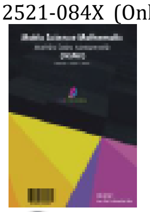
ISSN:2521-0831 (Print)
ISSN:2521-084X (Online)

Contents List available at RAZI Publishing
Matriks Sains Matematik (MSMK)
Journal Homepage: http://www.razipublishing.com/journals/matriks-sains-matematik
https://doi.org/10.26480/msmk.02.2017.01.03



CONSTRUCTION OF RIGHT NUCLEAR SQUARE LOOP

Amir Khan[1*], Tahir Khan[2], Hidayat Ullah Khan[2], Gul Zaman[2]

[1]Department of Mathematics and Statistics, University of Swat, Khyber Pakhtunkhwa, Pakistan
[2]Department of Mathematics, University of Malakand, Khyber Pakhtunkhwa, Pakistan
**Corresponding Author email: amir.maths@gmail.com*

ARTICLE DETAILS

ARTICLE HISTORY:

Received 2 July 2017
Accepted 4 October 2017
Available online 1 November 2017

KEYWORDS:

Right nuclear square loop,
Construction of loop, C-loops,
Factor set.

ABSTRACT

Right nuclear square loops are loops satisfying $y\ zz)) = (xy)(zz)$. We construct an infinite family of non-associative non-commutative right nuclear square loops whose smallest member is of order 12 .

1. INTRODUCTION

A groupoid $(Q,\cdot)$ is a quasigroup if, for each $a,b \in Q$, the equations $ax = b, ya = b$ have unique solutions where $x, y \in Q$ [1]. A loop is a quasigroup with an identity element e such that $x*e = x = e*x$ [2-3]. The left nucleus of a loop Q is $N_\lambda = \{l \in Q : l(xy) = (lx)y$ for every $x, y \in Q\}$. The right nucleus of a loop Q is the set $N_\rho = \{r \in Q : (xy)r = x(yr)$ for every $x, y \in Q\}$, and middle nucleus of Q is $N_\mu = \{m \in Q : (ym)x = y(mx)$ for every $x, y \in Q\}$. The nucleus of Q is the set $N = N_\rho \cap N_\lambda \cap N_\mu$. A loop $(L,*)$ is termed a right nuclear square loop if every square element, i.e., every element of the form $x*x$, is in the right nucleus [6-9]. In other words, the following identity is satisfied for all $x, y, z \in L$:

$$x*(y*(z*z)) = (x*y)*(z*z)$$

Every C-loop is right nuclear square loop [5]. In this paper, we construct right nuclear square loops of order 12 belongs to an infinite family of non-associative noncommutative right nuclear square loops constructed here for the first time.

2. CONSTRUCTION OF RIGHT NUCLEAR SQUARE LOOP

Let H be a multiplicative group with identity element 1, and A be an additively abelian group with identity element 0. Any map

$$\theta : H \times H \to A$$

Satisfying

$$\theta(1, g) = \theta(g, 1) = 0 \text{ for every } g \in H$$

is called a factor set [10-12]? When $\theta : H \times H \to A$ is a factor set, we can define multiplication on $H \times A$ by

$$(g,a)(h,b) = (gh, a + b + \theta(g,h)) \qquad (1)$$

The resulting groupoid is clearly a loop with neutral element $(1,0)$. It will be denoted by (H, A, θ). Additional properties of (H, A, θ) can be enforced by additional requirements on θ.
We construct right nuclear square loop with the help of two groups such that one is multiplicative group and other is additive abelian group.
Lemma 1. *Let $\theta : H \times H \to A$ be a factor set. Then (H, A, θ) is a right nuclear square loop if and only if*

$$\theta(h,k^2) + \theta(g,hk^2) = \theta(g,h) + \theta(gh,k^2) \text{ for all } g,\ h \in H \quad (2)$$

Proof. By definition the loop (H, A, θ) is right nuclear square loop if and only if

$$\begin{aligned}
&(g,a)[(h,b)((k,c)(k,c))] = [(g,a)(h,b][(k,c)(k,c)] \\
&\Rightarrow (g,a)[(h,b)(k^2, 2c + \theta(k,k)] \\
&\quad = [gh, a + b + \theta(g,h)][k^2, 2c + \theta(k,k)] \\
&\Rightarrow (g,a)[hk^2, b + 2c + \theta(k,k) + \theta(h,k^2)] \\
&\quad = [(gh)k^2, a + b + 2c + \theta(k,k) + \theta(g,h) + \theta(gh,k^2)] \\
&\Rightarrow [g(hk^2), a + b + 2c + \theta(k,k) + \theta(h,k^2) + \theta(g,hk^2)] \\
&\quad = [(gh)k^2, a + b + 2c + \theta(k,k) + \theta(g,h) + \theta(gh,k^2)]
\end{aligned}$$

comparing both sides we get

$$\theta(h,k^2) + \theta(g,hk^2) = \theta(g,h) + \theta(gh,k^2)$$

We call a factor set θ satisfying (2) a right nuclear square factor set.

Proposition 1 *Let A be an abelian group of order n where $n>2$, and $\beta \in A$ is an element of order bigger than 2. Let $H=\{1,x,x^2,x^3\}$ be the Cyclic group with identity element 1. Define*

$$\theta : H\times H \to A$$

By

$$\theta(a,b)=\begin{cases}\beta, & \text{if} \quad (a,b)=(x^3,x^3),(x^2,x^2)\\ -\beta, & \text{if} \quad (a,b)=(x,x^2),(x^3,x^2),(x,x^3)\\ 0, & \text{if} \quad \text{otherwise}\end{cases}$$

then $L=(H,A,\theta)$ is a non-associative and noncommutative right nuclear square loop with nucleus $N(L)=\{(1,a) : a\in A\}$.

Proof. The map θ is clearly right nuclear square factor set. It can be shown as follows

θ	1	x	x^2	x^3
1	0	0	0	0
x	0	0	$-\beta$	$-\beta$
x^2	0	0	β	0
x^3	0	0	$-\beta$	β

To show that $L=(H,A,\theta)$ is right nuclear square loop, we verify equation (2) as follows.

Case i : Since θ is a factor set so there is nothing to prove when $g,h,k=1$.

Case ii : when $g=x$, Equation (2) becomes

$$\theta(h,k^2)+\theta(x,hk^2)=\theta(x,h)+\theta(xh,k^2) \qquad (3)$$

Put $h=x$ in Equation (3) we get

$$\theta(x,k^2)+\theta(x,xk^2)=\theta(x,x)+\theta(x^2,k^2).$$

If $k=1$, then $\theta(x,x)=\theta(x,x)$.

If $k=x$, then $\theta(x,x^2)+\theta(x,x^3)=\theta(x,x)+\theta(x^2,x^2)$. This implies $-\beta-\beta=-\beta$.

If $k=x^2$, then $\theta(x,1)+\theta(x,x)=\theta(x,x)+\theta(x^2,1)$. This implies $0=0$.

If $k=x^3$, then $\theta(x,x^2)+\theta(x,x^3)=\theta(x,x^3)+\theta(x^2,x^2)$. This implies $-\beta-\beta=\beta$.

Put $h=x^2$ in Equation (3) we get

$$\theta(x^2,k^2)+\theta(x,x^2k^2)=\theta(x,x^2)+\theta(x^3,k^2)$$

If $k=1$, then $\theta(x,x^2)=\theta(x,x^2)$.

If $k=x$, then $\theta(x^2,x^2)+\theta(x,1)=\theta(x,x^2)+\theta(x^3,x^2)$. This implies $\beta=-\beta-\beta$.

If $k=x^2$, then $\theta(x^2,1)+\theta(x,x^2)=\theta(x,x^2)+\theta(x^3,1)$. This implies $-\beta=-\beta$.

If $k=x^3$, then $\theta(x^2,x^2)+\theta(x,1)=\theta(x,x^2)+\theta(x^3,x^2)$. This implies $\beta=-\beta-\beta$.

Put $h=x^3$ in Equation (3) we get

$$\theta(x^3,k^2)+\theta(x,x^3k^2)=\theta(x,x^3)+\theta(1,k)$$

If $k=1$, then $\theta(x,x^3)=\theta(x,x^3)$.

If $k=x$, then $\theta(x^3,x^2)+\theta(x,x)=\theta(x,x^3)$. This implies $-\beta=-\beta$.

If $k=x^2$, then $\theta(x^3,1)+\theta(x,x^3)=\theta(x,x^3)$. This implies $-\beta=-\beta$.

If $k=x^3$, then $\theta(x^3,x^2)+\theta(x,x)=\theta(x,x^3)$. This implies $-\beta=-\beta$ and hence all are true.

Case iii : When $g=x^2$, then Equation (2) becomes

$$\theta(h,k^2)+\theta(x^2,hk^2)=\theta(x^2,h)+\theta(x^2h,k^2) \qquad (4)$$

Put $h=x$ in Equation (4) we get

$$\theta(x,k^2)+\theta(x^2,xk^2)=\theta(x^2,x)+\theta(x^3,k^2)$$

If $k=1$, then $\theta(x^2,x)=\theta(x^2,x)$

If $k=x$, then $\theta(x,x^2)+\theta(x^2,x^3)=\theta(x^2,x)+\theta(x^3,x^2)$. This implies $-\beta=-\beta$.

If $k=x^2$, then $\theta(x,1)+\theta(x^2,x)=\theta(x^2,x)+\theta(x^3,1)$. This implies $-\beta=-\beta$.

If $k=x^3$, then $\theta(x,x^2)+\theta(x^2,x^3)=\theta(x^2,x)+\theta(x^3,x^2)$. This implies $-\beta=-\beta$.

Put $h=x^2$ in Equation (4) we get

$$\theta(x^2,k^2)+\theta(x^2,x^2k^2)=\theta(x^2,x^2)+\theta(1,k^2)$$

If $k=1$, then $\theta(x^2,x^2)=\theta(x^2,x^2)$.

If $k=x$, then $\theta(x^2,x^2)+\theta(x^2,1)=\theta(x^2,x^2)$. This implies $\beta=\beta$.

If $k=x^2$, then $\theta(x^2,1)+\theta(x^2,x^2)=\theta(x^2,x^2)$. This implies $\beta=\beta$.

If $k=x^3$, then $\theta(x^2,x^2)+\theta(x^2,1)=\theta(x^2,x^2)$. This implies $\beta=\beta$.

Put $h=x^3$ in Equation (4) we get

$$\theta(x^3,k^2)+\theta(x^2,x^3k^2)=\theta(x^2,x^3)+\theta(x,k^2).$$

If $k=1$, then $\theta(x^2,x^3)=\theta(x^2,x^3)$.

If $k=x$, then $\theta(x^3,x^2)+\theta(x^2,x)=\theta(x^2,x^3)+\theta(x,x^2)$. This implies $-\beta=-\beta$.

If $k=x^2$, then $\theta(x^3,1)+\theta(x^2,x^3)=\theta(x^2,x^3)+\theta(x,1)$. This implies $0=0$.

If $k=x^3$, then $\theta(x^3,x^2)+\theta(x^2,x)=\theta(x^2,x^3)+\theta(x,x^2)$. This implies $-\beta=-\beta$ and hence all are true.

Case iv: when $g=x^3$, then Equation (2) becomes

$$\theta(h,k^2)+\theta(x^3,hk^2)=\theta(x^3,h)+\theta(x^3h,k^2) \qquad (5)$$

Put $h=x$ in Equation (5) we get

$$\theta(x,k^2)+\theta(x^3,xk^2)=\theta(x^3,x)+\theta(1,k^2)$$

If $k=1$, then $\theta(x^3,x)=\theta(x^3,x)$.

If $k=x$, then $\theta(x,x^2)+\theta(x^3,x^3)=\theta(x^3,x)$. This implies $0=0$.

If $k=x^2$, then $\theta(x,1)+\theta(x^3,x)=\theta(x^3,x)$. This implies $0=0$.

If $k=x^3$, then $\theta(x,x^2)+\theta(x^3,x^3)=\theta(x^3,x)$. This implies $0=0$.

Put $h=x^2$ in Equation (5) we get

$$\theta(x^2,k^2)+\theta(x^3,x^2k^2)=\theta(x^3,x^2)+\theta(x,k^2).$$

If $k=1$, then $\theta(x^3,x^2)=\theta(x^3,x^2)$.

If $k=x$, then $\theta(x^2,x^2)+\theta(x^3,1)=\theta(x^3,x^2)+\theta(x,x^2)$. This implies $\beta=-\beta-\beta$.

If $k=x^2$, then $\theta(x^2,1)+\theta(x^3,x^2)=\theta(x^3,x^2)+\theta(x,1)$. This implies $-\beta=-\beta$.

If $k=x^3$, then $\theta(x^2,x^2)+\theta(x^3,1)=\theta(x^3,x^2)+\theta(x,x^2)$. This implies $\beta=-\beta-\beta$.

Put $h=x^3$ in Equation (5) we get

$$\theta(x^3,k^2)+\theta(x^3,x^3k^2)=\theta(x^3,x^3)+\theta(x^2,k^2).$$

If $k=1$, then $\theta(x^3,x^3)=\theta(x^3,x^3)$.

If $k=x$, then $\theta(x^3,x^2)+\theta(x^3,x)=\theta(x^3,x^3)+\theta(x^2,x^2)$. This implies $-\beta=\beta+\beta$.

If $k = x^2$, then $\theta(x^3,1)+\theta(x^3,x^3)=\theta(x^3,x^3)+\theta(x^2,1)$. This implies $\beta=\beta$.

If $k = x^3$, then $\theta(x^3,x^2)+\theta(x^3,x)=\theta(x^3,x^3)+\theta(x^2,x^2)$. This implies $-\beta=\beta+\beta$. Hence all are true.

Now we show that $L=(H,A,\theta)$ is not associative. For this consider $b\in A$. As

$$\begin{aligned}(x,b)\left((x^3,b)(x,b)\right) &= (x,b)(1,2b)\\ &= (x,3b)\\ &\neq (x,3b-\beta)\\ &= (1,2b-\beta)(x,b)\\ &= \left((x^3,b)(x,b)\right)(x,b).\end{aligned}$$

Hence $L=(H,A,\theta)$ is non-associative right nuclear square loop.

Also L is not commutative because

$$\begin{aligned}(x^3,b)(x^2,b) &= (x,2b-\beta)\\ &\neq (x,2b)\\ &= (x^2,b)(x^3,b).\end{aligned}$$

Now it remains to show that $N(L)=\{(1,b):b\in A\}$. For this consider

$$\begin{aligned}\left((g,b)(1,b)\right)(h,c) &= (g,b)\left((1,b)(h,c)\right)\\ &= \left(g,b+b+\theta(g,1)\right)(h,c)\\ &= (g,b)\left(h,b+c+\theta(1,h)\right)\\ &= (g,b+b+0)(h,c)\\ &= (g,b)(h,b+c+0)\\ &= \left(gh,b+b+c+\theta(g,h)\right)\\ &= \left(gh,b+b+c+\theta(g,h)\right).\end{aligned}$$

Which is true, so $(1,b)\in N_\theta(L)$.

Similarly, we can show that

$$(1,b)\in N_\lambda(L) \text{ and } (1,b)\in N_\rho(L).$$

Hence $(1,b)\in N(L)$ and this implies $N(L)=\{(1,b):b\in A\}$. Which is the required result.

Example 1 *The smallest group A satisfying the assumptions of Proposition (1) is the 3-element cyclic group $\{0,1,2\}$. Following the construction given in Proposition (1) and taking $\beta=2$, we get the following non-associative noncommutative right nuclear square loop of order 12.*

$\cdot$	$(1,0)$	$(1,1)$	$(1,2)$	$(x,0)$	$(x,1)$	$(x,2)$	$(x^2,0)$	$(x^2,1)$	$(x^2,2)$	$(x^3,0)$	$(x^3,1)$	$(x^3,2)$
$(1,0)$	$(1,0)$	$(1,1)$	$(1,2)$	$(x,0)$	$(x,1)$	$(x,2)$	$(x^2,0)$	$(x^2,1)$	$(x^2,2)$	$(x^3,0)$	$(x^3,1)$	$(x^3,2)$
$(1,1)$	$(1,1)$	$(1,2)$	$(1,0)$	$(x,1)$	$(x,2)$	$(x,0)$	$(x^2,1)$	$(x^2,2)$	$(x^2,0)$	$(x^3,1)$	$(x^3,2)$	$(x^3,0)$
$(1,2)$	$(1,2)$	$(1,0)$	$(1,1)$	$(x,2)$	$(x,0)$	$(x,1)$	$(x^2,2)$	$(x^2,0)$	$(x^2,1)$	$(x^3,2)$	$(x^3,0)$	$(x^3,1)$
$(x,0)$	$(x,0)$	$(x,1)$	$(x,2)$	$(x^2,0)$	$(x^2,1)$	$(x^2,2)$	$(x^3,1)$	$(x^3,2)$	$(x^3,0)$	$(1,1)$	$(1,2)$	$(1,0)$
$(x,1)$	$(x,1)$	$(x,2)$	$(x,0)$	$(x^2,1)$	$(x^2,2)$	$(x^2,0)$	$(x^3,2)$	$(x^3,0)$	$(x^3,1)$	$(1,2)$	$(1,0)$	$(1,1)$
$(x,2)$	$(x,2)$	$(x,0)$	$(x,1)$	$(x^2,2)$	$(x^2,0)$	$(x^2,1)$	$(x^3,0)$	$(x^3,1)$	$(x^3,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$(x^2,0)$	$(x^2,0)$	$(x^2,1)$	$(x^2,2)$	$(x^3,0)$	$(x^3,1)$	$(x^3,2)$	$(1,2)$	$(1,0)$	$(1,1)$	$(x,0)$	$(x,1)$	$(x,2)$
$(x^2,1)$	$(x^2,1)$	$(x^2,2)$	$(x^2,0)$	$(x^3,1)$	$(x^3,2)$	$(x^3,0)$	$(1,0)$	$(1,1)$	$(1,2)$	$(x,1)$	$(x,2)$	$(x,0)$
$(x^2,2)$	$(x^2,2)$	$(x^2,0)$	$(x^2,1)$	$(x^3,2)$	$(x^3,0)$	$(x^3,1)$	$(1,1)$	$(1,2)$	$(1,0)$	$(x,2)$	$(x,0)$	$(x,1)$
$(x^3,0)$	$(x^3,0)$	$(x^3,1)$	$(x^3,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(x,1)$	$(x,2)$	$(x,0)$	$(x^2,2)$	$(x^2,0)$	$(x^2,1)$
$(x^3,1)$	$(x^3,1)$	$(x^3,2)$	$(x^3,0)$	$(1,1)$	$(1,2)$	$(1,0)$	$(x,2)$	$(x,0)$	$(x,1)$	$(x^2,0)$	$(x^2,1)$	$(x^2,2)$
$(x^3,2)$	$(x^3,2)$	$(x^3,0)$	$(x^3,1)$	$(1,2)$	$(1,0)$	$(1,1)$	$(x,0)$	$(x,1)$	$(x,2)$	$(x^2,1)$	$(x^2,2)$	$(x^2,0)$

·	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	0	4	5	3	7	8	6	10	11	9
2	2	0	1	5	3	4	8	6	7	11	9	10
3	3	4	5	6	7	8	10	11	9	1	2	0
4	4	5	3	7	8	6	11	9	10	2	0	1
5	5	3	4	8	6	7	9	10	11	0	1	2
6	6	7	8	9	10	11	2	0	1	3	4	5
7	7	8	6	10	11	9	0	1	2	4	5	3
8	8	6	7	11	9	10	1	2	0	5	3	4
9	9	10	11	0	1	2	4	5	3	8	6	7
10	10	11	9	1	2	0	5	3	4	6	7	8
11	11	9	10	2	0	1	3	4	5	7	8	6

The above example is verified with the help of GAP (Group Algorithm Program) package [4].

REFERENCES

[1] Bruck, R. H. 1958. A Survey of Binary Systems, Ergebnisse der Mathematik und Ihrer Grenzgebiete, New Series, 20, Springer.

[2] Kinyon, M. K., Pula, J., and Vojtechovsky, P. 2009. Admissible Orders of Jordan Loops, Journal of Combinatorial Designs 17 (2), 103-118.

[3] McCrimmon, K. 2004. A Taste of Jordan Algebras, Universitext, Springer.

[4] Nagy, G. P., and Vojtechovsky, P. 2005. LOOPS: Computing with quasigroups and loops in GAP,version 1.0.0, computational package for GAP; http://www.math.du.edu/loops.

[5] Philips, J. D., and Vojtechovsky, P. 2006. C-loops: an introduction. Publicationes Mathematicae Debrecen 68 (1-2), 115-137.

[6] Pula, K. 2008. Power of elements in Jordan loops. Commentationes Mathematicae Universitatis Carolinae, 49 (2), 291-299.

[7] Slaney, J., and Ali, A. 2008. Generating loops with the inverse property, Sutcliffe G., Colton S., Schulz S. (eds.); Proceedings of ESARM, 55-66.

[8] Slaney, J. 1994. FINDER, finite domain enumerator: System description. In Proceedings of the twelfth Conference on Automated Deduction(CADE-12), pages 798-801,1994.

[9] Kandasamy, W. B. V. 2002. Smarandache Loops, American Research Press, Rehoboth.

[10] Khan, A., Shah, M., and Ali, A. 2013. Construction of Middle Nuclear Square Loops. The Journal of Prime Research in Mathematics, 9, 72-78.

[11] Khan, A., Shah, M., and Ali, A. 2013. A Construction Of Right Alternative Loop, International Journal of Algebra and Statistics, 2 (2), 29-32.

[12] Khan, A., Shah, M., Ali, A., Muhammad, F. 2014. On Commutative Quasigroup, International Journal of Algebra and Statistics, 3, 42-45.