Existence and Uniqueness of Common Fixed Point for Mappings Satisfying Integral Type Contractive Conditions in G-Metric Spaces

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ABSTRACT

We establish common fixed point results for two self-mappings satisfying integral type contractive conditions in the framework of complete G-metric spaces. The established results generalize and extend many results in the existing literature. Moreover, we provide some comparative examples to show that our results are substantial improvements of some known results in literature.

1. INTRODUCTION

Fixed point theory is one of the most active research subject in the development of nonlinear analysis. It is used in pure and applied mathematics and play a key part in nonlinear functional analysis as well as used for proving the existence theorems for nonlinear differential and integral equations. One of the main result in this area is the Banach contraction principle proved by Banach, which says that any contraction self-mappings on a complete metric space has a unique fixed point. Because of its importance for mathematical theory, this Principle has been extended and generalized in many directions and the references cited therein (Abbas et al., 2011; Suzuki, 2008; Zamfrescu, 1972).

In 2002, an integral version of the Banach fixed point theorem (Branciari, 2002). This result was more generalized and extended by many authors for the existence of fixed points and common fixed points for numerous mapping satisfying integral type contraction (Aliouche, 2006; Dsoudi and Merghadi, 2008; Khojesteh et al., 2010; Rhoades, 2003; Suuki, 2006; Liu and Kang, 2000). In recent years, there has been increasing interest in the study of fixed points of mappings satisfying integral type contraction. Recently, a studied common fixed point theorem for integral type contraction in generalized metric spaces (Ayadi, 2012).

The aims of this manuscript is to study the existence and uniqueness of common fixed points for contractive mappings of integral type in the set-up of generalized metric spaces. Our results generalize the G-metric spaces and carry the results in metric spaces to G-metric spaces (Ayadi, 2012; Liu et al., 2011). Throughout this paper $\mathbb{N}$ denotes the set of positive integers, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+$ is the set of non-negative real numbers and

$$\Phi = \left\{ \begin{array}{l} \phi \mid \phi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ where } \phi \text{ is Lebesgue integrable, summable} \\ \text{on each compact subset of } \mathbb{R}^+ \text{ and } \int_0^\varepsilon \phi(t)dt > 0 \text{ for each } \varepsilon > 0 \end{array} \right\}.$$

1.   Preliminaries
      Before going to the main results, we recall some definitions and results from the existing literature.

Definition 2.1 (Manro et al., 2013; Mustafa and Sims, 2006). Let X be a non-empty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following conditions:

1) $G(x, y, z) = 0$ implies that $x = y = z$ for all $x, y, z \in X$;

2) $0 < G(x, x, y)$ for all $x, y, \in X$ with $x \neq y$;

3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;

4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);

5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$

Then it is called G-metric on X and the pair (X, G) is called G-metric space.

Example 2.2. Let $X = [0, \infty)$ and $G : X \times X \times X \to \mathbb{R}^+$ be the function defined as follow

$G(x, y, z) = \max\{| x - y |, | y - z |, | z - x |\}$ for all $x, y, z \in X$. Then G is G-metric on X.

Proposition 2.3 (Mustafa and Sims, 2006). Let (X, G) be a G-metric space. The following are equivalent:

1) $\Omega_n \mathfrak{c}$ is G-convergent to x;

2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$

3) $G(x_n, x, x) \to 0$ as $n \to \infty$

4) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$

Muhammad Shoaib[1], Muhammad Sarwar[1*], Sultan Hussain[2] and Gohar Ali[3] Existence and Uniqueness of Common Fixed Point for Mappings Satisfying Integral Type Contractive Conditions in G-Metric Spaces Matriks Sains Matematik(MSMK) 1(1) (2017) 01-08

2

Proposition 2.4 (Sintunavarat and Kumam., 2012) Let f and g be weakly compatible self-mappings on a set X. If f and g have a unique point of coincidence $\xi = f\psi = g\psi$, then ξ is the unique common fixed point of f and g.

Proposition 2.5 (Sintunavarat and Kumam., 2012) Let (X,G) be a G-metric space. Then following statement holds:

1) $|G(x,y,z) - G(x,y,a)| \leq \max\{G(a,z,z), G(z,a,a)\}$,

2) $G(x,y,y) \leq 2G(y,x,x)$.

Lemma 2.6 (Liu et al., 2011). Let $\phi \in \Phi$ and $\{r_n\}_{n \in \mathbb{N}}$ is a non-negative sequence with $\lim_{n \to \infty} r_n = a$. Then

$$\lim_{n \to \infty} \int_0^{r_n} \phi(t)dt = \int_0^a \phi(t)dt.$$

Lemma 2.7 (Liu et al., 2011). Let $\phi \in \Phi$ and $\{r_n\}_{n \in \mathbb{N}}$ is a non-negative sequence. Then

$$\lim_{n \to \infty} \int_0^{r_n} \phi(t)dt = 0 \Leftrightarrow \lim_{n \to \infty} r_n = 0.$$

Definition 2.8 (Sintunavarat and Kumam, 2012). Let S and T be self-mappings of a nonempty set X.

1) A point $x \in X$ is said to be a fixed point of T if $Tx = x$.

2) A point $x \in X$ is said to be a coincidence point of S and T if $Sx = Tx$ and we call the point $w = Sx = Tx$ a point of coincidence of S and T.

3) A point $x \in X$ is said to be a common fixed point of S and T if $x = Sx = Tx$.

Definition 2.9 (Abbas et al., 2011; Aliouche, 2006). Let X be a non-empty set the mappings $T, f : X \to X$ are said to be weakly compatible if they commute at their coincidence point (i.e., $Tfx = fTx$ whenever $Tx = fx$).

Theorem 2.10. (Branclari, 2002). If T be a self-map of a complete metric space (X,d) such that for all $x, y \in X$

$$\int_0^{d(Tx,Ty)} \phi(t)dt \leq \eta \int_0^{d(x,y)} \phi(t)dt, \ \eta \in (0,1),$$

where, $\phi \in \Phi$. Then T has a unique fixed point in X.

In 2011, a researcher proved the following results which generalized Theorem 2.10 in metric spaces (Liu et al., 2011).

Theorem 2.11. If T be a mapping from a complete metric space (X,d) to itself such that for all $x, y \in X$

$$\int_0^{d(Tx,Ty)} \phi(t)dt \leq \alpha(d(x,y)) \int_0^{d(x,y)} \phi(t)dt,$$

where, $\phi \in \Phi$ and $\alpha : [0,\infty) \to [0,1)$ is a function with $\limsup_{s \to t} \alpha(s) < 1, \ \forall t > 0$. Then T has a unique fixed point in X.

Theorem 2.12. If T be a mapping from a complete metric space (X,d) to itself such that for all $x, y \in X$

$$\int_0^{d(Tx,Ty)} \phi(t)dt \leq \alpha(d(x,y) \int_0^{d(x,Tx)} \phi(t)dt + \beta(d(x,y) \int_0^{d(y,Ty)} \phi(t)dt,$$

where, $\phi \in \Phi$ and $\alpha, \beta : [0,+\infty) \to [0,1)$ is a function with $\limsup_{s \to t} \frac{\alpha(s)}{1 - \beta(s)} < 1 \ \forall t > 0$. Then T has a unique fixed point in X.

In 2014, generalized the above theorem for integral type contraction and proved the following theorem in metric spaces (Liu et al., 2014a).

Theorem 2.13. If T be a mapping from a complete metric space (X,d) to itself such that for all $x, y \in X$

$$\int_0^{d(Tx,Ty)} \phi(t)dt \leq \alpha(d(x,y)) \int_0^{d(x,Tx)} \phi(t)dt + \beta(d(x,y)) \int_0^{d(y,Ty)} \phi(t)dt$$
$$+ \gamma(d(x,y)) \int_0^{d(x,y)} \phi(t)dt,$$

For all $x, y \in X$ where, $\phi \in \Phi$ and $\alpha, \beta, \gamma : [0,+\infty) \to [0,1)$ is a function with

$$\alpha(t) + \beta(t) + \gamma(t) < 1 \ \forall t \in \mathbb{R}^+,$$
$$\limsup_{s \to t}[\alpha(s) + \beta(s) + \gamma(s)] < 1 \ \forall t > 0.$$

Then T has a unique fixed point in X.

Recently, a researcher has established the following common fixed point theorems in generalized metric spaces (Ayadi, 2012).

Theorem 2.14. Let (X,G) be complete G-metric space and $f, g : X \to X$ be a mapping such that

$$G(fx, fy, fz) \leq \alpha G(gx, gy, gz),$$

for all $x, y, z \in X$ where $\alpha \in [0,1)$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Theorem 2.15. Let (X,G) be complete G-metric space and $f, g : X \to X$ be a mapping such that

$$\int_0^{G(fx, fy, fz)} \phi(t)dt \leq \alpha \int_0^{(G(gx, gy, gz))} \phi(t)dt,$$

for all $x, y, z \in X$ where $\alpha \in [0,1)$ and $\phi \in \Phi$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

2. Main results

Theorem 3.1. Let (X,G) be complete G-metric space and $f, g : X \to X$ be a mapping such that

$$\int_0^{G(fx, fy, fz)} \phi(t)dt \leq \alpha(G(gx, gy, gz)) \int_0^{(G(gx, gy, gz))} \phi(t)dt,$$
$$(3.1)$$

for all $x, y, z \in X$, where $\phi \in \Phi$ and $\alpha : [0,+\infty) \to [0,1)$ is a function with $\alpha(t) < 1 \ \forall t \in \mathbb{R}^+$, $\limsup_{s \to t} \alpha(s) < 1$, $\forall t > 0$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let x_0 be arbitrary point in X. Since $f(X) \subset g(X)$ choose $x_1 \in X$ such that $gx_1 = fx_0$ choose $x_2 \in X$ such that $gx_2 = fx_1$ in last we choose $x_{n+1} \in X$ such that $y_n = gx_{n+1} = fx_n$. Let $G_n = G(y_n, y_{n+1}, y_{n+2})$ by using (3.1) we get

$$\int_0^{G_n} \phi(t)dt = \int_0^{G(fx_n, fx_{n+1}, fx_{n+2})} \phi(t)dt$$
$$\leq \alpha(G(gx_n, gx_{n+1}, gx_{n+2})) \int_0^{G(gx_n, gx_{n+1}, gx_{n+2})} \phi(t)dt$$
$$= \alpha(G(fx_{n-1}, fx_n, fx_{n+1})) \int_0^{G(fx_{n-1}, fx_n, fx_{n+1})} \phi(t)dt$$
$$= \alpha(G_{n-1}) \int_0^{G_{n-1}} \phi(t)dt$$
$$\leq \int_0^{G_{n-1}} \phi(t)dt, \forall n \in \mathbb{N}$$

$$(3.2)$$

which yields that

$$G_n \leq G_{n-1}, \forall n \in \mathbb{N}.$$
$$(3.3)$$

Thus, the sequence $\{G_n\}_{n \in \mathbb{N}}$ is decreasing. Consequently, them exist a constant c such that

$$\lim_{n \to \infty} G_n = c.$$
$$(3.4)$$

We claim that $c = 0$. If $c > 0$, then from (3.2) we have

$$\int_0^{G_n} \phi(t)dt \leq \alpha(G_{n-1}) \int_0^{G_{n-1}} \phi(t)dt.$$

Taking limit sup and using equation (3.4) and Lemma 2.6 we get

$$0 < \int_0^c \phi(t)dt = \limsup_{n \to \infty} \int_0^{G_n} \phi(t)dt \leq \limsup_{n \to \infty}(\alpha(G_{n-1}) \int_0^{G_{n-1}} \phi(t)dt)$$
$$\leq \limsup_{n \to \infty} \alpha(G_{n-1}) \limsup_{n \to \infty} \int_0^{G_{n-1}} \phi(t)dt < \int_0^c \phi(t)dt.$$

Which is contradiction. Hence

$$\lim_{n \to \infty} G_n = 0.$$
$$(3.5)$$

Next, we prove that $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $g(X)$.

Muhammad Shoaib[1], Muhammad Sarwar[1*], Sultan Hussain[2] and Gohar Ali[3] Existence and Uniqueness of Common Fixed Point for Mappings Satisfying Integral Type Contractive Conditions in G-Metric Spaces Matriks Sains Matematik(MSMK) 1(1) (2017) 01-08

3

Suppose $\{y_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and for each positive integer k, there are positive integers $m(k)$ and $n(k)$ with $n(k) > m(k) > k$ such that

$$G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \geq \varepsilon. \tag{3.6}$$

Corresponding to $m(k)$, we can choose $n(k)$ such that it is the smallest integer with $n(k) > m(k)$ and satisfying (3.6). Then

$$G(y_{n(k)}, y_{m(k)-1}, y_{m(k)-1}) < \varepsilon. \tag{3.7}$$

Then we have

$$\varepsilon \leq G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \leq G(y_{n(k)}, y_{m(k)-1}, y_{m(k)-1}) + G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}).$$
$$< \varepsilon + G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}).$$

Letting $k \to \infty$ and using (3.5) we get

$$\lim_{k\to\infty} G(y_{m(k)}, y_{m(k)}, y_{n(k)}) = \varepsilon. \tag{3.8}$$

Next by using Proposition 2.5 we can write

$$|G(y_{m(k)}, y_{m(k)}, y_{n(k)}) - G(y_{m(k)}, y_{m(k)}, y_{n(k)+1})|$$
$$\leq \max\{G(y_{n(k)+1}, y_{n(k)}, y_{n(k)}), G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1})\}$$
$$\leq 2G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+2}) = 2G_n(k); \tag{3.9}$$

$$|G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)+1}) - G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)})|$$
$$\leq \max\{G(y_{m(k)}, y_{m(k)+1}, y_{m(k)+1}), G(y_{m(k)+1}, y_{m(k)}, y_{m(k)})\}$$
$$\leq 2G(y_{m(k)}, y_{m(k)+1}, y_{m(k)+2}) = 2G_m(k); \tag{3.10}$$

$$|G(y_{m(k)+1}, y_{m(k)+1}, y_{n(k)+1}) - G(y_{m(k)+1}, y_{m(k)+1}, y_{n(k)+2})| \tag{3.11}$$
$$\leq \max\{G(y_{n(k)+2}, y_{n(k)+1}, y_{n(k)+1}), G(y_{n(k)+1}, y_{n(k)+2}, y_{n(k)+2})\}$$
$$\leq 2G(y_{n(k)+1}, y_{n(k)+2}, y_{n(k)+3}) = 2G_{n(k)+1}.$$

By letting $k \to \infty$ and taking into account (3.5), (3.8), (3.9), (3.10) and (3.11) we get the following system

$$\left.\begin{array}{l} \lim_{k\to\infty} G(y_{m(k)}, y_{m(k)}, y_{n(k)}) = \varepsilon, \\ \lim_{k\to\infty} G(y_{m(k)}, y_{m(k)}, y_{n(k)+1}) = \varepsilon, \\ \lim_{k\to\infty} G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)+1}) = \varepsilon, \\ \lim_{k\to\infty} G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)}) = \varepsilon, \\ \lim_{k\to\infty} G(y_{m(k)+1}, y_{m(k)+1}, y_{n(k)+2}) = \varepsilon. \end{array}\right\} \tag{3.12}$$

Using (3.1) we get

$$\int_0^{G(y_{m(k)+1}, y_{m(k)+1}, y_{n(k)+2})} \phi(t)dt \tag{3.13}$$
$$\leq \alpha(G(y_{m(k)}, y_{m(k)}, y_{n(k)+1}))\int_0^{G(y_{m(k)}, y_{m(k)}, y_{n(k)+1})} \phi(t)dt \ \ \forall k \in \mathbb{N}.$$

Taking limit superior and using (3.12) we have

$$0 < \int_0^{\varepsilon} \phi(t)dt$$
$$= \limsup_{k\to\infty} \int_0^{G(y_{m(k)+1}, y_{m(k)+1}, y_{n(k)+2})} \phi(t)dt$$
$$\leq \limsup_{k\to\infty} \alpha(G(y_{m(k)}, y_{m(k)}, y_{n(k)+1}))\int_0^{G(y_{m(k)}, y_{m(k)}, y_{n(k)+1})} \phi(t)dt)$$
$$\leq \limsup_{k\to\infty} \alpha(G(y_{m(k)}, y_{m(k)}, y_{n(k)+1}))\int_0^{\varepsilon} \phi(t)dt < \int_0^{\varepsilon} \phi(t)dt.$$

Again contradiction arises. Thus $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $g(X)$ and since $g(X)$ is complete, there exists a point $r \in g(X)$ such that $\lim_{n\to\infty} y_n(x) = r$. So, there exist $w \in X$ such that $gw = r$. From Proposition 2.3 we have

$$\lim_{n\to\infty} G(gx_n, gx_n, gw) = \lim_{n\to\infty} G(gx_n, gw, gw) = 0. \tag{3.14}$$

Further we show that $gw = fw$. Suppose that $gw \neq fw$ then by using inequality, one can get

$$\int_0^{G(gx_n, fw, fw)} \phi(t)dt = \int_0^{G(fx_{n-1}, fw, fw)} \phi(t)dt$$
$$\leq \alpha(G(gx_{n-1}, gw, gw))\int_0^{G(gx_{n-1}, gw, gw)} \phi(t)dt$$

By taking $n \to \infty$ and using (3.14) we have,

$$\int_0^{G(gw, fw, fw)} \phi(t)dt \leq 0,$$

which contradict the fact that $gw \neq fw$. Thus $fw = gw$.

Uniqueness: We now show that f and g have a unique point of coincidence. Suppose that $f\xi = g\xi$ for some $\xi \in X$, the using inequality (3.1), we can write

$$\int_0^{G(g\xi, gw, gw)} \phi(t)dt = \int_0^{G(f\xi, fw, fw)} \phi(t)dt$$
$$\leq \alpha(G(g\xi, gw, gw))\int_0^{G(f\xi, fw, fw)} \phi(t)dt < \int_0^{G(g\xi, gw, gw)} \phi(t)dt$$

but this holds if $G(g\xi, gw, gw) = 0$. Hence $g\xi = gw$. Which gives f and g have a unique point of coincident. From Proposition 2.4 f and g have unique common fixed point.

Now we present some corollaries of Theorem 3.1.

Corollary 3.2. Let α, G be complete G-metric space and $f, g : X \to X$ be a mapping such that

$$\int_0^{G(fx, fy, fz)} \phi(t)dt \leq c\int_0^{G(gx, gy, gz)} \phi(t)dt,$$

for all $x, y, z \in X$ where $c \in [0,1)$ and $\phi \in \Phi$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. It follows by taking $\alpha(s) = c$ in Theorem 3.1.

Corollary 3.3. Let (X, G) be a complete G-metric space and $f, g : X \to X$ such that

$$G(fx, fy, fz) \leq \alpha(G(gx, gy, gz)),$$

for all $x, y, z \in X$ where $0 \geq \alpha < 1$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. It follows by taking $\phi(t) = 1$ and $\alpha(x) = \alpha$ in Theorem 3.1.

Corollary 3.4. Let (X, G) be a complete G-metric space and $f : X \to X$ be a mapping such that

$$\int_0^{G(fx, fy, fz)} \phi(t)dt \leq \alpha(G(x, y, z))\int_0^{G(x, y, z)} \phi(t)dt,$$

for all $x, y, z \in X$ where $\phi \in \Phi$ and $\alpha : [0, +\infty) \to [0,1)$ is a function with $\alpha(t) < 1 \ \forall t \in \mathbb{R}^+$, $\limsup_{s\to t}\alpha(s) < 1$, $\forall t > 0$. Then f has a unique fixed point in X.

Proof. The proof follows by taking $g(x) = x$ in Theorem 3.1.

Corollary 3.5. Let (X, G) be a complete G-metric space and $f, g : X \to X$ such that

$$G(fx, fy, fz) \leq \alpha(G(gx, gy, gz))(G(gx, gy, gz)),$$

for all $x, y, z \in X$ where $\alpha : [0, +\infty) \to [0,1)$ is a function with $\alpha(t) < 1 \ \forall t \in \mathbb{R}^+$, $\limsup_{s\to t}\alpha(s) < 1$, $\forall t > 0$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X, Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. It follows by taking $\phi(t) = 1$ in Theorem 3.1.

Corollary 3.6. Let (X, G) be complete G-metric space and $f : X \to X$ such that

$$G(fx, fy, fz) \leq \alpha(G(x, y, z))(G(x, y, z)),$$

for all $x, y, z \in X$ where $\alpha : [0, +\infty) \to [0,1)$ is a function with

Muhammad Shoaib[1], Muhammad Sarwar[1*], Sultan Hussain[2] and Gohar Ali[3] Existence and Uniqueness of Common Fixed Point for Mappings Satisfying Integral Type Contractive Conditions in G-Metric Spaces Matriks Sains Matematik(MSMK) 1(1) (2017) 01-08

4

$\alpha(t) < 1 \, \forall \, t \in \mathsf{R}^+$, $\limsup_{s \to t} \alpha(s) < 1$, $\forall \, t > 0$. Then f has a unique fixed point in X.

Proof. It follows by taking $g(x) = x$ and $\phi(t) = 1$ in Theorem 3.1.

Corollary 3.7. Let (X, G) be complete G-metric space and $f, g : X \to X$ such that

$$\int_0^{G(f^m x, f^m y, f^m z)} \phi(t) dt \le \alpha(G(g^m x, g^m y, g^m z)) \int_0^{(G(g^m x, g^m y, g^m z))} \phi(t) dt,$$

for all $x, y, z \in X$ where $\phi \in \Phi$ and $\alpha : [0, +\infty) \to [0, 1)$ is a function with $\alpha(t) < 1 \, \forall \, t \in \mathsf{R}^+$, $\limsup_{s \to t} \alpha(s) < 1$, $\forall \, t > 0$. If $f^m(X) \subset g^m(X)$ and $g(X)$ is a complete subspace of X, Then f and g have a unique point of coincidence in X. Moreover, if f^m and g^m are weakly compatible, then f and g have a unique common fixed point $\alpha \in X$.

Proof. It is follow from Theorem 3.1 that $f^m(x)$, $g^m(x)$ have a unique common fixed point C. Now $f(\alpha) = f(f^m(\alpha)) = f^{m+1}(\alpha) = f^m(f(\alpha))$, $g(\alpha) = g(g^m(\alpha)) = g^{m+1}(\alpha) = g^m(g(\alpha))$ implies that $f\alpha$, $g\alpha$ are also fixed point for $f^m(x)$, $g^m(x)$. Since the common fixed point of $f^m(x)$, $g^m(x)$ is unique. Hence, $\alpha = f\alpha = g\alpha$.

Theorem 3.8. Let (X, G) be complete G-metric space and $f, g : X \to X$ be a mapping such that for all $x, y, z \in X$

$$\int_0^{G(fx, fy, fz)} \phi(t) dt \le \alpha(G(gx, gy, gz)) \int_0^{(G(gx, fx, gx))} \phi(t) dt \quad (3.15)$$

$$+ \beta(G(gx, gy, gz)) \int_0^{(G(gy, fy, gx))} \phi(t) dt + \gamma(G(gx, gy, gz)) \int_0^{(G(gx, gy, gz))} \phi(t) dt,$$

where $\phi \in \Phi$ and $\alpha, \beta, \gamma : [0, +\infty) \to [0, 1)$ are functions such that

$$\alpha(t) + \beta(t) + \gamma(t) < 1 \, \forall \, t \in \mathsf{R}^+, \quad (3.16)$$

and

$$\limsup_{s \to t} [\alpha(t) + \beta(t) + \gamma(t)] < 1 \, \forall \, t > 0. \quad (3.17)$$

If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let x_0 be arbitrary point in X. Since $f(X) \subset g(X)$ choose $x_1 \in X$ such that $gx_1 = fx_0$ choose $x_2 \in X$ such that $gx_2 = fx_1$ in last we choose $x_{n+1} \in X$ such that $y_n = gx_{n+1} = fx_n$. Let $G_n = G(y_n, y_{n+1}, y_{n+2})$ by using (3.15)

$$\int_0^{G_n} \phi(t) dt = \int_0^{G(fx_n, fx_{n+1}, fx_{n+2})} \phi(t) dt$$

$$\le \alpha(G(gx_n, gx_{n+1}, gx_{n+2})) \int_0^{G(gx_n, fx_n, gx_n)} \phi(t) dt$$

$$+ \beta(G(gx_n, gx_{n+1}, gx_{n+2})) \int_0^{G(gx_{n+1}, fx_{n+1}, gx_n)} \phi(t) dt$$

$$+ \gamma(G(gx_n, gx_{n+1}, gx_{n+2})) \int_0^{G(gx_n, gx_{n+1}, gx_{n+2})} \phi(t) dt,$$

Using symmetric property of G -metric and $G(x, x, y) \le G(x, y, z)$ we get

$$\int_0^{G_n} \phi(t) dt \le \alpha(G_{n-1}) \int_0^{G_{n-1}} \phi(t) dt + \beta(G_{n-1}) \int_0^{G_{n-1}} \phi(t) dt +$$

$$\gamma(G_{n-1}) \int_0^{G_{n-1}} \phi(t) dt = \left(\alpha(G_{n-1}) + \gamma(G_{n-1}) + \beta(G_{n-1}) \right) \int_0^{G_{n-1}} \phi(t) dt$$

$$\le \int_0^{G_{n-1}} \phi(t) dt, \quad \forall \, n \in \mathsf{N}.$$

Thus

$$G_n \le G_{n-1}, \forall \, n \in \mathsf{N}. \quad (3.18)$$

Which shows that the sequence $\{G_n\}_{n \in \mathsf{N}}$ is decreasing. Consequently, them exist a constant c such that

$$\lim_{n \to \infty} G_n = c. \quad (3.19)$$

We show that $c = 0$, Otherwise if $c > 0$ then from above we have

$$\int_0^{G_n} \phi(t) dt \le \left(\alpha(G_{n-1}) + \beta(G_{n-1}) + \gamma(G_{n-1}) \right) \int_0^{G_{n-1}} \phi(t) dt$$

Taking limit superior and using (3.17) and (3.19) we get

$$0 < \int_0^c \phi(t) dt = \limsup_{n \to \infty} \int_0^{G_n} \phi(t) dt$$

$$\le \limsup_{n \to \infty} \left(\alpha(G_{n-1}) + \beta(G_{n-1}) + \gamma(G_{n-1}) \right) \limsup_{n \to \infty} \int_0^{G_{n-1}} \phi(t) dt$$

$$< \int_0^c \phi(t) dt.$$

Which is contradiction. Hence

$$\lim_{n \to \infty} G_n = 0. \quad (3.20)$$

Next, we prove that $\{y_n\}_{n \in \mathsf{N}}$ is a Cauchy sequence in $g(X)$. Suppose $\{y_n\}_{n \in \mathsf{N}}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and for each positive integer k, there are positive integers $m(k)$ and $n(k)$ with $n(k) > m(k) > k$ such that

$$G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \ge \varepsilon. \quad (3.21)$$

Corresponding to $m(k)$, we can choose $n(k)$ such that it is the smallest integer with $n(k) > m(k)$ and satisfying (3.21). Then

$$G(y_{n(k)}, y_{m(k)-1}, y_{m(k)-1}) < \varepsilon. \quad (3.22)$$

Then we have

$$\varepsilon \le G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \le G(y_{n(k)}, y_{m(k)-1}, y_{m(k)-1}) + G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}).$$

$$< \varepsilon + G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}).$$

Setting $k \to \infty$ and using (3.21) we have,

$$\lim_{k \to \infty} G(y_{m(k)}, y_{m(k)}, y_{n(k)}) = \varepsilon. \quad (3.23)$$

Now by using Proposition 2.5 we get the next three inequalities

$$| G(y_{m(k)}, y_{m(k)}, y_{n(k)}) - G(y_{m(k)}, y_{m(k)}, y_{n(k)+1}) |$$

$$\le \max\{G(y_{n(k)+1}, y_{n(k)}, y_{n(k)}), G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1})\}$$

$$\le 2G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+2}) = 2G_n(k); \quad (3.24)$$

$$| G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)+1}) - G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)}) |$$

$$\le \max\{G(y_{m(k)+1}, y_{m(k)}, y_{m(k)}), G(y_{m(k)}, y_{m(k)+1}, y_{m(k)+1})\}$$

$$\le 2G(y_{m(k)}, y_{m(k)+1}, y_{m(k)+2}) = 2G_m(k); \quad (3.25)$$

Muhammad Shoaib[1], Muhammad Sarwar[1*], Sultan Hussain[2] and Gohar Ali[3] Existence and Uniqueness of Common Fixed Point for Mappings Satisfying Integral Type Contractive Conditions in G-Metric Spaces Matriks Sains Matematik(MSMK) 1(1) (2017) 01-08

5

And

$$| G(y_{m(k)+1}, y_{m(k)+1}, y_{n(k)+1}) - G(y_{m(k)+1}, y_{m(k)+1}, y_{n(k)+2}) |$$
$$\leq \max\{G(y_{n(k)+1}, y_{n(k)+1}, y_{n(k)+2}), G(y_{n(k)+2}, y_{n(k)+1}, y_{n(k)+1}\}$$
$$\leq 2G(y_{n(k)+1}, y_{n(k)+1}, y_{n(k)+3}) = 2G_{n(k)+1}.$$
$$(3.26)$$

Letting $k \to \infty$ and combining (3.20), (3.23), (3.24), (3.25) and (3.26) one can get

$$\left.\begin{array}{r} \lim\limits_{k \to \infty} G(y_{m(k)}, y_{m(k)}, y_{n(k)}) = \varepsilon, \\ \lim\limits_{k \to \infty} G(y_{m(k)}, y_{m(k)}, y_{n(k)+1}) = \varepsilon, \\ \lim\limits_{k \to \infty} G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)+1}) = \varepsilon, \\ \lim\limits_{k \to \infty} G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)}) = \varepsilon, \\ \lim\limits_{k \to \infty} G(y_{m(k)+1}, y_{m(k)+1}, y_{n(k)+2}) = \varepsilon. \end{array}\right\}$$
$$(3.27)$$

Using (3.15) we can write

$$\int_0^{G(y_{m(k)+1}, y_{m(k)+1}, y_{n(k)+2})} \phi(t)dt$$
$$\leq \alpha(G(y_{m(k)}, y_{m(k)}, y_{n(k)+1})) \int_0^{G(y_{m(k)}, y_{m(k)+1}, y_{m(k)})} \phi(t)dt$$
$$+ \beta(G(y_{m(k)}, y_{m(k)}, y_{n(k)+1})) \int_0^{G(y_{m(k)}, y_{m(k)+1}, y_{m(k)})} \phi(t)dt$$
$$+ \gamma(G(y_{m(k)}, y_{m(k)}, y_{n(k)+1})) \int_0^{G(y_{m(k)}, y_{m(k)}, y_{n(k)+1})} \phi(t)dt, \; \forall \, k \in \mathsf{N}.$$
$$(3.28)$$

Also

$$G(y_{m(k)}, y_{m(k)+1}, y_{m(k)}) = G(y_{m(k)}, y_{m(k)}, y_{m(k)+1})$$
$$\leq G(y_{m(k)}, y_{m(k)+1}, y_{m(k)+2}).$$
$$(3.29)$$

Taking limit superior of (3.28) and using (3.20), (3.27) and (3.29) we get

$$0 < \int_0^\varepsilon \phi(t)dt = \lim\sup_{k \to \infty} \int_0^{G(y_{m(k)+1}, y_{m(k)+1}, y_{n(k)+2})} \phi(t)dt$$
$$\leq \lim\sup_{k \to \infty}(\gamma(G(y_{m(k)}, y_{m(k)}, y_{n(k)+1})) \int_0^{G(y_{m(k)}, y_{m(k)}, y_{n(k)+1})} \phi(t)dt)$$
$$\leq \lim\sup_{k \to \infty} \gamma(G(y_{m(k)}, y_{m(k)}, y_{n(k)+1})) \lim\sup_{k \to \infty} \int_0^{G(y_{m(k)}, y_{m(k)}, y_{n(k)+1})} \phi(t)dt$$
$$\leq \lim\sup_{k \to \infty}(\gamma(G(y_{m(k)}, y_{m(k)}, y_{n(k)+1}) \int_0^\varepsilon \phi(t)dt < \int_0^\varepsilon \phi(t)dt$$

which is contradiction. Thus $\{y_n\}_{n \in \mathsf{N}}$ is a Cauchy sequence in $g(X)$ and since $g(X)$ is complete, there exists a point $r \in g(X)$ such that $\lim_{n \to \infty} y_n(x) = r$. So, there exist $w \in X$ such that $gw = r$. From Proposition 2.3 we have

$$\lim_{n \to \infty} G(gx_n, gx_n, gw) = \lim_{n \to \infty} G(gx_n, gw, gw) = 0.$$
$$(3.30)$$

Now we will show that $gw = fw$. Suppose that $gw \neq fw$ by (3.15), we have

$$\int_0^{G(gx_n, fw, gw)} \phi(t)dt = \int_0^{G(fx_{n-1}, fw, fw)} \phi(t)dt$$
$$\leq \alpha(G(gx_{n-1}, gw, gw) \int_0^{G(gx_{n-1}, fx_{n-1}, gx_{n-1})} \phi(t)dt$$
$$+ \beta(G(gx_{n-1}, gw, gw)) \int_0^{G(gw, fw, gx_{n-1})} \phi(t)dt$$
$$+ \gamma(G(gx_{n-1}, gw, gw) \int_0^{G(gx_{n-1}, gw, gw)} \phi(t)dt$$

By taking $n \to \infty$ and using (3.30) we get

$$\int_0^{G(gw, fw, gw)} \phi(t)dt \leq \beta(G(gx_{n-1}, gw, gw) \int_0^{G(gw, fw, gw)} \phi(t)dt$$

which contradict the fact that $fw \neq gw$. Hence $gw = fw$.

Uniqueness: By following the lines in the proof of Theorem 3.1 we conclude the uniqueness of fixed point.

Theorem 3.8 yields the following corollaries:

Corollary 3.9. Let (X, G) be complete G-metric space and $f, g : X \to X$ be a mapping such that for all $x, y, z \in X$

$$\int_0^{G(fx, fy, fz)} \phi(t)dt \leq \alpha(G(gx, gy, gz)) \left(\int_0^{(G(gx, fx, gx))} \phi(t)dt + \int_0^{(G(gy, fy, gx))} \phi(t)dt + \int_0^{(G(gx, gy, gz))} \phi(t)dt \right),$$

where $\phi \in \Phi$ and $\alpha : [0, +\infty) \to [0,1)$ is a function with $\alpha(t) < 1$ for all $t \in \mathsf{R}^+$ and $\lim\sup_{s \to t}[\alpha(s)] < 1$ for all $t > 0$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. It follows by taking $\alpha(s) = \beta(s) = \gamma(s)$ in Theorem 3.8.

Corollary 3.10. Let (X, G) be complete G-metric space and $f, g : X \to X$ be a mapping such that for all $x, y, z \in X$.

$$\int_0^{G(fx, fy, fz)} \phi(t)dt \leq a \int_0^{(G(gx, fx, gx))} \phi(t)dt + b \int_0^{(G(gy, fy, gx))} \phi(t)dt + c \int_0^{(G(gx, gy, gz))} \phi(t)dt,$$

where $\phi \in \Phi$ and $a + b + c < 1$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. It follows by taking $\alpha(s) = a, \beta(s) = b, \gamma(s) = c$ in Theorem 3.8.

Corollary 3.11. Let (X, G) be complete G-metric space and $f, g : X \to X$ be a mapping such that for all $x, y, z \in X$

$$\int_0^{G(fx, fy, fz)} \phi(t)dt \leq \alpha(G(x, y, z)) \int_0^{(G(x, fx, x))} \phi(t)dt + \beta(G(x, y, z)) \int_0^{(G(y, fy, x))} \phi(t)dt + \gamma(G(x, y, z)) \int_0^{(G(x, y, z))} \phi(t)dt,$$

where $\phi \in \Phi$ and $\alpha, \beta, \gamma : [0, +\infty) \to [0,1)$ are functions such that $\alpha(t) + \beta(t) + \gamma(t) < 1, \; \forall \, t \in \mathsf{R}^+$ and $\lim\sup_{s \to t}[\alpha(s) + \beta(s) + \gamma(s)] < 1, \; \forall \, t > 0$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. The proof follows by taking $g(x) = x$ in Theorem 3.8.

Corollary 3.12. Let (X, G) be complete G-metric space and $f, g : X \to X$ be a mapping such that for all $x, y, z \in X$

$$\int_0^{G(f^m x, f^m y, f^m z)} \phi(t)dt \leq \alpha(G(g^m x, g^m y, g^m z)) \int_0^{(G(g^m x, f^m x, g^m x))} \phi(t)dt + \beta(G(g^m x, g^m y, g^m z)) \int_0^{(G(g^m y, f^m y, g^m x))} \phi(t)dt + \gamma(G(g^m x, g^m y, g^m z)) \int_0^{(G(g^m x, g^m y, g^m z))} \phi(t)dt,$$

where $\phi \in \Phi$ and $\alpha, \beta, \gamma : [0, +\infty) \to [0,1)$ are functions such that $\alpha(t) + \beta(t) + \gamma(t) < 1, \; \forall \, t \in \mathsf{R}^+$ and $\lim\sup_{s \to t}[\alpha(s) + \beta(s) + \gamma(s)] < 1, \; \forall \, t > 0$. If $f^m(X) \subset g^m(X)$ and

Muhammad Shoaib[1], Muhammad Sarwar[1*], Sultan Hussain[2] and Gohar Ali[3] Existence and Uniqueness of Common Fixed Point for Mappings Satisfying Integral Type Contractive Conditions in G-Metric Spaces Matriks Sains Matematik(MSMK) 1(1) (2017) 01-08

6

$g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. The proof is same as of the Corollary 3.7.

Corollary 3.13. Let (X, G) be complete G-metric space and $f, g : X \to X$ be a mapping such that for all $x, y, z \in X$

$$G(fx, fy, fz) \le \alpha(G(gx, gy, gz))(G(gx, fx, gx))$$
$$+ \beta(G(gx, gy, gz))(G(gy, fy, gx)) + \gamma(G(gx, gy, gz))(G(gx, gy, gz)),$$

where $\alpha, \beta, \gamma : [0, +\infty) \to [0,1)$ are function such that $\alpha(t) + \beta(t) + \gamma(t) < 1, \forall t \in \mathsf{R}^+$ and $\limsup_{s \to t}[\alpha(s) + \beta(s) + \gamma(s)] < 1, \forall t > 0$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. The proof follows by taking $\phi(t) = 1$ in Theorem 3.8.

Corollary 3.14. Let (X, G) be complete G-metric space and $f, g : X \to X$ be a mapping such that for all $x, y, z \in X$
$G(fx, fy, fz) \le a(G(gx, fx, gx)) + b(G(gy, fy, gx)) + c(G(gx, gy, gz))$,
where $a + b + c < 1$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. The proof follows by taking $\phi(t) = 1$ and $\alpha(s) = a, \beta(s) = b, \gamma(s) = c$ in Theorem 3.8.

Similar to Theorem 3.8 we can prove the following theorem.

Theorem 3.15. Let (X, G) be complete G-metric space and $f, g : X \to X$ be a mapping such that for all $x, y, z \in X$
$$\int_0^{G(fx, fy, fz)} \phi(t)dt \le \alpha(G(gx, gy, gz))\int_0^{(G(gx, fx, gx))} \phi(t)dt$$
$$+ \beta(G(gx, gy, gz))\int_0^{(G(gy, fy, gx))} \phi(t)dt,$$

where $\phi \in \Phi$ and $\alpha, \beta : [0, +\infty) \to [0,1)$ is a function with $\alpha(t) + \beta(t) < 1, \forall t \in \mathsf{R}^+$ and $\limsup_{s \to t}[\alpha(s) + \beta(s)] < 1, \forall t > 0$. If $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Example 3.16. Let $X = [0,1]$ and defined $G : X \times X \times X \to \mathsf{R}^+$ by
$$G(x, y, z) = |x - y| + |y - z| + |z - x| \, \forall \, x, y, z \in X.$$
Then (X, G) is complete G-metric space. Let $f, g : X \to X$, $\phi : \mathsf{R}^+ \to \mathsf{R}^+$ and $\alpha : \mathsf{R}^+ \to [0,1)$ define by
$$f(x) = \frac{x}{9}, \quad \forall \, x \in [0,1];$$
$$g(x) = \frac{x}{3}, \quad \forall \, x \in [0,1];$$
$$\phi(t) = 2t, \quad \forall \, t \in [0, \infty);$$
$$\alpha(t) = \frac{1}{(t+3)}, \quad \forall \, t \in [0, \infty).$$

Clearly $f(X) \subset g(X)$ also f and g are weakly compatible. Now we have

$$\int_0^{G(fx, fy, fz)} \phi(t)dt = (G(fx, fy, fz))^2 = \frac{(G(x, y, z))^2}{81}$$

$$\le \frac{(G(x, y, z))^2}{9((G(x, y, z)) + 3)} = \alpha(G(gx, gy, gz))\int_0^{G(gx, gy, gz)} \phi(t)dt.$$

Thus all conditions of Theorem 3.1 are satisfied. Hence f and g have a unique common fixed point which is 0.

The following example shows that Theorem 3.1 is more general than Theorem 14.

Example 3.17. Let $X = [0, \frac{1}{2}]$ and defined $G : X \times X \times X \to \mathsf{R}^+$ by
$$G(x, y, z) = |x - y| + |y - z| + |z - x| \, \forall \, x, y, z \in X.$$
Then (X, G) is complete G-metric space. Let $f, g : X \to X$, $\phi : \mathsf{R}^+ \to \mathsf{R}^+$ and $\alpha : \mathsf{R}^+ \to [0,1)$ defined by
$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } 0 \le x \le \frac{1}{4}, \\ 0, & \text{if } \frac{1}{4} < x \le \frac{1}{2}. \end{cases}$$
$$g(x) = \begin{cases} x, & \text{if } 0 \le x \le \frac{1}{4}, \\ \frac{1}{2}, & \text{if } \frac{1}{4} < x \le \frac{1}{2}. \end{cases}$$
$$\phi(t) = \begin{cases} e^{-t}, & \text{if } t > 1, \\ 0, & \text{otherwise.} \end{cases}$$
$$\alpha(t) = \frac{1}{2(t+4)}, \, t \in [0, +\infty).$$

Clearly $f(X) \subset g(X)$ also f and g are weakly compatible. Now for all $x, y, z \in X$ $G(fx, fy, fz) \le \frac{3}{4}$ and hence
$$\int_0^{G(fx, fy, fz)} \phi(t)dt = 0.$$

Clearly, all conditions of Theorem 3.1 are satisfied. So f and g have a unique common fixed point.

On the other hand, if we take $x = \frac{1}{4}$ and $y = z = \frac{1}{2}$, then evidently
$$G\left(f(\tfrac{1}{4}), f(\tfrac{1}{2}), f(\tfrac{1}{2})\right) = \left|f(\tfrac{1}{4}) - f(\tfrac{1}{2})\right| + \left|f(\tfrac{1}{2}) - f(\tfrac{1}{2})\right| + \left|f(\tfrac{1}{2}) - f(\tfrac{1}{4})\right| = \frac{1}{2},$$
and
$$G\left(g(\tfrac{1}{4}), g(\tfrac{1}{2}), g(\tfrac{1}{2})\right) = \left|g(\tfrac{1}{4}) - g(\tfrac{1}{2})\right| + \left|g(\tfrac{1}{2}) - g(\tfrac{1}{2})\right| + \left|g(\tfrac{1}{2}) - g(\tfrac{1}{4})\right| = \frac{1}{2}.$$

So we have the following bound

$$G(fx, fy, fz) > k(G(gx, gy, gz)).$$

From the inequality, it is clear that Theorem th1.51 fail to guaranty that f and g have a unique common fixed point.

Example 3.18. Let $X = [0,1]$ and defined $G : X \times X \times X \to \mathsf{R}^+$ by
$$G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}, \, \forall \, x, y, z \in X.$$
Then (X, G) is complete G-metric space. Let $f, g : X \to X$, $\phi : \mathsf{R}^+ \to \mathsf{R}^+$ and $\alpha, \beta, \gamma : [0, +\infty) \to [0,1)$ defined by
$$f(x) = \begin{cases} \frac{\sqrt{x}}{12}, & \text{if } 0 \le x < \frac{1}{2}, \\ \frac{\sqrt{x}}{6}, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$
$$g(x) = \begin{cases} \frac{\sqrt{x}}{4}, & \text{if } 0 \le x < \frac{1}{2}, \\ \frac{\sqrt{x}}{2}, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$
$$\phi(t) = 3t^2, \quad \forall \, t \in [0, \infty),$$
$$\alpha(t) = \beta(t) = \gamma(t) = \frac{1}{(t^2 + 2)^2}, \, \forall \, t \in \mathsf{R}^+,$$

clearly $f(X) \subset g(X)$ also f and g are weakly compatible and

$g(X)$ are complete subspace. Without loss of generality, we suppose that $x \leq y \leq z$ for all $x, y, z \in X$. To check the validity of (3.15) we distinguish the following cases:

Case 1. If $x, y, z \in [0, \frac{1}{2})$, then

$$\int_0^{G(fx, fy, fz)} \phi(t)dt = (G(fx, fy, fz))^3 = (\max\{|fx - fy|, |fy - fz|, |fz - fx|\})^3$$

$$= |\frac{\sqrt{z}}{12} - \frac{\sqrt{x}}{12}|^3 = \frac{1}{27}|\frac{\sqrt{z}}{4} - \frac{\sqrt{x}}{4}|^3$$

$$\leq \frac{1}{27}(|\frac{\sqrt{x}}{6}|^3 + |\frac{\sqrt{y}}{6}|^3 + |\frac{\sqrt{z}}{4} - \frac{\sqrt{x}}{4}|^3)$$

$$\leq \frac{1024}{4225}(|\frac{\sqrt{x}}{6}|^3 + |\frac{\sqrt{y}}{6}|^3 + |\frac{\sqrt{z}}{4} - \frac{\sqrt{x}}{4}|^3)$$

$$\leq \frac{1}{(|\frac{\sqrt{z}}{4} - \frac{\sqrt{x}}{4}|^2) + 2)^2}(|\frac{\sqrt{x}}{6}|^3 + |\frac{\sqrt{y}}{6}|^3 + |\frac{\sqrt{z}}{4} - \frac{\sqrt{x}}{4}|^3)$$

$$= \alpha(G(gx, gy, gz))\int_0^{(G(gx, fx, gx))} \phi(t)dt + \beta(G(gx, gy, gz))\int_0^{(G(gy, fy, gx))} \phi(t)dt$$

$$+ \gamma(G(gx, gy, gz))\int_0^{(G(gx, gy, gz))} \phi(t)dt.$$

Case 2. If $x, y, z \in [\frac{1}{2}, 1]$, then we can write

$$\int_0^{G(fx, fy, fz)} \phi(t)dt = (G(fx, fy, fz))^3 = (\max\{|fx - fy|, |fy - fz|, |fz - fx|\})^3$$

$$= |\frac{\sqrt{z}}{6} - \frac{\sqrt{x}}{6}|^3 = \frac{1}{27}|\frac{\sqrt{z}}{2} - \frac{\sqrt{x}}{2}|^3$$

$$\leq \frac{1}{27}(|\frac{\sqrt{x}}{3}|^3 + |\frac{\sqrt{y}}{3}|^3 + |\frac{\sqrt{z}}{2} - \frac{\sqrt{x}}{2}|^3)$$

$$\leq \frac{4}{25}(|\frac{\sqrt{x}}{3}|^3 + |\frac{\sqrt{y}}{3}|^3 + |\frac{\sqrt{z}}{2} - \frac{\sqrt{x}}{2}|^3)$$

$$\leq \frac{1}{(|\frac{\sqrt{z}}{2} - \frac{\sqrt{x}}{2}|^2) + 2)^2}(|\frac{\sqrt{x}}{3}|^3 + |\frac{\sqrt{y}}{3}|^3 + |\frac{\sqrt{z}}{2} - \frac{\sqrt{x}}{2}|^3)$$

$$= \alpha(G(gx, gy, gz))\int_0^{(G(gx, fx, gx))} \phi(t)dt + \beta(G(gx, gy, gz))\int_0^{(G(gy, fy, gx))} \phi(t)dt$$

$$+ \gamma(G(gx, gy, gz))\int_0^{(G(gx, gy, gz))} \phi(t)dt.$$

Case 3. If $x \in [0, \frac{1}{2})$ and $y, z \in [\frac{1}{2}, 1]$, then

$$g(x) = \begin{cases} \frac{\sqrt{x}}{4}, & \text{if } 0 \leq x < \frac{1}{2}, \\ \frac{\sqrt{x}}{2}, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

$$\phi(t) = 3t^2, \quad \forall\, t \in [0, \infty),$$

$$\alpha(t) = \beta(t) = \gamma(t) = \frac{1}{(t^2 + 2)^2}, \quad \forall\, t \in \mathbb{R}^+,$$

clearly $f(X) \subset g(X)$ also f and g are weakly compatible and $g(X)$ are complete subspace. Without loss of generality, we suppose that $x \leq y \leq z$ for all $x, y, z \in X$. To check the validity of (3.15) we distinguish the following cases:

Case 1. If $x, y, z \in [0, \frac{1}{2})$, then

$$\int_0^{G(fx, fy, fz)} \phi(t)dt = (G(fx, fy, fz))^3 = (\max\{|fx - fy|, |fy - fz|, |fz - fx|\})^3$$

$$= |\frac{\sqrt{z}}{12} - \frac{\sqrt{x}}{12}|^3 = \frac{1}{27}|\frac{\sqrt{z}}{4} - \frac{\sqrt{x}}{4}|^3$$

$$\leq \frac{1}{27}(|\frac{\sqrt{x}}{6}|^3 + |\frac{\sqrt{y}}{6}|^3 + |\frac{\sqrt{z}}{4} - \frac{\sqrt{x}}{4}|^3)$$

$$\leq \frac{1024}{4225}(|\frac{\sqrt{x}}{6}|^3 + |\frac{\sqrt{y}}{6}|^3 + |\frac{\sqrt{z}}{4} - \frac{\sqrt{x}}{4}|^3)$$

$$\leq \frac{1}{(|\frac{\sqrt{z}}{4} - \frac{\sqrt{x}}{4}|^2) + 2)^2}(|\frac{\sqrt{x}}{6}|^3 + |\frac{\sqrt{y}}{6}|^3 + |\frac{\sqrt{z}}{4} - \frac{\sqrt{x}}{4}|^3)$$

$$= \alpha(G(gx, gy, gz))\int_0^{(G(gx, fx, gx))} \phi(t)dt + \beta(G(gx, gy, gz))\int_0^{(G(gy, fy, gx))} \phi(t)dt$$

$$+ \gamma(G(gx, gy, gz))\int_0^{(G(gx, gy, gz))} \phi(t)dt.$$

Case 2. If $x, y, z \in [\frac{1}{2}, 1]$, then we can write

$$\int_0^{G(fx, fy, fz)} \phi(t)dt = (G(fx, fy, fz))^3 = (\max\{|fx - fy|, |fy - fz|, |fz - fx|\})^3$$

$$= |\frac{\sqrt{z}}{6} - \frac{\sqrt{x}}{6}|^3 = \frac{1}{27}|\frac{\sqrt{z}}{2} - \frac{\sqrt{x}}{2}|^3$$

$$\leq \frac{1}{27}(|\frac{\sqrt{x}}{3}|^3 + |\frac{\sqrt{y}}{3}|^3 + |\frac{\sqrt{z}}{2} - \frac{\sqrt{x}}{2}|^3)$$

$$\leq \frac{4}{25}(|\frac{\sqrt{x}}{3}|^3 + |\frac{\sqrt{y}}{3}|^3 + |\frac{\sqrt{z}}{2} - \frac{\sqrt{x}}{2}|^3)$$

$$\leq \frac{1}{(|\frac{\sqrt{z}}{2} - \frac{\sqrt{x}}{2}|^2) + 2)^2}(|\frac{\sqrt{x}}{3}|^3 + |\frac{\sqrt{y}}{3}|^3 + |\frac{\sqrt{z}}{2} - \frac{\sqrt{x}}{2}|^3)$$

$$= \alpha(G(gx, gy, gz))\int_0^{(G(gx, fx, gx))} \phi(t)dt + \beta(G(gx, gy, gz))\int_0^{(G(gy, fy, gx))} \phi(t)dt$$

$$+ \gamma(G(gx, gy, gz))\int_0^{(G(gx, gy, gz))} \phi(t)dt.$$

Case 3. If $x \in [0, \frac{1}{2})$ and $y, z \in [\frac{1}{2}, 1]$, then

$$\int_0^{G(fx, fy, fz)} \phi(t)dt = (G(fx, fy, fz))^3$$

$$= |\frac{\sqrt{x}}{12} - \frac{\sqrt{z}}{6}|^3 = \frac{1}{27}|\frac{\sqrt{x}}{4} - \frac{\sqrt{z}}{2}|^3$$

$$\leq \frac{4}{25}(|\frac{\sqrt{x}}{6}|^3 + |\frac{\sqrt{x}}{4} - \frac{\sqrt{y}}{2}|^3 + |\frac{\sqrt{x}}{4} - \frac{\sqrt{z}}{2}|^3)$$

$$\leq \frac{1}{(|(\frac{\sqrt{x}}{4} - \frac{\sqrt{z}}{2}|^2) + 2)^2}(|\frac{\sqrt{x}}{6}|^3 + |\frac{\sqrt{x}}{4} - \frac{\sqrt{y}}{2}|^3 + |\frac{\sqrt{x}}{4} - \frac{\sqrt{z}}{2}|^3)$$

$$= \alpha(G(gx, gy, gz))\int_0^{(G(gx, fx, gx))} \phi(t)dt + \beta(G(gx, gy, gz))\int_0^{(G(gy, fy, gx))} \phi(t)dt$$

$$+ \gamma(G(gx, gy, gz))\int_0^{(G(gx, gy, gz))} \phi(t)dt.$$

Clearly all conditions of Theorem 3.8 are satisfied. Thus f and g have a unique common fixed point which is 0.

Remarks

- Corollary 3.2 and Corollary 3.3 are the results of H (Ayadi, 2012).
- Corollary 3.3 and Corollary 3.11 generalize the results of respectively (Liu et al., 2011; Liu et al., 2014b).

Muhammad Shoaib[1], Muhammad Sarwar[1*], Sultan Hussain[2] and Gohar Ali[3] Existence and Uniqueness of Common Fixed Point for Mappings Satisfying Integral Type Contractive Conditions in G-Metric Spaces Matriks Sains Matematik(MSMK) 1(1) (2017) 01-08

8

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