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INEQUALITIES FOR SOME BASIC HYPERGEOMETRIC FUNCTIONS

Abstract. We establish conditions for the discrete versions of logarithmic concavity and convexity of the higher order regularized basic hypergeometric functions with respect to the simultaneous shift of all its parameters. For a particular case of Heine's basic hypergeometric function, we prove logarithmic concavity and convexity with respect to the bottom parameter. We, further, establish a linearization identity for the generalized Turánian formed by a particular case of Heine's basic hypergeometric function. Its $q = 1$ case also appears to be new.

Key words: *basic hypergeometric function, log-convexity, log-concavity, multiplicative concavity, generalized Turánian, q-hypergeometric identity*

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1. Introduction. We will use the standard definition of the q-shifted factorial [2, (1.2.15)]:

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \ \ n \in \mathbb{N}. \tag{1}$$

This definition works for any complex a and q; but in this paper we confine ourselves to the case $0 < q < 1$. Under this restriction, we can also define

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n,$$

where the limit can be shown to exist as a finite number for all complex a. The q-gamma function is given by [2, (1.10.1)], [4, (21.16)]

$$\Gamma_q(z) = (1-q)^{1-z}\frac{(q;q)_\infty}{(q^z;q)_\infty} \tag{2}$$

for $|q|<1$ and all complex z such that $q^{z+k}\neq 1$ for $k\in\mathbb{N}_0$.

In our recent paper [7], we studied logarithmic concavity and convexity for a generic series with respect to a parameter contained in the argument of q-shifted factorial or q-gamma function. Namely, we have considered the series of the form

$$f(\mu;x)=\sum\nolimits_{k=0}^{\infty}f_k\phi_k(\mu)x^k, \tag{3}$$

where $\phi_k(\mu)$ is one of the functions $(q^{\mu};q)_k$, $[(q^{\mu};q)_k]^{-1}$, $\Gamma_q(\mu+k)$ or $\Gamma_q^{-1}(\mu+k)$, and f_k is a (typically non-negative) real sequence independent of μ and x. Our results were given in terms of the sign of the generalized Turánian

$$\Delta_f(\alpha,\beta;x)=f(\mu+\alpha;x)f(\mu+\beta;x)-f(\mu;x)f(\mu+\alpha+\beta;x)=\sum_{m=0}^{\infty}\delta_m x^m \tag{4}$$

and its power series coefficients δ_m. Note that inequality $\Delta_f(\alpha,\beta;x)\geqslant 0$ for μ, $\mu+\alpha$, $\mu+\beta$ and $\mu+\alpha+\beta$ in some interval and $\alpha,\beta\geqslant 0$ is equivalent to the logarithmic concavity of $\mu\to f(\mu;x)$ in that interval. Natural examples of the series (3) come from the realm of the basic (or q-) hypergeometric functions: we discussed seven such examples in [7].

The purpose of this paper is to prove new inequalities for the basic hypergeometric functions. Most of them cannot be derived from the general theorems presented in [7]. The key results deal with the case when the functions $\phi_k(\mu)$ from (3) are given by

$$\phi_k(\mu)=\frac{\prod_{j=1}^{t}\Gamma_q(a_j+\mu)}{\prod_{j=1}^{s}\Gamma_q(b_j+\mu)}\frac{\prod_{j=1}^{t}(q^{a_j+\mu};q)_k}{\prod_{j=1}^{s}(q^{b_j+\mu};q)_k}$$

and, thus, go well beyond the nomenclature given above. Furthermore, we establish several related identities, some of them seem to be new. We remark here, that while identities for the basic hypergeometric functions form an immense subject leaving no hope for making any reasonable survey of, inequalities for and convexity-like properties of these functions seem to be a rather rare species. We are only aware of a handful of papers dealing with this topic. Besides our paper [7], there are the works of Baricz, Raghavendar, and Swaminathan [1], Mehrez [10], and Mehrez and Sitnik [11]. An inequality for the basic hypergeometric function of a different type was also established by Zhang in [16].

2. Results for the generalized q-hypergeometric series. Throughout the paper, we use the following short-hand notation for products and sums. Given a vector $\mathbf{a} = (a_1, \dots, a_m)$ and a scalar μ, write

$$\mathbf{a} + \mu = (a_1 + \mu, \dots, a_m + \mu), \quad q^{\mathbf{a}} = q^{a_1} \cdots q^{a_m},$$

$$(1 - q^{\mathbf{a}}) = (1 - q^{a_1}) \cdots (1 - q^{a_m}),$$

$$(\mathbf{a}; q)_n = (a_1, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,$$

$$\Gamma_q(\mathbf{a}) = \Gamma_q(a_1) \cdots \Gamma_q(a_m),$$

where n may take the value ∞. The generalized q-hypergeometric series is defined by [2, formula (1.2.22)]

$${}_t\phi_s\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| q; z\right) = {}_t\phi_s(\mathbf{a}; \mathbf{b}; q; z) = \sum_{n=0}^{\infty} \frac{(\mathbf{a}; q)_n}{(\mathbf{b}; q)_n (q; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t} z^n, \quad (5)$$

where $\mathbf{a} = (a_1, \dots, a_t)$, $\mathbf{b} = (b_1, \dots, b_s)$, $t \leqslant s + 1$, and the series converges for all z if $t \leqslant s$ and for $|z| < 1$ if $t = s + 1$ [2, section 1.2]. As the base q will remain fixed throughout the paper, we will omit q from the above notation, and write ${}_t\phi_s(\mathbf{a}; \mathbf{b}; z)$ for the right-hand side of (5).

In this section, we will deal with the function

$$g(\mu; x) = \frac{\Gamma_q(\mathbf{a} + \mu)}{\Gamma_q(\mathbf{b} + \mu)} {}_t\phi_s\left(\begin{matrix}q^{\mathbf{a}+\mu}\\ q^{\mathbf{b}+\mu}\end{matrix}\middle| (q - 1)^{1+s-t} x\right), \quad (6)$$

where ${}_t\phi_s$ is defined in (5). The main result will be formulated in terms of the generalized Turánian $\Delta_g(\alpha, \beta; x)$ defined in (4). Its power series coefficients are denoted by γ_m:

$$\Delta_g(\alpha, \beta; x) := g(\mu + \alpha; x) g(\mu + \beta; x) - g(\mu; x) g(\mu + \alpha + \beta; x) = \sum_{m=0}^{\infty} \gamma_m x^m. \quad (7)$$

We will give sufficient conditions for its positivity and negativity, as well as for its power series coefficients γ_m. To this end, we need the standard definition of the elementary symmetric polynomials [12, 3.F]:

$$e_k(c_1, \dots, c_r) = \sum_{1 \leqslant j_1 < j_2 \cdots < j_k \leqslant r} c_{j_1} c_{j_2} \cdots c_{j_k}, \quad k = 1, \dots, r,$$

and $e_0(c_1, \dots, c_r) = 1$. The following theorem is a q-analogue of [6, Theorem 3].

Theorem 1. *For a given $0 < q < 1$ and non-negative vectors $\mathbf{a}$ and $\mathbf{b}$, set $\mathbf{c} = (q^{-a_1} - 1, \dots, q^{-a_t} - 1)$ and $\mathbf{d} = (q^{-b_1} - 1, \dots, q^{-b_s} - 1)$. The following statements hold for $\Delta_g(\alpha, \beta; x)$ defined in (7):*

(a) If $s \leqslant t \leqslant s + 1$ and the conditions

$$\frac{e_t(\mathbf{c})}{e_s(\mathbf{d})} \leqslant \frac{e_{t-1}(\mathbf{c})}{e_{s-1}(\mathbf{d})} \leqslant \dots \leqslant \frac{e_{t-s+1}(\mathbf{c})}{e_1(\mathbf{d})} \leqslant e_{t-s}(\mathbf{c}) \tag{8}$$

are satisfied, then $\Delta_g(\alpha, \beta; x) \leqslant 0$ for $x \geqslant 0$, $\beta \geqslant 0$ and $\alpha \in \mathbb{N}$. Moreover, if $\alpha \leqslant \beta + 1$, then the power series coefficients γ_m of $x \to \Delta_g(\alpha, \beta; x)$ are non-positive. In particular, the inverse Turán type inequality $g(\mu + 1; x)^2 \leqslant g(\mu; x)g(\mu + 2; x)$ holds for $\mu \geqslant 0$ and fixed $x > 0$ in the domain of convergence.

(b) If $t \leqslant s$ and the conditions

$$\frac{e_s(\mathbf{d})}{e_t(\mathbf{c})} \leqslant \frac{e_{s-1}(\mathbf{d})}{e_{t-1}(\mathbf{c})} \leqslant \cdots \leqslant \frac{e_{s-t+1}(\mathbf{d})}{e_1(\mathbf{c})} \leqslant e_{s-t}(\mathbf{d}) \tag{9}$$

are satisfied, then $\Delta_g(\alpha, \beta; x) \geqslant 0$ for $x \geqslant 0$, $\beta \geqslant 0$ and $\alpha \in \mathbb{N}$. Moreover, if $\alpha \leqslant \beta + 1$, then the power series coefficients γ_m of $x \to \Delta_f(\alpha, \beta; x)$ are non-negative. In particular, the Turán type inequality $g(\mu + 1; x)^2 \geqslant g(\mu; x)g(\mu + 2; x)$ holds for $\mu \geqslant 0$ and any fixed $x > 0$.

In many situations complicated conditions (8) or (9) can be replaced by slightly stronger but simpler majorization conditions: for given vectors $\mathbf{c}, \mathbf{d} \in \mathbb{R}^t$, the weak supermajorization $\mathbf{d} \prec^W \mathbf{c}$ means that [12, Definition A.2]

$$0 < c_1 \leqslant c_2 \leqslant \cdots \leqslant c_t, \;\; 0 < d_1 \leqslant d_2 \leqslant \cdots \leqslant d_t,$$

$$\sum_{i=1}^{k} c_i \leqslant \sum_{i=1}^{k} d_i \;\; \text{for} \;\; k = 1, 2 \dots, t.$$

Denote the size of a vector $\mathbf{a}$ by $|\mathbf{a}|$. We have demonstrated the following result in [6, Lemma 4]:

Lemma 1. *Let $\mathbf{c} \in \mathbb{R}^t$, $\mathbf{d} \in \mathbb{R}^s$ be positive vectors, $t \geqslant s$, and suppose that there exists $\mathbf{c}' \subset \mathbf{c}$, $|\mathbf{c}'| = s$, such that $\mathbf{d} \prec^W \mathbf{c}'$. Then inequalities (8) hold. Similarly, inequalities (9) hold if $t \leqslant s$ and $\mathbf{c} \prec^W \mathbf{d}'$ for some subvector $\mathbf{d}' \subset \mathbf{d}$ of size t.*

Recall that a non-negative function f defined on an interval I is called completely monotonic there if it has derivatives of all orders and $(-1)^n f^{(n)}(x) \geqslant 0$ for $n \in \mathbb{N}_0$ and $x \in I$, see [15, Defintion 1.3]. We need the following simple lemma. See [7, Lemma 2] and [7, Remark, p. 340] for a proof.

Lemma 2. *Suppose $\phi(x) = \sum_{k\geqslant 0} \phi_k x^k$ converges for $|x| < R$ with $0 < R \leqslant \infty$ and $\phi_k \geqslant 0$. Then $x \to \phi(x)$ is multiplicatively convex and $y \to \phi(1/y)$ is completely monotonic (and, hence, is log-convex) on $(1/R, \infty)$; so there exists a non-negative measure τ supported on $[0, \infty)$ such that*

$$\phi(x) = \int\limits_{[0,\infty)} e^{-(1/x-1/R)t}\tau(dt).$$

If $R = \infty$, this measure is given by

$$\tau(dt) = \phi_0 \mathbf{1}_0 + \left(\sum\nolimits_{m=1}^{\infty} \frac{\phi_m t^m}{(m-1)!}\right) dt,$$

where $\mathbf{1}_0$ is the unit mass concentrated in the origin.

We, further, remark that for $R = \infty$ the function $\phi(1/y)$ satisfies the conditions of [9, Theorem 1.1] and, hence, enjoys all the properties stated in that theorem. Application of Lemma 2 leads immediately to the following corollary to Theorem 1.

Corollary 1. *Suppose that conditions of Theorem 1(b) are satisfied. Then the $x \mapsto \Delta_g(\alpha, \beta; x)$ defined in (7) is multiplicatively convex on $(0, \infty)$, while the function $y \mapsto \Delta_g(\alpha, \beta; 1/y)$ is completely monotonic (and, therefore, log-convex) on $(0, \infty)$, so that*

$$\Delta_g(\alpha, \beta; x) = \int\limits_{[0,\infty)} e^{-t/x}\tau(dt),$$

where the non-negative measure τ is given by

$$\tau(dt) = \gamma_0 \mathbf{1}_0 + \left(\sum\nolimits_{m=1}^{\infty} \frac{\gamma_m t^m}{(m-1)!}\right) dt.$$

Here $\mathbf{1}_0$ stands for the unit mass concentrated in the origin and γ_m are defined in (7).

If conditions of Theorem 1(a) are satisfied and $t = s$, similar statements hold for $-\Delta_g(\alpha, \beta; x)$.

We conclude this section with two examples of applications of Theorem 1.

Example 1. Suppose $0 < a_1 \leqslant a_2$, $0 < b_1 \leqslant b_2$ and

$$\begin{aligned} &q^{-b_1} \leqslant q^{-a_1} \;\Leftrightarrow\; b_1 \leqslant a_1, \\ &q^{-b_1} + q^{-b_2} \leqslant q^{-a_1} + q^{-a_2}. \end{aligned} \tag{10}$$

Then, by Lemma 1 and Theorem 1(b) (with $t = s = 2$), the generalized Turánian $\Delta_g(\alpha, \beta; x)$ with

$$g(\mu; x) = \frac{\Gamma_q(a_1+\mu)\Gamma_q(a_2+\mu)}{\Gamma_q(b_1+\mu)\Gamma_q(b_2+\mu)}\, {}_2\phi_2\left(\begin{matrix} q^{a_1+\mu}, q^{a_2+\mu} \\ q^{b_1+\mu}, q^{b_2+\mu} \end{matrix} \middle| (q-1)x\right)$$

is non-negative for $x \geqslant 0$, $\beta \geqslant 0$ and $\alpha \in \mathbb{N}$. Moreover, if $\alpha \leqslant \beta + 1$, then the power series coefficients of $x \to \Delta_g(\alpha, \beta; x)$ are all non-negative. If both inequalities (10) are reversed, then $x \to \Delta_g(\alpha, \beta; x)$ has non-positive power series coefficients for $x \geqslant 0$, $\beta \geqslant 0$ and $\mathbb{N} \ni \alpha \leqslant \beta + 1$ and $\Delta_g(\alpha, \beta; x) \leqslant 0$ holds without the restriction $\alpha \leqslant \beta + 1$.

Example 2. Suppose $0 < a_1 \leqslant a_2 \leqslant a_3$, $0 < b_1 \leqslant b_2$ and

$$\begin{aligned} &q^{-a_1} \leqslant q^{-b_1} \;\Leftrightarrow\; a_1 \leqslant b_1, \\ &q^{-a_1} + q^{-a_2} \leqslant q^{-b_1} + q^{-b_2}. \end{aligned}$$

Then, by Lemma 1 and Theorem 1(a) (with $t = 3$, $s = 2$), the generalized Turánian $\Delta_g(\alpha, \beta; x)$ with

$$g(\mu; x) = \frac{\Gamma_q(a_1+\mu)\Gamma_q(a_2+\mu)\Gamma_q(a_3+\mu)}{\Gamma_q(b_1+\mu)\Gamma_q(b_2+\mu)}\, {}_3\phi_2\left(\begin{matrix} q^{a_1+\mu}, q^{a_2+\mu}, q^{a_3+\mu} \\ q^{b_1+\mu}, q^{b_2+\mu} \end{matrix} \middle| x\right)$$

is non-positive for $x \geqslant 0$, $\beta \geqslant 0$ and $\alpha \in \mathbb{N}$. Moreover, if $\alpha \leqslant \beta + 1$, then the power series coefficients of $x \to \Delta_g(\alpha, \beta; x)$ are all non-positive.

3. Results for Heine's q-hypergeometric series. Before stating the results, we need some definitions. The (first) q-exponential is defined by [2, formula (18)]

$$e_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k}, \qquad |z| < 1. \tag{11}$$

Jackson gave the following two q-analogues of the Bessel functions [2, Exersice 1.24, p. 30]:

$$J_\alpha^{(1)}(y) = \frac{(y/2)^\alpha (q^{\alpha+1};q)_\infty}{(q;q)_\infty} {}_2\phi_1(0,0;q^{\alpha+1};-y^2/4), \quad |y|<2, \tag{12}$$

$$J_\alpha^{(2)}(y) = \frac{(y/2)^\alpha (q^{\alpha+1};q)_\infty}{(q;q)_\infty} {}_0\phi_1(-;q^{\alpha+1};-y^2 q^{\alpha+1}/4), \quad y \in \mathbb{C}. \tag{13}$$

The lemma below is a slight modification of the product formula due to Rahman [14, formula (1.20)]. We present it in this section as it may be of independent interest.

Lemma 3. *The following q-identity holds when both sides are well defined*

$$\begin{aligned} &{}_2\phi_1(0,0;q^\nu;z){}_2\phi_1(0,0;q^\eta;z) = \\ &\qquad = e_q(z){}_4\phi_3\left(\begin{matrix} q^{(\nu+\eta-1)/2}, q^{(\nu+\eta)/2}, -q^{(\nu+\eta-1)/2}, -q^{(\nu+\eta)/2} \\ q^\nu, q^\eta, q^{\nu+\eta-1} \end{matrix} \;\middle|\; z\right). \end{aligned} \tag{14}$$

Proof. The identity given in [14, formula (1.20)] can be written in terms of the second q-Bessel function $J_\alpha^{(2)}$ as follows:

$$\begin{aligned} J_\alpha^{(2)}(y)J_\beta^{(2)}(y)/(-y^2/4;q)_\infty &= \frac{y^{\alpha+\beta}}{(2(1-q))^{\alpha+\beta}\Gamma_q(\alpha+1)\Gamma_q(\beta+1)} \times \\ &\times {}_4\phi_3\left(\begin{matrix} q^{(\alpha+\beta+1)/2}, q^{(\alpha+\beta+2)/2}, -q^{(\alpha+\beta+1)/2}, -q^{(\alpha+\beta+2)/2} \\ q^{\alpha+1}, q^{\beta+1}, q^{\alpha+\beta+1} \end{matrix} \;\middle|\; -\frac{y^2}{4}\right). \end{aligned}$$

Using the definitions (12), (13) and the connection formula [2, Exercise 1.24]

$$J_\alpha^{(2)}(y) = (-y^2/4;q)_\infty J_\alpha^{(1)}(y)$$

we arrive at (14) on writing $\nu=\alpha+1$, $\eta=\beta+1$, $z=-y^4/4$ and applying the summation formula [14, (1.14)], [2, (1.3.15)] $e_q(z) = \dfrac{1}{(z;q)_\infty}$. □

Equating the coefficients at equal powers of z in (14), we immediately obtain

Corollary 2. *For arbitrary indeterminates ν, η and $m \in \mathbb{N}$, the identity*

$$\sum_{k=0}^{m} \frac{1}{(q^{\nu};q)_k(q^{\eta};q)_{m-k}(q;q)_k(q;q)_{m-k}} =$$
$$= \sum_{k=0}^{m} \frac{(q^{(\nu+\eta-1)/2},q^{(\nu+\eta)/2},-q^{(\nu+\eta-1)/2},-q^{(\nu+\eta)/2};q)_k}{(q^{\nu},q^{\eta},q^{\nu+\eta-1};q)_k(q;q)_k(q;q)_{m-k}} \tag{15}$$

holds true.

The following theorem can be established by applying [7, Theorem 4]. Here we will furnish a simple and independent direct proof based on Lemma 3.

Theorem 2. *Suppose*

$$f(\mu;x) = {}_2\phi_1(0,0;q^{\mu};x) = \sum_{n=0}^{\infty} \frac{x^n}{(q^{\mu};q)_n(q;q)_n}.$$

Then the power series coefficients of the generalized Turánian

$$\Delta_f(\alpha,\beta;x) = f(\mu+\alpha;x)f(\mu+\beta;x) - f(\mu;x)f(\mu+\alpha+\beta;x)$$

are negative for all $\mu,\alpha,\beta > 0$. *In particular,* $\mu \to f(\mu;x)$ *is log-convex on* $[0,\infty)$ *for* $0 < x < 1$.

Remark. *The function appearing in the above theorem can also be written in terms of the (first) modified q-Bessel function introduced by Ismail* [3, (2.5)] *and rediscovered by Olshanetskii and Rogov* [13, section 3.1], *as follows:*

$$ {}_2\phi_1(0,0;q^{\mu};x) = \frac{(1-q)^{\mu-1}\Gamma_q(\mu)}{x^{(\mu-1)/2}} I^{(1)}_{\mu-1}(2\sqrt{x};q). \tag{16}$$

The basic hypergeometric series treated in the theorem below is a particular case of the generic series considered in [7, Theorem 3]. However, the conclusion we draw here is stronger than that of [7, Theorem 3], where an additional restriction $\alpha \in \mathbb{N}$, $\alpha \leqslant \beta + 1$ is imposed.

Theorem 3. *Suppose*

$$\tilde{f}(\mu;x) = \frac{{}_2\phi_1(0,0;q^{\mu};x)}{\Gamma_q(\mu)} = \sum_{n=0}^{\infty} \frac{[x/(1-q)]^n}{\Gamma_q(\mu+n)(q;q)_n}.$$

Then the power series coefficients of the generalized Turánian

$$\Delta_{\tilde{f}}(\alpha,\beta;x) = \tilde{f}(\mu+\alpha;x)\tilde{f}(\mu+\beta;x) - \tilde{f}(\mu;x)\tilde{f}(\mu+\alpha+\beta;x)$$

are positive for $\mu, \alpha, \beta > 0$. *In particular, the function* $\mu \to \tilde{f}(\mu; x)$ *is log-concave on* $[0, \infty)$ *for* $0 < x < 1$.

In [7, Example 1], we announced, without a proof, that the modified q-Bessel function $I_\nu^{(1)}(y; q)$ is log-concave with respect to ν. In view of (16), Theorem 3 immediately leads to

Corollary 1. *For each fixed* $y \in (0, 2)$, *the function*

$$\nu \to I_\nu^{(1)}(y; q)$$

is log-concave on $(-1, \infty)$.

Finally, we present a linearization identity for the product difference of Heine's q-hypergeometric functions ${}_2\phi_1$, which seems to be new.

Theorem 4. *For* $\alpha \in \mathbb{N}$, *the identity*

$$\begin{aligned}(q^{\mu+\beta}; q)_\alpha {}_2\phi_1\left(\begin{matrix} q, 0 \\ q^{\mu+\alpha} \end{matrix} \middle| x\right) {}_2\phi_1\left(\begin{matrix} q, 0 \\ q^{\mu+\beta} \end{matrix} \middle| x\right) - \\ - (q^{\mu}; q)_\alpha {}_2\phi_1\left(\begin{matrix} q,0 \\ q^{\mu} \end{matrix} \middle| x\right) {}_2\phi_1\left(\begin{matrix} q,0 \\ q^{\mu+\alpha+\beta} \end{matrix} \middle| x\right) = \\ = \sum_{j=0}^{\alpha-1} \Bigg\{ (q^{\mu+1+j}; q)_{\alpha-1-j} (q^{\mu+\alpha+\beta-1-j}; q)_{1+j} {}_2\phi_1\left(\begin{matrix} q,0 \\ q^{\mu+1+j} \end{matrix} \middle| x\right) - \\ - (q^{\mu+j}; q)_{\alpha-j} (q^{\mu+\alpha+\beta-j}; q)_j {}_2\phi_1\left(\begin{matrix} q,0 \\ q^{\mu+\alpha+\beta-j} \end{matrix} \middle| x\right) \Bigg\}\end{aligned} \tag{17}$$

holds for any values of parameters for which both sides make sense.

Taking the limit $q \to 1$ in the above theorem, we arrive at an identity generalizing [8, Theorem 3]:

Corollary 1. *For* $\alpha \in \mathbb{N}$, *the identity*

$$\begin{aligned}(\mu+\beta)_\alpha {}_1F_1\left(\begin{matrix} 1 \\ \mu+\alpha \end{matrix} \middle| x\right) {}_1F_1\left(\begin{matrix} 1 \\ \mu+\beta \end{matrix} \middle| x\right) - \\ - (\mu)_\alpha {}_1F_1\left(\begin{matrix} 1 \\ \mu \end{matrix} \middle| x\right) {}_1F_1\left(\begin{matrix} 1 \\ \mu+\alpha+\beta \end{matrix} \middle| x\right) = \\ = \sum_{j=0}^{\alpha-1} \Bigg\{ (\mu+1+j)_{\alpha-1-j} (\mu+\alpha+\beta-1-j)_{1+j} {}_1F_1\left(\begin{matrix} 1 \\ \mu+1+j \end{matrix} \middle| x\right) -\end{aligned}$$

$$-(\mu+j)_{\alpha-j}(\mu+\alpha+\beta-j)_{j\,1}F_1\left(\begin{matrix}1\\ \mu+\alpha+\beta-j\end{matrix}\middle|x\right)\Bigg\} \qquad (18)$$

holds for values of parameters for which both sides make sense. Here $(\mu)_\alpha$ *is the Pochhammer symbol and* ${}_1F_1$ *stands for the Kummer (or confluent hypergeometric) function* [2, (1.2.16)].

Proof. Substitute $(q-1)x$ instead of x in (17) and divide both sides of by $(q-1)^\alpha$. Now, taking the limit as $q\to 1^-$, yields (18). □

4. Proofs. We will repeatedly use the identity

$$\frac{\Gamma_q(x+k)}{\Gamma_q(x)}=\frac{(q^x;q)_k}{(1-q)^k} \qquad (19)$$

that is valid for any real $x\neq 0,-1,-2,\ldots$ and $k\in\mathbb{N}_0$; this is easy to verify with the help of the definition (2).

In order to give a proof of Theorem 1, we need to do some preliminary work. Define the rational function

$$R_{t,\,s}(y)=\frac{\prod_{k=1}^{t}(c_k+y)}{\prod_{k=1}^{s}(d_k+y)} \qquad (20)$$

with non-negative parameters c_k and d_k. Elementary symmetric polynomials $e_k(\mathbf{c})=e_k(c_1,\ldots,c_r)$, $k=0,1,\ldots,r$, have been defined in (11). We need the following lemma.

Lemma 4. [6, Lemma 3] *If* $t\geqslant s$ *and conditions* (8) *are satisfied, then the function* $R_{t,\,s}(y)$ *is monotone increasing on* $(0,\infty)$*. If* $t\leqslant s$ *and conditions* (9) *are satisfied, then the function* $R_{t,\,s}(y)$ *is monotone decreasing on* $(0,\infty)$.

The two propositions below can be found in [5, Lemmas 2 and 3].

Lemma 5. *Let* f *be a non-negative-valued function defined on* $[0,\infty)$ *and* $\Delta_f(\alpha,\beta)=f(\mu+\alpha)f(\mu+\beta)-f(\mu)f(\mu+\beta+\alpha)\geqslant 0$ *for* $\alpha=1$ *and all* $\mu,\beta\geqslant 0$*. Then* $\Delta_f(\alpha,\beta)\geqslant 0$ *for all* $\alpha\in\mathbb{N}$ *and* $\mu,\beta\geqslant 0$*. If the inequality is strict in the hypothesis of the lemma, then it is also strict in the conclusion.*

Lemma 6. *Let* f *be defined by the formal series*

$$f(\mu;x)=\sum_{k=0}^{\infty}f_k(\mu)x^k,$$

where the coefficients $f_k(\mu)$ are continuous non-negative functions. Assume that $\Delta_f(1,\beta;x)$ defined in (4) has non-negative (non-positive) coefficients at all powers of x for $\mu,\beta \geqslant 0$. Then $\Delta_f(\alpha,\beta;x)$ has non-negative (non-positive) coefficients at powers of x for all $\alpha \in \mathbb{N}$, $\alpha \leqslant \beta+1$ and $\mu \geqslant 0$.

Proof of Theorem 1. According to Lemmas 5 and 6, it suffices to consider the case $\alpha = 1$, $\beta \geqslant 0$. We have:

$$\frac{\Gamma_q(\mathbf{b}+\mu)\Gamma_q(\mathbf{b}+\mu+\beta)}{\Gamma_q(\mathbf{a}+\mu)\Gamma_q(\mathbf{a}+\mu+\beta)(1-q)^{s-t}}\Delta_g(1,\beta;x) =$$

$$=\frac{(1-q^{\mathbf{a}+\mu})}{(1-q^{\mathbf{b}+\mu})}{}_t\phi_s\left(\begin{matrix}q^{\mathbf{a}+\mu+1}\\ q^{\mathbf{b}+\mu+1}\end{matrix}\middle|(q-1)^{1+s-t}x\right){}_t\phi_s\left(\begin{matrix}q^{\mathbf{a}+\mu+\beta}\\ q^{\mathbf{b}+\mu+\beta}\end{matrix}\middle|(q-1)^{1+s-t}x\right)-$$

$$-\frac{(1-q^{\mathbf{a}+\mu+\beta})}{(1-q^{\mathbf{b}+\mu+\beta})}{}_t\phi_s\left(\begin{matrix}q^{\mathbf{a}+\mu}\\ q^{\mathbf{b}+\mu}\end{matrix}\middle|(q-1)^{1+s-t}x\right){}_t\phi_s\left(\begin{matrix}q^{\mathbf{a}+\mu+\beta+1}\\ q^{\mathbf{b}+\mu+\beta+1}\end{matrix}\middle|(q-1)^{1+s-t}x\right)=$$

$$=\sum_{j=1}^{\infty}\frac{(q^{\mathbf{a}+\mu};q)_j}{(q^{\mathbf{b}+\mu};q)_j}\left[q^{\binom{j-1}{2}}\right]^{1+s-t}\frac{((1-q)^{1+s-t}x)^{j-1}}{(q;q)_{j-1}}\times$$

$$\times\sum_{j=0}^{\infty}\frac{(q^{\mathbf{a}+\mu+\beta};q)_j}{(q^{\mathbf{b}+\mu+\beta};q)_j}\left[q^{\binom{j}{2}}\right]^{1+s-t}\frac{((1-q)^{1+s-t}x)^{j}}{(q;q)_{j}}-$$

$$-\sum_{j=0}^{\infty}\frac{(q^{\mathbf{a}+\mu};q)_j}{(q^{\mathbf{b}+\mu};q)_j}\left[q^{\binom{j}{2}}\right]^{1+s-t}\frac{((1-q)^{1+s-t}x)^{j}}{(q;q)_{j}}\times$$

$$\times\sum_{j=1}^{\infty}\frac{(q^{\mathbf{a}+\mu+\beta};q)_j}{(q^{\mathbf{b}+\mu+\beta};q)_j}\left[q^{\binom{j-1}{2}}\right]^{1+s-t}\frac{((1-q)^{1+s-t}x)^{j-1}}{(q;q)_{j-1}}=$$

$$=\sum_{m=1}^{\infty}((1-q)^{1+s-t}x)^{m-1}\times$$

$$\times\sum_{k=0}^{m}\frac{(q^{\mathbf{a}+\mu};q)_k(q^{\mathbf{a}+\mu+\beta};q)_{m-k}(1-q^k)}{(q^{\mathbf{b}+\mu};q)_k(q^{\mathbf{b}+\mu+\beta};q)_{m-k}(q;q)_k(q;q)_{m-k}}\left[q^{\left(\binom{k-1}{2}+\binom{m-k}{2}\right)}\right]^{1+s-t}-$$

$$-\sum_{m=1}^{\infty}((1-q)^{1+s-t}x)^{m-1}\times$$

$$\times\sum_{k=0}^{m}\frac{(q^{\mathbf{a}+\mu};q)_k(q^{\mathbf{a}+\mu+\beta};q)_{m-k}(1-q^{m-k})}{(q^{\mathbf{b}+\mu};q)_k(q^{\mathbf{b}+\mu+\beta};q)_{m-k}(q;q)_k(q;q)_{m-k}}\left[q^{\left(\binom{k}{2}+\binom{m-k-1}{2}\right)}\right]^{1+s-t}=$$

$$= \sum_{m=1}^{\infty}((1-q)^{1+s-t}x)^{m-1}\sum_{k=0}^{m}\frac{(q^{\mathbf{a}+\mu};q)_k(q^{\mathbf{a}+\mu+\beta};q)_{m-k}}{(q^{\mathbf{b}+\mu};q)_k(q^{\mathbf{b}+\mu+\beta};q)_{m-k}(q;q)_k(q;q)_{m-k}}\times$$
$$\times\left\{(1-q^k)\left[q^{\left(\binom{k-1}{2}+\binom{m-k}{2}\right)}\right]^{1+s-t}-(1-q^{m-k})\left[q^{\left(\binom{k}{2}+\binom{m-k-1}{2}\right)}\right]^{1+s-t}\right\}.$$

Applying the Gauss pairing to the finite inner sum, the last expression becomes

$$\sum_{m=1}^{\infty}((1-q)^{1+s-t}x)^{m-1}\sum_{0\leqslant k\leqslant m/2}\frac{1}{(q;q)_k(q;q)_{m-k}}\times$$
$$\times\left\{(1-q^{m-k})\left[q^{\left(\binom{k}{2}+\binom{m-k-1}{2}\right)}\right]^{1+s-t}-(1-q^k)\left[q^{\left(\binom{k-1}{2}+\binom{m-k}{2}\right)}\right]^{1+s-t}\right\}\times$$
$$\times\left[\frac{(q^{\mathbf{a}+\mu};q)_{m-k}(q^{\mathbf{a}+\mu+\beta};q)_k}{(q^{\mathbf{b}+\mu};q)_{m-k}(q^{\mathbf{b}+\mu+\beta};q)_k}-\frac{(q^{\mathbf{a}+\mu};q)_k(q^{\mathbf{a}+\mu+\beta};q)_{m-k}}{(q^{\mathbf{b}+\mu};q)_k(q^{\mathbf{b}+\mu+\beta};q)_{m-k}}\right].$$

The apparent (for even m) unpaired middle term vanishes due to the factor

$$(1-q^{m-k})\left[q^{\left(\binom{k}{2}+\binom{m-k-1}{2}\right)}\right]^{1+s-t}-(1-q^k)\left[q^{\left(\binom{k-1}{2}+\binom{m-k}{2}\right)}\right]^{1+s-t},$$

which is equal to zero for $k=m-k$ and is non-negative for $k\leqslant m-k$, $1+s-t\geqslant 0$ because, clearly, $1-q^{m-k}\geqslant 1-q^k$ and

$$\left[q^{\left(\binom{k}{2}+\binom{m-k-1}{2}\right)}\right]^{1+s-t}\geqslant\left[q^{\left(\binom{k-1}{2}+\binom{m-k}{2}\right)}\right]^{1+s-t}$$
$$\Leftrightarrow\ q^{\left(\binom{k}{2}+\binom{m-k-1}{2}\right)}\geqslant q^{\left(\binom{k-1}{2}+\binom{m-k}{2}\right)}$$
$$\Leftrightarrow\ \binom{k}{2}+\binom{m-k-1}{2}\leqslant\binom{k-1}{2}+\binom{m-k}{2}\ \Leftrightarrow\ 2(2k-m)\leqslant 0.$$

It remains to show that the factor in brackets has a constant sign. Indeed, for $k\leqslant m-k$ we can rearrange this factor as follows:

$$\frac{(q^{\mathbf{a}+\mu};q)_{m-k}(q^{\mathbf{a}+\mu+\beta};q)_k}{(q^{\mathbf{b}+\mu};q)_{m-k}(q^{\mathbf{b}+\mu+\beta};q)_k}-\frac{(q^{\mathbf{a}+\mu};q)_k(q^{\mathbf{a}+\mu+\beta};q)_{m-k}}{(q^{\mathbf{b}+\mu};q)_k(q^{\mathbf{b}+\mu+\beta};q)_{m-k}}=$$
$$=\frac{(q^{\mathbf{a}+\mu};q)_k(q^{\mathbf{a}+\mu+\beta};q)_k}{(q^{\mathbf{b}+\mu};q)_k(q^{\mathbf{b}+\mu+\beta};q)_k}\left\{\prod\nolimits_{j=k}^{m-k-1}h(q^{\mu+j})-\prod\nolimits_{j=k}^{m-k-1}h(q^{\mu+\beta+j})\right\}.$$

Here the function $h(z)$ is defined for $0 < z < 1$ by

$$h(z) = \frac{(1-zq^{\mathbf{a}})}{(1-zq^{\mathbf{b}})} = \frac{\prod_{j=1}^{t}(1-zq^{a_j})}{\prod_{j=1}^{s}(1-zq^{b_j})} = \frac{\prod_{j=1}^{t} q^{a_j} \prod_{j=1}^{t}(q^{-a_j}-z)}{\prod_{j=1}^{s} q^{b_j} \prod_{j=1}^{s}(q^{-b_j}-z)} =$$

$$= \frac{\prod_{j=1}^{t} q^{a_j} \prod_{j=1}^{t}((q^{-a_j}-1)+(1-z))}{\prod_{j=1}^{s} q^{b_j} \prod_{j=1}^{s}((q^{-b_j}-1)+(1-z))} =$$

$$= \frac{\prod_{j=1}^{t} q^{a_j} \prod_{j=1}^{t}(c_j+y)}{\prod_{j=1}^{s} q^{b_j} \prod_{j=1}^{s}(d_j+y)} = \frac{q^{\mathbf{a}}}{q^{\mathbf{b}}} R_{t,\,s}(y),$$

where $c_j = q^{-a_j} - 1 > 0$, $d_j = q^{-b_j} - 1 > 0$, $y = 1 - z > 0$ and $R_{t,s}$ is given in (20). Note, that the function $y(z)$ is decreasing. The claims (a) and (b) now follow by Lemma 4 in view of $\mu, \beta \geqslant 0$. $\square$

Proof of Theorem 2. Indeed, if

$$\Delta_f(\alpha,\beta;x) = f(\mu+\alpha;x)f(\mu+\beta;x) - f(\mu;x)f(\mu+\alpha+\beta;x) = \sum_{m=0}^{\infty} \delta_m x^m,$$

then, by using the Cauchy product and Corollary 2,

$$\delta_m =$$

$$= \sum_{k=0}^{m} \frac{1}{(q;q)_k (q;q)_{m-k}} \left\{ \frac{1}{(q^{\mu+\alpha};q)_k (q^{\mu+\beta};q)_{m-k}} - \frac{1}{(q^{\mu};q)_k (q^{\mu+\alpha+\beta};q)_{m-k}} \right\} =$$

$$= \sum_{k=0}^{m} \frac{(q^{(2\mu+\alpha+\beta-1)/2}, q^{(2\mu+\alpha+\beta)/2}, -q^{(2\mu+\alpha+\beta-1)/2}, -q^{(2\mu+\alpha+\beta)/2};q)_k}{(q^{2\mu+\alpha+\beta-1};q)_k (q;q)_k (q;q)_{m-k}} \times$$

$$\times \left\{ \frac{1}{(q^{\mu+\alpha}, q^{\mu+\beta};q)_k} - \frac{1}{(q^{\mu}, q^{\mu+\alpha+\beta};q)_k} \right\}.$$

It remains to note that, according to definition (1), the inequality

$$\frac{1}{(q^{\mu+\alpha},q^{\mu+\beta};q)_k} < \frac{1}{(q^{\mu}, q^{\mu+\alpha+\beta};q)_k}$$

for $\mu, \alpha, \beta > 0$ is implied by

$$(1-q^{\mu+j})(1-q^{\mu+\alpha+\beta+j}) < (1-q^{\mu+\alpha+j})(1-q^{\mu+\beta+j}) \;\Leftrightarrow\; q^{\alpha}+q^{\beta} < 1+q^{\alpha+\beta}$$

for $j = 0, 1, \ldots, k-1$. The last inequality is straightforward. $\square$

Proof of Theorem 3. We have

$$\begin{aligned}\Delta_{\tilde f}(\alpha,\beta;x) &= \tilde f(\mu+\alpha;x)\tilde f(\mu+\beta;x)-\tilde f(\mu;x)\tilde f(\mu+\alpha+\beta;x)=\\ &=\sum_{m=0}^{\infty}[x/(1-q)]^m\tilde\delta_m.\end{aligned}$$

Using the Cauchy product and Corollary 2, the coefficients $\tilde\delta_m$ are computed as follows:

$$\tilde\delta_m=\sum_{k=0}^{m}\frac{1}{(q;q)_k(q;q)_{m-k}}\times$$

$$\times\left\{\frac{1}{\Gamma_q(\mu+\alpha+k)\Gamma_q(\mu+\beta+m-k)}-\frac{1}{\Gamma_q(\mu+k)\Gamma_q(\mu+\alpha+\beta+m-k)}\right\}=$$

$$=\sum_{k=0}^{m}\frac{(1-q)^m}{(q;q)_k(q;q)_{m-k}}\left\{\frac{[\Gamma_q(\mu+\alpha)\Gamma_q(\mu+\beta)]^{-1}}{(q^{\mu+\alpha};q)_k(q^{\mu+\beta};q)_{m-k}}-\frac{[\Gamma_q(\mu)\Gamma_q(\mu+\alpha+\beta)]^{-1}}{(q^{\mu};q)_k(q^{\mu+\alpha+\beta};q)_{m-k}}\right\}=$$

$$=\sum_{k=0}^{m}\frac{(1-q)^m(q^{(2\mu+\alpha+\beta-1)/2},q^{(2\mu+\alpha+\beta)/2},-q^{(2\mu+\alpha+\beta-1)/2},-q^{(2\mu+\alpha+\beta)/2};q)_k}{(q^{2\mu+\alpha+\beta-1};q)_k(q;q)_k(q;q)_{m-k}}\times$$

$$\times\left\{\frac{[\Gamma_q(\mu+\alpha)\Gamma_q(\mu+\beta)]^{-1}}{(q^{\mu+\alpha},q^{\mu+\beta};q)_k}-\frac{[\Gamma_q(\mu)\Gamma_q(\mu+\alpha+\beta)]^{-1}}{(q^{\mu},q^{\mu+\alpha+\beta};q)_k}\right\}=\frac{(1-q)^{2\mu+\alpha+\beta-2}}{[(q;q)_\infty]^2}\times$$

$$\times\sum_{k=0}^{m}\frac{(1-q)^m(q^{(2\mu+\alpha+\beta-1)/2},q^{(2\mu+\alpha+\beta)/2},-q^{(2\mu+\alpha+\beta-1)/2},-q^{(2\mu+\alpha+\beta)/2};q)_k}{(q^{2\mu+\alpha+\beta-1};q)_k(q;q)_k(q;q)_{m-k}}\times$$

$$\times\left\{\frac{(q^{\mu+\alpha},q^{\mu+\beta};q)_\infty}{(q^{\mu+\alpha},q^{\mu+\beta};q)_k}-\frac{(q^{\mu},q^{\mu+\alpha+\beta};q)_\infty}{(q^{\mu},q^{\mu+\alpha+\beta};q)_k}\right\},$$

where we applied (19) in the first equality and (15) in the second. The expression in braces on the right-hand side is immediately seen to reduce to

$$(q^{\mu+\alpha+k},q^{\mu+\beta+k};q)_\infty-(q^{\mu+k},q^{\mu+\alpha+\beta+k};q)_\infty.$$

To prove its positivity, it suffices to show that each factor in the first term is greater than that in the second. Indeed, for any $j\in\mathbb{N}_0$

$$(1-q^{\mu+\alpha+k+j})(1-q^{\mu+\beta+k+j})>(1-q^{\mu+k+j})(1-q^{\mu+\alpha+\beta+k+j}),$$

which is seen by expanding both sides and applying the elementary inequality $u+v<1+uv$ valid for $u,v\in(0,1)$. $\square$

The summation formula contained in the following lemma was established in [7, Lemma 8].

Lemma 7. *The following identity holds:*

$$\sum_{k=0}^{m}\left(\frac{1}{\Gamma_q(k+\mu+1)\Gamma_q(m-k+\mu+\beta)}-\frac{1}{\Gamma_q(k+\mu)\Gamma_q(m-k+\mu+\beta+1)}\right)=$$

$$=\frac{\left(q^{\mu+\beta};q\right)_{m+1}-\left(q^{\mu};q\right)_{m+1}}{\Gamma_q(\mu+m+1)\Gamma_q(\mu+\beta+m+1)(1-q)^{m+1}}. \tag{21}$$

Proof of Theorem 4. First, consider the case $\alpha=1$. Using (19), we have:

$$\frac{1}{\Gamma_q(\mu+1)\Gamma_q(\mu+\beta)}{}_2\phi_1\left(\left.\begin{matrix}q,0\\q^{\mu+1}\end{matrix}\right|x\right){}_2\phi_1\left(\left.\begin{matrix}q,0\\q^{\mu+\beta}\end{matrix}\right|x\right)-$$

$$-\frac{1}{\Gamma_q(\mu)\Gamma_q(\mu+1+\beta)}{}_2\phi_1\left(\left.\begin{matrix}q,0\\q^{\mu}\end{matrix}\right|x\right){}_2\phi_1\left(\left.\begin{matrix}q,0\\q^{\mu+1+\beta}\end{matrix}\right|x\right)=$$

$$=\frac{1}{\Gamma_q(\mu+1)\Gamma_q(\mu+\beta)}\sum_{k=0}^{\infty}\frac{x^k}{(q^{\mu+1};q)_k}\sum_{n=0}^{\infty}\frac{x^n}{(q^{\mu+\beta};q)_n}-$$

$$-\frac{1}{\Gamma_q(\mu)\Gamma_q(\mu+1+\beta)}\sum_{k=0}^{\infty}\frac{x^k}{(q^{\mu};q)_k}\sum_{n=0}^{\infty}\frac{x^n}{(q^{\mu+1+\beta};q)_n}=$$

$$=\sum_{m=0}^{\infty}\frac{x^m}{(1-q)^m}\sum_{k=0}^{m}\left\{\frac{(1-q)^m}{(q^{1+\mu};q)_k\,\Gamma_q(1+\mu)\,(q^{\mu+\beta};q)_{m-k}\,\Gamma_q(\mu+\beta)}-\right.$$

$$\left.-\frac{(1-q)^m}{(q^{\mu};q)_k\,\Gamma_q(\mu)\,(q^{\mu+1+\beta};q)_{m-k}\,\Gamma_q(\mu+1+\beta)}\right\}=$$

$$=\sum_{m=0}^{\infty}\frac{x^m}{(1-q)^m}\sum_{k=0}^{m}\left\{\frac{1}{\Gamma_q(k+\mu+1)\Gamma_q(m-k+\mu+\beta)}-\right.$$

$$\left.-\frac{1}{\Gamma_q(k+\mu)\Gamma_q(m-k+\mu+\beta+1)}\right\}.$$

We can now apply Lemma 7 to the expression in the right-hand side to get

$$\sum_{m=0}^{\infty}\frac{x^m}{(1-q)^{2m+1}}\left\{\frac{\left(q^{\mu+\beta};q\right)_{m+1}-\left(q^{\mu};q\right)_{m+1}}{\Gamma_q(\mu+m+1)\Gamma_q(\mu+\beta+m+1)}\right\}=$$

$$
\begin{aligned}
&= \frac{1}{\Gamma_q(\mu+1)\Gamma_q(\mu+\beta)}\sum_{m=0}^{\infty}\frac{(q;q)_m}{(q^{\mu+1},q;q)_m}x^m- \\
&- \frac{1}{\Gamma_q(\mu)\Gamma_q(\mu+1+\beta)}\sum_{m=0}^{\infty}\frac{(q;q)_m}{(q^{\mu+\beta+1},q;q)_m}x^m = \\
&= \frac{1}{\Gamma_q(\mu+1)\Gamma_q(\mu+\beta)}{}_2\phi_1\left(\begin{matrix}q,0\\q^{\mu+1}\end{matrix}\middle|x\right)- \\
&\qquad - \frac{1}{\Gamma_q(\mu)\Gamma_q(\mu+1+\beta)}{}_2\phi_1\left(\begin{matrix}q,0\\q^{\mu+\beta+1}\end{matrix}\middle|x\right),
\end{aligned}
$$

where (19) has been applied in the first equality. Hence, the theorem is proved for $\alpha = 1$.

Next, define the function

$$
h(\mu) := \frac{1}{\Gamma_q(\mu)}{}_2\phi_1\left(\begin{matrix}q,0\\q^{\mu}\end{matrix}\middle|x\right).
$$

In terms of this function, we compute the generalized Turánian as follows:

$$
\begin{aligned}
&h(\mu+\alpha)h(\mu+\beta) - h(\mu)h(\mu+\alpha+\beta) = \\
&\quad = [h(\mu+\alpha)h(\mu+\beta) - h(\mu+\alpha-1)h(\mu+\beta+1)]+ \\
&\quad + [h(\mu+\alpha-1)h(\mu+\beta+1) - h(\mu+\alpha-2)h(\mu+\beta+2)]+ \\
&\quad + \cdots + [h(\mu+2)h(\mu+\beta+\alpha-2) - h(\mu+1)h(\mu+\alpha+\beta-1)]+ \\
&\quad + [h(\mu+1)h(\mu+\beta+\alpha-1) - h(\mu)h(\mu+\alpha+\beta) =] \\
&= \sum_{j=0}^{\alpha-1}\left\{\frac{1}{\Gamma_q(\mu+1+j)\Gamma_q(\mu+\alpha+\beta-1-j)}{}_2\phi_1\left(\begin{matrix}q,0\\q^{\mu+1+j}\end{matrix}\middle|x\right)-\right. \\
&\qquad \left. - \frac{1}{\Gamma_q(\mu+j)\Gamma_q(\mu+\alpha+\beta-j)}{}_2\phi_1\left(\begin{matrix}q,0\\q^{\mu+\alpha+\beta-j}\end{matrix}\middle|x\right)\right\},
\end{aligned}
$$

where we applied the $\alpha = 1$ case established above to each bracketed expression. It remains to multiply throughout by $\Gamma_q(\mu+\alpha)\Gamma_q(\mu+\alpha+\beta)$ to complete the proof of the theorem. □

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