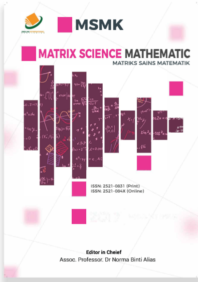
ZIBELINE INTERNATIONAL PUBLISHING

Matriks Sains Matematik (MSMK)

DOI : http://doi.org/10.26480/msmk.01.2018.18.23



ISSN: 2521-0874 (Print)
ISSN: 2521-0505 (online)
CODEN : MSMADH

CrossMark

RIGHT PURE UNI-SOFT IDEALS OF ORDERED SEMIGROUPS

Raees Khan[1,2*], Asghar Khan[2], M. Uzair Khan[1,2], Hidayat Ullah Khan[3]

[1]Department of Mathematics and Statistics, Bacha Khan University Charsadda, Khyber Pakhtunkhwa, Pakistan.
[2]Department of Mathematics, Abdul Wali Khan University Mardan, Khyber Pakhtunkhwa, Pakistan.
[3]Department of Mathematics, University of Malakand, Khyber Pakhtunkhwa, Pakistan.
**Corresponding Author email: raeeskhatim@yahoo.com*

ARTICLE DETAILS

Article History:

Received 05 November 2017
Accepted 07 December 2017
Available online 18 January 2018

ABSTRACT

In this paper, we initiate the study of pure uni-soft ideals in ordered semigroups. The soft version of right pure ideals in ordered semigroups is considered which is an extension of the concept of right pure ideal in ordered semigroups. We also give the main result for right pure uni-soft ideals in ordered semigroups and characterize right weakly regular ordered semigroups in terms of right pure uni-soft ideals.

KEYWORDS

Ordered semigroup, Soft set, Right pure uni-soft ideals, Weakly regular ordered semigroups

1. INTRODUCTION

The theory of fuzzy sets, which was initially introduced in 1965 by Zadeh, has been applied to many mathematical branches [1]. The study of fuzzy algebraic structures is an interesting research topic of fuzzy set theory. We noticed that the relationships between the fuzzy sets and algebraic structures have been already considered in [2-5]. Recently, studied the structure of fuzzy ideals in ordered semigroups, and provided some interesting results [5,6].

Soft set theory is basically the generalization of fuzzy set theory and originally proposed in 1999 by a Russian mathematician Molodstov [7]. Since then, soft set theory is widely investigated from the theoretical point of view and for their applications to many branches of pure and applied mathematics [8-12]. One of the main reason which attracts researches towards soft set theory is its uniqueness. Nowadays many scholars have studied different aspects of soft sets [11, 13-15]. It is worth pointing out that the concept of soft set as a generalization of fuzzy set [13]. A theory of soft set on ordered semigroups has been recently developed. In other studies, a group researchers applied the theory of soft set to ordered semigroups and introduced the notion of soft union ideals in ordered semigroups [21]. Since then, several researchers conducted the researches on the properties of soft ordered semigroups, and obtained some important results. For more details, the reader is referred to [16-21].

In this paper, we introduce the concept of right pure uni-soft ideal of an ordered semigroup. We identify those ordered semigroups for which each uni-soft ideal is idempotent. We also characterize those ordered semigroups for which each uni-soft two-sided ideal is right weakly pure uni-soft ideal

2. PRELIMINARIES

In this section, we recall the following definitions and results for subsequent use.

Definition 1. *By an ordered semigroup we mean a structure* $(S,\cdot,\leq)$ *such that:*

(1) $(S,\cdot)$ is a semigroup.

(2) $(S,\leq)$ is a poset.

(3) $(\forall a,b,x \in S)$ $(a \leq b \Rightarrow ax \leq bx$ and $xa \leq xb)$.

For $A \subseteq S$, we denote

$(A] := \{t \in S \;:\; t \leq h \text{ for some } h \in A\}$.

For $A, B \subseteq S$, we have $AB := \{ab \;:\; a \in A, b \in B\}$. A non-empty subset A of an ordered semigroup S is called a *subsemigroup* of S if $A^2 \subseteq A$.

Definition 2. *A left (resp. right) ideal* A *of an ordered semigroup* S *is a non-empty subset of* S *satisfying the following conditions:*

(1) $a \in A,\; a \geq b \in S,$ implies $b \in A$,

(2) $SA \subseteq A$ (resp. $AS \subseteq A$).

An ideal of S is a non-empty subset of S which is both a left and a right ideal of S.

We denote by $R(a)$ (resp. $L(a), I(a)$) the right (resp. left, two-sided ideal,) ideal of S generated by $a(a \in S)$, respectively. We have $R(a) = (a \cup aS]$, $L(a) = (a \cup Sa]$, $I(a) = (a \cup Sa \cup aS \cup SaS]$ [10]. A subset A of an ordered semigroup S is called *idempotent* if $A.A = A$.

Definition 3. *Let* I *be an ideal of* S. *Then* I *is called right pure in* S, *if for each* $x \in I$ *there exist* $y \in I$ *such that* $x \leq xy$. *Equivalently:* $x \in (xI]$ *for every* $x \in I$.

Definition 4. *An ordered semigroup* S *is called left (resp. right) weakly-*

regular if for every $a \in S$ *there exist* $x, y \in S$ *such that* $a \leq xaya$ *(resp.* $a \leq axay$ *)* [3,6,22].

Equivalent Definitions:

(1) $a \in ((Sa)^2]$ (resp. $a \in ((aS)^2]$ $\forall a \in S$ and

(2) $A \subseteq ((SA)^2]$ (resp. $A \subseteq ((AS)^2]$ $\forall A \subseteq S$.

If S is left weakly regular and right weakly regular then it is called *weakly regular*. Thus if S is *commutative* and weakly regular, then S is regular.

Definition 5. *Let* U *be a common universe,* E *be a set of parameters and* $A \subseteq E$ [7,13]. *Then a pair* (f_A, A) *is called a soft set over* U*, where* f_A *is a mapping given by* $f_A : E \to P(U)$*, such that* $f_A(x) = \phi$ *if* $x \notin A$. *Hence* f_A *is also called an approximation function.*

A soft set f_A over U can be represented by the set of orderd pairs $f_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\}$. It is clear from Definition 5, that a soft set is a *parameterized family* of subsets of U. Note that the set of all soft sets over U will be denoted $S(U)$.

Definition 6. [7, 13]. *Let* $f_A, f_B \in S(U)$. *we define the relation* $\tilde{\subseteq}$ *between* f_A *and* f_B*, the union and intersection of* f_A *and* f_B*, respectively, as*

(1) $f_A \tilde{\subseteq} f_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.

(2) $f_A \tilde{\cup} f_B = f_{A \cup B}$ is defined by $(f_A \tilde{\cup} f_B)(x) = f_A(x) \cup f_B(x)$ for all $x \in E$.

(3) $f_A \tilde{\cap} f_B = f_{A \cap B}$, is defined by $(f_A \tilde{\cap} f_B)(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.

For $x \in S$, we define

$$A_x = \{(y, z) \in S \times S \mid x \leq yz\}.$$

Let (f_S, S) and (g_S, S) be two soft sets over U. Then, the soft intersection-union product, denoted by $f_S \lozenge g_S$, is defined by

$$f_S \lozenge g_S : S \to P(U), x \mapsto \begin{cases} \bigcap_{(y,z) \in A_x} \{f_S(y) \cup g_S(z)\}, & \text{if} \quad A_x \neq \phi, \\ U, & \text{if} \quad A_x = \phi. \end{cases}$$

for all $x \in S$. One can easily prove that "◊" on $S(U)$ is well defined and the set ($S(U)$, ◊, $\tilde{\subseteq}$) forms an ordered semigroup [10].

For an ordered semigroup, the soft sets ϕ_S and T_S of S over U are defined as follows:

$$\phi_S : S \to P(U), x \mapsto \phi_S(x) = \phi,$$
$$\mathrm{T}_S : S \to P(U), x \mapsto \mathrm{T}_S(x) = U \text{ for all } x \in S.$$

Clearly, the soft set ϕ_S (resp. T_S) of an ordered semigroup S over U is the *least* (resp., the *greatest*) element of the ordered semigroup ($S(U)$, ◊, $\tilde{\subseteq}$). The soft set ϕ_S is the *null* element of ($S(U)$, ◊, $\tilde{\subseteq}$) (that is $f_S \lozenge \phi_S = \phi_S \lozenge f_S = \phi_S$ and $\phi_S \tilde{\subseteq} f_S$ for every $f_S \in S(U)$. The soft set (T_S, S) is called the *whole soft set* over U, where $\mathrm{T}_S(x) = U$ for all $x \in S$.

For a non-empty subset A of S, the *characteristic soft set* (χ_A, A) over U is a soft set defined as follows:

$$\chi_A : S \to P(U), x \mapsto \begin{cases} U, & \text{if} \quad x \in A, \\ \phi, & \text{if} \quad x \in S \setminus A. \end{cases}$$

For a non-empty subset A of S and $\varepsilon, \delta \in P(U)$ with $\varepsilon \subseteq \delta$, define a soft set $\chi_A^{(\varepsilon,\delta)}$ as follows:

$$\chi_A^{(\varepsilon,\delta)} : S \to, x \mapsto \begin{cases} \varepsilon, & \text{if} \quad x \in A, \\ \delta, & \text{if} \quad x \notin A, \end{cases}$$

The soft set $(\chi_A^{(\varepsilon,\delta)}, S)$ is called $(\varepsilon,\delta)-$characteristic uni-soft set over U. The $(\varepsilon,\delta)-$characteristic uni-soft set $(\chi_A^{(\varepsilon,\delta)}, S)$ with $\varepsilon = \phi$ and $\delta = U$ is called the characteristic soft set, and is denoted by (χ_A^c, S).
For a soft set (f_S, S) over U and a subset δ of U, the δ-exclusive set of (f_S, S), denoted by $e_S(f_S;\delta)$, is defined to be the set $e_A(f_S;\delta) = \{x \in A \mid f_S(x) \subseteq \delta\}$.

Definition 7. *Let* (f_S, S) *be a soft set over* U .[23]. *Then*

(1) (f_S, S) is called a *uni-soft semigroup* over U, if

$$f_S(xy) \subseteq f_S(x) \cup f_S(y), \forall x, y \in S.$$

(2) (f_S, S) is called a *uni-soft left (resp., right) ideal* over U if
  (i) $x \leq y \Rightarrow f_S(x) \subseteq f_S(y)$,
  (ii) $(\forall x, y \in S)$ $(f_S(xy) \subseteq f_S(y))$ (resp., $f_S(xy) \subseteq f_S(x)$).

If a soft set (f_S, S) over U is both a uni-soft left ideal and a uni-soft right ideal over U, we say that (f_S, S) is a uni-soft two-sided ideal over U.

Definition 8. *A soft set* (f_S, S) *of an ordered semigroup* $(S, \cdot, \leq)$ *over* U *is called idempotent if* ($f_S \lozenge f_S$, S) $= (f_S, S)$.

Lemma 1. [10]. *For any soft sets* (f_S, S), (g_S, S) *and* (h_S, S) *over* U*, we have*

(1) $(f_S \tilde{\cup} (g_S \tilde{\cap} h_S), S) = (f_S \tilde{\cup} g_B, S) \tilde{\cap} (f_S \tilde{\cup} h_B, S)$,

(2) $(f_S \tilde{\cap} (g_S \tilde{\cup} h_S), S) = (f_S \tilde{\cap} g_B, S) \tilde{\cup} (f_S \tilde{\cap} h_B, S)$,

(3) $(f_S \lozenge (g_S \tilde{\cup} h_S), S) = (f_S \lozenge g_S, S) \tilde{\cup} (f_S \lozenge h_S, S)$.

(4) $(f_S \lozenge (g_S \tilde{\cap} h_S), S) = (f_S \lozenge g_S, S) \tilde{\cap} (f_S \lozenge h_S, S)$.

Proposition 1. [23]. *For subset* A *and* B *of* S*, the following properties hold*

(i) $(\chi_A^c \lozenge \chi_A^c, S) = (\chi_{(AB]}^c, S)$,

(ii) $(\chi_A^c \tilde{\cup} \chi_A^c, S) = (\chi_{A \tilde{\cup} B}^c, S)$.

Lemma 2. [8]. *A soft set* (f_S, S) *is a uni-soft semigroup over* U *if and only if*

$$(f_S \lozenge f_S, S) \tilde{\supseteq} (f_S, S)$$

Proposition 2. *Let* S *be an ordered semigroup* [23]. *Then for every uni-soft right ideal* (f_S, S) *and uni-uni-soft left ideal* (g_S, S) *of* S *over* U*, we have*

$$(f_S \lozenge g_S, S) \tilde{\supseteq} (f_S \tilde{\cup} g_S, S)$$

Proposition 3. *Let S be an ordered semigroup, (f_S, S) a uni-soft left (resp., right) ideal of S over U. Then $\phi_S \lozenge f_S \tilde{\supseteq} f_S$ (resp., $f_S \lozenge \phi_S \tilde{\supseteq} f_S$)* [23].

Proposition 4. *Let S be an ordered semigroup, (f_S, S) and (g_S, S) are uni-soft ideals over U. Then $(f_S \tilde{\cup} g_S, S)$ is a uni-soft ideal over U.*

Proof. Suppose that (f_S, S) and (g_S, S) are uni-soft ideals over U. Let $x, y \in S$ be such that $x \leq y$. Then

$$(f_S \tilde{\cup} g_S)(x) = f_S(x) \cup g_S(x) \subseteq f_S(y) \cup g_S(y) = (f_S \tilde{\cup} g)(y)$$

Also, by Lemma 1 and Proposition 3, we have

$$(f_S \tilde{\cup} g_S) \lozenge \phi_S = (f_S \lozenge \phi_S) \tilde{\cup} (g_S \lozenge \phi_S) \tilde{\supseteq} f_S \tilde{\cup} g_S.$$

Thus $f_S \tilde{\cup} g_S$ is uni--soft right ideal of S over U. Similarly, we have

$$\phi_S \lozenge (f_S \tilde{\cup} g_S) = (\phi_S \lozenge f_S) \tilde{\cup} (\phi_S \lozenge g_S) \tilde{\supseteq} f_S \tilde{\cup} g_S.$$

Hence, $f_S \tilde{\cup} g_S$ is uni--soft left ideal of S over U. Therefore, $(f_S \tilde{\cup} g_S, S)$ *is a uni-soft ideal over U.*

Theorem 1. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then for a non-empty subset A of S, the following conditions are equivalent:*

(1) A is a right (resp., left, two-sided) ideal of S.

(2) The (ε, δ)-characteristic soft set $(\chi_S^{(\varepsilon,\delta)}, S)$ is a uni-soft right (resp., left, two-sided) ideal of S over U, where $\varepsilon, \delta \subseteq P(U)$ with $\varepsilon \subset \delta$

Proof. Suppose that A is a right ideal of S and $x, y \in S$. Let $\varepsilon, \delta \subseteq P(U)$ with $\varepsilon \subset \delta$ If $x \notin A$ implies that $\chi_S^{(\varepsilon,\delta)}(x) = \delta$, then obviously $\chi_S^{(\varepsilon,\delta)}(x) = \delta \supseteq \chi_S^{(\varepsilon,\delta)}(xy)$. If $x \in A$, then $\chi_S^{(\varepsilon,\delta)}(x) = \varepsilon$ and $xy \in AS \subseteq A$. Hence

$$\chi_S^{(\varepsilon,\delta)}(xy) = \varepsilon = \chi_S^{(\varepsilon,\delta)}(x).$$

Let $x, y \in S$ be such that $x \leq y$. If $y \in A$ then $\chi_S^{(\varepsilon,\delta)}(y) = \varepsilon$. Since A is a right ideal of S, so $x \in A$. Hence $\chi_S^{(\varepsilon,\delta)}(x) = \varepsilon = \chi_S^{(\varepsilon,\delta)}(y)$. If $y \notin A$ then $\chi_S^{(\varepsilon,\delta)}(y) = \delta$, then obviously

$$\chi_S^{(\varepsilon,\delta)}(x) \subseteq \delta = \chi_S^{(\varepsilon,\delta)}(y).$$

Therefore, $(\chi_S^{(\varepsilon,\delta)}, S)$ is a uni-soft right ideal of S over U.

Conversely, Assume that (ε, δ)-characteristic soft set $(\chi_S^{(\varepsilon,\delta)}, S)$ is a uni-soft right ideal of S over U, for any $\varepsilon, \delta \subseteq P(U)$ with $\varepsilon \subset \delta$
Let a be any element of AS. Then $a = xy$ for some $y \in S$ and $x \in A$. Then

$$\chi_S^{(\varepsilon,\delta)}(a) = \chi_S^{(\varepsilon,\delta)}(xy) \subseteq \chi_S^{(\varepsilon,\delta)}(x) = \varepsilon.$$

Thus $a \in A$, which means that $AS \subseteq A$.
Furthermore, let $x, y \in S$ with $x \leq y \in A$, then $\chi_S^{(\varepsilon,\delta)}(y) = \varepsilon$. By hypothesis we have

$$\chi_S^{(\varepsilon,\delta)}(x) \subseteq \chi_S^{(\varepsilon,\delta)}(y) = \varepsilon.$$

Thus $\chi_S^{(\varepsilon,\delta)}(x) = \varepsilon$, which follows that $x \in A$. Therefore, A is a right ideal of S.

Lemma 3. [23]. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then for a non-empty subset A of S, the following conditions are equivalent:*

(1) A is a right (resp., left, two-sided) ideal of S.

(2) The soft set (χ_A^c, S) over U is an uni-soft right (resp., left, two-sided) ideal over U.

3. RIGHT PURE UNI-SOFT IDEALS

In this section, we define right pure uni-soft ideals and prove that the notion of a right pure uni-soft ideal of S over U is an extension of ordinary right pure ideal of S.

Definition 9. *A uni-soft ideal g_s of an ordered semigroup S over U is called right pure uni-soft ideal of S over U if $f_s \tilde{\cup} g_s = f_S \lozenge g_S$ for each uni-soft right ideal f_S of S over U.*

Lemma 4. *An ideal of an ordered semigroup S is right pure if and only if $R \cap I = (RI]$ for every right ideal R of S.*

Proof. Suppose that I is an ideal of S and R a right ideal of S. Then

$$(RI] \subseteq (SI] \subseteq (I] = I$$

And

$$(RI] \subseteq (RS] \subseteq (R] = R.$$

Hence $(RI] \subseteq R \cap I$. Let $a \in R \cap I$, then $a \in R$ and $a \in I$. Since I is right pure, so there exists $b \in I$ such that $a \leq ab$. But $ab \in RI$. So $a \in (RI]$. Thus $R \cap I \subseteq (RI]$. Hence $R \cap I = (RI]$.

Conversely, assume that $R \cap I = (RI]$ for every right ideal R of S. Let $a \in I$. Take R the right ideal of S generated by a, that is, $R = (a \cup aS]$. Then $R \cap I = (RI]$ implies $a \in (RI]$, where $(RI] = ((a \cup aS]I] \subseteq ((a \cup aS)I] \subseteq (aI]$. Thus there exists $b \in I$ such that $a \leq ab$. Hence I is a right pure in S.

The following proposition shows that the notion of a right pure uni-soft ideal is an extension of an ordinary right pure ideal of S.

Proposition 5. *Let A be an ideal of an ordered semigroup S. Then the following conditions are equivalent:*

(1) A is right (resp. left) pure in S

(2) The soft set (χ_A^c, S) over U is right (resp. left) pure uni-soft ideal over U.

Proof. $(1) \Rightarrow (2)$: Suppose that A is right pure in S. Since A is an ideal of S, then by Lemma 3, the soft set (χ_A^c, S) over U is a uni-soft ideal over U. To prove that soft set (χ_A^c, S) over U is right pure uni-soft ideal over U, we show that $f_S \tilde{\cup} \chi_A^c = f_S \lozenge \chi_A^c$ for each uni-soft right ideal (f_S, S) over U. Let $a \in S$, and $A_a \neq \phi$. Then

$$(f_S \lozenge \chi_A^c)(a) = \bigcap_{a \leq xy} \{f_S(x) \cup \chi_A^c(y)\}$$

$$\supseteq \bigcap_{a \leq xy} \{f_S(xy) \cup \chi_A^c(xy)\}$$

$$\supseteq f_S(a) \cup \chi_A^c(a)$$

$$= f_S \cup \chi_A^c)(a).$$

This implies that $f_S \tilde{\cup} \chi_A^c \tilde{\subseteq} f_S \lozenge \chi_A^c$.

Now if $a \notin A$, then $\chi_A^c(a) = U$. Thus $(f_S \cup \chi_A^c)(a) = U \supseteq (f_S \lozenge \chi_A^c)(a)$.

Consider the case when $a \in A$. Since A is right pure in S, so for each $a \in A$ there exist $b \in A$ such that $a \leq ab$. We have $\chi_A^c(b) = \phi$ (since $b \in A$), therefore

$$\begin{aligned}(f_S \cup \chi_A^c)(a) &= f_S(a) \cup \chi_A^c(a)\\ &= f_S(a) \cup \chi_A^c(b)\\ &\supseteq \bigcap_{a \leq db} \{f_S(d) \cup \chi_A^c(b)\}\\ &= (f_S \lozenge \chi_A^c)(a).\end{aligned}$$

which implies that $f_S \tilde{\cup} \chi_A^c \tilde{\supseteq} f_S \lozenge \chi_A^c$. Thus $f_S \tilde{\cup} \chi_A^c = f_S \lozenge \chi_A^c$.
Conversely, assume that soft set (χ_A^c, S) over U is right pure uni-soft ideal over U. Then we show that A is right pure in S, that is $R \cap A = (RA]$, for each right ideal R of S. Let $x \in R \cap A$, implies that $x \in R$ and $x \in A$. Since R is a right ideal of S, then by Lemma 3, the soft set (χ_R^c, S) over U is uni-soft right ideal over U, then $\chi_R^c(x) = \phi$. By hypothesis, we have

$$(\chi_R^c \lozenge \chi_A^c)(x) = (\chi_R^c \cup \chi_A^c)(x) = \chi_R^c(x) \cup \chi_A^c(x) = \phi$$

Which follows that $\chi_R^c \lozenge \chi_A^c = \phi$ by Proposition 1, we have $\chi_{(RA]}^c(x) = \phi$, which means that $x \in (RA]$. That is $R \cap A \subseteq (RA]$ and clearly $(RA] \subseteq R \cap A$. Hence $R \cap A = (RA]$ and so A is right pure in S.

Similarly, we can prove the following theorem.

Theorem 2. *Let A be an ideal of an ordered semigroup S. Then the following conditions are equivalent:*

(1) A is pure in S,

(2) The soft set (χ_A^c, S) over U is pure uni-soft ideal over U.

Proposition 6. *A non-empty subset A of a an ordered semigroup S is a right pure ideal of S if and only if the soft set f_S defined by*

$$f_S = \begin{cases} \phi, & \text{if } x \in A,\\ \delta, & \text{if } x \notin A.\end{cases}$$

is a right pure uni-soft of S over U, where $\delta \subseteq U$.
Proof. Suppose that A is a right pure ideal of S. We show that f_S is a right pure uni-soft ideal of S over U, that is $g_S \tilde{\cup} f_S = g_S \lozenge f_S$ for every uni-soft right ideal g_S of S over U. Let $a \notin A$, then $f_S(a) = \delta$ and so

$$(g_S \cup f_S)(a) = \delta \supseteq (g_S \lozenge f_S)(a).$$

If $a \in A$, since A is a right pure ideal of S. Then there exist $b \in A$ such that $a \leq ab$. Thus $(a,b) \in A_a$. Hence

$$\begin{aligned}(g_S \cup f_S)(a) &= g_S(a) \cup f_S(a)\\ &= g_S(a) \cup f_S(b)\\ &\supseteq \bigcap_{a \leq db} \{g_S(d) \cup f_S(b)\}\\ &= (g_S \lozenge f_S)(a).\end{aligned}$$

Which means that $g_S \tilde{\cup} f_S \supseteq g_S \lozenge f_S$. And by Proposition 2, it follows that $g_S \tilde{\cup} f_S \subseteq g_S \lozenge f_S$. Hence $g_S \tilde{\cup} f_S = g_S \lozenge f_S$. Therefore, f_S is a right pure uni-soft ideal of S over U.

Conversely, assume that f_S is a right pure uni-soft ideal of S over U. Then f_S is a uni-soft right ideal of S over U. We show that A is a right pure ideal of S, that is $R \cap A = (RA]$, for every right ideal R of S. Let $x \in R \cap A$, implies that $x \in R$ and $x \in A$. Since R is a right ideal of S, then by Lemma 3, the soft set (χ_R^c, S) over U is uni-soft right ideal over U. By hypothesis , we have

$$(\chi_R^c \lozenge f_S)(x) = (\chi_R^c \cup f_S)(x) = \chi_R^c(x) \cup f_S(x) = \phi$$

Which follows that $(\chi_R^c \lozenge f_S)(x) = \phi$, by Proposition 1, we have $\chi_{(RA]}^c(x) = \phi$. Which means that $x \in (RA]$. That is $R \cap A \subseteq (RA]$ and clearly $(RA] \subseteq R \cap A$. Hence $R \cap A = (RA]$ and so A is right pure ideal of S.

Proposition 7. *Let $(S, \cdot, \leq)$ be an ordered semigroup, (f_S, S) a uni-soft right ideal and (g_S, S) a uni-soft ideal over U. Then the soft product $(g_S \lozenge f_S, S)$ of (f_S, S) and (g_S, S) is a uni-soft right ideal over U.*

Proof. Let (f_S, S) be a uni-soft right ideal and (g_S, S) a uni-soft ideal over U. Let $x, y, z \in S$. Then

$$(f_S \lozenge g_S)(x) = \bigcap_{(p,q) \in A_x} \{f_S(p) \cup g_S(q)\}$$

Since $x \leq pq$, hence $xy \leq (pq)y = p(qy)$ and $(p, qy) \in A_{xy}$. Thus, $A_{xy} \neq \phi$ and we have

$$\begin{aligned}(f_S \lozenge g_S)(x) &= \bigcap_{(p,q) \in A_x} \{f_S(p) \cup g_S(q)\}\\ &\supseteq \bigcap_{(p,qy) \in A_{xy}} \{f_S(p) \cup g_S(qy)\}\\ &\supseteq \bigcap_{(a,b) \in A_{xy}} \{f_S(a) \cup g_S(b)\}\\ &= (f_S \lozenge g_S)(xy).\end{aligned}$$

Thus, $(f_S \lozenge g_S)(xy) \subseteq (f_S \lozenge g_S)(x)$.

Let $x, y \in S$ be such that $x \leq y$. Then $(f_S \lozenge g_S)(x) \subseteq (f_S \lozenge g_S)(y)$. In fact, if $(p,q) \in A_y$, then $y \leq pq$ and we have $x \leq y \leq pq$ it follows that $(p,q) \in A_x$ and hence $A_y \subseteq A_x$. If $A_x = \phi$, then $A_y = \phi$ and we have $f_S \lozenge g_S)(y) = U \supseteq (f_S \lozenge g_S)(x)$. If $A_x \neq \phi$, then $A_y \neq \phi$ and we have

$$\begin{aligned}(f_S \lozenge g_S)(y) &= \bigcap_{(p,q) \in A_y} \{f_S(p) \cup g_S(q)\}\\ &\supseteq \bigcap_{(p,q) \in A_x} \{f_S(p) \cup g_S(q)\}\\ &= (f_S \lozenge g_S)(x).\end{aligned}$$

Hence $(f_S \lozenge g_S)(x) \subseteq (f_S \lozenge g_S)(y)$ for all $x, y \in S$ with $x \leq y$. Therefore, the soft product $(g_S \lozenge f_S, S)$ of (f_S, S) and (g_S, S) is a uni-soft right ideal over U.

Proposition 8. *Let $(S,\cdot,\leq)$ be an ordered semigroup, f_{S_1} and f_{S_2} are right pure uni-soft ideals over U. Then $f_{S_1}\tilde{\cup} f_{S_2}$ is also a right pure uni-soft ideal over U.*

Proof. Let f_{S_1} and f_{S_2} be right pure uni-soft ideals of an ordered semigroup S over U. We have to show that $f_{S_1}\tilde{\cup} f_{S_2}$ is a right pure uni-soft ideal of S over U. That is, for each uni-soft right ideal g_S of S over U, we have

$$g_S \lozenge (f_{S_1}\tilde{\cup} f_{S_2}) = g_S\tilde{\cup}(f_{S_1}\tilde{\cup} f_{S_2})$$

Since f_{S_1} is a right pure uni-soft ideal of S over U. It follows that $f_{S_1} \lozenge f_{S_2} = f_{S_1}\tilde{\cup} f_{S_2}$. Therefore,

$$g_S \lozenge (f_{S_1}\tilde{\cup} f_{S_2}) = g_S \lozenge (f_{S_1}\lozenge f_{S_2}) \qquad \text{(i)}$$

Since f_{S_1} is a right pure uni-soft ideal of S over U, so

$$g_S\tilde{\cup}(f_{S_1}\tilde{\cup} f_{S_2}) = (g_S\tilde{\cup} f_{S_1})\tilde{\cup} f_{S_2} = (g_S \lozenge f_{S_1})\tilde{\cup} f_{S_2} \qquad \text{(ii)}$$

Now by Proposition 7, it follows that, $g_S \lozenge f_{S_1}$ is a uni-soft right ideal of S. Thus (ii) gives us

$$g_S\tilde{\cup}(f_{S_1}\tilde{\cup} f_{S_2}) = (g_S\tilde{\cup} f_{S_1})\tilde{\cup} f_{S_2}$$

$$= (g_S \lozenge f_{S_1})\tilde{\cup} f_{S_2}$$

$$= (g_S \lozenge f_{S_1})\lozenge f_{S_2}$$

$$= g_S \lozenge (f_{S_1}\lozenge f_{S_2})$$

(by the associativity of the operation "◊")
Thus (i) and (ii) give us,

$$g_S \lozenge (f_{S_1}\tilde{\cup} f_{S_2}) = g_S\tilde{\cup}(f_{S_1}\tilde{\cup} f_{S_2})$$

Therefore, $f_{S_1}\tilde{\cup} f_{S_2}$ is a right pure uni-soft ideal of S over U.

Proposition 9. *Intersection of any collection of uni-soft bi-ideals of right pure uni-soft ideal of an ordered semigroup S over U is a right pure uni-soft ideal of S over U.*

Proof. Let $\{f_{S_i} : i \in I\}$ is a family of right pure uni-soft ideals of S *over* U. We have to show that $\bigcap_{i\in I} f_{S_i}$ is also a right pure uni-soft ideal of S *over* U. That is we have to show that

$$g_S\tilde{\cup}(\bigcap_{i\in I} f_{S_i}) = g_S \lozenge (\bigcap_{i\in I} f_{S_i})$$

for any uni-soft right ideal g_S of S over U. Since f_{S_i} is a pure uni-soft ideals of S over U. Therefore,

$$g_S\tilde{\cup}(\bigcap_{i\in I} f_{S_i}) = \bigcap_{i\in I}(g_S\tilde{\cup} f_{S_i}) = \bigcap_{i\in I}(g_S \lozenge f_{S_i}).$$

Now for each $a \in S$ and $A \neq \phi$ we have

$$g_S \lozenge (\bigcap_{i\in I} f_{S_i})(a) = \bigsqcap_{(y,z)\in A_a} \{g_S(y) \cup (\bigsqcap_{i\in I} f_{S_i})(z)\}$$

$$= \bigsqcap_{(y,z)\in A_a} \{\bigsqcap_{i\in I}(g_S(y) \cup f_{S_i}(z)\}$$

$$= \bigsqcap_{i\in I} [\bigsqcap_{(y,z)\in A_a} \{(g_S(y) \cup f_{S_i}(z)\}]$$

$$= \bigsqcap_{i\in I} (g_S \lozenge f_{S_i})(a).$$

Hence, $g_S \lozenge (\bigcap_{i\in I} f_{S_i})(a) = g_S\tilde{\cup}(\bigcap_{i\in I} f_{S_i})$ and therefore $\bigcap_{i\in I} f_{S_i}$ is a right pure uni-soft ideal of S over U.

4. RIGHT WEAKLY REGULAR ORDERED SEMIGROUPS

In this section we characterize right pure uni-soft ideal of a right weakly regular ordered semigroups [24-27]. We prove that in aright weakly regular ordered semigroups every uni-soft ideal of S over U is a pure uni-soft ideal.

Proposition 10. *Let $(S,\cdot,\leq)$ be a right weakly regular ordered semigroup. Then every ideal of S is right pure ideal of S.*

Proposition 11. *Let $(S,\cdot,\leq)$ be a right weakly regular ordered semigroup. Then every uni-soft ideal (g_S, S) over U is right pure uni-soft ideal of S over U*

Proof. Suppose that S be a right weakly regular ordered semigroup and (g_S, S) a uni-soft ideal of S over U. We show that (g_S, S) over U is right pure uni-soft ideal of S over U, That is $f_S\tilde{\cup} g_S = f_S \lozenge g_S$, for each uni-soft right ideal (f_S, S) of S over U. Since S right weakly regular, there exist $x, y \in S$, such that $a \leq axay$. Then $(ax, ay) \in A_a$. Since $A_a \neq \phi$. we have

$$(f_S \lozenge g_S)(a) = \bigsqcap_{(p,q)\in A_a} \{f_S(p) \cup g_S(q)\} \subseteq f_S(ax) \cup g_S(ay)$$

Since (f_S, S) is a uni-soft right ideal and (g_S, S) is a uni-soft ideal of S over U. We have

$$f_S(ax) \subseteq f_S(a) \text{ and } g_S(ay) \subseteq g_S(a).$$

Thus

$$(f_S \lozenge g_S)(a) \subseteq f_S(ax) \cup g_S(ay)$$

$$\subseteq f_S(a) \cup g_S(a)$$

$$= (f_S \cup g_S)(a)$$

Therefore, $(f_S \lozenge g_S, S) \tilde{\subseteq} (f_S\tilde{\cup} g_S, S)$. On the other hand, by Proposition 2, we have $(f_S \lozenge g_S, S) \tilde{\supseteq} (f_S\tilde{\cup} g_S, S)$. Therefore, $(f_S \lozenge g_S, S) = (f_S\tilde{\cup} g_S, S)$.

Theorem 3. *Let $(S,\cdot,\leq)$ be an ordered semigroup. Then the following conditions are equivalent:*

(1) S is right weakly regular.

(2) Every uni-soft right ideal of S over U is idempotent.

(3) Every uni-soft ideal of S over U is a pure uni-soft ideal.

Proof. $(1) \Leftrightarrow (2)$: Follows from [10].

$(1) \Rightarrow (3)$: Let (f_S, S) be a uni-soft ideal of S over U. we show that (f_S, S) is a pure uni-soft ideal, that is $g_S\tilde{\cup} f_S = g_S \lozenge f_S$, for every uni-soft right ideal of S over U. Let $a \in S$. Since S is right weakly regular, there exist $x, y \in S$, such that $a \leq axay$. Thus $(ax, ay) \in A_a$.

Since $A_a \neq \phi$ we have

$$(g_S \cup f_S)(a) = g_S(a) \cup f_S(a)$$

$$\supseteq g_S(ax) \cup f_S(ay)$$

$$\supseteq \bigcap_{(ax,ay)\in A_a} \{g_S(ax) \cup f_S(ay)\}$$

$$\supseteq \bigcap_{(p,q)\in A_a} \{g_S(p) \cup f_S(q)\}$$

$$= (g_S \lozenge f_S)(a).$$

Which means that $g_S \tilde{\cup} f_S \tilde{\supseteq} g_S \lozenge f_S$. And from Proposition 3, we have the reverse inclusion $g_S \lozenge f_S \tilde{\supseteq} g_S \tilde{\cup} f_S$. Hence $g_S \tilde{\cup} f_S = g_S \lozenge f_S$, that is (f_S, S) is a pure uni-soft ideal over U.

$(3) \Rightarrow (1)$: Let $a \in S$. We show that $a \in ((aS)^2]$. Let $I = (a \cup Sa \cup aS \cup SaS]$ be an ideal generated by a. Then by Lemma 3, the soft (χ_I^c, S) over U is a uni-soft right ideal of S over U, hence by (3), (χ_I^c, S) is a pure uni-soft ideal. Thus, by the Theorem 2, I is pure in S. Since $a \in I$ and I is pure in S, therefore there exist $b \in I$ such that $a \leq ab$. This means that

$$a \in (aI] = (a(a \cup Sa \cup aS \cup SaS)]$$
$$= (aa \cup aSa \cup aaS \cup aSaS].$$

Which means that $a \in ((aS)^2]$. Hence S is right weakly regular ordered semigroup.

REFERENCES

[1] Zadeh, L.A. 1965. Fuzzy sets. Information and Control, 8, 338-35.

[2] Bashir, S., Shabir, M., Rehman, N. 2012. Pure Fuzzy Ideals in Ternary Semigroups. International Journal of Algebra and Statistics, 1 (2), 1-7.

[3] Kehayopulu, N., Tsingelis, M. 2006. Regular ordered semigroups in terms of fuzzy subsets. Information Sciences, 176 (24), 3675-3693.

[4] Kehayopulu, N. 1990. Remark on ordered semigroups. Japanese Journal of Mathematics, 35 (6), 1061-1063.

[5] Khan, A., Jun, Y.B., Shabir, M. 2008. Fuzzy ideals in ordered semigroups-I. Quasigroups Related Systems, 16 (2), 207-220.

[6] Shabir, M., Khan, A. 2010. Characterizations of ordered semigroups by the properties of their fuzzy ideals. Computers & Mathematics with Applications, 59 (1), 539-549.

[7] Shabir, M., Khan, A. 2011. On fuzzy quasi-ideals of ordered semigroups. Bulletin of the Malaysian Mathematical Sciences Society, 34 (1), 87-102.

[8] Jun, Y.B., Lee, K.J., Khan, A. 2010. Soft ordered semigroups. Mathematical Logic Quarterly, 56 (1), 42--50.

[9] Atagun, A.O., Sezgin, A. 2011. Soft substructures of rings, fields and modules. Computers and Mathematics with Applications, 61 (3), 592-601.

[10] Khan, A., Jun, Y.B., Shah, S.I.A., Khan, R. 2016. Applications of soft-union sets in ordered semigroups via SU quasi-ideals. Journal of Intelligent and Fuzzy Systems, 30 (1), 97-107.

[11] Khan, R., Khan, A., Khan, M. U. 2017. Certain characterizations of ordered semigroups by uni-soft generalized bi-ideals. Journal of Applied Environmental and Biological Sciences, 7 (10), 19-31.

[12] Maji, P.K., Roy, A.R., Biswas, R. 2002. An application of soft sets in a decision-making problem. Computers and Mathematics with Applications, 44 (2), 1077-1083.

[13] Cagman, N., Enginoglu, S. 2010. Soft set theory and uni--int decision making. European Journal of Operational Research, 207 (2), 848-855.

[14] Maji, P.K., Biswas, R., Roy, A.R. 2003. Soft set theory. Computers and Mathematics with Applications, 45 (3), 555-562.

[15] Aktas, H., Cağman, N. 2007. Soft sets and soft groups. Information Sciences, 177 (13), 2726-2735.

[16] Feng, F., Ali, M.I., Shabir, M. 2013. Soft relations applied to semigroups. Filomat, 27 (7), 1183-1196.

[17] Chen, D., Tsang, E.C.C., Yeung, D.S., Wang, X. 2005. The parameterization reduction of soft sets and its applications. Computers and Mathematics with Applications, 49 (5-6), 757-763.

[18] Feng, F., Jun, Y.B., Zhao, X. 2008. Soft semirings. Computers and Mathematics with Applications, 56 (10), 2621-2628.

[19] Sezgin, A., Atagun, A., Cagman, N. 2012. Union soft substructures of near-rings and near-ring modules. Neural Computing and Applications, 21 (1), 221-229.

[20] Sezgin, A., Atagun, A., Cagman, N. 2012. Soft intersection near-rings with applications. Neural Computing and Applications, 21 (1), 133-143.

[21] Atagun, A., Sezgin, A. 2011. Soft substructures of rings, fields and modules. Computers and Mathematics with Applications, 61 (1), 592-601.

[22] Kehayopulu, N. 1991. On fuzzy semigroups. Information Sciences, 53 (2), 203-236.

[23] Jun, Y.B., Song, S.Z., Muhiuddin, G. 2014. Concave soft sets, critical soft points, and union-soft ideals of ordered semigroups. The Scientific World Journal, Article ID 467968, 11 pages.

[24] Bashir, S., Shabir, M. 2009. Pure ideals in ternary semigroups. Quasigroups and Related Systems, 17 (2), 149-160.

[25] Shabir, M., Khan, A. 2010. Characterizations of ordered semigroups by the properties of their fuzzy ideals. Computers and Mathematics with Applications, 59 (1), 539-549.

[26] Khan, A., Khan, R., Jun, Y.B. 2017. Uni-soft structure applied to ordered semigroups. Soft Computing, 21 (4), 1021-1030. DOI 10.1007/s00500-015-1837-8.

[27] Ali, M.I., Feng, F., Liu, X., Min, W. K., Shabir, M. 2009. On some new operations in soft set theory. Computers and Mathematics with Applications, 57 (2), 1547-1553.

Cite the Article: Raees Khan, Asghar Khan, M. Uzair Khan, Hidayat Ullah Khan (2018). Right Pure Uni-Soft Ideals Of Ordered Semigroups. *Matriks Sains Matematik*, 2(1) : 18-23.