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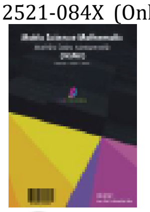

ON LEFT ALTERNATIVE LOOPS

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ABSTRACT

Left alternative loops are loops satisfying $x(xy)=(xx)y$. We construct an infinite family of non-associative non-commutative left alternative loops whose smallest member is of order 8.

1. INTRODUCTION

A groupoid $(Q,\cdot)$ is a quasigroup if, for each $a,b\in Q$, the equations $ax=b, ya=b$ have unique solutions where $x,y\in Q$ [1]. A loop is a quasigroup with an identity element e such that $x*e=x=e*x$. The left nucleus of a loop Q is $N_\lambda=\{l\in Q : l(xy)=(lx)y\,\forall\, x,y\in Q\}$. The right nucleus of a loop Q is the set $N_\rho=\{r\in Q : (xy)r=x(yr)\,\forall\, x,y\in Q\}$, and middle nucleus of Q is $N_\mu=\{m\in Q : (ym)x=y(mx)\,\forall\, x,y\in Q\}$. The nucleus of Q is the set $N=N_\rho\cap N_\lambda\cap N_\mu$ [2, 3].

A loop $(L,*)$ is termed as left alternative loop if the following identity is satisfied for all $x,y,z\in L$:

$$x*(x*y)=(x*x)*y.$$

Every C-loop and Moufang loop is left alternative loop [4]. In this paper, we construct left alternative loops of order 8 belongs to an infinite family of non-associative non-commutative left alternative loops constructed here for the first time.

2. CONSTRUCTION OF LEFT ALTERNATIVE LOOP

Let G be a multiplicative group with neutral element 1, and A an abelian group written additively with neutral element 0 [5-7]. Any map

$$\mu : G\times G\to A,$$

satisfying

$$\mu(1,g)=\mu(g,1)=0 \text{ for every } g\in G,$$

is called a factor set. When $\mu : G\times G\to A$ is a factor set, we can define multiplication on $G\times$ by

$$(g,a)(h,b)=(gh,a+b+\mu(g,h)).$$

The resulting groupoid is clearly a loop with neutral element $(1,0)$. It will be denoted by (G,A,μ). Additional properties of (G,A,μ) can be enforced by additional requirements on μ.

We construct left alternative loop with the help of two groups such that one is multiplicative group and other is additive abelian group [8-11].

Lemma 1. *Let $\mu : G\times G\to A$ be a factor set. Then (G,A,μ) is a left alternative loop if and only if*

$$\mu(g,g)+\mu(g^2,h)=\mu(g,h)+\mu(g,gh)\;\forall\; g,\, h\in G. \qquad (1)$$

Proof. By definition the loop (G,A,μ) is left alternative loop if and only if

$$[(g,a)(g,a)](h,b)=(g,a)[(g,a)(h,b)]$$
$$\Rightarrow (g^2,2a+\mu(g,g))(h,b)=(g,a)\times(gh,a+b+\mu(g,h))$$
$$\Rightarrow (g^2h,2a+b+\mu(g,g)+\mu(g^2,h))=(g(gh),2a+b+\mu(g,h)+\mu(g,gh)),$$

comparing both sides we get

$$\mu(g,g)+\mu(g^2,h)=\mu(g,h)+\mu(g,gh).$$

Hence the result follows.

We call a factor set satisfying (1) a left alternative factor set.

Proposition 1 *Let $n\geq 2$ be an integer. Let A be an abelian group of order n, and $\alpha\in A$ an element of order bigger than 1. Let $G=\{1,u,v,w\}$ be the Klein group with neutral element 1. Define*

$$\mu : G\times G\to A,$$

by

$$\mu(a,b)=\begin{cases}\alpha, & \text{if } (a,b)=(u,u),(u,w)\\ 0, & \text{if otherwise.}\end{cases}$$

Then $L=(G,A,\mu)$ is a non-flexible (hence non-associative) and non-commutative left alternative loop with $N(L)=\{(1,a)\ :\ a\in A\}$.

Proof. The map μ is clearly a factor set. It can be depicted as follows

μ	1	u	v	w
1	0	0	0	0
u	0	α	0	α
v	0	0	0	0
w	0	0	0	0

To show that $L=(G,A,\mu)$ is a left alternative loop, we verify Equation (1) as follows.

case 1 There is nothing to prove when $g,h=1$.

case 2 when $g=u$, Equation (1) becomes

$$\mu(u,u)=\mu(u,h)+\mu(u,uh).$$

If $h=u$, then $\mu(u,u)=\mu(u,u)+\mu(u,1)\Rightarrow\alpha=\alpha.$
If $h=v$, then $\mu(u,u)=\mu(u,v)+\mu(u,w)\Rightarrow\alpha=\alpha.$
If $h=w$, then $\mu(u,u)=\mu(u,w)+\mu(u,v)\Rightarrow\alpha=\alpha.$

case 3 when $g=v$, Equation (1) becomes

$$\mu(v,v)=\mu(v,h)+\mu(v,vh).$$

If $h=u$, then $\mu(v,v)=\mu(v,u)+\mu(v,w)\Rightarrow 0=0.$
If $h=v$, then $\mu(v,v)=\mu(v,v)+\mu(v,1)\Rightarrow 0=0.$
If $h=w$, then $\mu(v,v)=\mu(v,w)+\mu(v,u)\Rightarrow 0=0.$

case 4 when $g=w$, Equation (1) becomes

$$\mu(w,w)=\mu(w,h)+\mu(w,wh)$$

If $h=u$, then $\mu(w,w)=\mu(w,u)+\mu(w,v)\Rightarrow 0=0.$
If $h=v$, then $\mu(w,w)=\mu(w,v)+\mu(w,u)\Rightarrow 0=0.$
If $h=w$, then $\mu(w,w)=\mu(w,w)+\mu(w,1)\Rightarrow 0=0.$

Associativity:

$$(u,\alpha)((v,\alpha)(u,\alpha))=(u,\alpha)(w,0)=(v,0)$$

And

$$((u,\alpha)(v,\alpha))(u,\alpha)=(w,0)(u,\alpha)=(u,\alpha)$$

This implies

$$(u,\alpha)((v,\alpha)(u,\alpha))\neq((u,\alpha)(v,\alpha))(u,\alpha).$$

It implies that $L=(G,A,\mu)$ is non-flexible and hence non-associative.
Commutativity:

$$(u,\alpha)(w,\alpha)\neq(w,\alpha)(u,\alpha).$$

It implies that $L=(G,A,\mu)$ is non-commutative.
Now it remains to show that $N(L)=\{(1,a):\ a\in A\}$. For this consider

$$\begin{aligned}((g,b)(1,a))(h,c)&=(g,b)((1,a)(h,c))\\&=(g,b+a+\mu(g,1))(h,c)\\&=(g,b)(h,a+c+\mu(1,h))\\&=(g,b+a+0)(h,c)\\&=(g,b)(h,a+c+0)\\&=(gh,b+a+c+\mu(g,h))\\&=(gh,b+a+c+\mu(g,h)).\end{aligned}$$

Which is true, so $(1,a)\in N_\mu(L)$.
Similarly, we can show that

$$(1,a)\in N_\lambda(L) \text{ and } (1,a)\in N_\rho(L),$$

Hence $(1,a)\in N(L)$. This implies $N(L)=\{(1,a)\ :\ a\in A\}$.
Which is the required result.

Example 1. *The smallest group A satisfying the assumptions of Proposition 1 is the 2-element cyclic group* $\{0,1\}$. *The construction of Proposition 1 with* $\alpha=1$ *then gives rise to the smallest non-commutative non-associative left alternative loop of order 8.*

·	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	1	0	6	7	5	4
3	3	2	0	1	7	6	4	5
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

We verified the above example with the help of GAP (Group Algorithm Program) package [12]

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