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LEFT DOUBLE DISPLACEMENT SEMIGROUP: A FIRST RESULT

Nisar Ahmad[1*], Mutahir Ali[1], Farhad Ali[1], Arif Mehmood Khattak[2]

[1]Department of Mathematics, Kohat University of Science and Technology (KUST), Kohat KPK, Pakistan
[2]Department of Mathematics and Statistics, Riphah International University Islamabad, Pakistan
**Corresponding Author email: nisarnn22@yahoo.com*

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ABSTRACT

In the present article a new type of algebraic structures named as left double displacement semigroup (LDD-semigroup). The structure is enhanced toward its left double displacement group (LDD-group) and discovered some useful results about these structures. The name of the notion is due to double displacement reactions in chemistry because they have same pattern of elements arrangement.

KEYWORDS

Groupoid, Semigroup, Left Almost Semigroup, Abel Grassedman Groupoid, Left Double Displacement Semigroup, Left Double Displacement Group.

1. INTRODUCTION

1.1 Semigroups and Groups

In literature semigroup is consider in the advanced topic of algebra. Basically, it is a magma which hold associative property. Mostly the binary operation of a semigroup is denoted by $\lambda.\mu$, or simply $\lambda\mu$. In literature associativity is formally expressed as $(\lambda\mu)\nu = \lambda(\mu\nu)$ for all λ, μ and ν in the semigroup. The collection of positive integers is the example of semigroup under the usual addition and multiplication. The semigroup structure is enhanced toward group, ring, module, vector space etc. The next subsection is about left almost semigroup and left almost group and their related results.

1.2 LA-semigroup and its Enhancement

Kazim and Naseeruddin introduced the idea of left almost semigroup. A groupoid T is said to be a left almost semigroup or an LA-semigroup if all of its elements satisfy the left invertive law: $(\lambda\mu)\nu = (\nu\mu)\lambda$, for all $\lambda, \mu, \nu \in T$ [1]. In some articles this structure is also known as an AG-groupoid because it also satisfies the Abel-Grassmann's property: $(\lambda\mu)(\nu\delta) = (\lambda\nu)(\mu\delta)$, for all $\lambda, \mu, \nu, \delta \in T$ [2]. Some beautiful results about this structure was studied in [3]. Moreover, Mushtaq and Kamran extended this structure to its left almost groups in [5]. Left almost groups is an LA-semigroup which have left identity and each element have left inverse. Some properties of LA-group are very similar to an abelian group, as like it is interesting to know that, we can make a factor for an LA-group by any of its sub-LA-group. We know that if **G** is a group and K is its subgroup, then $K(\lambda\mu) = (K\lambda)(K\mu)$ if K is normal in **G**. But in LA-group the condition of normality is obvious [6]. This structure was explored toward LA-ring and LA-module etc in [7].

2. MAIN RESULTS

The new algebraic structure defined in this article is due to the inspiration of the pattern in a double displacement reaction which is huge applicable knowledge in chemistry. A double displacement reaction is such type of reaction in which two reactants exchange ions to form two new compounds. Although we are interested in the pattern of the reaction which is mathematically and specially in abstract algebra can be viewed as $(AB)(CD) = (CB)(AD)$. From the pattern of the chemical reaction $AB + CD \to CB + AD$ we named the law $(AB)(CD) = (CB)(AD)$ as left double displacement law (LDD-law). Similarly, we can define the right double displacement law (RDD-law) as $(AB)(CD) = (AD)(CB)$.

Definition 2.1. A groupoid D is said to be left double displacement semigroup (LDD-semigroup) if its elements satisfies the left double displacement law (LDD-law) i.e for all λ, μ, ν, δ in D satisfies $(\lambda\mu)(\nu\delta) = (\nu\mu)(\lambda\delta)$.

Definition 2.2. A groupoid D is said to be right double displacement semigroup (RDD-semigroup) if its elements satisfies the right double displacement law (RDD-law) i.e for all λ, μ, ν, δ in D satisfies $(\lambda\mu)(\nu\delta) = (\lambda\mu)(\nu\mu)$.

Here we only study about LDD-semigroup where RDD-semigroup is left as open problem for researchers.

Definition 2.3. An LDD-semigroup D is said to be LDD-monoid if it has left identity element "ℓ" such that

$$\ell\lambda = \lambda \text{ for all } \lambda \in D$$

Definition 2.4. An LDD-semigroup D is said to be LDD-group if it holds the following axioms:

1. There exists an element $\in \ell D$ such that $\lambda = \lambda.\ell$ for all $\in \lambda D$, i.e. consists left identity.

2. For all $\in \lambda D$ there exist an element $\lambda^{-1} \in D$ such that $\lambda^{-1}.\lambda = \ell$, i.e, consists left inverses.

In the following we have given some results in the comparison of group theory.

Theorem 2.5. *Let D be LDD-semigroup with right identity ℓ, then D is an LDD-moniod.*

Proof. Suppose λ belong to D, then

$$\begin{aligned}\ell\lambda &= (\ell.\ell)\lambda \\ &= (\ell.\ell)(\lambda.\ell) \\ &= (\lambda.\ell)(\ell.\ell) \\ &= \lambda.\end{aligned}$$

Remark 2.6. Although we know that in semigroup which contains right identity, but it is not necessary to have a left identity.

Theorem 2.7. Let D be LDD-semigroup with right identity ℓ then D is a commutative semigroup.

Proof. First, we check for commutativity:
Suppose $\lambda,\ \mu,\ \nu$ belong D then

$$\begin{aligned}\lambda\mu &= (\lambda.\mu)\ell \\ &= (\lambda.\mu)(\ell.\ell) \\ &= (\ell.\mu)(\lambda.\ell) \\ &= \mu\lambda.\end{aligned}$$

Now we check for associativity

$$\begin{aligned}(\lambda.\mu)\nu &= (\lambda.\mu)(\nu.\ell) \\ &= (\nu.\mu).(\lambda.\ell) \\ &= (\mu.\nu).(\lambda) \\ &= \lambda.\ (\mu.\nu).\end{aligned}$$

Due to commutativity every left inverse is right inverse, so all the axioms of abelian groups are satisfied.

Remark 2.8. Although we know that each group contains right identity but it not necessary to be commutative.

Theorem 2.9. In an LDD-group D the left identity is right identity.

Proof. Let $\ell \in D$ be the left identity we will show that is right identity also,

$$\begin{aligned}(\lambda.\ell) &= (\ell.\lambda).(\lambda^{-1}.\lambda) \\ &= (\lambda^{-1}.\lambda).(\ell.\lambda) \\ &= (\ell).(\ell.\lambda) \\ &= \lambda.\end{aligned}$$

Which shows that in LDD-group left identity become right identity.

Remark 2.10. By the above result we can conclude that every LDD-group is abelian group.

Remark 2.11. It is interesting to note that every LDD-group is group, but every LDD-semigroup is not semigroup.

As LDD-semigroup is a new structure in literature so a lot of examples has been provided to clarify the definition.

Example 2.12. The set of integers Z under binary operation " . " defined as $\lambda.\mu = \mu^2$ where $\lambda,\ \mu,\ \nu,\ \delta$ belong to Z is LDD-semigroup because " . " satisfies $(\lambda\mu)(\nu\delta) = (\nu\mu)(\lambda\delta)$ i.e

$$\begin{aligned}(\lambda.\mu).(\nu.\delta) &= (\mu^2).(\delta^2) \\ (\lambda.\mu).(\nu.\delta) &= \delta^4 \\ (\nu.\mu).(\lambda.\delta) &= (\mu^2).(\delta^2) \\ (\nu.\mu).(\lambda.\delta) &= \delta^4.\end{aligned}$$

Remark 2.13. In example 2.12 we can see that it is not a semigroup, but it is LDD-semigroup. Clearly, we have

$$\begin{aligned}\lambda.\ (\mu.\nu) &= \lambda.\mu^2 \\ \lambda.\ (\mu.\nu) &= (\mu^2)^2 \\ \lambda.\ (\mu.\nu) &= \mu^4 \\ (\lambda.\mu).\nu &= \mu^2.c \\ (\lambda.\mu).\nu &= \nu^2\end{aligned}$$

so $\lambda.(\mu.\nu) \neq (\lambda.\mu).\nu$. Which shows that associative law does not hold.

Next if we see commutative law here:

$$\begin{aligned}\lambda.\mu &= \mu^2 \\ b.\lambda &= \mu^2 \\ \lambda.\mu &\neq \mu.\lambda\end{aligned}$$

so commutative law does not hold.

Example 2.14. If $\mathbf{M}$ is the set of square matrices of order n under binary operation " . " defined as $\mathbf{A.B} = \mathbf{B}^t$ (where $\mathbf{B}^t$ denotes the usual transpose of a matrix $\mathbf{B}$) and $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{E}$ belong to $\mathbf{M}$ is LDD-semigroup because " . " satisfies $(\mathbf{A.B}).(\mathbf{C.E}) = (\mathbf{C.B}).(\mathbf{A.E})$ i.e:

$$\begin{aligned}(\mathbf{A.B}).(\mathbf{C.E}) &= (\mathbf{B}^t).(\mathbf{E}^t) \\ &= (\mathbf{E}^t)^t \\ (\mathbf{C.B}).(\mathbf{A.E}) &= (\mathbf{B}^t).(\mathbf{E}^t) \\ &= (\mathbf{E}^t)^t.\end{aligned}$$

Remark 2.15. Again, in example 2.14 we can see that it is not a semigroup, but it is LDD-semigroup for associative law we have

$$\begin{aligned}\mathbf{A}.(\mathbf{B.C}) &= \mathbf{A.B}^t \\ \mathbf{A}.(\mathbf{B.C}) &= (\mathbf{B}^t)^t \\ (\mathbf{A.B}).\mathbf{C} &= \mathbf{B}^t.\mathbf{C} \\ (\mathbf{A.B}).\mathbf{C} &= \mathbf{C}^t\end{aligned}$$

so $\mathbf{A}.(\mathbf{B.C}) \neq (\mathbf{A.B}).\mathbf{C}$ and hence associative law does not hold.
Similarly, for commutative law we have:

$$\begin{aligned}\mathbf{A.B} &= \mathbf{B}^t \\ \mathbf{B.A} &= \mathbf{A}^t.\end{aligned}$$

Hence $\mathbf{A.B} \neq \mathbf{B.A}$ so commutative law also not hold.

Example 2.16. The binary operation " . " defined as $\lambda.\mu = \ln\mu$ where $\lambda,\ \mu,\ \nu,\ \delta$ belong to N (set of positive real numbers) is LDD-semigroup because " . " satisfies LLD-law i.e $(\lambda.\mu).(\nu.\delta) = (\nu.\mu).(\lambda.\delta)$.

$$\begin{aligned}(\lambda.\mu).(\nu.\delta) &= (\ln\mu).(\ln\delta) \\ &= \ln(\ln\delta) \\ (\nu.\mu).(\lambda.\delta) &= (\ln\mu).(\ln\delta) \\ &= \ln(\ln\delta).\end{aligned}$$

Remark 2.17. Again, in Example 2.16 we can see that it is not a semigroup, but it is LDD-semigroup. Here for associative law $\lambda.(\mu.\nu) = (\lambda.\mu).\nu$ we have:

$$\begin{aligned}\lambda.(\mu.\nu) &= \lambda.\ln\nu \\ \lambda.(\mu.\nu) &= \ln(\ln\nu) \\ (\lambda.\mu).\nu &= \ln\mu.\nu \\ (\lambda.\mu).\nu &= \ln\nu.\end{aligned}$$

Hence $\lambda.(\mu.\nu) \neq (\lambda.\mu).\nu$, so associative law does not hold. For commutative law we have:

$$\begin{aligned}\lambda.\mu &= \ln\mu \\ \mu.\lambda &= \ln\lambda \\ \lambda.\mu &\neq \mu.\lambda\end{aligned}$$

so commutative law does not hold.

Example 2.18. The binary operation " . " defined as $\lambda.\mu = \mu^2 + 1$ where $\lambda,\ \mu,\ \nu,\ \delta$ belong to R (set of all real numbers) is LDD-semigroup because " . " satisfies left double displacement law i.e

$$(\lambda.\mu).(\nu.\delta) = (\mu^2+1).(\delta^2+1)$$
$$= (d^2+1)^2+1$$
$$(\nu.\mu).(\lambda.\delta) = (\mu^2+1).(\delta^2+1)$$
$$= (\delta^2+1)^2+1.$$

Remark 2.19. Again, in Example 2.18 we can see that it is not a semigroup, but it is LDD-semigroup i.e $\lambda.(\mu.c) = (\lambda.\mu).\nu$ does not hold here.

$$\lambda.(\mu.\nu) = \lambda.(\nu^2+1)$$
$$\lambda.(\mu.\nu) = (\nu^2+1)^2+1$$
$$(\lambda.\mu).\nu = (\mu^2+1).\nu$$
$$(\lambda.\mu).\nu = \nu^2+1.$$

Clearly $\lambda.(\mu.\nu) \neq (\lambda.\mu).\nu$ so associative law does not hold. Also, here commutative law also not hold i.e

$$\lambda.\mu = \mu^2+1$$
$$\mu.\lambda = \lambda^2+1.$$

Hence $\lambda.\mu \neq \mu.\lambda$. So commutative law does not hold.

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