Categories and
General Algebraic Structures
with Applications
Volume 9, Number 1, July 2018, 59-75.
WWW.CGASA.ir

Pointfree topology version of image of real-valued continuous functions

A. Karimi Feizabadi*, A.A. Estaji, and M. Robat Sarpoushi

Abstract. Let $\mathcal{R}L$ be the ring of real-valued continuous functions on a frame L as the pointfree version of $C(X)$, the ring of all real-valued continuous functions on a topological space X. Since $C_c(X)$ is the largest subring of $C(X)$ whose elements have countable image, this motivates us to present the pointfree version of $C_c(X)$. The main aim of this paper is to present the pointfree version of image of real-valued continuous functions in $\mathcal{R}L$. In particular, we will introduce the pointfree version of the ring $C_c(X)$. We define a relation from $\mathcal{R}L$ into the power set of $\mathbb{R}$, namely *overlap*. Fundamental properties of this relation are studied. The relation overlap is a pointfree version of the relation defined as $\mathrm{Im}(f) \subseteq S$ for every continuous function $f : X \to \mathbb{R}$ and $S \subseteq \mathbb{R}$.

1 Introduction

As is well known, $C(X)$ denotes the ring of all real-valued continuous functions on a topological space X. Undoubtedly, the book *Rings of Continuous Functions* written by Gillman and Jerison is the best reference to study the

* Corresponding Author
Keywords: Frame, ring of real-valued continuous functions, countable image, f-ring.
Mathematics Subject Classification[2010]: 06D22, 13A15, 54C05, 54C30.
Received: 18 March 2017, Accepted: 7 June 2017
ISSN Print: 2345-5853 Online: 2345-5861

rings of continuous functions [14]. In [13], $C_c(X)$, the subalgebra of $C(X)$, consisting of functions with countable image is studied. It turns out that $C_c(X)$, although not isomorphic to any $C(Y)$ in general, enjoys most of the important properties of $C(X)$. This subalgebra has recently received some attention, see [6, 16–18].

The concept of a frame, or pointfree topology, is a generalization of the classical topology. The ring of real-valued continuous functions on a frame, that is, $\mathcal{R}L$, as the pointfree version of the ring $C(X)$, has been studied prior to 1996 by some authors such as R.N. Ball and A.W. Hager in [1]. A systematic and indepth study of the ring of real continuous functions in pointfree topology was undertaken by B. Banaschewski in 1997 (see [2, 4, 5]). Also, [3, 7, 15, 19] are valuable references on the subject of frames and the ring $\mathcal{R}L$.

In this paper, we introduce the pointfree version of image of real-valued continuous functions in the ring of real-valued continuous functions on a frame, namely, $\mathcal{R}_\lambda L$. In particular, we will have $\mathcal{R}_c L$ as the pointfree version of the ring $C_c(X)$. For this, we use the subsets of $\mathbb{R}$. One may think that we should use the sublocales of the frame $\mathcal{L}(\mathbb{R})$ instead of the subsets of $\mathbb{R}$. In reply, we say that countability image of a continuous function by its very nature deals with number of points of its range, and is not a topological concept. In other words, the countability image of a continuous function does not seem to lend itself to localic interpretation because it is about the number of points in a set.

This paper is organized as follows. In Section 2, we review some basic notions and properties of frames and the pointfree version of the ring of real-valued continuous functions.

In Section 3, we define the concept of *overlap* for $\alpha \in \mathcal{R}L$ (Definition 3.1). To do this, we introduce an onto (quotient) frame map $i : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}S$ given by $i(p, q) = \{s \in S : p < s < q\}$, where $S \subseteq \mathbb{R}$ is taken as a subspace of $\mathbb{R}$ with usual topology and $\mathfrak{O}S$ is the frame of open subsets of S. For every $\alpha \in \mathcal{R}L$ and $S \subseteq \mathbb{R}$, we show that α is an overlap of S if and only if $\breve{\alpha}$ is a frame map, where $\breve{\alpha} : \mathfrak{O}S \to L$ is given by $\breve{\alpha}(U) = \bigvee\{\alpha(v) : v \in \mathcal{L}(\mathbb{R}), i(v) \subseteq U\}$ (see Theorem 3.8). Also, for every continuous function $f : X \to \mathbb{R}$ and $S \subseteq \mathbb{R}$, we show that $f_\tau : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}X$ is an overlap of S if and only if $\mathrm{Im}(f) \subseteq S$ if and only if there exists a continuous function $g : X \to S$ such that $f(x) = g(x)$ for every $x \in X$ (see Proposition 3.11).

In Section 4, we introduce the ring $\mathcal{R}_\lambda L$ as the pointfree version of the image of real-valued continuous functions.

2 Preliminaries

Here, we recall some definitions and results from the literature on frames and the pointfree topology version of the ring of continuous real-valued functions. Our references for frames are [15] and [19].

A *frame* is a complete lattice L in which the distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

holds for all $x \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by $\top$ and $\bot$, respectively. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}X$.

A *frame homomorphism* (*or frame map*) is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element.

An element $p \in L$ is said to be *prime* if $p < \top$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. A lattice ordered ring A is called an *f-ring*, if $(f \wedge g)h = fh \wedge gh$ for every $f, g \in A$ and every $0 \leq h \in A$.

Recall the contravariant *functor* Σ from **Frm** to the category **Top** of topological spaces which assigns to each frame L its *spectrum* ΣL of prime elements with $\Sigma_a = \{p \in \Sigma L : a \not\leq p\}$ $(a \in L)$ as its open sets.

An element a of a frame L is said to be *completely below* b, written $a \prec\!\prec b$, if there exists a sequence $\{c_q\}$, $q \in \mathbb{Q} \cap [0,1]$, where $c_0 = a,\ c_1 = b$, and $c_p \prec c_q$ if $p < q$ where $u \prec v$ means that $u^* \vee v = \top$. A frame L is called *completely regular* if each $a \in L$ is the join of elements completely below it.

Regarding the *frame of reals* $\mathcal{L}(\mathbb{R})$ and the f-ring $\mathcal{R}L$ of *continuous real functions* on L, we use the notations of [4] (see also [2]).

For every pair $(p,q) \in \mathbb{Q}^2$, put

$$\langle p, q\rangle := \{x \in \mathbb{Q} : p < x < q\} \quad \text{and} \quad \rrbracket p, q\llbracket := \{x \in \mathbb{R} : p < x < q\}.$$

Corresponding to every continuous operation $\diamond : \mathbb{Q}^2 \to \mathbb{Q}$ (in particular $+, ., \wedge, \vee$) we have an operation on $\mathcal{R}L$, denoted by the same symbol $\diamond$, defined by

$$\alpha \diamond \beta(p,q) = \bigvee \{\alpha(r,s) \wedge \beta(u,w) :< r,s > \diamond < u,w > \leq < p,q >\},$$

where $< r, s > \diamond < u, w > \leq < p, q >$ means that for each $r < x < s$ and $u < y < w$ we have $p < x \diamond y < q$. For every $r \in \mathbb{R}$, define the constant frame map $\mathbf{r} \in \mathcal{R}L$ by $\mathbf{r}(p, q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p, q) = \bot$.

Recall that a frame L is called *spatial* if there exists a topological space X such that $L \cong \mathfrak{O}X$. We have the next proposition.

Proposition 2.1. [10] *A frame L is spatial if and only if $\eta : L \to \mathfrak{O}\Sigma L$ by $\eta(a) = \Sigma_a$, for every $a \in L$, is an isomorphism in* **Frm**.

Here we recall the necessary notations, definitions, and results form [9]. Let $a \in L$ and $\alpha \in \mathcal{R}L$. The sets $\{r \in \mathbb{Q} : \alpha(-, r) \leq a\}$ and $\{s \in \mathbb{Q} : \alpha(s, -) \leq a\}$ are denoted by $L(a, \alpha)$ and $U(a, \alpha)$, respectively. For $a \neq \top$ it is obvious that for each $r \in L(a, \alpha)$ and $s \in U(a, \alpha)$, $r \leq s$. In fact, we have

Proposition 2.2. [9] *If $p \in \Sigma L$ and $\alpha \in \mathcal{R}L$, then $(L(p, \alpha), U(p, \alpha))$ is a Dedekind cut for a real number which is denoted by $\widetilde{p}(\alpha)$.*

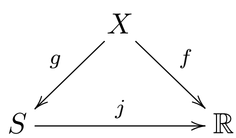
Proposition 2.3. [9] *If p is a prime element of a frame L, then there exists a unique map $\widetilde{p} : \mathcal{R}L \longrightarrow \mathbb{R}$ such that for each $\alpha \in \mathcal{R}L$, $r \in L(p, \alpha)$ and $s \in U(p, \alpha)$ we have $r \leq \widetilde{p}(\alpha) \leq s$.*

Let p be a prime element of L. Throughout this paper, for every $\alpha \in \mathcal{R}L$ we define $\alpha[p] = \widetilde{p}(\alpha)$ (see [11]). For every $\alpha : \mathcal{L}(\mathbb{R}) \to L$, we define $\overline{\alpha} : \Sigma L \to \mathbb{R}$ by $\overline{\alpha}(p) = \alpha[p]$, for $p \in \Sigma L$.

It is well known that the homomorphism $\tau : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}\mathbb{R}$ taking (p, q) to $\rrbracket p, q \llbracket$ is an isomorphism (see [4, Proposition 2]).

3 Overlap and its properties

For a topological space X, to say the image of a continuous function $f : X \to \mathbb{R}$ is contained in the set $S \subseteq \mathbb{R}$ is to say there is a morphism $X \xrightarrow{g} S$ in **Top** such that the triangle

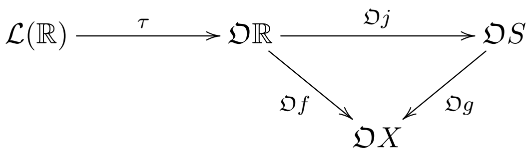


commutes, where j is the inclusion map. Our aim is to extend this notion to pointfree function rings, so that, for instance, we can have an analogue of the $\mathbb{R}$-subalgebra $C_c(X)$ of $C(X)$ whose elements are those functions with countable range.

Regarding the latter, the obvious hurdle is that "countability" is not a topological notion. It is thus not clear how one should define a function $\alpha \in \mathcal{R}L$ to have "countable range". So to obviate this, we, in effect, apply the open-set functor

$$\mathfrak{O} : \mathbf{Top} \to \mathbf{Frm}$$

to the triangle above to obtain the commutative diagram



in **Frm**, after adjoining the morphism $\mathcal{L}(\mathbb{R}) \xrightarrow{\tau} \mathfrak{O}\mathbb{R}$ which maps a generator (p, q) to the open interval $\{x \in \mathbb{R} : p < x < q\}$. Now, starting with an arbitrary $\alpha \in \mathcal{R}L$, we define the concept of "overlapping". We then show that, for any $f \in C(X)$ and $S \subseteq \mathbb{R}$,

$$\mathrm{Im}(f) \subseteq S \quad \Longleftrightarrow \quad \mathfrak{O}f \text{ is an overlap of } S;$$

thus justifying that this is a "correct" extension of the notion of image for pointfree real-valued functions.

In what follows, L, S and $i : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}S$, denote a frame, a subspace of $\mathbb{R}$ with usual topology, and the onto (quotient) frame map, such that for every $p, q \in \mathbb{Q}$, $i(p, q) = \tau(p, q) \cap S$, respectively.

Definition 3.1. For $\alpha \in \mathcal{R}L$ and $S \subseteq \mathbb{R}$, we say that α is an *overlap of* S (denoted by $\alpha \blacktriangleleft S$) if

$$i(u) \subseteq i(v) \text{ implies } \alpha(u) \leq \alpha(v),$$

for every $u, v \in \mathcal{L}(\mathbb{R})$.

Proposition 3.2. *If $\alpha \in \mathcal{R}L$, then it is not an overlap of $\emptyset$.*

Proof. Suppose that $\alpha \blacktriangleleft \emptyset$. Now, we assume that $p, q, r, s \in \mathbb{Q}$, $p < q$ and $r < s$. Since $\tau(p, q) \cap \emptyset = \emptyset = \tau(r, s) \cap \emptyset$, we conclude that $\alpha(p, q) = \alpha(r, s)$.

It follows that $\alpha(p,q) = \bigvee\{\alpha(r,s) : r,s \in \mathbb{Q}\} = \top$. Now, if $p,q,r,s \in \mathbb{Q}$ and $p < q < r < s$, then

$$\bot = \alpha((p,q) \wedge (r,s)) = \alpha(p,q) \wedge \alpha(r,s) = \top,$$

which is a contradiction. □

Definition 3.3. For any $\alpha \in \mathcal{R}L$ and any $S \subseteq \mathbb{R}$, we say that α is a *weakly overlap* of S (denoted by $\alpha \lhd S$) if

$$i(p,q) = i(r,s) \text{ implies } \alpha(p,q) = \alpha(r,s),$$

for every $p,q,r,s \in \mathbb{Q}$.

Example 3.4. Let $\mathrm{Id} : \mathbb{Q} \to \mathbb{R}$ be the identity map. Then $\alpha : \mathfrak{O}\mathbb{R} \to \mathfrak{O}\mathbb{Q}$ is a frame map such that $\alpha(p,q) = \tau(p,q) \cap \mathbb{Q}$. Let $S = \mathbb{R} \setminus \{0\}$. Clearly, $\alpha \lhd S$. Now, if $0 \in \tau(p,q)$ and $p,q \in \mathbb{Q}$, then

$$i(p,q) = \tau(p,q) \cap S \subseteq (\tau(p,0) \cup \tau(0,q)) \cap S = i((p,0) \vee (0,q))$$

and $\alpha(p,q) \not\leq \alpha((p,0) \vee (0,q))$. Thus, α is not an overlap of S.

It is clear that $\alpha \blacktriangleleft S$ implies $\alpha \lhd S$, but the previous example shows that the converse need not hold.

Lemma 3.5. *For any $\alpha \in \mathcal{R}L$ and any $S \subseteq \mathbb{R}$, the following statements are equivalent:*

(1) $\alpha \blacktriangleleft S$.

(2) $i(u) = i(v)$ *implies* $\alpha(u) = \alpha(v)$, *for any* $u,v \in \mathcal{L}(\mathbb{R})$.

(3) $i(p,q) = i(v)$ *implies* $\alpha(p,q) = \alpha(v)$, *for every* $v \in \mathcal{L}(\mathbb{R})$ *and* $p,q \in \mathbb{Q}$.

(4) $i(p,q) \subseteq i(v)$ *implies* $\alpha(p,q) \leq \alpha(v)$, *for any* $v \in \mathcal{L}(\mathbb{R})$ *and any* $p,q \in \mathbb{Q}$.

Proof. (1) ⇒ (2) ⇒ (3) are obviously.

For (3) ⇒ (4), suppose that $i(p,q) \subseteq i(v)$. So

$$i(p,q) = i(p,q) \cap i(v) = i((p,q) \wedge v).$$

By (3), $\alpha(p,q) = \alpha((p,q) \wedge v)$, and hence $\alpha(p,q) \leq \alpha(v)$.

Finally, to show (4) $\Rightarrow$ (1), let $u, v \in \mathcal{L}(\mathbb{R})$ such that $i(u) \subseteq i(v)$. Let $(p, q) \leq u$ where $p, q \in \mathbb{Q}$. Hence $i(p, q) \subseteq i(u) \subseteq i(v)$, so, by (4), $\alpha(p, q) \leq \alpha(v)$. Therefore,

$$\alpha(u) = \alpha(\bigvee_{(p,q)\leq u} (p, q)) = \bigvee_{(p,q)\leq u} \alpha(p, q) \leq \alpha(v).$$

$\square$

Definition 3.6. For $\alpha \in \mathcal{R}L$ and $S \subseteq \mathbb{R}$, define $\breve{\alpha} : \mathfrak{O}S \to L$ by

$$\breve{\alpha}(U) = \bigvee\{\alpha(v) : v \in \mathcal{L}(\mathbb{R}), i(v) \subseteq U\}.$$

It is clear that $\breve{\alpha}(U) = \bigvee\{\alpha(p, q) : \tau(p, q) \cap S \subseteq U\}$.

Lemma 3.7. *For $\alpha \in \mathcal{R}L$ and $S \subseteq \mathbb{R}$,*

(1) *$\breve{\alpha}$ is an order preserving map such that for every $u \in \mathcal{L}(\mathbb{R})$, $\alpha(u) \leq \breve{\alpha}(i(u))$.*

(2) *$\breve{\alpha}i = \alpha$ if and only if $\alpha \blacktriangleleft S$.*

Proof. (1) is clear.

To show (2), first suppose that $\breve{\alpha}i = \alpha$ and $i(u) \subseteq i(v)$. So

$$\alpha(u) = \breve{\alpha}i(u) \leq \breve{\alpha}i(v) = \alpha(v).$$

Therefore, $\alpha \blacktriangleleft S$. Conversely, suppose that $\alpha \blacktriangleleft S$. Let $u \in \mathcal{L}(\mathbb{R})$. We have

$$\begin{aligned} \breve{\alpha}(i(u)) &= \bigvee\{\alpha(v) : v \in \mathcal{L}(\mathbb{R}), i(v) \subseteq i(u)\} \\ &\leq \bigvee\{\alpha(v) : v \in \mathcal{L}(\mathbb{R}), \alpha(v) \leq \alpha(u)\} \\ &= \alpha(u). \end{aligned}$$

So, by (1), $\breve{\alpha}i = \alpha$. $\square$

In the proof of one of the implications in the upcoming theorem we will use the fact that if M is a regular frame and $f, g : M \to L$ are frame maps such that $f(x) \leq g(x)$ for all $x \in M$, then $f = g$.

Theorem 3.8. *For any $\alpha \in \mathcal{R}L$ and any $S \subseteq \mathbb{R}$, the following statements are equivalent:*

(1) *$\alpha \blacktriangleleft S$.*

(2) *$\breve{\alpha}i = \alpha$.*

(3) *$\breve{\alpha}$ is a frame map.*

Proof. (1) $\Leftrightarrow$ (2). It follows from Lemma 3.7.

(2) $\Rightarrow$ (3). This is because, $i : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}S$ is an onto frame map and $\breve{\alpha}$ is a well-defined function.

Finally, to see (3) $\Rightarrow$ (2), note that for every $u \in \mathcal{L}(\mathbb{R})$, by Lemma 3.7(1), $(\breve{\alpha} i)(u) \geq \alpha(u)$. Since $\mathcal{L}(\mathbb{R})$ is a regular frame and $\breve{\alpha} i, \alpha : \mathcal{L}(\mathbb{R}) \to L$ are two frame maps, we conclude that $\breve{\alpha} i = \alpha$. □

Corollary 3.9. *For any $\alpha \in \mathcal{R}L$ and any $S \subseteq \mathbb{R}$, the following statements are equivalent*:

(1) $\alpha \blacktriangleleft S$.

(2) *For every* $\{(p_i, q_i)\}_{i \in I}, \{(r_j, s_j)\}_{j \in J} \subseteq \mathbb{Q} \times \mathbb{Q}$, *if*

$$\bigcup_{i \in I} \tau(p_i, q_i) \cap S = \bigcup_{j \in J} \tau(r_j, s_j) \cap S,$$

then $\bigvee_{i \in I} \alpha(p_i, q_i) = \bigvee_{j \in J} \alpha(r_j, s_j)$.

(3) *There exists a unique frame map* $\beta : \mathfrak{O}S \to L$ *such that* $\beta i = \alpha$.

Proof. By Theorem 3.8, it is evident. □

In what follows, for $f \in C(X)$, the frame map

$$f^{-1} \circ \tau : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}X$$

is denoted by f_τ. Note that for $p < q$ in $\mathbb{Q}$,

$$f_\tau(p, q) = \{x \in X : p < f(x) < q\}.$$

Lemma 3.10. *For every $f \in C(X)$, if $Im(f) \subseteq S \subseteq \mathbb{R}$, then $f_\tau \blacktriangleleft S$.*

Proof. Let $p, q \in \mathbb{Q}$ and $u \in \mathcal{L}(\mathbb{R})$. If $\tau(p, q) \cap S \subseteq i(u)$, then

$$\begin{aligned} x \in f_\tau(p, q) &\Rightarrow f(x) \in \tau(p, q) \cap \mathrm{Im}(f) \subseteq \tau(u) \cap S \cap \mathrm{Im}(f) \\ &\Rightarrow x \in f_\tau(u). \end{aligned}$$

Therefore, $f_\tau \blacktriangleleft S$. □

Proposition 3.11. *Let $S \subseteq \mathbb{R}$ and $f \in C(X)$. Then the following statements are equivalent*:

(1) $f_\tau \blacktriangleleft S$.

(2) *There exists a continuous function $g : X \to S$ such that $f(x) = g(x)$, for every $x \in X$.*

(3) $Im(f) \subseteq S$.

Proof. (1) $\Rightarrow$ (3). Suppose that $\mathrm{Im}(f) \nsubseteq S$. Then there exists $x \in X$ such that $y = f(x) \in \mathrm{Im}(f)\setminus S$. Let $p, q \in \mathbb{Q}$ and $p < y < q$. There exist sequences $\{p_n\}_{n\in\mathbb{N}}, \{q_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ such that $p_n \longrightarrow y$, $q_n \longrightarrow y$ and for every $n \in \mathbb{N}$, $p < p_n < y < q_n < q$. Hence

$$\tau(p,q) \cap S = \bigcup_{n\in\mathbb{N}} (\tau(p,p_n) \cup \tau(q_n,q)) \cap S.$$

By Corollary 3.9, $x \in f_\tau(p,q) = \bigvee_{n\in\mathbb{N}}(f_\tau(p,p_n) \cup f_\tau(q_n,q))$ and it follows that there is $n \in \mathbb{N}$ such that $x \in f_\tau(p,p_n) \cup f_\tau(q_n,q)$, which is a contradiction.

(3) $\Rightarrow$ (1). By Lemma 3.10, it is clear.

(3) $\Leftrightarrow$ (2). It is evident. □

Lemma 3.12. *Let p be a prime element of L. For $\alpha \in \mathcal{R}L$ and $t \in \mathbb{R}$, $\alpha[p] \neq t$ if and only if $\bigvee\{\alpha(-,r) \vee \alpha(s,-) : r,s \in \mathbb{Q}, r < t < s\} \nleq p$.*

Proof. Suppose that $\alpha[p] \neq t$, assume that $\alpha[p] > t$. Hence, there is a rational number r such that $\alpha[p] > r > t$. Thus, by [9, Lemma 3.1], $r \in L(p,\alpha)$, and so, by the definition of $L(p,\alpha)$, $\alpha(-,r) \leq p$. Now, if

$$\bigvee\{\alpha(-,r) \vee \alpha(s,-) : r,s \in \mathbb{Q}, r < t < s\} \leq p,$$

we have

$$\top = \alpha(-,r) \vee \bigvee\{\alpha(-,r) \vee \alpha(s,-) : r,s \in \mathbb{Q}, r < t < s\} \leq p \vee p = p,$$

which contradicts p being a prime element. Therefore,

$$\bigvee\{\alpha(-,r) \vee \alpha(s,-) : r,s \in \mathbb{Q}\} \nleq p.$$

The case $\alpha[p] < t$ is proved similarly.

Conversely, suppose that $\alpha[p] = t$. So, by [9, Lemma 3.1], for every two rationals $r < t < s$, we have $r \in L(p,\alpha)$ and $s \in U(p,\alpha)$. Hence $\alpha(-,r) \vee \alpha(s,-) \leq p$, by the definition of $L(p,\alpha)$ and $U(p,\alpha)$. Thus,

$$\bigvee\{\alpha(-,r) \vee \alpha(s,-) : r,s \in \mathbb{Q}, r < t < s\} \leq p,$$

which contradicts the assumption. □

Proposition 3.13. *For every $\alpha \in \mathcal{R}L$ and $S \subseteq \mathbb{R}$, if $\alpha \blacktriangleleft S$, then $Im(\overline{\alpha}) \subseteq S$.*

Proof. Suppose that $\mathrm{Im}(\overline{\alpha}) \nsubseteq S$. Then there exists $p \in \Sigma L$ such that $\overline{\alpha}(p) = t \in \mathrm{Im}(\overline{\alpha}) \setminus S$. By Lemma 3.12,

$$\bigvee\{\alpha(-,r) \vee \alpha(s,-) : r,s \in \mathbb{Q}, r < t < s\} \leq p.$$

Since $t \notin S$, we conclude that

$$\bigcup\{\tau(r,s) \cap S : r,s \in \mathbb{Q}\} = S = \bigcup\{\tau(-,r) \cap S \vee \tau(s,-) \cap S : r,s \in \mathbb{Q}, r < t < s\}.$$

By Corollary 3.9,

$$\top = \bigvee\{\alpha(r,s) : r,s \in \mathbb{Q}\} = \bigvee\{\alpha(-,r) \vee \alpha(s,-) : r,s \in \mathbb{Q}, r < t < s\} \leq p,$$

which is a contradiction. □

Corollary 3.14. *For any $t \in \mathbb{R}$, the following statements are equivalent:*
(1) *$t \in S$.*
(2) *$\boldsymbol{t} \blacktriangleleft S$, where $\boldsymbol{t} \in \mathcal{R}L$.*

Proof. (1) ⇒ (2). Let $t \in S$ and $u, v \in \mathcal{L}(\mathbb{R})$ with $i(u) \subseteq i(v)$. If $t \in i(u)$, then $\mathbf{t}(u) = \mathbf{t}(v) = \top$ and if $t \notin i(u)$, then $\mathbf{t}(u) = \bot$. Therefore, $\mathbf{t}(u) \leq \mathbf{t}(v)$, which gives that $\mathbf{t} \blacktriangleleft S$.

(2) ⇒ (1). Suppose that $\mathbf{t} \blacktriangleleft S$. So, by Proposition 3.13, $\mathrm{Im}(\overline{\mathbf{t}}) = \{t\} \subseteq S$, that is, $t \in S$. □

Lemma 3.15. *Let L be a spatial frame. For any $\alpha \in \mathcal{R}L$ and the frame isomorphism $\eta : L \to \mathfrak{O}(\Sigma L)$ by $\eta(a) = \Sigma_a$, we have $\eta\alpha = \overline{\alpha}_\tau$.*

Proof. Let $(p,q) \in \mathcal{L}(\mathbb{R})$. We have

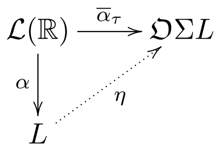
$$\eta\alpha(p,q) = \eta(\alpha(p,q)) = \Sigma_{\alpha(p,q)} = \{x \in \Sigma L : \alpha(p,q) \nleq x\}$$

and $\overline{\alpha}_\tau(p,q) = \{x \in \Sigma L : p < \overline{\alpha}(x) < q\}$. We show that

$$\Sigma_{\alpha(p,q)} = \{x \in \Sigma L : p < \alpha[x] < q\}.$$

Let $x \in \Sigma_{\alpha(p,q)}$, then $\alpha(p,q) \nleq x$. So $\alpha(-,p) \leq x$ and $\alpha(q,-) \leq x$, because x is prime and $\alpha(p,q) \wedge \alpha(-,p) = \bot \leq x$ and $\alpha(p,q) \wedge \alpha(q,-) = \bot \leq x$.

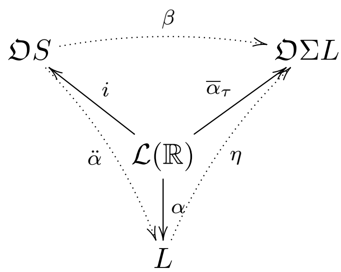
So $p \in L(x, \alpha)$ and $q \in U(x, \alpha)$. Hence $p < \alpha[x] < q$. Thus $x \in \overline{\alpha}_\tau(p, q)$. Therefore, $\eta\alpha(p, q) \leq \overline{\alpha}_\tau(p, q)$ for all $p, q \in \mathbb{Q}$. Hence $\eta\alpha = \overline{\alpha}_\tau$, by the regularity of $\mathcal{L}(\mathbb{R})$. Consequently, $\eta\alpha = \overline{\alpha}_\tau$ and the following diagram is commutative:



□

Proposition 3.16. *Let L be a spatial frame. Then the converse of the Proposition* 3.13 *holds.*

Proof. Let L be a spatial frame and $\mathrm{Im}(\overline{\alpha}) \subseteq S$. Then, by Proposition 3.11, $\overline{\alpha}_\tau \blacktriangleleft S$. Now, by Corollary 3.9, there exists a unique frame map $\beta : \mathfrak{O}S \to \mathfrak{O}\Sigma L$ such that $\beta i = \overline{\alpha}_\tau$. Also, since L is spatial, we have the isomorphism $\eta : L \to \mathfrak{O}\Sigma L$ with $\eta(a) = \Sigma_a$. Now, define $\ddot{\alpha} : \mathfrak{O}S \to L$ by $\ddot{\alpha} = \eta^{-1}\beta$. See the following diagram:



By Corollary 3.9, it is sufficient to show that $\ddot{\alpha}i$ is a unique frame map such that $\ddot{\alpha}i = \alpha$. To do this, let $(p, q) \in \mathcal{L}(\mathbb{R})$. So, by Lemma 3.15, we have

$$\begin{aligned} \ddot{\alpha}i(p, q) &= \ddot{\alpha}(i(p, q)) \\ &= \eta^{-1}\beta(i(p, q)) \\ &= \eta^{-1}(\beta i)(p, q) \\ &= \eta^{-1}\overline{\alpha}_\tau(p, q) \\ &= \alpha(p, q). \end{aligned}$$

Also, since the frame map β is unique, it follows that $\ddot{\alpha}$ is unique. □

Remark 3.17. Recall from [8] that for an infinite cardinal number k, then X is a (Tychonoff) space of weight at most k. This means that X has a basis for its topology of cardinality at most k. Moreover, let $\mathcal{I}$ be a k^+-complete ideal of subsets of X. This means that $\mathcal{I}$ is an ideal of subsets of X which has the following property: if $\mathcal{A} \subseteq \mathcal{I}$ and $|\mathcal{A}| \leq k$, then $\bigcup \mathcal{A} \in \mathcal{I}$. Now, let $L = \mathfrak{O}X$. We define a relation $\sqsubseteq$ on L as follows: for $U, V \in L$ we put

$$U \sqsubseteq V \quad \text{if and only if} \quad U \setminus V \in \mathcal{I}.$$

Next, an equivalence relation $\sim$ on L is defined by

$$U \sim V \quad \text{if and only if} \quad U \sqsubseteq V \text{ and } V \sqsubseteq U.$$

For $U \in L$, we let $[U]$ denote its $\sim$-equivalence class. Now, put $M = L/\sim$, and define a partial order $\leq$ on M by

$$[U] \leq [V] \quad \text{if and only if} \quad U \sqsubseteq V.$$

This definition is well defined and M is a completely regular frame with bottom $[\emptyset] = \{U \in \mathfrak{O}X : U \in \mathcal{I}\}$ and top $[X] = \{U \in \mathfrak{O}X : X \setminus U \in \mathcal{I}\}$. For more details see [8].

Let $\alpha \in \mathcal{R}L$ and $\{S_i : i \in I\}$ be a family of subsets of $\mathbb{R}$. In the following example, we show that if $\alpha \blacktriangleleft S_i$, for all $i \in I$, then α may not be an overlap of $\bigcap\{S_i : i \in I\}$.

Example 3.18. Consider $X = [0, 1]$ and $k = \aleph_0$. Let

$$\mathcal{I} = \{A \subseteq [0, 1] : \text{the measure of } A \text{ is zero}\}.$$

It is clear that $\mathcal{I}$ is a k^+-complete ideal of subsets of X. Now, let $\alpha : X \to \mathbb{R}$ be defined by $\alpha(x) = x$. Consider the frame map $\alpha_\tau : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}X$ defined by $\alpha_\tau(p, q) = \tau(p, q) \cap [0, 1]$. Now, let $L = \mathfrak{O}X$ and put $M = L/\sim$, where $\sim$ is the equivalence relation on L defined in Remark 3.17. Define $\beta : \mathcal{L}(\mathbb{R}) \to M$ by

$$\beta(u) = [\alpha_\tau(u)] = [\tau(u) \cap [0, 1]].$$

Let c be an arbitrary element of $\mathcal{I}$. Let $S_c = [0, 1] \setminus c$. We claim that $\beta \blacktriangleleft S_c$. Let $u, v \in \mathcal{L}(\mathbb{R})$ and $i(u) \subseteq i(v)$. Then

$$\tau(u) \cap [0, 1] \cap S_c \subseteq \tau(v) \cap [0, 1] \cap S_c,$$

which follows that

$$\tau(u) \cap [0,1] \setminus \tau(v) \cap [0,1] \subseteq c.$$

Since $c \in \mathcal{I}$, then

$$(\tau(u) \cap [0,1]) \setminus (\tau(v) \cap [0,1]) \in \mathcal{I}.$$

Hence, by Remark 3.17,

$$\tau(u) \cap [0,1] \sqsubseteq \tau(v) \cap [0,1],$$

which follows that

$$[\tau(u) \cap [0,1]] \leq [\tau(v) \cap [0,1]].$$

Therefore, $\beta(u) \leq \beta(v)$. Thus, $\beta \blacktriangleleft S_c$. Also, we have $\bigcap_{c\in\mathcal{I}} S_c = \emptyset$. Hence, by Proposition 3.2, β is not an overlap of $\bigcap\{S_c : c \in \mathcal{I}\} = \emptyset$.

Proposition 3.19. *Let* $\alpha : \mathcal{L}(\mathbb{R}) \to L$ *and* $\beta : L \to M$ *be frame maps.*
(1) *If* $\alpha \blacktriangleleft S$ *then* $\beta \circ \alpha \blacktriangleleft S$.
(2) *If* β *is a monomorphism and* $\beta \circ \alpha \blacktriangleleft S$, *then* $\alpha \blacktriangleleft S$.

Proof. (1) Let $u, v \in \mathcal{L}(\mathbb{R})$ and $i(u) \subseteq i(v)$, then $\alpha(u) \leq \alpha(v)$. Therefore, $\beta \circ \alpha(u) \leq \beta \circ \alpha(v)$. Hence $\beta \circ \alpha \blacktriangleleft S$.
(2) Let $u, v \in \mathcal{L}(\mathbb{R})$ and $i(u) = i(v)$, then $\beta \circ \alpha(u) = \beta \circ \alpha(v)$. Since β is a monomorphism, $\alpha(u) = \alpha(v)$. □

Remark 3.20. In Proposition 3.19 (2), the condition that β is a monomorphism is necessary.

Example 3.21. In Example 3.18, for every $c \in \mathcal{I}$, $\beta \blacktriangleleft S_c = [0,1] \setminus c$, but α_τ is not an overlap of $S_c = [0,1] \setminus c$, because $\mathrm{Im}(\alpha) = [0,1]$.

4 The ring $\mathcal{R}_\lambda L$

Let S_1 and S_2 be subsets of $\mathbb{R}$. For the binary operations $\diamond = +, \cdot, \wedge, \vee : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we define

$$S_1 \diamond S_2 = \{a \diamond b : a \in S_1, b \in S_2\}.$$

Lemma 4.1. *Let S_1 and S_2 be subsets of $\mathbb{R}$ and $S_\diamond = S_1 \diamond S_2$, for any $\diamond \in \{+, \cdot, \wedge, \vee\}$. Let $r, s \in \mathbb{Q}$, $u \in \mathcal{L}(\mathbb{R})$ and $\diamond \in \{+, \cdot, \wedge, \vee\}$. If $\tau(r,s) \cap S_\diamond \subseteq \tau(u) \cap S_\diamond$, then*

$$A_i := \bigcup \{\tau(p,q) \cap S_i : p, q \in \mathbb{Q}\,,\ \tau(p,q) \diamond \tau(t,k) \subseteq \tau(r,s),\ \textit{for some}\ t, k \in \mathbb{Q}\}$$

is a subset of

$$B_i := \bigcup \{\tau(a,b) \cap S_i : a, b \in \mathbb{Q}\,,\ \tau(a,b) \diamond \tau(c,d) \subseteq \tau(u),\ \textit{for some}\ c, d \in \mathbb{Q}\},$$

for $i = 1, 2$.

Proof. Let $x \in A_1$. Then there exist $p, q, t, k \in \mathbb{Q}$ such that $x \in \tau(p,q) \cap S_1$ and $\tau(p,q) \diamond \tau(t,k) \subseteq \tau(r,s)$. Hence for every $y \in \tau(t,k) \cap S_2$, $x \diamond y \in \tau(r,s) \cap S_\diamond$. Thus, there exist sequences

$$\{p_n\}_{n\in\mathbb{N}}, \{q_n\}_{n\in\mathbb{N}}, \{t_n\}_{n\in\mathbb{N}}, \{k_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$$

such that $p_n, q_n \longrightarrow x$, $t_n, k_n \longrightarrow y$ and for every $n \in \mathbb{N}$,

$$\begin{gathered} p < p_n < p_{n+1} < x < q_{n+1} < q_n < q \text{ and} \\ t < t_n < t_{n+1} < y < k_{n+1} < k_n < k. \end{gathered}$$

Since $x \diamond y \in \tau(u)$, $p_n \diamond t_n \longrightarrow x \diamond y$ and $q_n \diamond k_n \longrightarrow x \diamond y$, we conclude that there exists $n \in \mathbb{N}$ such that

$$x \diamond y \in \tau(p_n, q_n) \diamond \tau(t_n, k_n) \subseteq \tau(u)$$

and $x \in \tau(p_n, q_n) \cap S_1$, which shows that $x \in B_1$. The case for $i = 2$ is proved similarly. □

Proposition 4.2. *Let S_1 and S_2 be subsets of $\mathbb{R}$. If $\alpha, \beta \in \mathcal{R}L$ such that $\alpha \blacktriangleleft S_1$ and $\beta \blacktriangleleft S_2$, then $\alpha \diamond \beta \blacktriangleleft S_1 \diamond S_2$, where $\diamond = +, \cdot, \wedge, \vee$.*

Proof. Let $S_\diamond = S_1 \diamond S_2$, $r, s \in \mathbb{Q}$ and $u \in \mathcal{L}(\mathbb{R})$. If $\tau(r,s) \cap S_\diamond \subseteq \tau(u) \cap S_\diamond$, then, by Lemma 4.1, we have

$$\begin{aligned} \alpha \diamond \beta(r,s) &= \bigvee \{\alpha(p,q) \wedge \beta(t,k) : < p, q > \diamond < t, k > \subseteq < r, s >\} \\ &\leq \bigvee \{\alpha(a,b) \wedge \beta(c,d) : < a, b > \diamond < c, d > \subseteq \tau(u)\} \\ &= \alpha \diamond \beta(u). \end{aligned}$$

Therefore, $\alpha \diamond \beta \blacktriangleleft S_\diamond$. □

Definition 4.3. Let λ be an infinite cardinal number and $\alpha \in \mathcal{R}L$. We say that α *has the pointfree λ-image* if there exists a subset $S \subseteq \mathbb{R}$ such that $|S| < \lambda$ and $\alpha \blacktriangleleft S$.

Corollary 4.4. *For every $\alpha \in \mathcal{R}L$ and $S \subseteq \mathbb{R}$, if $\lambda < \aleph_1$ (the first uncountable cardinal) and α has the pointfree λ-image, then $Im(\overline{\alpha})$ is countable.*

Proof. It follows from Proposition 3.13. □

Corollary 4.5. *Let $f \in C(X)$, then the following statements are equivalent*:

(1) *The frame map f_τ has the pointfree λ-image.*
(2) *$Im(f)$ is a subset of $\mathbb{R}$ with $|Im(f)| < \lambda$.*

Proof. It follows from Lemma 3.10 and Proposition 3.11. □

Remark 4.6. Let L be a frame such that $\Sigma L = \emptyset$. For every $\alpha \in \mathcal{R}L$, we have $\mathrm{Im}(\overline{\alpha}) = \emptyset$. By Proposition 3.2, countability of $\mathrm{Im}(\overline{\alpha})$ does not imply countability of pointfree image of α.

Definition 4.7. For every frame L, we put

$$\mathcal{R}_\lambda L = \{\alpha \in \mathcal{R}L : \alpha \text{ has the pointfree } \lambda\text{-image}\}.$$

For every $r \in \mathbb{R}$, if $S_r = \{r\}$, then $\mathbf{r} \blacktriangleleft S_r$. Therefore,

$$\{\mathbf{r} : r \in \mathbb{R}\} \subseteq \mathcal{R}_\lambda L.$$

Remark 4.8. If $\lambda > \aleph_1$, then $\mathcal{R}_\lambda L = \mathcal{R}L$, because for every $\alpha \in \mathcal{R}L$, $\alpha \blacktriangleleft \mathbb{R}$.

Corollary 4.9. *Let L be a frame. Then the set $\mathcal{R}_\lambda L$ is a sub-f-ring of $\mathcal{R}L$.*

Proof. By Proposition 4.2, it is evident. □

Remark 4.10. We have

$$\mathcal{R}_c L := \{\alpha \in \mathcal{R}L : \text{there exists a countable subset } S \text{ such that } \alpha \blacktriangleleft S\}$$

as the pointfree version of the ring $C_c(X)$, the subalgebra of $C(X)$, consisting of functions with countable image.

A study of z_c-ideals and prime ideals in the ring $\mathcal{R}_c L$ is done in [12].

Acknowledgement

The authors are grateful to Professor T. Dube for useful comments about this topic and thereby providing the stimulus for further thought, leading to a better understanding of this engaging subject. The authors also would like to thank the referee for helpful comments and suggestions on the manuscript especially for the begining of Section 3.

References

[1] Ball, R.N. and Hager, A.W., *On the localic Yoshida representation of an archimedean lattice ordered group with weak order unit*, J. Pure Appl. Algebra, 70 (1991), 17-43.

[2] Ball, R.N. and Walters-Wayland, J., *$C-$ and C^*- quotients on pointfree topology*, Dissertations Mathematicae (Rozprawy Mat), 412 Warszawa (2002), 62 pp.

[3] Banaschewski, B., *Pointfree topology and the spectra of f-rings*, Ordered algebraic structures, (Curacao 1995), Kluwer Acad. Publ. (1997), 123-148.

[4] Banaschewski, B., *The real numbers in pointfree topology*, Textos Mat. Sér. B 12, University of Coimbra, 1997.

[5] Banaschewski, B. and Gilmour, C.R.A., *Pseudocompactness and the cozero part of a frame*, Comment. Math. Univ. Carolin. 37 (1996), 577-587.

[6] Bhattacharjee, P., Knox, M.L., and McGovern, W.W., *The classical ring of quotients of $C_c(X)$*, Appl. Gen. Topol. 15(2) (2014), 147-154.

[7] Dube, T. and Ighedo, O., *On z-ideals of pointfree function rings*, Bull. Iran. Math. Soc. 40 (2014), 657-675.

[8] Dube, T., Iliadis, S., Van Mill, J., and Naidoo, I., *A Pseudocompact completely regular frame which is not spatial*, Order 31(1) (2014), 115-120.

[9] Ebrahimi, M.M. and Karimi Feizabadi, A., *Pointfree prime representation of real Riesz maps*, Algebra Universalis 54 (2005), 291-299.

[10] Ebrahimi, M.M. and Mahmoudi, M., "Frames", Technical Report, Department of Mathematics, Shahid Beheshti University, 1996.

[11] Estaji, A.A., Karimi Feizabadi, A., and Abedi, M., *Zero sets in pointfree topology and strongly z-ideals*, Bull. Iran. Math. Soc 41(5) (2015), 1071-1084.

[12] Estaji, A.A., Karimi Feizabadi, A., and Robat Sarpoushi, M., *z_c-Ideals and prime ideals in the ring $\mathcal{R}_cL$*, to appear in Filomat.

[13] Ghadermazi, M., Karamzadeh, O.A.S., and Namdari, M., *On the functionally countable subalgebra of $C(X)$*, Rend. Sem. Mat. Univ. Padova 129 (2013), 47-69.

[14] Gillman, L. and Jerison, M., "Rings of continuous functions", Springer-Verlag, 1976.

[15] Johnstone, P.T., "Stone spaces", Cambridge Univ. Press, 1982.

[16] Karamzadeh, O.A.S., Namdari, M., and Soltanpour, *On the locally functionally countable subalgebra of $C(X)$*, Appl. Gen. Topol. 16 (2015), 183-207.

[17] Karamzadeh, O.A.S. and Rostami, M., *On the intrinsic topology and some related ideals of $C(X)$*, Proc. Amer. Math. Soc. 93 (1985), 179-184.

[18] Namdari, M. and Veisi, A., *Rings of quotients of the subalgebra of $C(X)$ consisting of functions with countable image*, Inter. Math. Forum 7 (2012), 561-571.

[19] Picado, J. and Pultr, A., "Frames and Locales: topology without points", Birkhäuser/Springer, Basel AG, 2012.

Abolghasem Karimi Feizabadi, *Department of Mathematics, Gorgan Branch, Islamic Azad University, Gorgan, Iran.*

Email: akarimi@gorganiau.ac.ir; abolghasem.karimi.f@gmail.com

Ali Akbar Estaji, *Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran.*

Email: aaestaji@hsu.ac.ir, aa_estaji@yahoo.com

Maryam Robat Sarpoushi, *Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran.*

Email: m.sarpooshi@yahoo.com