sense that

$$\xi(\mathfrak{v}) = \bigwedge_{A \in \mathfrak{v}} \bigvee_{u \in A} u = \bigvee_{A \in \mathfrak{v}} \bigwedge_{u \in A} u,$$

for all $\mathfrak{v} \in U\mathcal{V}$. For the lattices $\mathcal{V} = \mathbf{2}$, $\mathcal{V} = [0,1]$, $\mathcal{V} = [0,\infty]$ and $\mathcal{V} = \mathcal{D}$ we denote the corresponding map $\xi : U\mathcal{V} \to \mathcal{V}$ by $\xi_{\mathbf{2}}$, $\xi_{[0,1]}$, $\xi_{[0,\infty]}$ and $\xi_{\mathcal{D}}$, respectively.

Proposition 3.3. *Let $\mathcal{V}$ be a completely distributive quantale and $\xi\colon U\mathcal{V} \to \mathcal{V}$ as in Theorem* 3.2. *Then $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ is compatible with finite suprema.*

Proof. Just apply Theorem 3.2 to the quantale $\mathcal{V}^{\mathrm{op}}$ with tensor product given by binary suprema $\vee$ in $\mathcal{V}$. Here we use the fact that also the lattice $\mathcal{V}^{\mathrm{op}}$ is completely distributive and therefore in particular a frame. $\square$

Example 3.4. According to the quantales introduced in Examples 2.1, and keeping in mind Examples 2.8, we have the following examples of ultrafilter theories.

(1) For $\mathcal{V} = \mathbf{2}$, the convergence of Theorem 3.2 corresponds to the discrete topology on $\mathbf{2}$. We denote this theory as $\mathcal{U}_{\mathbf{2}}$.

(2) For the quantales based on the lattices $[0,1]$ and $[0,\infty]$, the convergence of Theorem 3.2 corresponds to the usual Euclidean topology. We denote the corresponding theories by $\mathcal{U}_{\overleftarrow{[0,\infty]}_+}$, $\mathcal{U}_{\overleftarrow{[0,\infty]}_\wedge}$, $\mathcal{U}_{[0,1]_*}$, $\mathcal{U}_{[0,1]_\wedge}$, and $\mathcal{U}_{[0,1]_\odot}$, respectively.

(3) We will denote the ultrafilter theory based on the quantale $\mathcal{D}$ and on the the convergence of Theorem 3.2 by $\mathcal{U}_{\mathcal{D}}$.

Remark 3.5. For each of the quantales $\mathcal{V} = \mathbf{2}$, $\mathcal{V} = [0,1]$ and $\mathcal{V} = [0,\infty]$, the theory obtained from Theorem 3.2 is strict. However, we do not know if there is a strict ultrafilter theory involving the quantale $\mathcal{D}$ of distribution functions.

Definition 3.6. Let $\mathscr{U}_1 = (\mathbb{U}, \mathcal{V}_1, \xi_1)$ and $\mathscr{U}_2 = (\mathbb{U}, \mathcal{V}_2, \xi_2)$ be ultrafilter theories and $\varphi\colon \mathcal{V}_1 \to \mathcal{V}_2$ be a lax quantale morphism. Then φ is *compatible* with $\mathscr{U}_1$ and $\mathscr{U}_2$ whenever, for all $\mathfrak{v} \in U\mathcal{V}_1$, $\xi_2 \cdot U\varphi(\mathfrak{v}) \leq \varphi \cdot \xi_1(\mathfrak{v})$.

$$\begin{tikzcd}
U\mathcal{V}_1 \arrow[r, "U\varphi"] \arrow[d, "\xi_1"'] \arrow[dr, phantom, "\geq"] & U\mathcal{V}_2 \arrow[d, "\xi_2"] \\
\mathcal{V}_1 \arrow[r, "\varphi"'] & \mathcal{V}_2
\end{tikzcd}$$

For instance, for every ultrafilter theory $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$, the canonical map $i\colon \mathbf{2} \to \mathcal{V}$ (see Subsection 2.4) is a lax quantale morphism making the diagram

$$\begin{tikzcd} U\mathbf{2} \arrow[r, "Ui"] \arrow[d, "\xi_{\mathbf{2}}"'] & U\mathcal{V} \arrow[d, "\xi"] \\ \mathbf{2} \arrow[r, "i"'] & \mathcal{V} \end{tikzcd}$$

commutative; hence i is compatible with $\mathcal{U}_{\mathbf{2}}$ and $\mathcal{U}$.

Lemma 3.7. *Let $\mathscr{U} = (\mathbb{U}, \mathcal{V}, \xi)$ be an ultrafilter theory where k is the top element of $\mathcal{V}$. Then the right adjoint $p\colon \mathcal{V} \to \mathbf{2}$ of i is compatible with $\mathscr{U}$ and $\mathscr{U}_{\mathbf{2}}$.*

Proof. Let $\mathfrak{v} \in U\mathcal{V}$ and assume that $\xi_{\mathbf{2}}(Up(\mathfrak{v})) = 1$. Then $Up(\mathfrak{v}) = \dot{1}$ and therefore $\uparrow k \in \mathfrak{v}$. If k is the top-element of $\mathcal{V}$, then $\{k\} \in \mathfrak{v}$ and consequently $\xi(\mathfrak{v}) = k$. □

Lemma 3.8. *Let $\mathscr{U}_1 = (\mathbb{U}, \mathcal{V}_1, \xi_1)$ and $\mathscr{U}_2 = (\mathbb{U}, \mathcal{V}_2, \xi_2)$ be ultrafilter theories where ξ_1, ξ_2 are as in Theorem 3.2. Assume that $\varphi\colon \mathcal{V}_1 \to \mathcal{V}_2$ is a lax quantale morphism preserving codirected infima. Then φ is compatible with $\mathscr{U}_1$ and $\mathscr{U}_2$.*

Proof. Let $\mathfrak{v} \in U\mathcal{V}_1$. Then

$$\xi_2(U\varphi(\mathfrak{v})) = \bigwedge_{A\in\mathfrak{v}} \bigvee_{u\in A} \varphi(u) \leq \bigwedge_{A\in\mathfrak{v}} \varphi\left(\bigvee_{u\in A} u\right) = \varphi\left(\bigwedge_{A\in\mathfrak{v}} \bigvee_{u\in A} u\right) = \varphi(\xi_1(\mathfrak{v})).$$

□

The result above applies in particular when $\varphi\colon \mathcal{V}_1 \to \mathcal{V}_2$ is right adjoint. For instance, if $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ is an ultrafilter theory where $\mathcal{V}$ is completely distributive and $\xi\colon U\mathcal{V} \to \mathcal{V}$ is as in Theorem 3.2, then $p\colon \mathcal{V} \to \mathbf{2}$ is compatible with $\mathcal{U}$ and $\mathcal{U}_{\mathbf{2}}$ since $i \dashv p$.

Example 3.9. Recall the chain $O_\infty \dashv I_\infty \dashv P_\infty$ of adjoint lax quantale morphisms introduced in Example 2.15 (2). Since I_∞ and P_∞ are both right adjoints, they are compatible with the ultrafilter theories $\mathcal{U}_{\overleftarrow{[0,\infty]}_+}$ and $\mathcal{U}_{\mathcal{D}}$.

3.2 Extending the monad Given an ultrafilter theory $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$, we extend the functor $U : \mathsf{Set} \to \mathsf{Set}$ to a lax functor U_ξ on $\mathcal{V}$-Rel by putting $U_\xi X = UX$ for each set X and

$$U_\xi r : UX \times UY \longrightarrow \mathcal{V}$$
$$(\mathfrak{x}, \mathfrak{y}) \longmapsto \bigvee \left\{ \xi \cdot Ur(\mathfrak{w}) \;\middle|\; \mathfrak{w} \in U(X \times Y), U\pi_X(\mathfrak{w}) = \mathfrak{x}, U\pi_Y(\mathfrak{w}) = \mathfrak{y} \right\}$$

for each $\mathcal{V}$-relation $r : X \times Y \to \mathcal{V}$. The following result can be found in [36].

Theorem 3.10. *Let $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ be an ultrafilter theory. Then the following assertions hold.*

(1) *For each $\mathcal{V}$-relation $r\colon X \nrightarrow Y$, $U_\xi(r^\circ) = U_\xi(r)^\circ$ (and we write $U_\xi r^\circ$).*

(2) *For each function $f\colon X \to Y$, $Uf = U_\xi f$ and $(Uf)^\circ = U_\xi(f^\circ)$.*

(3) *For each $\mathcal{V}$-relation $r\colon X \nrightarrow Y$ and functions $f\colon A \to X$ and $g\colon Y \to Z$,*

$$U_\xi(g \cdot r) = Ug \cdot U_\xi r \quad \text{and} \quad U_\xi(r \cdot f) = U_\xi r \cdot Uf.$$

(4) *For all $\mathcal{V}$-relations $r\colon X \nrightarrow Y$ and $s\colon Y \nrightarrow Z$, $U_\xi s \cdot U_\xi r \le U_\xi(s \cdot r)$. We have even equality if $\mathcal{U}$ is a strict theory.*

(5) *Then e becomes an op-lax natural transformation $e\colon 1 \to U_\xi$ and m a natural transformation $m\colon U_\xi U_\xi \to U_\xi$, that is, for every $\mathcal{V}$-relation $r\colon X \nrightarrow Y$ we have*

$$e_Y \cdot r \le U_\xi r \cdot e_X, \qquad m_Y \cdot U_\xi U_\xi r = U_\xi r \cdot m_X.$$

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[d, "r"'] \arrow[dr, phantom, "\le"] & U_\xi X \arrow[d, "U_\xi r"] \\ Y \arrow[r, "e_Y"'] & U_\xi Y \end{tikzcd} \qquad \begin{tikzcd} U_\xi U_\xi X \arrow[r, "m_X"] \arrow[d, "U_\xi U_\xi r"'] & U_\xi X \arrow[d, "U_\xi r"] \\ U_\xi U_\xi Y \arrow[r, "m_Y"'] & U_\xi Y \end{tikzcd}$$

3.3 $\mathcal{V}$-categorical compact Hausdorff spaces Based on the lax extension of the Set-monad $\mathbb{U} = (U, m, e)$ to $\mathcal{V}$-Rel described in Subsection 3.2, the Set-monad $\mathbb{U}$ admits a natural extension to a monad on $\mathcal{V}$-Cat, in the sequel also denoted as $\mathbb{U} = (U, m, e)$ (see [72]). Here the functor $U\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ sends a $\mathcal{V}$-category (X, a_0) to $(UX, U_\xi a_0)$, and with this definition $e_X\colon X \to UX$ and $m_X\colon UUX \to UX$ become $\mathcal{V}$-functors for each $\mathcal{V}$-category X.

Definition 3.11. Let $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ be an ultrafilter theory. An Eilenberg–Moore algebra for the monad $\mathbb{U}$ on $\mathcal{V}$-Cat is called *$\mathcal{V}$-categorical compact Hausdorff space.*

Hence, a $\mathcal{V}$-categorical compact Hausdorff space can be described as a triple (X, a_0, α) where (X, a_0) is a $\mathcal{V}$-category and $\alpha\colon UX \to X$ is the convergence of a compact Hausdorff topology on X such that $\alpha\colon U(X, a_0) \to (X, a_0)$ is a $\mathcal{V}$-functor. For $\mathbb{U}$-algebras (X, a_0, α) and (Y, b_0, β), a map $f\colon X \to Y$ is a homomorphism $f\colon (X, a_0, \alpha) \to (Y, b_0, \beta)$ precisely if f preserves both structures, that is, whenever $f\colon (X, a_0) \to (Y, b_0)$ is a $\mathcal{V}$-functor and $f\colon (X, \alpha) \to (Y, \beta)$ is continuous. Since the extension U_ξ of U commutes with the involution $(-)^\circ$, with $X = (X, a_0, \alpha)$ also (X, a_0°, α) is a $\mathcal{V}$-categorical compact Hausdorff space. It follows from [36, Lemma 3.2] that the $\mathcal{V}$-category $(\mathcal{V}, \hom)$ combined with the $\mathbb{U}$-algebra structure ξ induces the $\mathcal{V}$-categorical compact Hausdorff space $\mathcal{V} = (\mathcal{V}, \hom, \xi)$.

Example 3.12. (1) Our motivating example is produced by $\mathcal{U} = \mathcal{U}_{\mathbf{2}}$. In this case, the objects of the Eilenberg-Moore category for the monad $\mathbb{U}$ on Ord are precisely the ordered compact Hausdorff spaces introduced in [59], and the homomorphisms are the monotone continuous map. We denote this category by OrdCH. We recall that an *ordered compact Hausdorff space X* is a set equipped with an order relation $\leq$ and a compact Hausdorff topology so that

$$\{(x, y) \mid x \leq y\} \subseteq X \times X$$

is closed with respect to the product topology. It is shown in [72] that this condition is equivalent to being an Eilenberg–Moore algebra for the ultrafilter monad on Ord.

(2) For $\mathcal{U} = \mathcal{U}_{\overleftarrow{[0,\infty]}_+}$, we put $\mathsf{MetCH} = \mathsf{Met}^{\mathbb{U}}$ and call an object of MetCH a *metric compact Hausdorff space.*

(3) Similarly, for $\mathcal{U} = \mathcal{U}_{\mathcal{D}}$, the objects of $\mathsf{ProbMet}^{\mathbb{U}}$ are called *probabilistic metric compact Hausdorff spaces.* The category $\mathsf{ProbMet}^{\mathbb{U}}$ will be represented by $\mathsf{ProbMetCH}$.

Proposition 3.13. *Let $\mathscr{U}_1 = (\mathbb{U}, \mathcal{V}_1, \xi_1)$ and $\mathscr{U}_2 = (\mathbb{U}, \mathcal{V}_2, \xi_2)$ be ultrafilter theories and $\varphi\colon \mathcal{V}_1 \to \mathcal{V}_2$ be a lax quantale morphism compatible with $\mathscr{U}_1$ and $\mathscr{U}_2$. Then, for every $\mathcal{V}_1$-category X, the identity map on the set UX is a $\mathcal{V}_2$-functor of type*

$$UB_\varphi(X) \longrightarrow B_\varphi U(X).$$

Proof. Let (X, a_0) be a $\mathcal{V}_1$-category. Then, since φ is compatible with the ultrafilter theories $\mathcal{U}_1$ and $\mathcal{U}_2$, for all $\mathfrak{x}, \mathfrak{y} \in UX$ we have

$$\begin{aligned} U_{\xi_2}(\varphi a_0)(\mathfrak{x}, \mathfrak{y}) &= \bigvee\{\xi_2 \cdot U\varphi \cdot Ua_0(\mathfrak{w}) \mid U\pi_1(\mathfrak{w}) = \mathfrak{x}, U\pi_2(\mathfrak{w}) = \mathfrak{y}\} \\ &\leq \bigvee\{\varphi \cdot \xi_1 \cdot Ua_0(\mathfrak{w}) \mid U\pi_1(\mathfrak{w}) = \mathfrak{x}, U\pi_2(\mathfrak{w}) = \mathfrak{y}\} \\ &\leq \varphi\left(\bigvee\{\xi_1 \cdot Ua_0(\mathfrak{w}) \mid U\pi_1(\mathfrak{w}) = \mathfrak{x}, U\pi_2(\mathfrak{w}) = \mathfrak{y}\}\right) \\ &= \varphi(U_{\xi_1} a_0)(\mathfrak{x}, \mathfrak{y}); \end{aligned}$$

which proves the claim. $\square$

Hence, the family of these maps defines a natural transformation

$$\begin{tikzcd} \mathcal{V}_1\text{-}\mathsf{Cat} \arrow[r, "B_\varphi"] \arrow[d, "U"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{V}_2\text{-}\mathsf{Cat} \arrow[d, "U"] \\ \mathcal{V}_1\text{-}\mathsf{Cat} \arrow[r, "B_\varphi"'] & \mathcal{V}_2\text{-}\mathsf{Cat} \end{tikzcd}$$

and, together with $B_\varphi \colon \mathcal{V}_1\text{-}\mathsf{Cat} \to \mathcal{V}_2\text{-}\mathsf{Cat}$, a monad morphism (see [63]) from the ultrafilter monad $\mathbb{U}$ on $\mathcal{V}_1$-Cat to the ultrafilter monad $\mathbb{U}$ on $\mathcal{V}_2$-Cat. As a result, we obtain the functor $B_\varphi \colon \mathcal{V}_1\text{-}\mathsf{Cat}^{\mathbb{U}} \to \mathcal{V}_2\text{-}\mathsf{Cat}^{\mathbb{U}}$ sending (X, a_0, α) to $(X, \varphi a_0, \alpha)$ and making the diagram

$$\begin{tikzcd} \mathcal{V}_1\text{-}\mathsf{Cat}^{\mathbb{U}} \arrow[r, "B_\varphi"] \arrow[d, "G^{\mathbb{U}}"'] & \mathcal{V}_2\text{-}\mathsf{Cat}^{\mathbb{U}} \arrow[d, "G^{\mathbb{U}}"] \\ \mathcal{V}_1\text{-}\mathsf{Cat} \arrow[r, "B_\varphi"'] & \mathcal{V}_2\text{-}\mathsf{Cat} \end{tikzcd}$$

commutative. In particular, for every completely distributive quantale $\mathcal{V}$ and ξ given by the formula in Theorem 3.2, the lax quantale morphism $p \colon \mathcal{V} \to \mathbf{2}$ induces the functor

$$B_p \colon \mathcal{V}\text{-}\mathsf{Cat}^{\mathbb{U}} \longrightarrow \mathsf{OrdCH}.$$

Example 3.14. We have seen in Example 3.9 that the lax quantale morphisms introduced in Example 2.15 (2) are compatible with the ultrafilter theories $\mathcal{U}_{\overleftarrow{[0,\infty]}_+}$ and $\mathcal{U}_{\mathcal{D}}$. As a consequence one has the adjoint functors

$$\begin{tikzcd} \mathsf{ProbMetCH} \arrow[r, bend left, "B_{P_\infty}"] \arrow[r, phantom, "\top"] & \mathsf{MetCH}. \arrow[l, bend left, "B_{I_\infty}"] \end{tikzcd}$$

3.4 $\mathcal{U}$-categories and $\mathcal{U}$-functors We have already mentioned in Section 1 that there is a close connection between ordered compact Hausdorff spaces and certain topological spaces. In this subsection we recall the definition of $\mathcal{U}$-categories as enriched substitutes of topological spaces. This notion has its roots in Barr's "relational algebras" [4]; an extensive presentation of the theory of $(\mathbb{T},\mathcal{V})$-categories (also called $(\mathbb{T},\mathcal{V})$-algebras), for a monad $\mathbb{T}$ and a quantale $\mathcal{V}$, can be found in [40].

Definition 3.15. A *$\mathcal{U}$-category* is a pair (X,a) consisting of a set X and a $\mathcal{V}$-relation $a\colon TX \to\!\!\!\!\!\mapsto X$ satisfying the lax Eilenberg–Moore axioms $1_X \le a\cdot e_X$ and $a\cdot U_\xi a \le a\cdot m_X$.

Expressed elementwise, these two conditions read as

$$k \le a(e_X(x),x) \quad \text{and} \quad U_\xi a(\mathfrak{X},\mathfrak{x})\otimes a(\mathfrak{x},x) \le a(m_X(\mathfrak{X}),x),$$

for all $\mathfrak{X}\in UUX$, $\mathfrak{x}\in UX$ and $x\in X$.

Definition 3.16. A function $f\colon X\to Y$ between $\mathcal{U}$-categories (X,a) and (Y,b) is a *$\mathcal{U}$-functor* whenever $f\cdot a\le b\cdot Uf$.

Since $f\dashv f^\circ$ in $\mathcal{V}$-Rel, this condition is equivalent to $a\le f^\circ\cdot b\cdot Uf$, and in pointwise notation the latter inequality becomes

$$a(\mathfrak{x},x)\le b(Uf(\mathfrak{x}),f(x)),$$

for all $\mathfrak{x}\in UX$, $x\in X$. The category of $\mathcal{U}$-categories and $\mathcal{U}$-functors is denoted by $\mathcal{U}$-Cat.

Example 3.17. (1) For $\mathcal{V}=\mathbf{2}$, a $\mathcal{U}_{\mathbf{2}}$-category is a set X equipped with a relation $\to\colon UX\to\!\!\!\!\!\mapsto X$ such that $e_x(x)\to x$ and, if $\mathfrak{X}\to\mathfrak{r}$ and $\mathfrak{r}\to x$, then $m_X(\mathfrak{X})\to x$. It is shown in [4] that these are precisely the convergence relations induced by topologies; in fact, the main result of [4] states that $\mathcal{U}_{\mathbf{2}}$-Cat is isomorphic to the category Top of topological spaces and continuous maps.

(2) The concept of approach space was introduced by Lowen in 1989 (see [52, 53]). It involves a set X and a map $\delta\colon PX \times X \to [0,\infty]$, called approach distance or distance map, satisfying:

(a) $\delta(\{x\},x) = 0$,

(b) $\delta(\varnothing,x) = \infty$,

(c) $\delta(A \cup B, x) = \min\{\delta(A,x), \delta(B,x)\}$,

(d) $\delta(A^{(\varepsilon)},x) + \varepsilon \geq \delta(A,x)$, with $A^{(\varepsilon)} = \{x \in X \mid \varepsilon \geq \delta(A,x)\}$,

for all $x \in X$, all $A, B \in PX$ and all $\varepsilon \in [0,\infty]$. A non-expansive map is a map $f\colon X \to Y$ between approach spaces (X,δ) and (Y,δ') subject to $\delta(A,x) \geqslant \delta'(f(A),f(x))$, for all $A \in PX$ and all $x \in X$.

It was proved in [14] that a $\overleftarrow{[0,\infty]}_+$-relation $a\colon UX \nrightarrow X$ is induced by an approach distance $\delta\colon PX \times X \to [0,\infty]$ if and only if

$$0 \geqslant a(\dot{x},x) \quad \text{and} \quad U_\xi a(\mathfrak{X},\mathfrak{x}) + a(\mathfrak{x},x) \geqslant a(m_X(\mathfrak{X}),x).$$

for all $x \in X$, all $\mathfrak{x} \in UX$ and all $\mathfrak{X} \in UUX$, or equivalently, if and only if (X,a) is a $\mathcal{U}_{\overleftarrow{[0,\infty]}_+}$-category. Moreover, non-expansive maps correspond precisely to $\mathcal{U}_{\overleftarrow{[0,\infty]}_+}$-functors, so that $\mathsf{App} \simeq \mathcal{U}_{\overleftarrow{[0,\infty]}_+}$-$\mathsf{Cat}$.

(3) For $\mathcal{V} = \overleftarrow{[0,\infty]}_\wedge \simeq [0,1]_\wedge$, $\mathcal{U}_{\overleftarrow{[0,\infty]}_\wedge}$-$\mathsf{Cat}$ can be identified with the subcategory of App whose objects (X,a) are the approach spaces satisfying additionally the condition

$$\max(U_\xi a(\mathfrak{X},\mathfrak{x}), a(\mathfrak{x},x)) \geqslant a(m_X(\mathfrak{X}),x),$$

for all $\mathfrak{X} \in UUX$, $\mathfrak{x} \in UX$ and $x \in X$.

(4) For $\mathcal{V} = [0,1]_\odot$, $\mathcal{U}_{[0,1]_\odot}$-$\mathsf{Cat}$ is the category whose objects are structures of the type (X,a) with $a\colon UX \nrightarrow X$ satisfying $1 \leqslant a(\dot{x},x)$ and

$$U_\xi a(\mathfrak{X},\mathfrak{x}) + a(\mathfrak{x},x) \geqslant 1 \implies U_\xi a(\mathfrak{X},\mathfrak{x}) + a(\mathfrak{x},x) \leqslant a(m_X(\mathfrak{X}),x) + 1.$$

(5) For $\mathcal{V} = \mathcal{D}$, $\mathcal{U}$-categories can be identified with probabilistic approach spaces. This is an example of a *quantale-valued approach space* studied in [48]. For the sake of simplicity we will represent $\mathcal{U}_\mathcal{D}$-Cat by $\mathsf{ProbApp}$.

Similarly to the situation for $\mathcal{V}$-categories, the canonical forgetful functor $O_{\mathcal{U}}\colon \mathcal{U}\text{-}\mathsf{Cat} \to \mathsf{Set}$ is topological (see [14]); which implies that the category $\mathcal{U}\text{-}\mathsf{Cat}$ is complete and cocomplete and $O_{\mathcal{U}}$ preserves limits and colimits. We denote the free $\mathcal{U}$-category over the one-element set 1 by $G = (1, k)$; here $k\colon U1 \nrightarrow 1$ is the $\mathcal{V}$-relation which sends the unique element of $U1 \times 1$ to k.

There are several functors connecting $\mathcal{U}$-categories with $\mathcal{V}$-categories and topological spaces. Firstly, we have a functor $\mathsf{Set}^{\mathbb{U}} \hookrightarrow \mathcal{U}\text{-}\mathsf{Cat}$ interpreting an Eilenberg–Moore algebra as a lax one. Furthermore, there is a forgetful functor $(-)_0\colon \mathcal{U}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ sending (X, a) to $(X, a_0 = a \cdot e_X)$ and leaving maps unchanged. We notice that the diagram

$$\begin{tikzcd}
\mathsf{Set}^{\mathbb{U}} \arrow[r] \arrow[d, "G^{\mathbb{U}}"'] & \mathcal{U}\text{-}\mathsf{Cat} \arrow[d, "(-)_0"] \\
\mathsf{Set} \arrow[r, "\text{discrete}"'] & \mathcal{V}\text{-}\mathsf{Cat}
\end{tikzcd}$$

commutes. Furthermore, by [36, Section 4], we have:

Proposition 3.18. *Assume that $\varphi\colon \mathcal{V}_1 \to \mathcal{V}_2$ is is a lax quantale morphism compatible with the ultrafilter theories $\mathscr{U}_1 = (\mathbb{U}, \mathcal{V}_1, \xi_1)$ and $\mathscr{U}_2 = (\mathbb{U}, \mathcal{V}_2, \xi_2)$. Then*

$$(X, a) \longmapsto (X, \varphi a) \qquad \text{and} \qquad f \longmapsto f$$

define a functor $B_\varphi\colon \mathscr{U}_1\text{-}\mathsf{Cat} \to \mathscr{U}_2\text{-}\mathsf{Cat}$.

By Proposition 3.18, for every ultrafilter theory $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$, the canonical map $i\colon \mathbf{2} \to \mathcal{V}$ (see Subsection 2.4) induces the functor

$$B_i\colon \mathsf{Top} \longrightarrow \mathcal{U}\text{-}\mathsf{Cat}.$$

interpreting a topological space X as the $\mathcal{U}$-category with structure

$$(\mathfrak{x}, x) \longmapsto \begin{cases} k & \text{if } \mathfrak{x} \to x \\ \bot & \text{else,} \end{cases}$$

for $\mathfrak{x} \in UX$ and $x \in X$. If, moreover, the right adjoint $p\colon \mathcal{V} \to \mathbf{2}$ of i is compatible with $\mathcal{U}$ and $\mathcal{U}_{\mathbf{2}}$ (see Lemmas 3.7 and 3.8), then p defines a functor

$$B_p\colon \mathcal{U}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Top}$$

which is right adjoint to B_i. Here B_p sends an $\mathcal{U}$-category (X,a) to the topological space X with convergence

$$UX \times X \xrightarrow{\ a\ } \mathcal{V} \xrightarrow{\ p\ } \mathbf{2};$$

that is, for $\mathfrak{x} \in UX$ and $x \in X$, $\mathfrak{x} \to x$ if and only if $k \leq a(\mathfrak{x}, x)$.

Example 3.19. The adjoint lax quantale morphisms $I_\infty \dashv P_\infty$ (see Example 2.15) are compatible with the ultrafilter theories $\mathcal{U}_{\overleftarrow{[0,\infty]_+}}$ and $\mathcal{U}_{\mathcal{D}}$. Therefore they induce the adjoint functors

$$\begin{tikzcd} \mathsf{ProbApp} \arrow[r, bend left, "B_{P_\infty}"] \arrow[r, phantom, "\top"] & \mathsf{App}. \arrow[l, bend left, "B_{I_\infty}"] \end{tikzcd}$$

3.5 Comparison with $\mathcal{U}$-categories It is shown in [72] that there is a canonical functor

$$K\colon (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{U}} \longrightarrow \mathcal{U}\text{-}\mathsf{Cat}$$

which associates to each $X = (X, a_0, \alpha)$ in $(\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{U}}$ the $\mathcal{U}$-category $KX = (X, a)$ where $a = a_0 \cdot \alpha$. Note that $(a_0 \cdot \alpha)_0 = a_0$, that is, the diagram

$$\begin{tikzcd} (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{U}} \arrow[r, "K"] \arrow[dr, "G^{\mathbb{U}}"'] & \mathcal{U}\text{-}\mathsf{Cat} \arrow[d, "(-)_0"] \\ & \mathcal{V}\text{-}\mathsf{Cat} \end{tikzcd} \tag{3.5.1}$$

commutes. Applying K to $\mathcal{V} = (\mathcal{V}, \hom, \xi)$ produces the $\mathcal{U}$-category $\mathcal{V} = (\mathcal{V}, \hom_\xi)$ where

$$\hom_\xi\colon U\mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V},\ (\mathfrak{v}, v) \longmapsto \hom(\xi(\mathfrak{v}), v).$$

Example 3.20. (1) For $\mathcal{U} = \mathcal{U}_{\mathbf{2}}$, one obtains the commutative diagram

$$\begin{tikzcd} \mathsf{OrdCH} \arrow[r, "K"] \arrow[dr, "G^{\mathbb{U}}"'] & \mathsf{Top} \arrow[d, "(-)_0"] \\ & \mathsf{Ord} \end{tikzcd}$$

Here every ordered compact Hausdorff space maps to a weakly sober, locally compact and stable topological space; assuming also the T0-axiom, these spaces are called *stably compact* (see [26]). We recall that a topological space X is called *weakly sober* whenever every irreducible closed subset of $A \subseteq X$ is the closure of a singleton $A = \overline{\{x\}}$; and X is called *stable* whenever the way-below relation on the lattice of opens of X is stable under finite intersection. It is also shown in [26] that the full subcategory of OrdCH defined by the separated orders is isomorphic to the category $\mathsf{StablyComp}$ of stably compact topological spaces and spectral maps. We also note that the space $K\mathbf{2}$ is the Sierpiński space $\mathbf{2} = \{0,1\}$ with $\{1\}$ closed.

(2) When we consider the ultrafilter theory $\mathcal{U} = \mathcal{U}_{\overleftarrow{[0,\infty]}_+}$, diagram (3.5.1) becomes

$$\begin{tikzcd} \mathsf{MetCH} \arrow[r, "K"] \arrow[dr, "G^{\mathbb{U}}"'] & \mathsf{App} \arrow[d, "(-)_0"] \\ & \mathsf{Met}. \end{tikzcd}$$

Here the space $K\overleftarrow{[0,\infty]}_+$ coincides with the "Sierpiński approach space" of [53, Example 1.8.33 (2)]. Similarly to the topological case, it is shown in [29] that separated metric compact Hausdorff spaces correspond precisely to stably compact approach spaces.

(3) For $\mathcal{U} = \mathcal{U}_{\mathcal{D}}$, we obtain the diagram

$$\begin{tikzcd} \mathsf{ProbMetCH} \arrow[r, "K"] \arrow[dr, "G^{\mathbb{U}}"'] & \mathsf{ProbApp} \arrow[d, "(-)_0"] \\ & \mathsf{ProbMet}. \end{tikzcd}$$

The functor $K\colon (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{U}} \to \mathcal{U}\text{-}\mathsf{Cat}$ is right adjoint, its left adjoint assigns to every $\mathcal{U}$-category the $\mathcal{V}$-categorical compact Hausdorff space $(UX, \hat{a}, m_X)$. Regarding this construction, we recall here from [15]:

Lemma 3.21. *For every $\mathcal{U}$-category (X,a), $\hat{a} := U_\xi a \cdot m_X^\circ$ is a $\mathcal{V}$-category structure on UX.*

We give now an alternative characterisation of the compatibility between the convergence and the $\mathcal{V}$-categorical structure of an Eilenberg–Moore algebra (X, a_0, α) in $\mathcal{V}\text{-}\mathsf{Cat}^{\mathbb{U}}$, which resembles the classical condition stating that "the order relation is closed in the product space".

Proposition 3.22. *For a $\mathcal{V}$-category (X, a_0) and a $\mathbb{U}$-algebra (X, α) with the same underlying set X, the following assertions are equivalent.*

(i) $\alpha\colon U(X, a_0) \to (X, a_0)$ *is a $\mathcal{V}$-functor.*

(ii) $a_0\colon (X, \alpha) \times (X, \alpha) \to (\mathcal{V}, \hom_\xi)$ *is an $\mathcal{U}$-functor.*

Proof. The first assertions is equivalent to

$$\forall \mathfrak{x}, \mathfrak{y} \in UX \,.\, U_\xi a_0(\mathfrak{x}, \mathfrak{y}) \leq a_0(\alpha(\mathfrak{x}), \alpha(\mathfrak{y})),$$

and, since

$$U_\xi a_0(\mathfrak{x}, \mathfrak{y}) = \bigvee \{\xi \cdot Ua_0(\mathfrak{w}) \mid \mathfrak{w} \in U(X \times X), U\pi_1(\mathfrak{w}) = \mathfrak{x}, U\pi_2(\mathfrak{w}) = \mathfrak{y}\},$$

this is equivalent to

$$\forall \mathfrak{x}, \mathfrak{y} \in UX, \forall \mathfrak{w} \in U(X \times X) \,.\, ((U\pi_1(\mathfrak{w}) = \mathfrak{x} \,\&\, U\pi_2(\mathfrak{w}) = \mathfrak{y}) \implies (\xi \cdot Ua_0(\mathfrak{w}) \leq a_0(\alpha(\mathfrak{x}), \alpha(\mathfrak{y})))).$$

On the other hand, the second statement translates to

$$\forall \mathfrak{w} \in U(X \times X), \forall \mathfrak{x}, \mathfrak{y} \in UX \,.\, ((U\pi_1(\mathfrak{w}) = \mathfrak{x} \,\&\, U\pi_2(\mathfrak{w}) = \mathfrak{y}) \implies (k \leq \hom(\xi \cdot Ua_0(\mathfrak{w}), a_0(\alpha(\mathfrak{x}), \alpha(\mathfrak{y}))))),$$

which proves the equivalence. □

From Proposition 3.22 we conclude immediately:

Lemma 3.23. *Let $\mathcal{U}$ be an ultrafilter theory so that $p\colon \mathcal{V} \to \mathbf{2}$ is compatible with $\mathcal{U}$ and $\mathcal{U}_{\mathbf{2}}$, and (X, a_0, α) be a $\mathcal{V}$-categorical compact Hausdorff space. Then, for all $x \in X$ and $u \in \mathcal{V}$, the closed balls*

$$\{y \in X \mid a_0(x, y) \geq u\} \quad \text{and} \quad \{y \in X \mid a_0(y, x) \geq u\}$$

with center x and radius u are closed with respect to the compact Hausdorff topology.

Proof. Applying the forgetful functor $B_p\colon \mathcal{U}\text{-}\mathsf{Cat} \to \mathsf{Top}$, we obtain that $a_0\colon (X, \alpha) \times (X, \alpha) \to B_p(\mathcal{V}, \hom_\xi)$ is continuous. For every $x \in X$, consider the continuous maps

$$X \xrightarrow{\langle 1_X, x\rangle} X \times X \xrightarrow{a_0} \mathcal{V} \quad \text{and} \quad X \xrightarrow{\langle x, 1_X\rangle} X \times X \xrightarrow{a_0} \mathcal{V}.$$

Then, for every $u \in \mathcal{V}$, the closed balls with center x and radius u are the preimages of the closed set $\uparrow u$ in $\mathcal{V}$. □

3.6 Convergences from $\mathcal{V}$-categories We recall from Corollary 2.22 that, for every quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$ where k is $\vee$-irreducible, we have the functor

$$L_{\mathcal{V}} \colon \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Top}$$

sending a $\mathcal{V}$-category (X, a_0) to X equipped with the L-closure of (X, a_0). We investigate now connections between $\mathcal{V}$-categorical and topological properties. Note that, if k is the top element of the quantale $\mathcal{V}$, then the projection maps $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ are $\mathcal{V}$-functors

$$\pi_1 \colon (X, a_0) \otimes (Y, b_0) \longrightarrow (X, a_0) \quad \text{and} \quad \pi_2 \colon (X, a_0) \otimes (Y, b_0) \longrightarrow (Y, b_0),$$

for all $\mathcal{V}$-categories (X, a_0) and (Y, b_0). Therefore, with the same proof as for [42, Corollary 5.8], we obtain:

Lemma 3.24. *Let $\mathcal{V}$ be a quantale where k is the top element and (X, a_0) be $\mathcal{V}$-category. For all $x, y \in X$,*

$$x \simeq y \iff (x, y) \in \overline{\Delta} \text{ in } (X, a_0) \otimes (X, a_0).$$

Hence, (X, a_0) is separated if and only if Δ is closed in $(X, a_0) \otimes (X, a_0)$.

In the sequel we will often require that the functor $L_{\mathcal{V}}$ is monoidal.

Definition 3.25. The functor $L_{\mathcal{V}} \colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Top}$ is *monoidal* if, for all $\mathcal{V}$-categories (X, a_0) and (Y, b_0), the identity map on $X \times Y$ is continuous of type

$$L_{\mathcal{V}}(X, a_0) \times L_{\mathcal{V}}(Y, b_0) \longrightarrow L_{\mathcal{V}}((X, a_0) \otimes (Y, b_0)).$$

Proposition 3.26. *Let $\mathcal{V} = (\mathcal{V}, \otimes, k)$ be a completely distributive quantale where k is the top element and $\vee$-irreducible in $\mathcal{V}$. We consider the ultrafilter theory $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ where $\xi \colon U\mathcal{V} \to \mathcal{V}$ is as in Theorem 3.2. Then the following assertions hold.*

(1) *The identity map on $\mathcal{V}$ is continuous of type $L_{\mathcal{V}}(\mathcal{V}, \hom) \to B_p(\mathcal{V}, \hom_\xi)$.*

(2) *Assume that $L_{\mathcal{V}}$ is monoidal. Then, for every $\mathcal{V}$-category (X, a_0), the topological space $L_{\mathcal{V}}(X, a_0)$ is Hausdorff if and only if (X, a_0) is separated.*

Proof. To see (1), we note that an ultrafilter $\mathfrak{x}$ converges to x in $L_{\mathcal{V}}(\mathcal{V}, \hom)$ if and only if, for all $A \in \mathfrak{x}$, $x \in \overline{A}$; that is,

$$k \le \bigvee_{z \in A} \hom(x, z) \otimes \hom(z, x).$$

On the other hand, $\mathfrak{x} \to x$ in $B_p(\mathcal{V}, \hom_\xi)$ is equivalent to

$$\forall A \in \mathfrak{x}\,.\,(\bigwedge A \le x).$$

Assume $\mathfrak{x} \to x$ in $L_\mathcal{V}(\mathcal{V}, \hom)$. For every $A \in \mathfrak{x}$, we calculate

$$\begin{aligned} x = \hom(k, x) \ge \hom(\bigvee_{z\in A} \hom(x,z) \otimes \hom(z,x), x) \\ = \bigwedge_{z\in A} \hom(\hom(x,z) \otimes \hom(z,x), x). \end{aligned}$$

Since k is the top element of $\mathcal{V}$,

$$z \otimes \hom(x,z) \otimes \hom(z,x) \le x \otimes \hom(x,z) \le x;$$

we obtain

$$\bigwedge_{z\in A} z \le \bigwedge_{z\in A} \hom(\hom(x,z) \otimes \hom(z,x), x) \le x.$$

Regarding (2), by Lemma 3.24, a $\mathcal{V}$-category (X, a_0) is separated if and only

$$\Delta_X \subseteq X \times X$$

is closed in $L_\mathcal{V}((X, a_0) \otimes (X, a_0))$. Hence, since $L_\mathcal{V}$ is monoidal, the assertion follows. $\square$

Definition 3.27. Let $\mathcal{V} = (\mathcal{V}, \otimes, k)$ be a quantale where k is $\vee$-irreducible. A $\mathcal{V}$-category X is called *compact* whenever the topological space $L_\mathcal{V}(X)$ is compact. The full subcategory of $\mathcal{V}\text{-}\mathsf{Cat}$ defined by all compact separated $\mathcal{V}$-categories is denoted by $\mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,comp}}$.

Theorem 3.28. *Let $\mathcal{V}$ be a completely distributive quantale where k is $\vee$-irreducible and the top element of $\mathcal{V}$ and assume that $L_\mathcal{V}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Top}$ is monoidal. Let $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ be the ultrafilter theory with $\xi\colon U\mathcal{V} \to \mathcal{V}$ as in Theorem 3.2. Then the functor $L_\mathcal{V}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,comp}} \to \mathsf{CompHaus}$ lifts to a functor $T_\mathcal{U}\colon \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,comp}} \to (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{U}}$ which commutes with the canonical* *forgetful functors*

$$\begin{tikzcd} & & \mathcal{V}\text{-}\mathsf{Cat} \\ \mathcal{V}\text{-}\mathsf{Cat}_{\mathrm{sep,comp}} \arrow[urr] \arrow[rr, "T_{\mathcal{U}}"] \arrow[drr, "L_{\mathcal{V}}"'] & & (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{U}} \arrow[u] \arrow[d] \\ & & \mathsf{CompHaus} \end{tikzcd}$$

to $\mathcal{V}$-Cat *and* $\mathsf{CompHaus}$.

Proof. Since the composite

$$L_{\mathcal{V}}(X,a_0)\times L_{\mathcal{V}}(X,a_0)\xrightarrow{\text{can}} L_{\mathcal{V}}((X,a_0)^{\mathrm{op}}\otimes(X,a_0))\xrightarrow{L_{\mathcal{V}}a_0} \\ \longrightarrow L_{\mathcal{V}}(\mathcal{V},\hom)\longrightarrow B_p(\mathcal{V},\hom_\xi)$$

is continuous, every separated compact $\mathcal{V}$-category becomes a $\mathcal{V}$-categorical compact Hausdorff space when equipped with the topology of $L_{\mathcal{V}}(X)$. $\square$

Proposition 3.29. *Let* $\mathcal{V}=(\mathcal{V},\otimes,k)$ *be a quantale where* k *is approximated. Then* $L_{\mathcal{V}}$ *is monoidal.*

Proof. Under the condition that k is approximated, the topology of a $\mathcal{V}$-category (X,a_0) is generated by the "symmetric open balls"

$$\mathrm{B_S}(x,u)=\{y\in X\mid u\ll a_0(x,y)\text{ and }u\ll a_0(y,x)\},$$

for $x\in X$ and $u\ll k$ (see Proposition 2.10 and [39, Remark 4.21]). Let now (X,a_0) and (Y,b_0) be $\mathcal{V}$-categories, $(x,y)\in X\times Y$ and $u\ll k$. By Proposition 2.10, there is some $v\ll k$ with $u\le v\otimes v$. Then $\mathrm{B_S}(x,v)\times \mathrm{B_S}(y,v)\subseteq \mathrm{B_S}((x,y),u)$; which proves the claim. $\square$

Remark 3.30. By the results of this subsection, every compact separated (probabilistic) metric space is a (probabilistic) metric compact Hausdorff space. Put differently, the theory developed in this paper provides a new technique to study (probabilistic) metric spaces: instead of considering the topology *induced* by the metric, find a (compact Hausdorff) topology *compatible* with the metric. As we will see, this method is particularly useful in the study of Cauchy completeness (see Corollary 4.21).

4 Completeness from compactness

The central topic of this section is Cauchy completeness for $\mathcal{U}$-categories, a notion defined in terms of adjoint $\mathcal{U}$-distributors in [15]. We recall here the notion of $\mathcal{U}$-distributor and how to compose $\mathcal{U}$-distributors; unfortunately, only if the ultrafilter theory $\mathcal{U}$ is strict we are able to conclude that this composition is associative. In order to keep the quantale $\mathcal{D}$ of distribution functions included (see Remark 3.5), we avoid as much as possible assuming that $\mathcal{U}$ is strict. The lack of associativity forces us to be more careful in our treatment of adjunctions; in particular, adjoints need not be unique. We show, under some conditions on the quantale $\mathcal{V}$, that the corresponding $\mathcal{U}$-category of a $\mathcal{V}$-categorical compact Hausdorff space is Cauchy complete. Moreover, for strict theories $\mathcal{U}$, we prove that the forgetful functor $\mathcal{U}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ preserves Cauchy-completeness. Combining both results shows that the underlying $\mathcal{V}$-category of a $\mathcal{V}$-categorical compact Hausdorff space is Cauchy complete. In the last subsection we go a step further and study codirected completeness for $\mathcal{V}$-categories.

4.1 $\mathcal{U}$-distributors A $\mathcal{V}$-relation of the form $\varphi\colon UX \to\!\!\!\!\!\!\mapstochar\;\; Y$ is called a *$\mathcal{U}$-relation* and think of φ as an arrow from X to Y, and write $\varphi\colon X \rightharpoonup\!\!\!\!\!\mapstochar\;\; Y$. Composition is given by *Kleisli composition*:

$$\psi \circ \varphi := \psi \cdot U_\xi \varphi \cdot m_X^\circ,$$

$$\begin{tikzcd} UX \arrow[d, "\varphi", "|" marking] & UY \arrow[d, "\psi", "|" marking] & \longmapsto & UX \arrow[r, "m_X^\circ", "|" marking] \arrow[ddr, "\psi\circ\varphi"', "|" marking] & UUX \arrow[d, "U_\xi\varphi", "|" marking] \\ & & & & UY \arrow[d, "\psi", "|" marking] \\ Y & Z & & & Z \end{tikzcd}$$

for all $\varphi\colon X \rightharpoonup\!\!\!\!\!\mapstochar\;\; Y$ and $\psi\colon Y \rightharpoonup\!\!\!\!\!\mapstochar\;\; Z$. One easily verifies

$$\varphi \circ e_X^\circ = \varphi \cdot U e_X^\circ \cdot m_X^\circ = \varphi$$

and

$$e_X^\circ \circ \varphi = e_X^\circ \cdot U_\xi \varphi \cdot m_X^\circ \geq \varphi \cdot e_{UX}^\circ \cdot m_X^\circ = \varphi,$$

for all $\mathcal{U}$-relations $\varphi\colon X \relbar\joinrel\rightharpoonup Y$; that is, e_X° is a lax identity for the Kleisli composition. Moreover:

Theorem 4.1. *For composable $\mathcal{U}$-relations we have*

$$\varphi \circ (\psi \circ \gamma) \geq (\varphi \circ \psi) \circ \gamma,$$

with equality if $\mathcal{U}$ is a strict theory.

Proof. See [35, Subsection 2.1]. □

Remark 4.2. In the language of $\mathcal{U}$-relations, an $\mathcal{U}$-category (X, a) consists of a set X and an $\mathcal{U}$-relation $a\colon X \relbar\joinrel\rightharpoonup X$ satisfying $e_X^\circ \leq a$ and $a \circ a \leq a$.

Definition 4.3. A $\mathcal{U}$-relation $\varphi\colon X \relbar\joinrel\rightharpoonup Y$ between $\mathcal{U}$-categories $X = (X, a)$ and $Y = (Y, b)$ is a *$\mathcal{U}$-distributor*, written as $\varphi\colon X \circ\!\!\!\rightharpoonup Y$, whenever $\varphi \circ a \leq \varphi$ and $b \circ \varphi \leq \varphi$. In pointwise notation, $\varphi\colon X \relbar\joinrel\rightharpoonup Y$ is an $\mathcal{U}$-distributor if, for all $\mathfrak{x} \in UX$, all $\mathfrak{X} \in UUX$, all $y \in Y$ and all $\mathfrak{y} \in UY$,

$$U_\xi a(\mathfrak{X}, \mathfrak{x}) \otimes \varphi(\mathfrak{x}, y) \leq \varphi(m_X(\mathfrak{X}), y) \quad \text{and} \quad U_\xi \varphi(\mathfrak{X}, \mathfrak{y}) \otimes b(\mathfrak{y}, y) \leq \varphi(m_X(\mathfrak{X}), y).$$

In other words, an $\mathcal{U}$-distributor $\varphi\colon X \circ\!\!\!\rightharpoonup Y$ comes with a *right action* of the $\mathcal{U}$-relation a and a *left action* of b. This perspective motivates the designations *bimodule* or *module* used by some authors. Note that we always have $\varphi \circ a \geq \varphi$ and $b \circ \varphi \geq \varphi$, so that the $\mathcal{U}$-distributor conditions above are in fact equalities which make the $\mathcal{U}$-structures identities for the composition of $\mathcal{U}$-distributors.

Remark 4.4. In general, $\mathcal{U}$-distributors do not compose. However, this property is guaranteed by assuming that the ultrafilter theory is strict.

The following result establishes a connection between $\mathcal{U}$-distributors and $\mathcal{U}$-functors and generalises slightly [15, Theorem 4.3].

Theorem 4.5. *Let (X, a) and (Y, b) be $\mathcal{U}$-categories, and $\varphi\colon UX \relbar\joinrel\rightarrow Y$ be a $\mathcal{V}$-relation. Then the following assertions are equivalent.*

(i) *The $\mathcal{V}$-relation φ is an $\mathcal{U}$-distributor $\varphi\colon X \circ\!\!\!\rightharpoonup Y$.*

(ii) $\varphi\colon (UX, \hat{a})^{\mathrm{op}} \times (Y, 1_Y) \to (\mathcal{V}, \hom)$ *is a $\mathcal{V}$-functor and* $\varphi\colon (UX, m_X) \times (Y, b) \to (\mathcal{V}, \hom_\xi)$ *is an $\mathcal{U}$-functor.*

Proof. First note that φ is an $\mathcal{U}$-distributor if and only if

$$\varphi \cdot \hat{a} \leq \varphi \quad \text{and} \quad b \cdot U_{\xi}\varphi \leq \varphi \cdot m_X.$$

The first inequality above means precisely that, for all $y \in Y$ and all $\mathfrak{x}, \mathfrak{y} \in UX$,

$$\varphi(\mathfrak{x}, y) \otimes \hat{a}(\mathfrak{y}, \mathfrak{x}) \leq \varphi(\mathfrak{y}, y),$$

which in turn is equivalent to

$$\hat{a}(\mathfrak{y}, \mathfrak{x}) \leq \hom(\varphi(\mathfrak{x}, y), \varphi(\mathfrak{y}, y)).$$

Consequently, $\varphi \cdot \hat{a} \leq \varphi$ if and only if, for all $y \in Y$,

$$\varphi(-, y) \colon (UX, \hat{a})^{\mathrm{op}} \to (\mathcal{V}, \hom)$$

is a $\mathcal{V}$-functor; which is the case if and only if $\varphi \colon (UX, \hat{a})^{\mathrm{op}} \times (Y, 1_Y) \to (\mathcal{V}, \hom)$ is a $\mathcal{V}$-functor.

Secondly, $b \cdot U_{\xi}\varphi \leq \varphi \cdot m_X$ if and only if, for all $\mathfrak{X} \in UUX$, $\mathfrak{y} \in UY$ and $y \in Y$,

$$b(\mathfrak{y}, y) \otimes U_{\xi}\varphi(\mathfrak{X}, \mathfrak{y}) \leq \varphi(m_X(\mathfrak{X}), y),$$

and this inequality is equivalent to

$$\bigvee \{ b(\mathfrak{y}, y) \otimes \xi U\varphi(\mathfrak{W}) \mid \mathfrak{W} \in U(UX \times Y), U\pi_1(\mathfrak{W}) = \mathfrak{X}, U\pi_2(\mathfrak{W}) = \mathfrak{y} \} \\ \leq \varphi(m_X(\mathfrak{X}), y).$$

The latter holds if and only if, for all $\mathfrak{W} \in U(UX \times Y)$, $\mathfrak{x} \in UX$ and $y \in Y$ with $m_X(U\pi_1(\mathfrak{W})) = \mathfrak{x}$,

$$b(U\pi_2(\mathfrak{W}), y) \leq \hom(\xi U\varphi(\mathfrak{W}), \varphi(\mathfrak{x}, y)).$$

Hence, $b \cdot U_{\xi}\varphi \leq \varphi \cdot m_X$ is equivalent to $\varphi \colon (UX, m_X) \times (Y, b) \to (\mathcal{V}, \hom_{\xi})$ being an $\mathcal{U}$-functor. $\square$

In the sequel we will consider in particular $\mathcal{U}$-distributors with domain or codomain G. For an $\mathcal{U}$-category $X = (X, a)$, an $\mathcal{U}$-relation $\varphi \colon 1 \rightharpoonup X$ is an $\mathcal{U}$-distributor $\varphi \colon G \circ\!\!\rightharpoonup X$ if and only if, for all $x \in X$ and all $\mathfrak{r} \in UX$,

$$U_{\xi}\varphi(\mathfrak{r}) \otimes a(\mathfrak{r}, x) \leq \varphi(x).$$

Similarly, an $\mathcal{U}$-relation $\psi\colon X \relbar\joinrel\rightharpoonup 1$ is an $\mathcal{U}$-distributor $\psi\colon X \circ\!\!\rightharpoonup G$ if and only if, for all $\mathfrak{r} \in UX$ and all $\mathfrak{X} \in UUX$,

$$U_\xi a(\mathfrak{X}, \mathfrak{r}) \otimes \psi(\mathfrak{r}) \leq \psi(m_X(\mathfrak{X})) \quad \text{and} \quad U_\xi \psi(\mathfrak{X}) \leq \psi(m_X(\mathfrak{X})).$$

Let (X, a) and (Y, b) be $\mathcal{U}$-categories. Each map $f\colon X \to Y$ induces $\mathcal{U}$-relations

$$f_\circledast = b \cdot Uf\colon X \relbar\joinrel\rightharpoonup Y \quad \text{and} \quad f^\circledast = f^\circ \cdot b\colon Y \relbar\joinrel\rightharpoonup X;$$

moreover, one has $b \circ f_\circledast \leq f_\circledast$ and $f^\circledast \circ b \leq f^\circledast$. These $\mathcal{U}$-relations are actually $\mathcal{U}$-distributors precisely when f is an $\mathcal{U}$-functor.

Lemma 4.6. *The following are equivalent, for $\mathcal{U}$-categories (X, a) and (Y, b) and a map $f\colon X \to Y$.*

(i) *f is an $\mathcal{U}$-functor $f\colon (X, a) \to (Y, b)$.*

(ii) *$f_\circledast$ is an $\mathcal{U}$-distributor, that is, $f_\circledast \circ a \leq f_\circledast$.*

(iii) *$f^\circledast$ is an $\mathcal{U}$-distributor, that is, $a \circ f^\circledast \leq f\circledast$.*

Proof. See [15, Subsection 3.6]. □

Lemma 4.7. *Let $f\colon A \to X$ and $g\colon Y \to B$ be $\mathcal{U}$-functors and $\varphi\colon X \circ\!\!\rightharpoonup Y$ be an $\mathcal{U}$-distributor. Then*

$$\varphi \circ f_\circledast = \varphi \cdot Uf \qquad \textit{and} \qquad g^\circledast \circ \varphi = g^\circ \cdot \varphi$$

are $\mathcal{U}$-distributors.

Proof. See [15, Proposition 3.6]. □

Similarly to the case of $\mathcal{V}$-categories, the local order of $\mathcal{V}$-Rel allows us to consider $\mathcal{U}$-Cat as an ordered category: for $\mathcal{U}$-functors $f, g\colon X \to Y$,

$$\begin{aligned} f \leq g \iff f^\circledast \leq g^\circledast &\iff g_\circledast \leq f_\circledast \\ &\iff f^* \leq g^*. \end{aligned}$$

In particular, every $\mathcal{U}$-category X has an underlying order where $x \leq y$ whenever $x^{\circledast} \leq y^{\circledast}$, for all $x, y \in X$; which in turn is equivalent to $k \leq a(\dot{x}, y)$. This construction defines a functor

$$\widetilde{O}_{\mathcal{U}} \colon \mathcal{U}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Ord},$$

and the diagrams

$$\begin{tikzcd}
\mathcal{U}\text{-}\mathsf{Cat} \arrow[r, "(-)_0"] \arrow[d, "B_p"'] \arrow[dr, "\widetilde{O}_{\mathcal{U}}"] & \mathcal{V}\text{-}\mathsf{Cat} \arrow[d, "B_p"] \\
\mathsf{Top} \arrow[r, "(-)_0"'] & \mathsf{Ord}
\end{tikzcd}$$

commute. A $\mathcal{U}$-category (X, a) is *separated* (see [42]) whenever the underlying ordered set $\widetilde{O}_{\mathcal{U}}(X, a)$ is separated. We note that $(-)_0 \colon \mathcal{U}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ sends separated $\mathcal{U}$-categories to separated $\mathcal{V}$-categories.

4.2 Adjoint $\mathcal{U}$-distributors In this subsection we study the important notion of adjoint $\mathcal{U}$-distributor. We employ here the usual definition of adjunction in an ordered category; however, some extra caution is needed since $\mathcal{U}$-distributors in general do not compose.

Definition 4.8. Let $X = (X, a)$ and $Y = (Y, b)$ be $\mathcal{U}$-categories. A pair of $\mathcal{U}$-distributors $\varphi\colon X \mathrel{-\!\!\circ\!\!\rightharpoonup} Y$ and $\psi\colon Y \mathrel{-\!\!\circ\!\!\rightharpoonup} X$ form an *adjunction*, denoted as $\varphi \dashv \psi$, whenever their composites, $\varphi \circ \psi$ and $\psi \circ \varphi$, are $\mathcal{U}$-distributors and $a \leq \psi \circ \varphi$ and $\varphi \circ \psi \leq b$.

We hasten to remark that $f_{\circledast} \dashv f^{\circledast}$, for every $\mathcal{U}$-functor $f\colon (X, a) \to (Y, b)$. In fact, by [15, Proposition 3.6 (2), p. 188], $f^{\circledast} \circ f_{\circledast}$ and $f_{\circledast} \circ f^{\circledast}$ are $\mathcal{U}$-distributors and

$$f_{\circledast} \circ f^{\circledast} = b \cdot Uf \cdot Uf^{\circ} \cdot U_{\xi} b \cdot m_Y^{\circ} \leq b \cdot U_{\xi} b \cdot m_Y^{\circ} = b$$

and

$$f^{\circledast} \circ f_{\circledast} = f^{\circ} \cdot b \cdot U_{\xi} b \cdot UUf \cdot m_X^{\circ} = f^{\circ} \cdot b \cdot U_{\xi} b \cdot m_Y^{\circ} \cdot Uf = f^{\circ} \cdot b \cdot Uf \geq f^{\circ} \cdot f \cdot a \geq a.$$

Similarly to the nomenclature for $\mathcal{V}$-categories, we call an $\mathcal{U}$-functor $f\colon (X, a) \to (Y, b)$ *fully faithful* whenever $f^{\circledast} \circ f_{\circledast} = a$, and *fully dense* whenever $f_{\circledast} \circ f^{\circledast} = b$.

In general, we are not able to prove unicity of left adjoints since composition of $\mathcal{U}$-distributors does not need to be associative. However, we can still prove that right adjoints are unique:

Proposition 4.9. *Let $\varphi\colon X \mathrel{\circ\!\!\!\rightharpoonup} Y$, $\psi\colon Y \mathrel{\circ\!\!\!\rightharpoonup} X$ and $\psi'\colon Y \mathrel{\circ\!\!\!\rightharpoonup} X$ be $\mathcal{U}$-distributors with $\varphi \dashv \psi$ and $\varphi \dashv \psi'$. Then $\psi = \psi'$.*

Proof. From $a \le \psi\circ\varphi$ we get $\psi' = a\circ\psi' \le (\psi\circ\varphi)\circ\psi' \le \psi\circ(\varphi\circ\psi') \le \psi\circ b = \psi$. Similarly, $\psi \le \psi'$, and we conclude that $\psi = \psi'$. $\square$

We now turn our attention to $\mathcal{U}$-distributors with domain or codomain G.

Lemma 4.10. *Let $X = (X, a)$ be an $\mathcal{U}$-category and $\varphi\colon G \mathrel{\circ\!\!\!\rightharpoonup} X$ and $\psi\colon X \mathrel{\circ\!\!\!\rightharpoonup} G$ be $\mathcal{U}$-distributors. Then the composites $\varphi \circ \psi$ and $\psi \circ \varphi$ are $\mathcal{U}$-distributors.*

Proof. Clearly, $\psi \circ \varphi\colon G \mathrel{\circ\!\!\!\rightharpoonup} G$ is an $\mathcal{U}$-distributor. To prove that $\varphi \circ \psi$ is indeed an $\mathcal{U}$-distributor of type $X \mathrel{\circ\!\!\!\rightharpoonup} X$, we verify first that

$$\varphi \circ \psi = \varphi \cdot e_1 \cdot e_1^\circ \cdot U\psi \cdot m_X^\circ = \varphi \cdot e_1 \cdot (e_1^\circ \circ \psi) = \varphi \cdot e_1 \cdot \psi.$$

Therefore

$$a\circ(\varphi\circ\psi) = a\cdot U\varphi\cdot Ue_1\cdot U\psi\cdot m_X^\circ \le a\cdot U\varphi\cdot m_1^\circ\cdot U\psi\cdot m_X^\circ = (a\circ\varphi)\cdot U\psi\cdot m_X^\circ = \varphi\circ\psi,$$

and $(\varphi \circ \psi) \circ a \le \varphi \circ (\psi \circ a) = \varphi \circ \psi$. $\square$

Therefore, when studying adjunctions of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\psi"] \arrow[r, phantom, "\top"] & G, \arrow[l, bend left, "\varphi"] \end{tikzcd}$$

we do not need to worry about the composites $\varphi\circ\psi$ and $\psi\circ\varphi$. Elementwise, $\varphi \dashv \psi$ translates to

$$k \le \bigvee_{\mathfrak{z}\in UX} \psi(\mathfrak{z}) \otimes \xi U\varphi(\mathfrak{z}) \quad \text{and} \quad \psi(\mathfrak{x}) \otimes \varphi(x) \le a(\mathfrak{x}, x),$$

for all $\mathfrak{x} \in UX$ and $x \in X$. We also point out that

- A map $\varphi\colon X \to \mathcal{V}$ (seen as an $\mathcal{U}$-relation $\varphi\colon G \mathrel{-\!\!\!\mapstochar\!\!\rightharpoonup} X$) is an $\mathcal{U}$-distributor $\varphi\colon G \mathrel{-\!\!\!\circ\!\!\rightharpoonup} X$ if and only if $a\circ\varphi \le \varphi$ if and only if $\varphi\colon X \to \mathcal{V}$ is a $\mathcal{U}$-functor (see Theorem 4.5) if and only if

$$\varphi(x) = \bigvee_{\mathfrak{x}\in UX} a(\mathfrak{x},x)\otimes \xi U\varphi(\mathfrak{x}).$$

- A $\mathcal{U}$-relation $\psi\colon X \mathrel{-\!\!\!\mapstochar\!\!\rightharpoonup} G$ is an $\mathcal{U}$-distributor $\psi\colon X \mathrel{-\!\!\!\circ\!\!\rightharpoonup} G$ if and only if $\psi\circ a \le \psi$ and $e_1^\circ \cdot U_\xi\psi\cdot m_X^\circ \le \psi$.

Proposition 4.11. *Let $\psi\colon X \mathrel{-\!\!\!\circ\!\!\rightharpoonup} G$, $\varphi\colon G \mathrel{-\!\!\!\circ\!\!\rightharpoonup} X$ and $\varphi'\colon G \mathrel{-\!\!\!\circ\!\!\rightharpoonup} X$ be $\mathcal{U}$-distributors with $\varphi \dashv \psi$ and $\varphi' \dashv \psi$. Then $\varphi = \varphi'$.*

Proof. We calculate

$$\varphi'(x) \le \bigvee_{\mathfrak{z}\in UX} \varphi'(x)\otimes\psi(\mathfrak{z})\otimes\xi U\varphi(\mathfrak{z}) \le \bigvee_{\mathfrak{z}\in UX} a(\mathfrak{x},x)\otimes\xi U\varphi(\mathfrak{z}) = \varphi(x). \qquad \square$$

4.3 Cauchy complete $\mathcal{U}$-categories

With the notion of adjunction of $\mathcal{U}$-distributors at our disposal, we come now to the concept of Cauchy completeness (called Lawvere completeness in [15]).

Definition 4.12. A $\mathcal{U}$-category $X = (X,a)$ is called *Cauchy complete* whenever every adjunction

$$\begin{tikzcd} X \arrow[r, bend left, "\psi", "\circ"{marking}] \arrow[r, phantom, "\top"] & G, \arrow[l, bend left, "\varphi", "\circ"{marking}] \end{tikzcd}$$

of $\mathcal{U}$-distributors is of the form $x_\circledast \dashv x^\circledast$, for some $x \in X$.

Note that $x_\circledast = a(\dot{x},-)$ and that $x^\circledast = a(-,x)$, so that $x_\circledast \dashv x^\circledast$ means, for all $\mathfrak{x}\in UX$ and $x' \in X$,

$$k \le \bigvee_{\mathfrak{z}\in UX} a(\mathfrak{z},x)\otimes U_\xi x_\circledast(\mathfrak{z}) \qquad \text{and} \qquad a(\mathfrak{x},x)\otimes a(\dot{x},x') \le a(\mathfrak{x},x').$$

Example 4.13. Various examples of Cauchy complete $\mathcal{U}$-categories are described in [15], we sketch here the principal facts.

(1) We have already seen that $\mathsf{Top} \simeq \mathcal{U}_{\mathbf{2}}\text{-}\mathsf{Cat}$. In this context, a $\mathcal{U}_{\mathbf{2}}$-distributor $\varphi\colon (X,a) \mathrel{\circ\!\!\!\rightharpoonup} (Y,b)$ is a relation $\varphi\colon UX \nrightarrow Y$ that satisfies, for all $y \in Y$, $\mathfrak{y} \in UY$, $\mathfrak{r} \in UX$ and $\mathfrak{X} \in UUX$,

$$\mathfrak{X} \to \mathfrak{r} \;\&\; \varphi(\mathfrak{r}, y) \implies \varphi(m_X(\mathfrak{X}), y)$$

and

$$U_\xi\varphi(\mathfrak{X}, \mathfrak{y}) \;\&\; \mathfrak{y} \to y \implies \varphi(m_X(\mathfrak{X}), y).$$

In particular, $\mathcal{U}_{\mathbf{2}}$-distributors of the form $\varphi\colon G \mathrel{\circ\!\!\!\rightharpoonup} X$ can be identified with relations $\varphi\colon 1 \nrightarrow X$ satisfying

$$\forall x \in X\, \forall \mathfrak{r} \in UX\,.\,(U_\xi\varphi(\mathfrak{r}) \;\&\; \mathfrak{r} \to x) \implies \varphi(x),$$

and a relation $\psi\colon UX \nrightarrow 1$ is a $\mathcal{U}_{\mathbf{2}}$-distributor $\psi\colon X \mathrel{\circ\!\!\!\rightharpoonup} G$ if and only if

$$(\mathfrak{X} \to \mathfrak{r} \;\&\; \psi(\mathfrak{r})) \leq \psi(m_X(\mathfrak{X})) \quad \text{and} \quad U_\xi\psi(\mathfrak{X}) \leq \psi(m_X(\mathfrak{X})),$$

for all $\mathfrak{X} \in UUX$ and $\mathfrak{r} \in UX$. Using Theorem 4.5, a $\mathcal{U}_{\mathbf{2}}$-distributor $\varphi\colon G \mathrel{\circ\!\!\!\rightharpoonup} X$ can be also seen as a continuous map $X \to \mathbf{2}$ into the Sierpiński space, which in turn can be interpreted as a closed subset $A \subseteq X$. A $\mathcal{U}$-distributor $\psi\colon X \mathrel{\circ\!\!\!\rightharpoonup} G$ is a map $UX \to \mathbf{2}$ which is continuous with respect to the Zariski closure on UX (an ultrafilter $\mathfrak{x} \in UX$ belongs to the closure of $\mathcal{B} \subseteq UX$ whenever $\bigcap \mathcal{B} \subseteq \mathfrak{r}$) and antitone with respect to the order relation where

$$\mathfrak{x} \leq \mathfrak{y} \text{ whenever } \forall A \in \mathfrak{x}\,.\,\overline{A} \in \mathfrak{y},$$

for all $\mathfrak{x}, \mathfrak{y} \in UX$. Such maps correspond precisely to subsets $\mathcal{A} \subseteq UX$ which are Zariski closed and down-closed with respect to the order relation defined above.

A pair of $\mathcal{U}$ -distributors forms an adjunction $\begin{tikzcd} X \arrow[r, bend left, "\psi", "\circ" marking] \arrow[r, phantom, "\top"] & G \arrow[l, bend left, "\varphi", "\circ" marking] \end{tikzcd}$ if and only if

$$\exists \mathfrak{x} \in UX\,.\,U_\xi\varphi(\mathfrak{x}) \;\&\; \psi(\mathfrak{x})$$

and

$$\forall x \in X\, \forall \mathfrak{x} \in UX\,.\,(\psi(\mathfrak{x}) \;\&\; \varphi(x)) \implies \mathfrak{r} \to x.$$

In terms of the corresponding subsets $A \subseteq X$ and $\mathcal{A} \subseteq UX$, these conditions read as

$$\exists \mathfrak{x} \in UX\,.\,(A \in \mathfrak{x} \;\&\; \mathfrak{x} \in \mathcal{A})$$

and

$$\forall x \in X\, \forall \mathfrak{x} \in UX\,.\,(\mathfrak{x} \in \mathcal{A} \;\&\; x \in A) \implies \mathfrak{x} \to x.$$

From this it follows that $\varphi\colon G \,{\circ\!\!\rightharpoonup}\, X$ is left adjoint if and only if the corresponding closed subset $A \subseteq X$ is irreducible. Consequently, a topological space X is Cauchy complete if and only if X is weakly sober.

(2) We consider now $\mathcal{U} = \mathcal{U}_{\overleftarrow{[0,\infty]}_+}$, and recall that $\mathcal{U}_{\overleftarrow{[0,\infty]}_+}$-$\mathsf{Cat} \simeq \mathsf{App}$. Here, a $\mathcal{U}_{\overleftarrow{[0,\infty]}_+}$-distributor $\varphi\colon (X,a) \,{\circ\!\!\rightharpoonup}\, (Y,b)$ is a $\overleftarrow{[0,\infty]}_+$-relation $\varphi\colon UX \nrightarrow Y$ subject to $\varphi \circ a \geqslant \varphi$ and $b \circ \varphi \geqslant \varphi$. These conditions express that, for all $y \in Y$, $\mathfrak{y} \in UY$, $\mathfrak{x} \in UX$ and $\mathfrak{X} \in UUX$,

$$U_\xi a(\mathfrak{X}, \mathfrak{x}) + \varphi(\mathfrak{x}, y) \geqslant \varphi(m_X(\mathfrak{X}), y)$$

and

$$U_\xi \varphi(\mathfrak{X}, \mathfrak{y}) + b(\mathfrak{y}, y) \geqslant \varphi(m_X(\mathfrak{X}), y).$$

A $\mathcal{U}_{\overleftarrow{[0,\infty]}_+}$-distributor of the type $\varphi\colon G \,{\circ\!\!\rightharpoonup}\, X$ can be seen as a $\mathcal{U}_{\overleftarrow{[0,\infty]}_+}$-functor $\varphi\colon X \to \overleftarrow{[0,\infty]}_+$ and it is characterised by

$$U_\xi \varphi(\mathfrak{x}) + a(\mathfrak{x}, x) \geq \varphi(x),$$

for $x \in X$ and $\mathfrak{x} \in UX$, and a $\mathcal{U}_{\overleftarrow{[0,\infty]}_+}$-distributor of the type $\psi\colon X \,{\circ\!\!\rightharpoonup}\, G$ is a mapping $UX \to \overleftarrow{[0,\infty]}_+$ that satisfies

$$U_\xi a(\mathfrak{X}, \mathfrak{x}) + \psi(\mathfrak{x}) \geqslant \psi(m_X(\mathfrak{X}))$$

and

$$U_\xi \psi(\mathfrak{X}) \geqslant \psi(m_X(\mathfrak{X})),$$

for all $\mathfrak{x} \in UX$ and $\mathfrak{X} \in UUX$. $\mathcal{U}_{\overleftarrow{[0,\infty]}_+}$-distributors form an adjunction of type $\begin{tikzcd} X \arrow[r, bend left, "\psi"] \arrow[r, phantom, "\top"] & G \arrow[l, bend left, "\varphi"] \end{tikzcd}$ if, for all $x \in X$ and $\mathfrak{X} \in UUX$,

$$0 \geqslant \bigwedge_{\mathfrak{x} \in UX} U_{\xi}\varphi(\mathfrak{x}) + \psi(\mathfrak{x})$$

and

$$\psi(m_X(\mathfrak{X})) + \varphi(x) \geqslant a(m_X(\mathfrak{X}), x).$$

Furthermore, $\mathcal{U}_{\overleftarrow{[0,\infty]}_+}$-distributors of type $\varphi\colon G \mathrel{-\!\!\circ\!\!\rightharpoonup} X$ are identified with closed variable sets. Here a variable set is a family $(A_v)_{v\in[0,\infty]}$ such that, for all $v \in [0,\infty]$, $A_v = \bigcap_{u>v} A_u$. Such a variable set is closed whenever, for all $u, v \in [0,\infty]$, $\{x \in X \mid d(A_u, x) \le v\} \subseteq A_{u+v}$, where $d(A_u, x) = \inf\{a(\mathfrak{x}, x) \mid \mathfrak{x} \in UA_u\}$. A $\mathcal{U}_{\overleftarrow{[0,\infty]}_+}$-distributor $\psi\colon X \mathrel{-\!\!\circ\!\!\rightharpoonup} G$ which is right adjoint to φ is induced by the variable set $\mathcal{A} = (\mathcal{A}_v)_{v\in[0,\infty]}$ with $\mathcal{A}_v = \{\mathfrak{x} \in UX \mid \forall u \in [0,\infty], \forall x \in A_u, a(\mathfrak{x}, x) \le u + v\}$. Such a variable set $\mathcal{A}$ corresponds to a right adjoint of φ if and only if A is irreducible, that is, for all $u \in [0,\infty]$ with $u > 0$, $UA_u \cap \mathcal{A} \neq \varnothing$. A $\mathcal{U}_{\overleftarrow{[0,\infty]}_+}$-distributor $\varphi\colon G \mathrel{-\!\!\circ\!\!\rightharpoonup} X$ is represented by $x \in X$ if and only if the induced variable set $A = (A_v)_{v\in[0,\infty]}$ is given by $A_v = \{y \in X \mid d(x, y) \le v\}$ for each $v \in [0,\infty]$. Therefore an approach space X is Cauchy complete if and only if each irreducible variable set is representable. Finally, this condition is equivalent to X being weakly sober in the sense of [3].

4.4 $\mathcal{U}$-distributors vs $\mathcal{U}$-functors In this subsection we will show that, under suitable conditions, every $\mathcal{U}$-category of the form $K(X, a_0, \alpha)$ is Cauchy complete, for (X, a_0, α) in $(\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{U}}$. For an $\mathcal{U}$-category $X = (X, a)$ and $M \subseteq X$, we define

$$\varphi_M(x) = \bigvee_{\mathfrak{z} \in UM} a(\mathfrak{z}, x),$$

for all $x \in X$. We can view φ_M as an $\mathcal{U}$-relation $\varphi_M\colon 1 \mathrel{-\!\!\!+\!\!\!\rightharpoonup} X$ given by $\varphi_M = a \cdot Ui \cdot U!^{\circ}$ (here $i\colon M \hookrightarrow X$ and $!\colon M \to 1$). It is easy to see that

φ_M is actually an $\mathcal{U}$-distributor $\varphi_M\colon G \relbar\!\!\!\!\circ\!\!\!\rightharpoonup X$, hence, $\varphi_M\colon X \to \mathcal{V}$ is an $\mathcal{U}$-functor. We also not that $\varphi_\varnothing = \bot$ and $\varphi_{A\cup B} = \varphi_A \vee \varphi_B$.

We import now from [41, Lemma 3.2 and Corollary 3.3]:

Proposition 4.14. *Let $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ be an ultrafilter theory where $\mathcal{V}$ is completely distributive and $\xi\colon U\mathcal{V} \to \mathcal{V}$ is as in Theorem 3.2. For every $\mathcal{U}$-category $X = (X, a)$, $\mathfrak{x} \in UX$ and $x \in X$, $a(\mathfrak{x}, x) = \bigwedge_{A\in\mathfrak{x}} \varphi_A(x)$.*

Next we analyse left adjoint $\mathcal{U}$-distributors $\varphi\colon G \relbar\!\!\!\!\circ\!\!\!\rightharpoonup X$.

Lemma 4.15. *Let $\mathcal{U}$ be an ultrafilter theory and $\varphi\colon G \relbar\!\!\!\!\circ\!\!\!\rightharpoonup X$ be a left adjoint $\mathcal{U}$-distributor with right adjoint $\psi\colon X \relbar\!\!\!\!\circ\!\!\!\rightharpoonup G$. Then, for every $\mathcal{U}$-distributor $\varphi'\colon G \relbar\!\!\!\!\circ\!\!\!\rightharpoonup X$,*

$$[\varphi, \varphi'] := \bigwedge_{x\in X} \hom(\varphi(x), \varphi'(x)) = \psi \circ \varphi'.$$

Proof. Recall that $[\varphi, \varphi'] = \varphi' \bullet\!\!- \varphi$ is the largest element $u \in \mathcal{V}$ with $\varphi(x) \otimes u \le \varphi'(x)$, for all $x \in X$ (see Subsection 2.3). From

$$\begin{aligned}\varphi(x) \otimes \bigvee_{\mathfrak{x}\in UX} \psi(\mathfrak{x}) \otimes \xi U\varphi'(\mathfrak{x}) &= \bigvee_{\mathfrak{x}\in UX} \varphi(x) \otimes \psi(\mathfrak{x}) \otimes \xi U\varphi'(\mathfrak{x})\\ &\le \bigvee_{\mathfrak{x}\in UX} a(\mathfrak{x}, x) \otimes \xi U\varphi'(\mathfrak{x}) = \varphi'(x)\end{aligned}$$

we get $\psi \circ \varphi' \le [\varphi, \varphi']$. On the other hand, from $\varphi \otimes u \le \varphi'$ we get

$$u \le \bigvee_{\mathfrak{x}\in UX} \psi(\mathfrak{x}) \otimes \xi U\varphi(\mathfrak{x}) \otimes u \le \bigvee_{\mathfrak{x}\in UX} \psi(\mathfrak{x}) \otimes \xi U(\varphi\otimes u)(\mathfrak{x}) \le \bigvee_{\mathfrak{x}\in UX} \psi(\mathfrak{x}) \otimes \xi U\varphi'(\mathfrak{x}).$$

□

Proposition 4.16. *Let $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ be an ultrafilter theory where $\mathcal{V}$ is completely distributive, ξ is as in Theorem 3.2, and k is the top element of $\mathcal{V}$.*

(1) *For every left adjoint $\mathcal{U}$-distributor $\varphi\colon G \relbar\!\!\!\!\circ\!\!\!\rightharpoonup X$,*

$$k \le \bigvee_{x\in X} \varphi(x).$$

(2) *If k is $\vee$-irreducible, then every left adjoint $\mathcal{U}$-distributor $\varphi\colon G \relbar\!\!\!\!\circ\!\!\!\rightharpoonup X$ is irreducible (that is: $\varphi \ne \bot$ and $\varphi \le \varphi_1 \vee \varphi_2$ implies $\varphi \le \varphi_1$ or $\varphi \le \varphi_2$).*

Proof. Regarding the first statement, first observe that

$$k \leq \bigvee_{\mathfrak{x} \in UX} \psi(\mathfrak{x}) \otimes \xi U\varphi(\mathfrak{x}) \leq \bigvee_{\mathfrak{x} \in UX} \xi U\varphi(\mathfrak{x}).$$

Let $u \ll k$. Then there is some $\mathfrak{x} \in UX$ with

$$u \leq \xi U\varphi(\mathfrak{x}) = \bigwedge_{A \in \mathfrak{x}} \bigvee_{x \in A} \varphi(x) \leq \bigvee_{x \in X} \varphi(x).$$

Regarding the second statement, we observe first that $\varphi \neq \bot$ since

$$\bot < k \leq \bigvee_{x \in X} \varphi(x).$$

Furthermore, by Lemma 4.15, $[\varphi, -]$ preserves finite suprema. Therefore, if $\varphi \leq \varphi_1 \vee \varphi_2$, then

$$k \leq [\varphi, \varphi_1 \vee \varphi_2] = [\varphi, \varphi_1] \vee [\varphi, \varphi_2].$$

Since k is $\vee$-irreducible, we conclude that $\varphi \leq \varphi_1$ or $\varphi \leq \varphi_2$. $\square$

The following result is inspired by [40, Lemma III.5.9.1] which in turn is motivated by [3, Proposition 5.7]

Proposition 4.17. *Let $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ be an ultrafilter theory where $\mathcal{V}$ is completely distributive, ξ is as in Theorem 3.2, and k is approximated and the top element of $\mathcal{V}$. Then every left adjoint $\mathcal{U}$-distributor $\varphi\colon G \nrightarrow X$ is of the form $\varphi = a(\mathfrak{x}, -)$, for some $\mathfrak{x} \in UX$.*

Proof. First note that from $\{u \in \mathcal{V} \mid u \ll k\}$ is directed it follows that k is $\vee$-irreducible (see [39, Remark 4.21]). For every $u \ll k$, put $A_u = \{x \in X \mid u \leq \varphi(x)\}$. By hypothesis, $A_u \neq \varnothing$. We claim that $\varphi \leq \varphi_{A_u}$. To see this, put $A = \{x \in X \mid \varphi(x) \leq \varphi_{A_u}(x)\}$. Since $\varphi_{A_u}(x) = k$ for every $x \in A_u$, it follows that $A_u \subseteq A$. Put $v = \bigvee\{\varphi(x) \mid x \notin A\}$, then $k \not\leq v$ since $u \ll v$. By construction, $\varphi \leq \varphi_{A_u} \vee v$. But $\varphi \leq v$ is impossible since $k \leq \bigvee_{x \in X} \varphi(x)$ and $k \not\leq v$, hence $\varphi \leq \varphi_{A_u}$.

The directed set $\mathfrak{f} = \{A_u \mid u \ll k\}$ is disjoint from the ideal $\mathfrak{j} = \{B \subseteq X \mid \varphi \not\leq \varphi_B\}$, hence there is some ultrafilter $\mathfrak{x} \in UX$ with $\mathfrak{f} \subseteq \mathfrak{x}$ and $\mathfrak{x} \cap \mathfrak{j} = \varnothing$. Therefore

$$\varphi \leq \bigwedge_{A \in \mathfrak{x}} \varphi_A = a(\mathfrak{x}, -)$$

and $\varphi(x) \geq a(\mathfrak{x}, x) \otimes \xi U\varphi(\mathfrak{x}) \geq a(\mathfrak{x}, x)$, for all $x \in X$. $\square$

Corollary 4.18. *Under the conditions of Proposition* 4.17*, every* $\mathcal{U}$*-category in the image of*

$$K\colon (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{U}} \longrightarrow \mathcal{U}\text{-}\mathsf{Cat}$$

is Cauchy complete. In particular, the $\mathcal{U}$*-category* $\mathcal{V}$ *is Cauchy complete.*

Proof. Given a left adjoint $\mathcal{U}$-distributor $\varphi\colon G \looparrowright X$, we have $\varphi = a(\mathfrak{x}, -) = a_0(\alpha(\mathfrak{x}), -)$. $\square$

For our next result, we recall that the forgetful functor $(-)_0\colon \mathcal{U}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ has a left adjoint $F\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{U}\text{-}\mathsf{Cat}$ sending a $\mathcal{V}$-category (X, a_0) to the $\mathcal{U}$-category $(X, e_X^\circ \cdot U_\xi a_0)$, and leaving maps unchanged.

Proposition 4.19. *Let* $\mathcal{U}$ *be an ultrafilter theory. Then the following assertions hold.*

(1) F *sends fully faithful* $\mathcal{V}$*-functors to fully faithful* $\mathcal{U}$*-functors.*

(2) *If* $\mathcal{U}$ *is strict, then* F *sends fully dense* $\mathcal{V}$*-functors to fully dense* $\mathcal{U}$*-functors.*

Proof. For a $\mathcal{V}$-functor $f\colon (X, a_0) \to (Y, b_0)$, we write

$$a = e_X^\circ \cdot U_\xi a_0 \qquad \text{and} \qquad b = e_Y^\circ \cdot U_\xi b_0$$

for the corresponding $\mathcal{U}$-structures. Assume first that $f\colon (X, a_0) \to (Y, b_0)$ is fully faithful. Then

$$f^\circledast \circ f_\circledast = f^\circ \cdot e_Y^\circ \cdot U_\xi b_0 \cdot Uf = e_X^\circ \cdot Uf^\circ \cdot U_\xi b_0 \cdot Uf = e_X^\circ \cdot U_\xi(f^\circ \cdot b_0 \cdot f) = a$$

Assume now that $\mathcal{U}$ is strict and f is fully dense. Now we calculate:

$$\begin{aligned} f_\circledast \circ f^\circledast &= b \cdot Uf \cdot Uf^\circ \cdot U_\xi b \cdot m_Y^\circ = e_Y^\circ \cdot U_\xi b_0 \cdot Uf \cdot Uf^\circ \cdot Ue_Y^\circ \cdot U_\xi U_\xi b_0 \cdot m_Y^\circ \\ &= e_Y^\circ \cdot U_\xi b_0 \cdot Uf \cdot Uf^\circ \cdot Ue_Y^\circ \cdot m_Y^\circ \cdot U_\xi b_0 = e_Y^\circ \cdot U_\xi b_0 \cdot Uf \cdot Uf^\circ \cdot U_\xi b_0 = b \quad \square \end{aligned}$$

Theorem 4.20. *Let* $\mathcal{U}$ *be a strict ultrafilter theory. Then* $(-)_0\colon \mathcal{U}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ *sends Cauchy complete* $\mathcal{U}$*-categories to Cauchy complete* $\mathcal{V}$*-categories.*

Proof. Just note that a $\mathcal{V}$-category (resp. $\mathcal{U}$-category) is Cauchy complete if and only if it is injective with respect to fully faithful and fully dense $\mathcal{V}$-functors (resp. $\mathcal{U}$-functors) as it was proven in [42, Theorems 3.10 and 5.11]. $\square$

Corollary 4.21. *Let $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ be a strict ultrafilter theory where $\mathcal{V}$ is completely distributive, ξ is as in Theorem 3.2, and k is approximated and the top element of $\mathcal{V}$. Then, for every (X, a_0, α) in $(\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{U}}$, the $\mathcal{V}$-category (X, a_0) is Cauchy complete. In particular, every compact separated $\mathcal{V}$-category is Cauchy complete.*

Corollary 4.22. *Let $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ be a strict ultrafilter theory. Then, for every $\mathcal{V}$-category (X, a_0), the $\mathcal{V}$-category (UX, Ua_0) is Cauchy complete.*

Proof. Just observe that (UX, Ua_0, m_X) is a $\mathcal{V}$-categorical compact Hausdorff space since it is the free Eilenberg–Moore algebra over (X, a_0). □

We stress that the topology of Ua_0 need not be compact. For instance, if a_0 is discrete, then so is Ua_0.

For $\mathcal{U} = \mathcal{U}_{\mathbf{2}}$, Corollary 4.21 is vacuous since every ordered set is Cauchy complete. As we already pointed out in Section 1, a stronger result holds in this case: the underlying order of a sober space is codirected complete. In the next subsection we proof a similar result for $\mathcal{U}$-categories, under additional conditions on the quantale $\mathcal{V}$.

Remark 4.23. A related study of properties of metric spaces via approach spaces can be found in [51]. Among other results, it is shown there that in the underlying metric of an approach space every *forward Cauchy sequence* converges (see [7, 76]). We will come back to this notion in the next subsection.

4.5 Codirected complete $\mathcal{V}$-categories In this subsection we look at Cauchy completeness of $\mathcal{V}$-categories from a different perspective, namely as
(co)completeness with respect to some choice of (co)limit weights. In this paper we need only very particular limits and colimits, therefore we refer for more information to [47, 71] and recall here only what we believe is essential for our paper.

As the starting point, we assume that a saturated class Φ of limit weights $\varphi\colon G \to\!\!\!\!\circ\, X$ is given; examples of such choices are given below. For each $\mathcal{V}$-category X, we write $\Phi(X)$ to denote the weights with codomain X. Moreover, we consider $\mathcal{V}\text{-}\mathsf{Dist}(G, X)$ as a $\mathcal{V}$-subcategory of $\mathcal{V}\text{-}\mathsf{Rel}(1, X) \simeq \mathcal{V}^X$

and $\Phi(X)$ as a $\mathcal{V}$-subcategory of $\mathcal{V}\text{-}\mathsf{Dist}(G,X)^{\mathrm{op}}$, this way the mapping

$$\hbar_X^\Phi\colon X \longrightarrow \Phi(X),\ x \longmapsto x_*$$

is a $\mathcal{V}$-functor. A $\mathcal{V}$-category X is called Φ-*complete* whenever $\hbar_X^\Phi$ has a right adjoint

$$\inf_X^\Phi\colon \Phi(X) \longrightarrow X.$$

Intuitively, $\inf_X^\Phi$ calculates the infimum of a limit weight $\varphi\colon G \nrightarrow X$. The assumption that Φ is saturated guarantees that each $\Phi(X)$ is Φ-complete; in fact, it is the free Φ-completion of X. Dually, notions of cocompleteness depend on a choice of a saturated class Ψ of colimit weights $\psi\colon X \nrightarrow G$. Then a $\mathcal{V}$-category X is Ψ-cocomplete if and only if the $\mathcal{V}$-functor

$$X \longrightarrow \Psi(X),\ x \longmapsto x^*$$

has a left adjoint. Here we consider $\Psi(X)$ as a $\mathcal{V}$-subcategory of $\mathcal{V}\text{-}\mathsf{Dist}(X,G)$.

Remark 4.24. For a saturated class Φ of limit weights, a $\mathcal{V}$-category X is Φ-complete if and only if there exists a $\mathcal{V}$-functor $I\colon \Phi(X) \to X$ with $I\hbar_X^\Phi \simeq 1_X$; such a $\mathcal{V}$-functor is necessarily right adjoint to $\hbar_X^\Phi$.

For instance,

$$\Phi = \{\text{all left adjoint } \mathcal{V}\text{-distributors } \varphi\colon G \nrightarrow X \text{ with domain } G\}$$

is a saturated class of limit weights, and a $\mathcal{V}$-category X is Φ-complete if and only if X is Cauchy complete. The following definition provides another important example of a saturated class of limit weights.

Definition 4.25. Let $\mathcal{V}$ be a quantale. A $\mathcal{V}$-distributor $\varphi_0\colon G \nrightarrow X$ with domain G is called *codirected* whenever the $\mathcal{V}$-functor

$$[\varphi_0,-]\colon \mathcal{V}\text{-}\mathsf{Dist}(G,X) \longrightarrow \mathcal{V}$$

preserves finite suprema and tensors; that is, for all $\varphi,\varphi'\colon G \nrightarrow X$ and all $u \in \mathcal{V}$,

$$[\varphi_0,\bot] = \bot,\quad [\varphi_0,\varphi\vee\varphi'] = [\varphi_0,\varphi]\vee[\varphi_0,\varphi']\quad [\varphi_0,u\otimes\varphi] = u\otimes[\varphi_0,\varphi].$$

We note that the class Φ_Δ of all codirected $\mathcal{V}$-distributors $\varphi\colon G \mathrel{-\!\!\circ\!\!\!\!\to} X$ is saturated (see [47]).

Definition 4.26. A $\mathcal{V}$-category X is called *codirected complete* whenever X is Φ_Δ-complete.

For a left adjoint $\mathcal{V}$-distributor $\varphi\colon G \mathrel{-\!\!\circ\!\!\!\!\to} X$ with right adjoint $\psi\colon X \mathrel{-\!\!\circ\!\!\!\!\to} G$, we have

$$[\varphi, -] = \psi \cdot -$$

since $\varphi \cdot - \dashv \psi \cdot -$ and $\varphi \cdot - \dashv [\varphi, -]$; which shows that $\varphi\colon G \mathrel{-\!\!\circ\!\!\!\!\to} X$ is codirected. Therefore every codirected complete $\mathcal{V}$-category is Cauchy complete.

Example 4.27. For $\mathcal{V} = \mathbf{2}$, we can interpret every $\mathbf{2}$-distributor $\varphi\colon G \mathrel{-\!\!\circ\!\!\!\!\to} X$ as an upclosed subset $A \subseteq X$ of X. Then A is codirected in the sense of Definition 4.25 if and only if A is codirected in the usual sense; that is, $A \neq \varnothing$ and, for all $x, y \in A$, there is some $z \in A$ with $z \leq x$ and $z \leq y$.

We recall now that, by Theorem 4.5, $\mathcal{U}$-distributors of type $G \mathrel{-\!\!\circ\!\!\!\!\rightharpoonup} X$ correspond to $\mathcal{U}$-functors $X \to \mathcal{V}$; and with this perspective we can consider $\mathcal{U}\text{-}\mathsf{Dist}(G, X)$ as a $\mathcal{V}$-subcategory of $\mathcal{V}\text{-}\mathsf{Dist}(G, X_0)$.

Proposition 4.28. *For every ultrafilter theory $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$, the inclusion $\mathcal{V}$-functor*

$$\mathcal{U}\text{-}\mathsf{Dist}(G, X) \longrightarrow \mathcal{V}\text{-}\mathsf{Dist}(G, X_0)$$

has a left adjoint

$$\overline{(-)}\colon \mathcal{V}\text{-}\mathsf{Dist}(G, X_0) \longrightarrow \mathcal{U}\text{-}\mathsf{Dist}(G, X).$$

Moreover, if $\mathcal{U}$ is pointwise strict and compatible with finite suprema, then $\mathcal{U}\text{-}\mathsf{Dist}(G, X)$ is closed in $\mathcal{V}\text{-}\mathsf{Dist}(G, X_0)$ under finite suprema and tensors.

Proof. By [36, Corollary 5.3], the $\mathcal{V}$-category $\mathcal{U}\text{-}\mathsf{Dist}(G, X)$ is closed in $\mathcal{V}\text{-}\mathsf{Dist}(G, X_0)$ under weighted limits. The additional conditions guarantee that the maps

$$t_u\colon \mathcal{V} \longrightarrow \mathcal{V} \qquad \text{and} \qquad \vee\colon \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$$

are $\mathcal{U}$-functors, for every $u \in \mathcal{V}$; which justifies the second claim. $\square$

Corollary 4.29. *Let $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ be a strict ultrafilter theory compatible with finite suprema so that k is approximated and the top element of $\mathcal{V}$. Then, for every codirected $\mathcal{V}$-distributor $\varphi\colon G \mathrel{\circ\!\!\to} X$, the $\mathcal{U}$-distributor $\overline{\varphi}\colon G \mathrel{\circ\!\!\rightharpoonup} X$ is left adjoint in $\mathcal{U}$-Dist.*

Proof. We recall first from Proposition 2.10 that, under these assumptions, k is $\vee$-irreducible. Using the adjunction of Proposition 4.28, the $\mathcal{V}$-functor

$$[\overline{\varphi}, -]\colon \mathcal{U}\text{-}\mathsf{Dist}(G, X) \longrightarrow \mathcal{V}$$

is equal to the composite

$$\mathcal{U}\text{-}\mathsf{Dist}(G, X) \longrightarrow \mathcal{V}\text{-}\mathsf{Dist}(G, X_0) \xrightarrow{[\varphi,-]} \mathcal{V},$$

and therefore $[\overline{\varphi}, -]$ preserves tensors and finite suprema. By [41, Propositions 2.15 and 3.5], $\overline{\varphi}\colon G \mathrel{\circ\!\!\rightharpoonup} X$ is left adjoint in $\mathcal{U}$-Cat. Note that the notation regarding distributors in [41] is dual to ours. □

Theorem 4.30. *Let $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ be a strict ultrafilter theory compatible with finite suprema where $\mathcal{V}$ is completely distributive, ξ is as in Theorem 3.2, and k is approximated and the top element of $\mathcal{V}$. Then, for every $\mathcal{V}$-categorical compact Hausdorff space (X, a_0, α), the $\mathcal{V}$-category (X, a_0) is codirected complete.*

Proof. Let $\varphi\colon G \mathrel{\circ\!\!\to} X$ be a codirected $\mathcal{V}$-distributor. By Corollaries 4.18 and 4.29, there is some $y \in X$ with
$\overline{\varphi} = y_{\circledast} = y_*$. Then, for every $x \in X$,

$$[\varphi, x_*] = [\overline{\varphi}, x_*] = [y_*, x_*] = a_0(x, y).$$

This proves that $y_X\colon X \to \Phi_\Delta(X)$ has a right adjoint in $\mathcal{V}$-Cat. □

We finish this subsection by exhibiting a connection with other accounts of "codirected complete metric spaces" which appear in the literature. Firstly, non-symmetric versions of Cauchy sequences and their limits are introduced in [69] and further studied in [7, 66]: a sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is called *forward-Cauchy* whenever

$$\forall \varepsilon > 0\, \exists N \in \mathbb{N}\, \forall n \geq m \geq N \,.\, d(x_m, x_n) < \varepsilon,$$

and $(x_n)_{n\in\mathbb{N}}$ is called *backward-Cauchy* whenever

$$\forall\varepsilon > 0\,\exists N \in \mathbb{N}\,\forall n \geq m \geq N\,.\, d(x_n, x_m) < \varepsilon.$$

The definitions above extend naturally to nets (see [21]), and in [74] it is shown that that forward-Cauchy nets in metric spaces correspond precisely to those $\overleftarrow{[0,\infty]}_+$-distributors $\psi\colon X \looparrowright G$ with the property that the $\mathcal{V}$-functor

$$\psi\cdot -\colon \overleftarrow{[0,\infty]}_+\text{-}\mathsf{Dist}(G, X) \longrightarrow \overleftarrow{[0,\infty]}_+,\ \varphi \longmapsto \psi\cdot\varphi$$

preserves finite meets. On the other hand, in [43] it is shown that these distributors do *not* coincide with forward-Cauchy nets for $\mathcal{V} = \overleftarrow{[0,\infty]}_\wedge$. Such $\overleftarrow{[0,\infty]}_+$-distributors are called *flat* in [74]; however, in this paper we deviate slightly from the notation of [74].

Definition 4.31. A $\mathcal{V}$-distributor $\psi\colon X \looparrowright G$ is called *flat* if

$$\psi\cdot -\colon \mathcal{V}\text{-}\mathsf{Dist}(G, X) \to \mathcal{V}$$

preserves finite infima and cotensors.

In order to compare these two notions of "directedness", we restrict our study to a certain type of quantales.

Definition 4.32. We call a quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$ a *Girard quantale* whenever $\mathcal{V}$ has a dualising element $D \in \mathcal{V}$; that is, for every $u \in \mathcal{V}$, $u = \hom(\hom(u, D), D)$.

This type of quantales is introduced in [80], we also refer to [77] for a study of categories enriched in a Girard quantale.

Example 4.33. The quantale $\mathbf{2} = \{0, 1\}$ and the quantale $[0, 1]$ with the Lukasiewicz tensor $\otimes = \odot$ are Girard quantales, with dualising object the bottom element 0.

For $\mathcal{V} = (\mathcal{V}, \otimes, k)$ being a Girard quantale with dualising element D, we write $u^\perp = \hom(u, D)$. As shown in [80], the operations $(-)^\perp$ and $\otimes$ allow us to determine the internal hom of $\mathcal{V}$: for all $u, v \in \mathcal{V}$,

$$\hom(u, v) = (u \otimes v^\perp)^\perp.$$

Lemma 4.34. *The map $(-)^\perp\colon \mathcal{V} \to \mathcal{V}^{\mathrm{op}}$ is a $\mathcal{V}$-functor. Hence, $\mathcal{V} \simeq \mathcal{V}^{\mathrm{op}}$ in $\mathcal{V}$-Cat.*

Proof. For all $u, v \in \mathcal{V}$ we have

$$\hom(u,v) \otimes \hom(v,D) \le \hom(u,D),$$

which is equivalent to $\hom(u,v) \le \hom(v^\perp, u^\perp)$. □

Hence, for every $\varphi\colon G \to X$ in $\mathcal{V}$-Dist, $\varphi^\perp(x) = \varphi(x)^\perp$ defines a $\mathcal{V}$-distributor $\varphi^\perp\colon X \to G$. Hence, the isomorphism of Lemma 4.34 induces a $\mathcal{V}$-isomorphism

$$(-)^\perp\colon \mathcal{V}\text{-}\mathsf{Dist}(G,X) \longrightarrow \mathcal{V}\text{-}\mathsf{Dist}(X,G)^{\mathrm{op}}.$$

Proposition 4.35. *Let $\mathcal{V} = (\mathcal{V}, \otimes, k)$ be a Girard quantale, X a $\mathcal{V}$-category and $\varphi_0\colon G \to X$ in $\mathcal{V}$-Dist. Then the diagram*

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathsf{Dist}(G,X) \arrow[r, "(-)^\perp"] \arrow[d, "{[\varphi_0,-]}"'] & \mathcal{V}\text{-}\mathsf{Dist}(X,G)^{\mathrm{op}} \arrow[d, "(-\cdot\varphi_0)^{\mathrm{op}}"] \\
\mathcal{V} \arrow[r, "(-)^\perp"'] & \mathcal{V}^{\mathrm{op}}
\end{tikzcd}$$

commutes.

Proof. Let $\varphi\colon G \to X$ be a $\mathcal{V}$-distributor. Then

$$\begin{aligned}
[\varphi_0, \varphi]^\perp &= \left(\bigwedge_{x\in X} \hom(\varphi_0(x), \varphi(x))\right)^\perp \\
&= \bigvee_{x\in X} \hom(\varphi_0(x), \varphi(x))^\perp \\
&= \bigvee_{x\in X} \varphi_0(x) \otimes \varphi(x)^\perp.
\end{aligned}$$ □

Corollary 4.36. *Let $\mathcal{V} = (\mathcal{V}, \otimes, k)$ be a Girard quantale. Then a $\mathcal{V}$-distributor $\varphi\colon G \to X$ is codirected if and only if the $\mathcal{V}$-functor*

$$-\cdot\varphi\colon \mathcal{V}\text{-}\mathsf{Dist}(X,G) \to \mathcal{V}$$

preserves cotensors and finite infima. Hence, X is codirected complete if and only if X^{op} is cocomplete with respect to all flat $\mathcal{V}$-distributors $\psi\colon X \to G$.

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