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Existence and stability analysis of nonlinear sequential coupled system of Caputo fractional differential equations with integral boundary conditions

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Abstract. In this paper we study existence and uniqueness of solutions for a coupled system consisting of fractional differential equations of Caputo type, subject to Riemann–Liouville fractional integral boundary conditions. The uniqueness of solutions is established by Banach contraction principle, while the existence of solutions is derived by Leray–Schauder's alternative. We also study the Hyers–Ulam stability of mentioned system. At the end, examples are also presented which illustrate our results.

1. Introduction

The subject of fractional calculus (calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, mainly due to its demonstrated applications in numerous fields of science and engineering. Historically, the first appearance of the concept of a fractional derivative was found in a letter by the famous mathematician Gottfried Leibniz (1646 – 1716) in 1695 to a French mathematician Guillaume de L'Hospital (1661 – 1704). Leibniz introduced the following symbol $\frac{d^n}{dt^n}f(t)$ which denotes the n^{th} order derivative of a function f with the hypothesis that $n \in \mathbb{N}$ and reported this to L'Hospital. So L'Hospital posed a question; what will be the derivative if $n = \frac{1}{2}$? Leibniz replied to him on September 13^{th}, 1695 and wrote: "This is an apparent paradox from which, one day useful consequences will be drawn" [8].

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In the last few decades, fractional differential equations have gained much attention due to extensive applications of these equations in the mathematical modeling of physical, engineering, biological phenomena and viscoelasticity etc, [13]. Several interesting and important results concerning the existence and uniqueness of solutions, stability properties of solutions, analytic and numerical methods of solutions for fractional differential equations can be found in the recent literature. Fractional-order operators are nonlocal in nature and take care of the hereditary properties of many phenomena and processes. Fractional calculus has also emerged as a powerful modeling tool for many real world problems, see [2, 6, 9, 17].

The study of coupled systems involving fractional differential equations is also important because these systems occur in various problems of applied nature. Coupled systems of fractional differential equations have also been investigated by many authors. Such systems appear naturally in many real world situations, for example, see[4]. Some recent results on the topic can be found in [5, 7, 19, 18, 23, 24].

Moreover, the theory of fractional order differential equations, involving different kinds of boundary conditions has been a field of interest in pure and applied sciences. Nonlocal conditions are used to describe certain features of applied mathematics and physics such as blood flow problems, cellular systems [1], chemical engineering, thermo-elasticity, underground water flow, population dynamics[10], and so on. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to [3, 21, 22, 26, 27, 29, 31, 30, 36, 37].

In 1940, Ulam asked: "Under what situation we can have a function(additive) near an approximate function(additive)"? see [12]. After twelve months, Hyers gave answer(partial) to Ulam [25] in the form of complete normed spaces. Since then, this concept of stability is known as Ulam–Hyers stability. Rassias [16] extended the mentioned concept of stability to general variables. For different approaches [14, 15, 20, 28, 34, 32, 33, 35].

In this paper, we study the nonlinear sequential coupled system of Caputo fractional differential equations with Riemann–Liouville fractional integral boundary conditions of the following form

$$\begin{cases} ({}^cD^q + k\,{}^cD^{q-1})x(t) = f(t, x(t), y(t)), \quad t \in [0,T],\ 2 < q \leq 3, \\ ({}^cD^p + k\,{}^cD^{p-1})y(t) = g(t, x(t), y(t)), \quad t \in [0,T],\ 2 < p \leq 3, \\ x(0) = 0, \qquad x(T) = \displaystyle\sum_{i=1}^{n} \alpha_i I^{\rho_i} y(\eta_i), \\ y(0) = 0, \qquad y(T) = \displaystyle\sum_{j=1}^{m} \beta_j I^{\gamma_j} y(\theta_j), \end{cases} \tag{1}$$

where ${}^cD^q$, ${}^cD^p$ denote the Caputo fractional derivatives of order p, q, I^{ρ_i}, I^{γ_j} are the Riemann–Liouville fractional integral of order $\rho_i, \gamma_j > 0$, $\eta_i, \theta_j \in (0,T)$, $k \in \mathbf{R}^+$, $f, g\colon [0,T] \times \mathbf{R}^2 \to \mathbf{R}$ and $\alpha_i, \beta_j \in \mathbf{R}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ are real constants such that

$$\sum_{i=1}^{n} \frac{\alpha_i \eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)} \cdot \sum_{j=1}^{m} \frac{\beta_j \theta_j^{\gamma_j+1}}{\Gamma(\gamma_j+2)} \neq T^2.$$

Here, we emphasize that the integral boundary conditions (1) can be understood in the sense that the value of the unknown function at an arbitrary position $\eta_i, \theta_j \in (0,T)$ is proportional to the Riemann–Liouville fractional integral of the unknown functions

$$\sum_{i=1}^{n} \alpha_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\rho_i-1}}{\Gamma(\rho_i)} y(s)ds, \quad \sum_{j=1}^{m} \beta_j \int_0^{\theta_j} \frac{(\theta_j - s)^{\gamma_j-1}}{\Gamma(\gamma_j)} y(s)ds,$$

where $\rho_i, \gamma_j > 0$. Further, for $\eta_i = \theta_j = 1$, the integral boundary conditions reduce to the usual form of a nonlocal integral conditions

$$\sum_{i=1}^{n} \alpha_i \int_0^{\eta_i} y(s)ds, \quad \sum_{j=1}^{m} \beta_j \int_0^{\theta_j} y(s)ds.$$

We show the existence of solutions for problem (1) by applying Leray–Schauder alternative criterion while uniqueness of solutions for (1) relies on Banach contraction mapping principle. The rest of the paper is organized as follows: In Section 2 we recall some preliminary concepts which we will need in the sequel. Section 3 contains the main results for problem (1). In Section 4, we present the Hyers–Ulam stability of problem (1).

2. Preliminaries and background materials

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs.

Definition 2.1
The Riemann–Liouville fractional integral of order $q > 0$ of a function $f\colon (0,\infty) \to \mathbf{R}$ is defined by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds,$$

provided the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2.2
The Caputo fractional derivative of order $q > 0$ for a function $f \in C^n[0,\infty)$ is defined by

$$^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)ds, \qquad n-1 < q < n,$$

where $n = [q]+1$, $[q]$ denotes the integer part of q and $\Gamma(.)$ is the Gamma function defined by

$$\Gamma(q) = \int_0^\infty e^{-s} t^{q-1} ds, \qquad q > 0.$$

Lemma 2.3 ([13])
Let $q > 0$ and $x \in C([0,T],\mathbf{R}) \cap L^1([0,T],\mathbf{R})$. Then the fractional differential equation

$$^cD^q x(t) = 0$$

has a unique solution

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{m-1} t^{m-1},$$

where $c_i \in \mathbf{R}$, $i = 1,2,\ldots,m-1$.

Lemma 2.4 ([13])
Let $q > 0$. Then for $x \in C([0,T],\mathbf{R}) \cap L^1([0,T],\mathbf{R})$ it holds

$$I^q\, {}^cD^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{m-1} t^{m-1},$$

where $c_i \in \mathbf{R}$, $i = 1,2,\ldots,m-1$, $m = -[-q]$.

Lemma 2.5
Given that $\phi, \psi \in C([0,T],\mathbf{R})$, the unique solution of the problem

$$\begin{cases} ({}^cD^q + k\,{}^cD^{q-1})x(t) = \phi(t), \quad t \in [0,T],\ 2 < q \le 3, \\ ({}^cD^p + k\,{}^cD^{p-1})y(t) = \psi(t), \quad t \in [0,T],\ 2 < p \le 3, \\ x(0) = 0, \qquad x(T) = \displaystyle\sum_{i=1}^{n} \alpha_i I^{\rho_i} y(\eta_i), \\ y(0) = 0, \qquad y(T) = \displaystyle\sum_{j=1}^{m} \beta_j I^{\gamma_j} y(\theta_j), \end{cases} \tag{2}$$

is

$$\begin{aligned} x(t) = \upsilon_1(t) + \frac{t}{\Omega}\Bigg[\sum_{i=1}^{n} \frac{\alpha_i \eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)} \Bigg(\sum_{j=1}^{m} \beta_j I^{\gamma_j} \int_0^{\theta j} e^{-k(\theta_j - s)} I^{q-1}\phi(\theta j) ds \\ - \int_0^T e^{-k(T-s)} I^{p-1}\psi(T) ds \Bigg) + T \sum_{i=1}^{n} \alpha_i I^{\rho_i} \int_0^{\eta_i} e^{-k(\eta_i - s)} I^{p-1}\psi(\eta_i) ds \\ - T \int_0^T e^{-k(T-s)} I^{q-1}\phi(T) ds \Bigg] + \int_0^t e^{-k(t-s)} I^{q-1}\phi(t) ds \end{aligned} \tag{3}$$

and

$$\begin{aligned} y(t) = \upsilon_2(t) + \frac{t}{\Omega}\Bigg[\sum_{j=1}^{m} \frac{\beta_j \theta_j^{\gamma_i+1}}{\Gamma(\gamma_j+2)} \Bigg(\sum_{i=1}^{n} \alpha_i I^{\rho_i} \int_0^{\eta_i} e^{-k(\eta_i - s)} I^{p-1}\psi(\eta_i) ds \\ - \int_0^T e^{-k(T-s)} I^{q-1}\phi(T) ds \Bigg) + T \sum_{j=1}^{m} \beta_j I^{\gamma_j} \int_0^{\theta j} e^{-k(\theta_j - s)} I^{q-1}\phi(\theta j) ds \\ - T \int_0^T e^{-k(T-s)} I^{q-1}\phi(T) ds \Bigg] + \int_0^t e^{-k(t-s)} I^{p-1}\psi(t) ds, \end{aligned} \tag{4}$$

where

$$\Omega := T^2 - \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)} \sum_{j=1}^{m} \frac{\beta_j \theta_j^{\gamma_j+1}}{\Gamma(\gamma_j+2)} \neq 0,$$

$$\begin{aligned}\upsilon_1(t) = A_0(e^{-kt}-1) + \frac{t}{\Omega}\Bigg[TB_0 \sum_{i=1}^{n} \alpha_i I^{\rho_i}(e^{-k\eta_i}-1) + \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)} \\ \times \left(A_0 \sum_{j=1}^{m} \beta_j I^{\gamma_j}(e^{-k\theta_j}-1) - B_0(e^{-kT}-1) \right) - TA_0(e^{-k\eta_i}-1) \Bigg]\end{aligned}$$

and

$$\begin{aligned}\upsilon_2(t) = B_0(e^{-kt}-1) + \frac{t}{\Omega}\Bigg[TA_0 \sum_{j=1}^{m} \beta_j I^{\gamma_j}(e^{-k\theta_j}-1) + \sum_{j=1}^{m} \frac{\beta_j \theta_j^{\gamma_j+1}}{\Gamma(\gamma_j+2)} \\ \times \left(B_0 \sum_{i=1}^{n} \alpha_i I^{\rho_i}(e^{-k\eta_i}-1) - A_0(e^{-kT}-1) \right) - TB_0(e^{-k\theta_j}-1) \Bigg].\end{aligned}$$

Proof. Writing the linear sequential fractional differential equations in (2) as

$$^cD^{q-1}(D+k)x(t) = \phi(t) \quad \text{and} \quad ^cD^{p-1}(D+k)y(t) = \psi(t)$$

and then applying the Riemann–Liouville integral operator I^{q-1} and I^{p-1} on both sides, followed by integration from 0 to t, we get

$$x(t) = A_0 e^{-kt} + A_1 + A_2 t + \int_0^t e^{-k(t-s)} I^{q-1}\phi(s))ds \tag{5}$$

and

$$y(t) = B_0 e^{-kt} + B_1 + B_2 t + \int_0^t e^{-k(t-s)} I^{p-1}\psi(s))ds, \tag{6}$$

where A_0, A_1, A_2, B_0, B_1 and B_2 are arbitrary constants and

$$I^{q-1}x(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} x(t)ds, \qquad I^{p-1}y(t) = \int_0^t \frac{(t-s)^{p-2}}{\Gamma(p-1)} y(t)ds,$$

for A_0, A_1, A_2, B_0, B_1 and B_2. The conditions $x(0) = 0$, $y(0) = 0$ imply that $A_1 = -A_0$, $B_1 = -B_0$. Taking the Riemann–Liouville fractional integral of order $\rho_i > 0$ for (5) and $\gamma_j > 0$ for (6) and using the property of the Riemann–Liouville fractional integral, we get

$$\begin{aligned}A_2 = \frac{1}{\Omega}\Bigg\{ \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)} \Bigg[\sum_{j=1}^{m} \beta_j I^{\gamma_j} \left(A_0(e^{-k\theta_j}-1) + \int_0^{\theta_j} e^{-k(\theta_j-s)} I^{q-1}\phi(\theta_j)ds \right) \\ - B_0(e^{-kT}-1) - \int_0^T e^{-k(T-s)} I^{p-1}\psi(T)ds \Bigg]\end{aligned}$$

$$+T\sum_{i=1}^{n}\alpha_i I^{\rho_i}\bigg(B_0(e^{-k\eta_i}-1)+\int_0^{\eta_i}e^{-k(\eta_i-s)}I^{p-1}\psi(\eta_i)ds\bigg)$$
$$-T\bigg(A_0(e^{-kT}-1)+\int_0^{T}e^{-k(T-s)}I^{q-1}\phi(T)ds\bigg)\bigg\}$$

and

$$B_2=\frac{1}{\Omega}\bigg\{\sum_{j=1}^{m}\frac{\beta_j\theta_j^{\gamma_j+1}}{\Gamma(\gamma_j+2)}\bigg[\sum_{i=1}^{n}\alpha_i I^{\rho_i}\bigg(B_0(e^{-k\eta_i}-1)+\int_0^{\eta_i}e^{-k(\eta_i-s)}I^{p-1}\psi(\eta_i)ds\bigg)$$
$$-A_0(e^{-kT}-1)-\int_0^{T}e^{-k(T-s)}I^{q-1}\phi(T)ds\bigg]$$
$$+T\sum_{j=1}^{m}\beta_j I^{\gamma_j}\bigg(A_0(e^{-k\theta_j}-1)+\int_0^{\theta_j}e^{-k(\theta_j-s)}I^{q-1}\phi(\theta_j)ds\bigg)$$
$$-T\bigg(B_0(e^{-kT}-1)+\int_0^{T}e^{-k(T-s)}I^{p-1}\psi(T)ds\bigg)\bigg\}.$$

Substituting the values of A_1, A_2, B_1 and B_2 in (5) and (6), we obtain the solutions (3) and (4).

3. Main results

Throughout this paper, for convenience, we use the following expression

$$I^w h(t,x(t),y(t))=\frac{1}{\Gamma(w)}\int_0^{v}(v-s)^{w-1}h(s,x(s),y(s))ds,$$

where $v\in\{t,T,\eta_i,\theta_j\}$, $w=\{p,q\}$ and $h=\{f,g\}$, $i=1,2,\dots,n$, $j=1,2,\dots,m$. Let $\mathcal{C}=C([0,T],\mathbf{R})$ denotes the Banach space of all continuous functions from $[0,T]$ to $\mathbf{R}$. Let us introduce the space $X=\{x(t):\ x(t)\in C^1([0,T])\}$ endowed with the norm $\|x\|=\sup\{|x(t)|,\ t\in[0,T]\}$. Obviously, $(X,\|.\|)$ is a Banach space. Also let $Y=\{y(t):\ y(t)\in C^1([0,T])\}$ be endowed with the norm $\|y\|=\sup\{|y(t)|,\ t\in[0,T]\}$. Clearly, the product space $(X\times Y,\|(x,y)\|)$ is a Banach space with the norm $\|(x,y)\|=\|x\|+\|y\|$. In view of Lemma 2.5, we define the operator $\mathcal{T}\colon X\times Y\to X\times Y$ by

$$\mathcal{T}(x,y)(t)=\begin{pmatrix}\mathcal{T}_1(x,y)(t)\\ \mathcal{T}_2(x,y)(t)\end{pmatrix},$$

where

$$\mathcal{T}_1(x,y)(t)=\upsilon_1(t)+\frac{t}{\Omega}\bigg[T\sum_{i=1}^{n}\alpha_i I^{\rho_i}\int_0^{\eta_i}e^{-k(\eta_i-s)}I^{p-1}g(s,x(s),y(s))(\eta_i)ds$$
$$+\sum_{i=1}^{n}\frac{\alpha_i\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\bigg(\sum_{j=1}^{m}\beta_j I^{\gamma_j}\int_0^{\theta j}e^{-k(\theta_j-s)}I^{q-1}f(x,x(s),y(s))(\theta j)ds$$

$$- \int_0^T e^{-k(T-s)} I^{p-1} g(s, x(s), y(s))(T) ds \Big)$$
$$- T \int_0^T e^{-k(T-s)} I^{q-1} f(x, x(s), y(s))(T) ds \Big]$$
$$+ \int_0^t e^{-k(t-s)} I^{q-1} f(x, x(s), y(s))(t) ds$$

and

$$\begin{aligned}\mathcal{T}_2(x,y)(t) = \upsilon_2(t) + \frac{t}{\Omega}\Big[& T \sum_{j=1}^{m} \beta_j I^{\gamma_j} \int_0^{\theta j} e^{-k(\theta_j - s)} I^{q-1} f(x, x(s), y(s))(\theta j) ds \\ & + \sum_{j=1}^{m} \frac{\beta_j \theta_j^{\gamma_i+1}}{\Gamma(\gamma_j+2)} \Big(\sum_{i=1}^{n} \alpha_i I^{\rho_i} \int_0^{\eta_i} e^{-k(\eta_i - s)} I^{p-1} g(x, x(s), y(s))(\eta_i) ds \\ & - \int_0^T e^{-k(T-s)} I^{q-1} f(x, x(s), y(s))(T) ds \Big) \\ & - T \int_0^T e^{-k(T-s)} I^{q-1} f(x, x(s), y(s))(T) ds \Big] \\ & + \int_0^t e^{-k(t-s)} I^{p-1} g(x, x(s), y(s))(t) ds.\end{aligned}$$

For the sake of convenience, we set

$$\begin{aligned}M_1 = \frac{T}{|\Omega|\Gamma(q)} \Big[& \sum_{i=1}^{n} \frac{|\alpha_i| \eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)} \sum_{j=1}^{m} |\beta_j| \theta_j^{q-1} I^{\gamma_j} \int_0^{\theta_j} e^{-k(\theta_j - s)} ds \\ & + T^q \int_0^T e^{-k(T-s)} ds \Big] + \frac{T^{q-1}}{\Gamma(q)} \int_0^T e^{-k(T-s)} ds,\end{aligned} \tag{7}$$

$$\begin{aligned}M_2 = \frac{T^2}{|\Omega|\Gamma(p)} \Big[& \sum_{i=1}^{n} |\alpha_i| \eta_i^{p-1} I^{\rho_i} \int_0^{\eta_i} e^{-k(\eta_i - s)} ds \\ & + T^{p-2} \sum_{i=1}^{n} \frac{|\alpha_i| \eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)} \int_0^T e^{-k(T-s)} ds \Big],\end{aligned} \tag{8}$$

$$\begin{aligned}M_3 = \frac{T^2}{|\Omega|\Gamma(q)} \Big[& \sum_{j=1}^{m} |\beta_j| \theta_j^{q-1} I^{\gamma_j} \int_0^{\theta_j} e^{-k(\theta_j - s)} ds \\ & + T^{q-2} \sum_{j=1}^{m} \frac{|\beta_i| \theta_j^{\gamma_j+1}}{\Gamma(\gamma_i+2)} \int_0^T e^{-k(T-s)} ds + T^{q-1} \int_0^T e^{-k(T-s)} ds \Big],\end{aligned} \tag{9}$$

$$\begin{aligned}M_4 = \frac{T}{|\Omega|\Gamma(p)} \Big[& \sum_{j=1}^{m} \frac{|\beta_j| \theta_j^{\gamma_j+1}}{\Gamma(\gamma_j+2)} \sum_{i=1}^{n} |\alpha_i| \eta_i^{p-1} I^{\rho_i} \int_0^{\eta_i} e^{-k(\eta_i - s)} ds \Big] \\ & + \frac{T^{p-1}}{\Gamma(p)} \int_0^T e^{-k(T-s)} ds\end{aligned} \tag{10}$$

and

$$M_0 = \min\{1-(M_1+M_3)k_1-(M_2+M_4)\lambda_1,\\ 1-(M_1+M_3)k_2-(M_2+M_4)\lambda_2\},$$

The first result is concerned with the existence and uniqueness of the solution for the problem (1) and is based on Banach contraction principle.

Theorem 3.1
Assume that $f, g\colon [0,T]\times\mathbf{R^2}\to\mathbf{R}$ are continuous functions and there exist constants m_i, n_i, $i=1,2$ such that for all $t\in[0,T]$ and $x_i, y_i\in\mathbf{R}$, $i=1,2$,

$$|f(t,x_2,y_2)-f(t,x_1,y_1)|\le m_1|x_2-x_1|+m_2|y_2-y_1|$$

and

$$|g(t,x_2,y_2)-g(t,x_1,y_1)|\le n_1|x_2-x_1|+n_2|y_2-y_1|.$$

In addition, assume that

$$(M_1+M_3)(m_1+m_2)+(M_2+M_4)(n_1+n_2)<1,$$

where M_i, $i=1,2,3,4$ are given by (7) – (10). Then the boundary value problem (1) has a unique solution.

Proof. Define $\sup_{t\in[0,1]} f(t,0,0)=N_1<\infty$ and $\sup_{t\in[0,1]} g(t,0,0)=N_2<\infty$ such that

$$r>\max\left\{\frac{M_4N_2+M_3N_1+|\upsilon_1(T)|}{1-(M_4n_1+M_3m_1+M_4n_2+M_3m_2)},\right.\\ \left.\frac{M_1N_1+M_2N_2+|\upsilon_2(T)|}{1-(M_2n_1+M_1m_1+M_2n_2+M_1m_2)}\right\}.$$

We show that $\mathcal{T}\mathbf{B_r}\subset\mathbf{B_r}$, where $\mathbf{B_r}=\{(x,y)\in X\times Y:\ \|(x,y)\|<r\}$.
For $(x,h)\in\mathbf{B_r}$, we have

$$\begin{aligned}&\big|\mathcal{T}_1(x,y)(t)\big|\\ &\quad=\sup_{t\in[0,T]}\Bigg\{\upsilon_1(t)+\frac{t}{\Omega}\Bigg[T\sum_{i=1}^{n}\alpha_i I^{\rho_i}\int_0^{\eta_i}e^{-k(\eta_i-s)}I^{p-1}g(s,x(s),y(s))(\eta_i)ds\\ &\qquad+\sum_{i=1}^{n}\frac{\alpha_i\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\Bigg(\sum_{j=1}^{m}\beta_j I^{\gamma_j}\int_0^{\theta j}e^{-k(\theta_j-s)}I^{q-1}f(x,x(s),y(s))(\theta j)ds\\ &\qquad-\int_0^{T}e^{-k(T-s)}I^{p-1}g(s,x(s),y(s))(T)ds\Bigg)\\ &\qquad-T\int_0^{T}e^{-k(T-s)}I^{q-1}f(x,x(s),y(s))(T)ds\Bigg]\\ &\qquad+\int_0^{t}e^{-k(t-s)}I^{q-1}f(x,x(s),y(s))(t)ds\Bigg\}\end{aligned}$$

$$
\begin{aligned}
&\le |\upsilon_1(T)| + \frac{T}{|\Omega|}\bigg[T\sum_{i=1}^{n}|\alpha_i| I^{\rho_i}\int_0^{\eta_i} e^{-k(\eta_i-s)} I^{p-1}(|g(s,x(s),y(s))\\
&\qquad\qquad - g(s,0,0)| + |g(s,0,0)|)(\eta_i)ds\\
&\quad + \sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\bigg(\sum_{j=1}^{m}|\beta_j| I^{\gamma_j}\int_0^{\theta j} e^{-k(\theta_j-s)} I^{q-1}(|f(x,x(s),y(s))\\
&\qquad\qquad - f(s,0,0)| + |g(s,0,0)|)(\theta j)ds\\
&\quad + \int_0^T e^{-k(T-s)} I^{p-1}(|g(s,x(s),y(s)) - g(s,0,0)| + g(s,0,0)|)(T)ds\bigg)\\
&\quad + T\int_0^T e^{-k(T-s)} I^{q-1}(|f(x,x(s),y(s) - f(s,0,0)| + |f(s,0,0)|))(T)ds\bigg]\\
&\quad + \int_0^T e^{-k(T-s)} I^{q-1}(|f(x,x(s) - f(s,0,0)| + |f(s,0,0)|), y(s))(T)ds\\
&\le |\upsilon_1(T)| + \frac{T}{|\Omega|}\bigg[T\sum_{i=1}^{n}|\alpha_i| I^{\rho_i}\int_0^{\eta_i} e^{-k(\eta_i-s)} I^{p-1}(n_1\|x\| + n_2\|y\| + N_2)(\eta_i)ds\\
&\quad + \sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\bigg(\sum_{j=1}^{m}|\beta_j| I^{\gamma_j}\int_0^{\theta j} e^{-k(\theta_j-s)} I^{q-1}(m_1\|x\|\\
&\qquad\qquad + m_2\|y\| + N_1)(\theta j)ds\\
&\quad + \int_0^T e^{-k(T-s)} I^{p-1}(n_1\|x\| + n_2\|y\| + N_2)(T)ds\bigg)\\
&\quad + T\int_0^T e^{-k(T-s)} I^{q-1}(m_1\|x\| + m_2\|y\| + N_1)(T)ds\bigg]\\
&\quad + \int_0^T e^{-k(T-s)} I^{q-1}(m_1\|x\| + m_2\|y\| + N_1), y(s))(T)ds\\
&= (m_1\|x\| + m_2\|y\| + N_1)\\
&\quad \cdot\bigg[\frac{T}{|\Omega|}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\sum_{j=1}^{m}|\beta_j| I^{\gamma_j}\int_0^{\theta j} e^{-k(\theta_j-s)} I^{q-1}(1)(\theta_j)ds\\
&\quad + \frac{T}{|\Omega|}T\int_0^T e^{-k(T-s)} I^{q-1}(1)(T)ds + \int_0^T e^{-k(T-s)} I^{q-1}(1)(T)ds\bigg]\\
&\quad + (n_1\|x\| + n_2\|y\| + N_2)\bigg[\frac{T}{|\Omega|}T\sum_{i=1}^{n}|\alpha_i| I^{\rho_i}\int_0^{\eta_i} e^{-k(\eta_i-s)} I^{p-1}(1)(\eta_i)ds\\
&\quad + \frac{T}{|\Omega|}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\int_0^T e^{-k(T-s)} I^{p-1}(1)(T)ds\bigg] + |\upsilon_1(T)|\\
&= (m_1\|x\| + m_2\|y\| + N_1)\\
&\quad \cdot\bigg[\frac{T}{|\Omega|\Gamma(q)}\bigg(\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\sum_{j=1}^{m}|\beta_j|\theta_j^{q-1} I^{\gamma_j}\int_0^{\theta_j} e^{-k(\theta_j-s)}ds
\end{aligned}
$$

$$
\begin{aligned}
&+ T^q \int_0^T e^{-k(T-s)}ds\Big) + \frac{T^{q-1}}{\Gamma(q)} \int_0^T e^{-k(T-s)}ds\Big] \\
&+ (n_1\|x\| + n_2\|y\| + N_2)\Big[\frac{T^2}{|\Omega|\Gamma(p)}\Big(\sum_{i=1}^{n} |\alpha_i|\eta_i^{p-1} I^{\rho_i} \int_0^{\eta_i} e^{-k(\eta_i-s)}ds \\
&+ T^{p-2} \sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)} \int_0^T e^{-k(T-s)}ds\Big)\Big] + |\upsilon_1(T)| \\
&= M_1(m_1\|x\| + m_2\|y\| + N_1) + M_2(n_1\|x\| + n_2\|y\| + N_2) + |\upsilon_1(T)| \\
&= (M_1 m_1 + M_2 n_1)\|x\|(+M_1 m_2 + M_2 n_2)\|y\| + M_1 N_1 + M_2 N_2 + |\upsilon_1(T)| \\
&\leq (M_1 m_1 + M_2 n_1 + M_1 m_2 + M_2 n_2)r + M_1 N_1 + M_2 N_2 + |\upsilon_1(T)| \leq r.
\end{aligned}
$$

In the same way, we can obtain that

$$
\begin{aligned}
&|\mathcal{T}_2(x,y)(t)| \\
&\leq (m_1\|x\| + m_2\|y\| + N_1)\Big[\frac{T^2}{|\Omega|\Gamma(q)}\Big(\sum_{j=1}^{m} |\beta_j|\theta_j^{q-1} I^{\gamma_j} \int_0^{\theta_j} e^{-k(\theta_j-s)}ds \\
&+ T^{q-2} \sum_{j=1}^{m} \frac{|\beta_i|\theta_j^{\gamma_j+1}}{\Gamma(\gamma_i+2)} \int_0^T e^{-k(T-s)}ds + T^{q-1}\int_0^T e^{-k(T-s)}ds\Big)\Big] \\
&+ (n_1\|x\| + n_2\|y\| + N_2) \\
&\cdot \Big[\frac{T}{|\Omega|\Gamma(p)}\Big(\sum_{j=1}^{m} \frac{|\beta_j|\theta_j^{\gamma_j+1}}{\Gamma(\gamma_j+2)} \sum_{i=1}^{n} |\alpha_i|\eta_i^{p-1} I^{\rho_i} \int_0^{\eta_i} e^{-k(\eta_i-s)}ds\Big) \\
&+ \frac{T^{p-1}}{\Gamma(p)} \int_0^T e^{-k(T-s)}ds\Big] + |\upsilon_2(T)| \\
&= (m_1\|x\| + m_2\|y\| + N_1)M_3 + (n_1\|x\| + n_2\|y\| + N_2)M_4 + |\upsilon_2(T)| \\
&= (M_4 n_1 + M_3 m_1)\|x\|(+M_4 n_2 + M_3 m_2)\|y\| + M_3 N_1 + M_4 N_2 + |\upsilon_2(T)| \\
&= (M_4 n_1 + M_3 m_1 + M_4 n_2 + M_3 m_2)r + M_3 N_1 + M_4 N_2 + |\upsilon_2(T)| \leq r.
\end{aligned}
$$

Consequently, $|\mathcal{T}(x,y)(t)| \leq r$. Now for $(x_2,y_2),(x_1,y_1) \in X \times Y$, and for any $t \in [0,T]$, we get

$$
\begin{aligned}
&|\mathcal{T}_1(x_2,y_2)(t) - \mathcal{T}_1(x_1,y_1)(t)| \\
&\leq \frac{T}{|\Omega|}\Big[T \sum_{i=1}^{n} |\alpha_i| I^{\rho_i} \int_0^{\eta_i} e^{-k(\eta_i-s)} I^{p-1}|g(s,x_2,y_2) - g(s,x_1,y_1)|(\eta_i)ds \\
&\quad + \sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\Big(\sum_{j=1}^{m} |\beta_j| I^{\gamma_j} \int_0^{\theta j} e^{-k(\theta_j-s)} I^{q-1}|f(s,x_2,y_2) \\
&\qquad - f(s,x_1,y_1)|(\theta j)ds \\
&\quad - \int_0^T e^{-k(T-s)} I^{p-1}|g(s,x_2,y_2) - g(s,x_1,y_1)|(T)ds\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- T\int_0^T e^{-k(T-s)}I^{q-1}|f(s,x_2,y_2)-f(s,x_1,y_1)|(T)ds\Bigg]\\
&\quad+\int_0^T e^{-k(T-s)}I^{q-1}|f(s,x_2,y_2)-f(s,x_1,y_1)|(T)ds\\
\leq&(m_1\|x_2-x_1\|+m_2\|y_2-y_1\|)\\
&\quad\cdot\Bigg[\frac{T}{|\Omega|\Gamma(q)}\Bigg(\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\sum_{j=1}^{m}|\beta_j|\theta_j^{q-1}I^{\gamma_j}\int_0^{\theta_j}e^{-k(\theta_j-s)}ds\\
&\quad+T^q\int_0^T e^{-k(T-s)}ds\Bigg)+\frac{T^{q-1}}{\Gamma(q)}\int_0^T e^{-k(T-s)}ds\Bigg]\\
&\quad+(n_1\|x_2-x_1\|+n_2\|y_2-y_1\|)\\
&\quad\cdot\Bigg[\frac{T^2}{|\Omega|\Gamma(p)}\Bigg(\sum_{i=1}^{n}|\alpha_i|\eta_i^{p-1}I^{\rho_i}\int_0^{\eta_i}e^{-k(\eta_i-s)}ds\\
&\quad+T^{p-2}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\int_0^T e^{-k(T-s)}ds\Bigg)\Bigg]\\
=&\,M_1(m_1\|x_2-x_1\|+m_2\|y_2-y_1\|)+M_2(n_1\|x_2-x_1\|+n_2\|y_2-y_1\|)\\
=&\,(M_1m_1+M_2n_1)\|x_2-x_1\|+(M_1m_2+M_2n_2)\|y_2-y_1\|.
\end{aligned}
$$

Consequently we obtain

$$
\begin{aligned}
&|\mathcal{T}_1(x_2,y_2)(t)-\mathcal{T}_1(x_1,y_1)(t)|\\
&\qquad\leq(M_1m_1+M_2n_1+M_1m_2+M_2n_2)[\|x_2-x_1\|+\|y_2-y_1\|].
\end{aligned}
\tag{11}
$$

Similarly,

$$
\begin{aligned}
&|\mathcal{T}_2(x_2,y_2)(t)-\mathcal{T}_2(x_1,y_1)(t)|\\
&\qquad\leq(M_3m_1+M_4n_1+M_3m_2+M_4n_2)[\|x_2-x_1\|+\|y_2-y_1\|].
\end{aligned}
\tag{12}
$$

It follows from (11) and (12) that

$$
\begin{aligned}
|\mathcal{T}(x_2,y_2)(t)-\mathcal{T}(x_1,y_1)(t)|\leq&[(M_1+M_3)(m_1+m_2)+(M_2+M_4)(n_1+n_2)]\\
&\cdot(\|x_2-x_1\|+\|y_2-y_1\|).
\end{aligned}
$$

Since $(M_1+M_3)(m_1+m_2)+(M_2+M_4)(n_1+n_2)<1$, therefore, $\mathcal{T}$ is a contraction operator. So, By Banach fixed point theorem, the operator $\mathcal{T}$ has a unique fixed point, which is the unique solution of problem (1). This completes the proof.

In the next result, we prove the existence of solutions for the problem (1) by applying Leray–Schauder alternative.

Lemma 3.2 (Leray–Schauder alternative, [11])
Let $F\colon E\to E$ be a completely continuous operator (i.e. a map that restricted to any bounded set in E is compact). Let

$$\aleph(F)=\{x\in E:\ x=\lambda F(x)\ \textit{for some}\ 0<\lambda<1\}.$$

Then either the set $\aleph(F)$ is unbounded, or F has at least one fixed point.

Theorem 3.3
Assume that there exist real constants k_i, $\lambda_i > 0$, $i = 1, 2$ and $k_0 > 0$, $\lambda_0 > 0$ such that for all $x_i \in \mathbf{R}$, $i = 1, 2$ we have

$$|f(t, x_1, x_2)| \leq k_0 + k_1|x_1| + k_2|x_2|$$
$$|g(t, x_1, x_2)| \leq \lambda_0 + \lambda_1|x_1| + \lambda_2|x_2|.$$

In addition, it is assumed that

$$[(M_1 + M_3)k_1 + (M_2 + M_4)\lambda_1] \leq 1 \quad \text{and} \quad [(M_1 + M_3)k_2 + (M_2 + M_4)\lambda_2] \leq 1,$$

where M_i for $i = 1, 2, 3, 4$ are given by (7) – (10). *Then there exists at least one solution for the boundary value problem* (1).

Proof. First we show that the operator $\mathcal{T} \colon X \times Y \to X \times Y$ is completely continuous. By continuity of functions f and g the operator $\mathcal{T}$ is continuous.

Let $\Theta \subset X \times Y$ be bounded. Then there exist positive constants L_1 and L_2 such that for all $(x, y) \in \Theta$,

$$|f(t, x(t), y(t))| \leq L_1 \quad \text{and} \quad |g(t, x(t), y(t))| \leq L_2.$$

Then for any $(x, y) \in \Theta$, we have

$$\begin{aligned}
&\|\mathcal{T}_1(x, y)(t)\| \\
&\quad \leq |\upsilon_1(T)| + \frac{T}{|\Omega|}\Bigg[T\sum_{i=1}^{n}\alpha_i I^{\rho_i}\int_0^{\eta_i} e^{-k(\eta_i - s)}I^{p-1}|g(s, x(s), y(s))|(\eta_i)ds \\
&\qquad + \sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i + 2)}\Bigg(\sum_{j=1}^{m}|\beta_j|I^{\gamma_j}\int_0^{\theta j} e^{-k(\theta_j - s)}I^{q-1}|f(x, x(s), y(s))|(\theta j)ds \\
&\qquad + \int_0^T e^{-k(T-s)}I^{p-1}|g(s, x(s), y(s))|(T)ds\Bigg) \\
&\qquad + T\int_0^T e^{-k(T-s)}I^{q-1}|f(x, x(s), y(s))|(T)ds\Bigg] \\
&\qquad + \int_0^T e^{-k(T-s)}I^{q-1}|f(x, x(s), y(s))|(T)ds \\
&\quad \leq |\upsilon_1(T)| + \Bigg[\frac{T}{|\Omega|\Gamma(q)}\Bigg(\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i + 2)}\sum_{j=1}^{m}|\beta_j|\theta_j^{q-1}I^{\gamma_j}\int_0^{\theta_j} e^{-k(\theta_j - s)}ds \\
&\qquad + T^q\int_0^T e^{-k(T-s)}ds\Bigg) + \frac{T^{q-1}}{\Gamma(q)}\int_0^T e^{-k(T-s)}ds\Bigg]L_1 \\
&\qquad + \frac{T^2}{|\Omega|\Gamma(p)}\Bigg[\sum_{i=1}^{n}|\alpha_i|\eta_i^{p-1}I^{\rho_i}\int_0^{\eta_i} e^{-k(\eta_i - s)}ds \\
&\qquad + T^{p-2}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i + 2)}\int_0^T e^{-k(T-s)}ds\Bigg]L_2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|\mathcal{T}_1(x,y)(t)\| \\
&\le |\upsilon_1(T)| + \Bigg[\frac{T}{|\Omega|\Gamma(q)}\Bigg(\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\sum_{j=1}^{m}|\beta_j|\theta_j^{q-1}I^{\gamma_j}\int_0^{\theta_j}e^{-k(\theta_j-s)}ds \\
&\quad + T^q\int_0^T e^{-k(T-s)}ds\Bigg) + \frac{T^{q-1}}{\Gamma(q)}\int_0^T e^{-k(T-s)}ds\Bigg]L_1 \\
&\quad + \frac{T^2}{|\Omega|\Gamma(p)}\Bigg[\sum_{i=1}^{n}|\alpha_i|\eta_i^{p-1}I^{\rho_i}\int_0^{\eta_i}e^{-k(\eta_i-s)}ds \\
&\quad + T^{p-2}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\int_0^T e^{-k(T-s)}ds\Bigg]L_2 \\
&= M_1L_1 + M_2L_2 + |\upsilon_1(T)|.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\|\mathcal{T}_2(x,y)(t)\| \\
&\le |\upsilon_2(T)| + \frac{T^2}{|\Omega|\Gamma(q)}\Bigg[\sum_{j=1}^{m}|\beta_j|\theta_j^{q-1}I^{\gamma_j}\int_0^{\theta_j}e^{-k(\theta_j-s)}ds \\
&\quad + T^{q-2}\sum_{j=1}^{m}\frac{|\beta_i|\theta_j^{\gamma_j+1}}{\Gamma(\gamma_i+2)}\int_0^T e^{-k(T-s)}ds + T^{q-1}\int_0^T e^{-k(T-s)}ds\Bigg]L_1 \\
&\quad + \Bigg[\frac{T}{|\Omega|\Gamma(p)}\Bigg(\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{\gamma_j+1}}{\Gamma(\gamma_j+2)}\sum_{i=1}^{n}|\alpha_i|\eta_i^{p-1}I^{\rho_i}\int_0^{\eta_i}e^{-k(\eta_i-s)}ds\Bigg) \\
&\quad + \frac{T^{p-1}}{\Gamma(p)}\int_0^T e^{-k(T-s)}ds\Bigg]L_2 \\
&= M_3L_1 + M_4L_2 + |\upsilon_2(T)|.
\end{aligned}$$

Thus, it follows from the above inequalities that the operator $\mathcal{T}$ is uniformly bounded.

Next, we show that $\mathcal{T}$ is equicontinuous. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&|\mathcal{T}_1(x(t_2),y(t_2)) - \mathcal{T}_1(x(t_1),y(t_1))| \\
&\le |\upsilon_1(t_2) - \upsilon_1(t_1)| \\
&\quad + \frac{|t_2-t_1|}{|\Omega|}\Bigg[T\sum_{i=1}^{n}|\alpha_i|I^{\rho_i}\int_0^{\eta_i}e^{-k(\eta_i-s)}I^{p-1}|g(s,x(s),y(s))|(\eta_i)ds \\
&\quad + \sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\Bigg(\sum_{j=1}^{m}|\beta_j|I^{\gamma_j}\int_0^{\theta j}e^{-k(\theta_j-s)}I^{q-1}|f(x,x(s),y(s))|(\theta j)ds \\
&\quad + \int_0^T e^{-k(T-s)}I^{p-1}|g(s,x(s),y(s))|(T)ds\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&- T\int_0^T e^{-k(T-s)} I^{q-1}|f(x,x(s),y(s))|(T)ds\Big] \\
&+ \int_0^{t_1} |(e^{-k(t_2-s)} - e^{-k(t_1-s)})| I^{q-1}|f(x,x(s),y(s))|(t_1)ds \\
&+ \int_{t_1}^{t_2} e^{-k(t_2-s)} I^{q-1}|f(x,x(s),y(s))|(t_2)ds \\
\leq\ & A_0(e^{-kt_1} - e^{-kt_2}) + \frac{|t_2-t_1|}{|\Omega|}\Big[TB_0\sum_{i=1}^{n}|\alpha_i| I^{\rho_i}|(e^{-k\eta_i}-1)| \\
&+ \sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\Big(A_0\sum_{j=1}^{m}|\beta_j| I^{\gamma_j}|(e^{-k\theta_j}-1)| \\
&+ B_0|(e^{-kT}-1)|\Big) + TA_0|(e^{-k\eta_i}-1)|\Big] \\
&+ \frac{|t_2-t_1|}{|\Omega|}\Big[T\sum_{i=1}^{n}|\alpha_i| I^{\rho_i}\int_0^{\eta_i} e^{-k(\eta_i-s)} I^{p-1}|g(s,x(s),y(s))|(\eta_i)ds \\
&+ \sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\Big(\sum_{j=1}^{m}|\beta_j| I^{\gamma_j}\int_0^{\theta j} e^{-k(\theta_j-s)} I^{q-1}|f(x,x(s),y(s))|(\theta j)ds \\
&+ \int_0^T e^{-k(T-s)} I^{p-1}|g(s,x(s),y(s))|(T)ds\Big) \\
&- T\int_0^T e^{-k(T-s)} I^{q-1}|f(x,x(s),y(s))|(T)ds\Big] \\
&+ \int_0^{t_1} |(e^{-k(t_2-s)} - e^{-k(t_1-s)})| I^{q-1}|f(x,x(s),y(s))|(t_1)ds \\
&+ \int_{t_1}^{t_2} e^{-k(t_2-s)} I^{q-1}|f(x,x(s),y(s))|(t_2)ds.
\end{aligned}
$$

Analogously, we can obtain

$$
\begin{aligned}
&|\mathcal{T}_2(x(t_2),y(t_2)) - \mathcal{T}_2(x(t_1),y(t_1))| \\
&\leq B_0(e^{-kt_1} - e^{-kt_2}) + \frac{|t_2-t_1|}{|\Omega|}\Big[\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{\gamma_j+1}}{\Gamma(\gamma_j+2)}\Big(B_0\sum_{i=1}^{n}|\alpha_i| I^{\rho_i}|(e^{-k\eta_i}-1)| \\
&+ A_0|(e^{-kT}-1)|\Big) + TA_0\sum_{j=1}^{m}|\beta_j| I^{\gamma_j}|(e^{-k\theta_j}-1)| + TB_0|(e^{-kT}-1)|\Big] \\
&+ \frac{|t_2-t_1|}{|\Omega|}\Big[\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{\gamma_i+1}}{\Gamma(\gamma_j+2)}\Big(\sum_{i=1}^{n}|\alpha_i| I^{\rho_i}\int_0^{\eta_i} e^{-k(\eta_i-s)} I^{p-1}|g(x,x(s),y(s))|(\eta_i)ds \\
&+ \int_0^T e^{-k(T-s)} I^{q-1}|f(x,x(s),y(s))|(T)ds\Big)
\end{aligned}
$$

$$+T\sum_{j=1}^{m}|\beta_j|I^{\gamma_j}\int_0^{\theta j}e^{-k(\theta_j-s)}I^{q-1}|f(x,x(s),y(s))|(\theta j)ds$$
$$+T\int_0^{T}e^{-k(T-s)}I^{q-1}|f(x,x(s),y(s))|(T)ds\Big]$$
$$+\int_0^{t_1}|(e^{-k(t_2-s)}-e^{-k(t_1-s)})|I^{p-1}|g(x,x(s),y(s))|(t_1)ds$$
$$+\int_{t_1}^{t_2}e^{-k(t_2-s)}I^{p-1}|g(x,x(s),y(s))|(t_2)ds.$$

Obviously, the right-hand sides of the above inequalities tend to zero independently of $f,g\in\mathbf{B_r}$ as $t_2-t_1\to 0$. Therefore, the operator $\mathcal{T}(x,y)$ is equicontinuous, and thus it is completely continuous.

Finally, it will be verified that the set

$$\aleph=\{(x,y)\in X\times Y:\ (x,y)=\lambda\mathcal{T}(x,y),\ 0\le\lambda\le 1\}$$

is bounded. Let $(x,y)\in\aleph$, then $(x,y)=\lambda\mathcal{T}(x,y)$. For any $t\in[0,T]$, we have $x(t)=\lambda\mathcal{T}_1(x,y)(t)$ and $y(t)=\lambda\mathcal{T}_2(x,y)(t)$. Then

$$|x(t)|\le|\upsilon_1(T)|+(k_0+k_1\|x\|+k_2\|y\|)$$
$$\cdot\Big[\frac{T}{|\Omega|\Gamma(q)}\Big(\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\sum_{j=1}^{m}|\beta_j|\theta_j^{q-1}I^{\gamma_j}\int_0^{\theta_j}e^{-k(\theta_j-s)}ds$$
$$+T^q\int_0^{T}e^{-k(T-s)}ds\Big)+\frac{T^{q-1}}{\Gamma(q)}\int_0^{T}e^{-k(T-s)}ds\Big]$$
$$+(\lambda_0+\lambda_1\|x\|+\lambda_2\|y\|)\Big[\frac{T^2}{|\Omega|\Gamma(p)}\Big(\sum_{i=1}^{n}|\alpha_i|\eta_i^{p-1}I^{\rho_i}\int_0^{\eta_i}e^{-k(\eta_i-s)}ds$$
$$+T^{p-2}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\int_0^{T}e^{-k(T-s)}ds\Big)\Big]$$

and

$$|y(t)|\le|\upsilon_2(T)|+(\lambda_0+\lambda_1\|x\|+\lambda_2\|y\|)$$
$$\cdot\Big[\frac{T}{|\Omega|\Gamma(p)}\Big(\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{\gamma_j+1}}{\Gamma(\gamma_j+2)}\sum_{i=1}^{n}|\alpha_i|\eta_i^{p-1}I^{\rho_i}\int_0^{\eta_i}e^{-k(\eta_i-s)}ds\Big)$$
$$+\frac{T^{p-1}}{\Gamma(p)}\int_0^{T}e^{-k(T-s)}ds\Big]$$
$$+(k_0+k_1\|x\|+k_2\|y\|)\Big[\frac{T^2}{|\Omega|\Gamma(q)}\Big(\sum_{j=1}^{m}|\beta_j|\theta_j^{q-1}I^{\gamma_j}\int_0^{\theta_j}e^{-k(\theta_j-s)}ds$$
$$+T^{q-2}\sum_{j=1}^{m}\frac{|\beta_i|\theta_j^{\gamma_j+1}}{\Gamma(\gamma_i+2)}\int_0^{T}e^{-k(T-s)}ds+T^{q-1}\int_0^{T}e^{-k(T-s)}ds\Big)\Big].$$

Hence we have

$$\|x(t)\| \le |\upsilon_1(T)| + (k_0 + k_1\|x\| + k_2\|y\|)M_1 + (\lambda_0 + \lambda_1\|x\| + \lambda_2\|y\|)M_2$$

and

$$\|y(t)\| \le |\upsilon_2(T)| + (\lambda_0 + (k_0 + k_1\|x\| + k_2\|y\|)M_3 + \lambda_1\|x\| + \lambda_2\|y\|)M_4,$$

which imply that

$$\begin{aligned}\|x(t)\| + \|y(t)\| = {} & |\upsilon_1(T)| + |\upsilon_2(T)| + (M_1 + M_3)k_0 + (M_2 + M_4)\lambda_0 \\ & + [(M_1 + M_3)k_1 + (M_2 + M_4)\lambda_1]\|x\| \\ & + [(M_1 + M_3)k_2 + (M_2 + M_4)\lambda_2]\|y\|.\end{aligned}$$

Consequently,

$$\|(x,y)\| \le \frac{(M_1 + M_3)k_0 + (M_2 + M_4)\lambda_0 + |\upsilon_1(T)| + |\upsilon_2(T)|}{M_0}$$

for any $t \in [0,T]$, where M_0 is defined by (5), which proves that $\aleph$ is bounded. Thus, by Lemma 3.2 the operator $\mathcal{T}$ has at least one fixed point. Hence, the boundary value problem (1) has at least one solution.

4. Hyers–Ulam stability of system (1)

This section is devoted to the investigation of Hyers–Ulam stability for our proposed system. Consider the following inequality:

$$\begin{cases} |({}^cD^q + k\,{}^cD^{q-1})x(t) - f(t,x(t),y(t))| \le \varepsilon_1, & t \in [0,T], \\ |({}^cD^p + k\,{}^cD^{p-1})y(t) - g(t,x(t),y(t))| \le \varepsilon_2, & t \in [0,T], \end{cases} \tag{13}$$

where $\varepsilon_1, \varepsilon_2$ are given two positive real numbers.

Definition 4.1
Problem (1) is Hyers–Ulam stable if there exist $M_i > 0$, $i = 1,2,3,4$ such that for given $\varepsilon_1, \varepsilon_2 > 0$ and for each solution $(x,y) \in C([0,T] \times \mathbf{R^2}, \mathbf{R})$ of inequality (13), there exists a solution $(x^*, y^*) \in C([0,T] \times \mathbf{R^2}, \mathbf{R})$ of problem (1) with

$$\begin{cases} |x(t) - x^*(t)| \le M_1\varepsilon_1 + M_2\varepsilon_2, & t \in [0,T], \\ |y(t) - y^*(t)| \le M_3\varepsilon_1 + M_4\varepsilon_2, & t \in [0,T]. \end{cases}$$

Remark 4.2
(x,y) is a solution of inequality (13) if there exist functions $Q_i \in C([0,T], \mathbf{R})$, $i = 1,2$ which depend upon x, y respectively, such that

- $|Q_1(t)| \le \varepsilon_1, \quad |Q_2(t)| \le \varepsilon_2, \quad t \in [0,T]]$.
- $\begin{cases} ({}^cD^q + k\,{}^cD^{q-1})x(t) = f(t,x(t),y(t)) + Q_1(t), & t \in [0,T], \\ ({}^cD^p + k\,{}^cD^{p-1})y(t) + g(t,x(t),y(t)) + Q_2(t), & t \in [0,T]. \end{cases}$

Remark 4.3

If (x, y) represent a solution of inequality (13), then (x, y) is a solution of following inequality

$$\begin{cases} |x(t) - x^*(t)| \leq M_1\varepsilon_1 + M_2\varepsilon_2, & t \in [0, T], \\ |y(t) - y^*(t)| \leq M_3\varepsilon_1 + M_4\varepsilon_2, & t \in [0, T]. \end{cases}$$

As from Remark 4.2, we have

$$\begin{cases} ({}^cD^q + k\,{}^cD^{q-1})x(t) = f(t, x(t), y(t)) + Q_1(t), & t \in [0, T], \\ ({}^cD^p + k\,{}^cD^{p-1})y(t) = g(t, x(t), y(t)) + Q_2(t), & t \in [0, T]. \end{cases}$$

With the help of Definition 4.1 and Remark 4.2 we verified Remark 4.3, in the following lines

$$\begin{aligned}
&|x(t) - \upsilon_1(t)| \\
&\quad = \Bigg| \frac{t}{\Omega} \Bigg[T \sum_{i=1}^{n} \alpha_i I^{\rho_i} \int_0^{\eta_i} e^{-k(\eta_i - s)} I^{p-1} g(s, x(s), y(s))(\eta_i) ds \\
&\qquad + \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{\rho_i+1}}{\Gamma(\rho_i + 2)} \Bigg(\sum_{j=1}^{m} \beta_j I^{\gamma_j} \int_0^{\theta j} e^{-k(\theta_j - s)} I^{q-1} f(s, x(s), y(s))(\theta j) ds \\
&\qquad - \int_0^{T} e^{-k(T-s)} I^{p-1} g(s, x(s), y(s))(T) ds \Bigg) \\
&\qquad - T \int_0^{T} e^{-k(T-s)} I^{q-1} f(s, x(s), y(s))(T) ds \Bigg] \\
&\qquad - \int_0^{t} e^{-k(t-s)} I^{q-1} f(s, x(s), y(s))(t) ds \Bigg] \Bigg| \\
&\quad \leq \frac{T}{|\Omega|} \Bigg[T \sum_{i=1}^{n} |\alpha_i| I^{\rho_i} \int_0^{\eta_i} e^{-k(\eta_i - s)} I^{p-1} |Q_2(t)|(\eta_i) ds \\
&\qquad + \sum_{i=1}^{n} \frac{|\alpha_i| \eta_i^{\rho_i+1}}{\Gamma(\rho_i + 2)} \Bigg(\sum_{j=1}^{m} |\beta_j| I^{\gamma_j} \int_0^{\theta j} e^{-k(\theta_j - s)} I^{q-1} |Q_1(t)|(\theta j) ds \qquad (14) \\
&\qquad + \int_0^{T} e^{-k(T-s)} I^{p-1} |Q_2(t)|(T) ds \Bigg) + T \int_0^{T} e^{-k(T-s)} I^{q-1} |Q_1(t)|(T) ds \Bigg] \\
&\qquad + \int_0^{T} e^{-k(T-s)} I^{q-1} |Q_1(t)|(T) ds \Bigg| \\
&\quad \leq \varepsilon_1 \Bigg[\frac{T}{|\Omega|\Gamma(q)} \Bigg(\sum_{i=1}^{n} \frac{|\alpha_i| \eta_i^{\rho_i+1}}{\Gamma(\rho_i + 2)} \sum_{j=1}^{m} |\beta_j| \theta_j^{q-1} I^{\gamma_j} \int_0^{\theta_j} e^{-k(\theta_j - s)} ds \\
&\qquad + T^q \int_0^{T} e^{-k(T-s)} ds \Bigg) + \frac{T^{q-1}}{\Gamma(q)} \int_0^{T} e^{-k(T-s)} ds \Bigg] \\
&\qquad + \varepsilon_2 \Bigg[\frac{T^2}{|\Omega|\Gamma(p)} \Bigg(\sum_{i=1}^{n} |\alpha_i| \eta_i^{p-1} I^{\rho_i} \int_0^{\eta_i} e^{-k(\eta_i - s)} ds
\end{aligned}$$

$$+ T^{p-2}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\int_0^T e^{-k(T-s)}ds\Big)\Big]$$
$$= M_1\varepsilon_1 + M_2\varepsilon_2.$$

By the same method we can obtain that

$$|y(t) - y^*(t)| \le M_3\varepsilon_1 + M_4\varepsilon_2, \tag{15}$$

where M_i, $i = 1, 2, 3, 4$ are given by (7)-(10). Hence Remark 4.3 is verified, with the help of (14) and (15). Thus the nonlinear sequential coupled system of Caputo fractional differential equations is Hyers–Ulam stable and consequently, the system (1) is Hyers–Ulam stable.

5. Examples

Example 5.1
Consider the following system of coupled Caputo fractional differential equations with Riemann–Liouville fractional integral boundary conditions

$$\begin{cases} ({}^cD^{5/2}+2{}^cD^{3/2})x(t) \\ \qquad = \dfrac{e^{t^2}}{(t+7)^2}\dfrac{|x(t)|}{(1+|x(t)|)} + \dfrac{\sin^2(2\pi t)}{(3e^t+1)^2}\dfrac{|y(t)|}{1+|y(t)|} + \dfrac{1}{3}, \quad t\in[0,2], \\ ({}^cD^{7/3}+2{}^cD^{4/3})y(t) \\ \qquad = \dfrac{1}{24}\cos x(t) + \dfrac{1}{(t+6)^2}\sin y(t) + 1, \quad t\in[0,2], \\ x(0)=0, \qquad x(2) = \dfrac{3}{2}I^{1/3}y(2/3) + \sqrt{2}I^{3/7}y(4/3), \\ y(0)=0, \qquad y(2) = \sqrt{3}I^{1/4}x(1/2) + \dfrac{1}{2}I^{4/7}x(1) + 2I^{7/10}x(3/2). \end{cases} \tag{16}$$

Here $q = 5/2$, $p = 7/3$, $n = 2$, $m = 3$, $T = 2$, $\alpha_1 = 3/2$, $\alpha_2 = \sqrt{2}$, $\beta_1 = \sqrt{3}$, $\beta_2 = 1/2$, $\beta_3 = 2$, $\rho_1 = 1/3$, $\rho_2 = 3/7$, $\gamma_1 = 1/4$, $\gamma_2 = 4/7$, $\gamma_3 = 7/10$, $\eta_1 = 2/3$, $\eta_2 = 4/3$, $\theta_1 = 1/2$, $\theta_2 = 1$, $\theta_3 = 3/2$ and

$$f(t,x,y) = (e^{t^2}|x|)/(t+7)^2)(1+|x|) + (\sin(2\pi t)|y|)/(3e^t+1)^2(1+|y|) + 1/3$$
$$g(t,x,y) = (\cos x/25) + (\sin y)/((t+6)^2) + 1.$$

Since
$$|f(t,x_2,y_2) - f(t,x_1,y_1)| \le (1/49)|x_2-x_1| + (1/16)|y_2-y_1|$$
and
$$|g(t,x_2,y_2) - g(t,x_1,y_1)| \le (1/25)|x_2-x_1| + (1/36)|y_2-y_1|$$
and we can find

$$\Omega = T^2 - \sum_{i=1}^{n}\frac{\alpha_i\eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)}\sum_{j=1}^{m}\frac{\beta_j\theta_j^{\gamma_j+1}}{\Gamma(\gamma_j+2)} \simeq -8.442 \ne 0.$$

With the given values, it is found that $m_1 = 1/49$, $m_2 = 1/16$, $n_1 = 1/25$, $n_2 = 1/36$, $M_1 \simeq 3.358$, $M_2 \simeq 1.795$, $M_3 \simeq 3.303$, $M_4 \simeq 2.331$, and

$$(M_1 + M_3)(m_1 + m_2) + (M_2 + M_4)((n_1 + n_2) \simeq 0.825 < 1.$$

Thus all the conditions of Theorem 3.1 are satisfied. Therefore, by the conclusion of Theorem 3.1, the problem (16) has a unique solution on $[0, 2]$. Further, it is also straightforward to prove the problem (16) is Hyers–Ulam stable.

Example 5.2
Consider the following coupled fractional integral boundary conditions

$$\begin{cases} ({}^cD^{9/4} + \frac{1}{2}{}^cD^{5/4})x(t) \\ \qquad = \dfrac{|x(t)|}{(t+3)^4(1+|x(t)|)} + \dfrac{1}{63(1+y^2(t))} + \dfrac{1}{18}, \quad t \in [0,4], \\ ({}^cD^{12/5} + \frac{1}{2}{}^cD^{7/5})y(t) \\ \qquad = \dfrac{\sin(2\pi x(t))}{172\pi} + \dfrac{1}{10\sqrt{t+4}} + \dfrac{|y(t)|}{60(1+|y(t)|)}, \quad t \in [0,4], \\ x(0) = 0, \qquad x(4) = \dfrac{3}{2} I^{7/10} y(1/2) + \sqrt{5} I^{3/7} y(2/3), \\ y(0) = 0, \qquad y(4) = \sqrt{7} I^{3/4} x(5/4) + \dfrac{11}{7} I^{9/8} x(2/3). \end{cases} \tag{17}$$

Here $q = 9/4$, $p = 12/5$, $n = 2$, $m = 2$, $T = 4$, $\alpha_1 = 3/2$, $\alpha_2 = 27$, $\beta_1 = \sqrt{7}$, $\beta_2 = 11/7$, $\rho_1 = 7/10$, $\rho_2 = 3/7$, $\gamma_1 = 3/4$, $\gamma_2 = 9/8$, $\eta_1 = 1/2$, $\eta_2 = 2/3$, $\theta_1 = 5/4$, $\theta_2 = 2/3$ and

$$f(t,x,y) = |x(t)|/(t+3)^4(1+|x(t)|) + 1/63(1+y^2(t)) + 1/18$$
$$g(t,x,y) = sin(2\pi x(t))/172\pi + 1/10\sqrt{t+4} + |y(t)|/60(1+|y(t)|).$$

Since

$$|f(t,x_2,y_2) - f(t,x_1,y_1)| \le (1/81)|x_2 - x_1| + (1/63)|y_2 - y_1|$$

and

$$|g(t,x_2,y_2) - g(t,x_1,y_1)| \le (1/86)|x_2 - x_1| + (1/60)|y_2 - y_1|$$

and we can find

$$\Omega = T^2 - \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{\rho_i+1}}{\Gamma(\rho_i+2)} \sum_{j=1}^{m} \frac{\beta_j \theta_j^{\gamma_j+1}}{\Gamma(\gamma_j+2)} \simeq -28.38879 \neq 0.$$

With the given values, it is found that $m_1 = 1/81$, $m_2 = 1/63$, $n_1 = 1/86$, $n_2 = 1/60$, $M_1 \simeq 14.38186$, $M_2 \simeq 5.15674$, $M_3 \simeq 7.46746$, $M_4 \simeq 3.61879$, and

$$(M_1 + M_3)(m_1 + m_2) + (M_2 + M_4)((n_1 + n_2) \simeq 0.86485 < 1.$$

Thus all the conditions of Theorem 3.1 are satisfied. Therefore, by the conclusion of Theorem 3.1, the problem (17) has a unique solution on $[0, 4]$. Further, it is also straightforward to prove the problem (17) is Hyers–Ulam stable.

6. Conclusion

We discussed the existence and stability of nonlinear sequential coupled system of Caputo fractional differential equations with Riemann–Liouville fractional integral boundary conditions. The existence and uniqueness of solutions is relies on Banach contraction principle, while the existence of solutions is established by applying Leray–Schauder's alternative. Finally examples are presented to illustrate the main results.

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