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Research Paper

Free Vibration Analysis of Quintic Nonlinear Beams using Equivalent Linearization Method with a Weighted Averaging

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Abstract. In this paper, the equivalent linearization method with a weighted averaging proposed by Anh (2015) is applied to analyze the transverse vibration of quintic nonlinear Euler-Bernoulli beams subjected to axial loads. The proposed method does not require small parameter in the equation which is difficult to be found for nonlinear problems. The approximate solutions are harmonic oscillations, which are compared with the previous analytical solutions and the exact solutions. Comparisons show the accuracy of the present solutions. The impact of nonlinear terms on the dynamical behavior of beams and the effect of the initial amplitude on frequencies of beams are investigated. Furthermore, the effect of the axial force and the length of beams on frequencies are studied.

Keywords: Equivalent linearization method, Weighted averaging, Non-linear vibration, Euler-Bernoulli beam.

1. Introduction

Analyzing the nonlinear vibration of beams is one of the most important issues in structural engineering. Applications such as high-rise buildings, long-span bridges, and aerospace vehicles have necessitated the study of their dynamic behavior at large amplitudes. As the amplitude of oscillation increases, these structures are subjected to non-linear vibrations which often lead to material fatigue and structural damage. These effects become more significant around the natural frequencies of the system. Therefore, it is very important to provide an accurate analysis towards the understanding of the non-linear vibration characteristics of these structures. Vibration problems of beams have recently been investigated by many researchers with different boundary conditions and hypotheses. These researches predict the nonlinear frequency of beams which are very important for the design of many engineering structures.

Siddiqui et al. [1] studied large free vibrations of a beam carrying a moving mass. Bayat et al. [2] used Max-Min Approach (MMA) and Homotopy Perturbation Method (HPM) to obtain the natural frequency and the corresponding displacement of tapered beams. Bennouna and White [3] investigated the effects of large vibration amplitude on the fundamental mode shape of a clamped-clamped uniform beam. The issue of the new elastic terms discovered in the nonlinear dynamic model of an enhanced nonlinear 3D Euler-Bernoulli beam was discussed by Zohoor and Khorsandijou [4]. While the elastic orientation is negligible, the nonlinear dynamic model governing tension-compression, torsion and two spatial bending is presented. Sedighi et al. [5] attempted to investigate the dynamical analysis of beam vibrations in the presence of preload discontinuity and proposed an innovative accurate equivalent function for this well-known nonlinearity. Sedighi et al. [6] have obtained analytical expressions for geometrically nonlinear vibration of Euler–Bernoulli beam using He's Parameter Expanding Method (HPEM), with deadzone nonlinear boundary condition, by introducing a novel and efficient equivalent function. Miguel et al. [7] studied a steel cantilever beam with a tip mass, which was tested at the Group of Applied Mechanics/Federal University of Rio Grande do Sul. The Variational Iteration Method (VIM) and Parametrized Perturbation Method (PPM) have been used by Barari et al. to investigate non-linear vibration of Euler-Bernoulli beams subjected to axial loads [8]. The non-linear non-planar

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dynamic responses of a near-square cantilevered geometrically imperfect (i.e., slightly curved) beam under harmonic primary resonance based on excitation with a one-to-one internal resonance was investigated by Aghababaei et al. [9]. Li and Hua studied the effects of shear deformation on the free vibration of elastic beams with general boundary conditions [10]. Various finite element formulations of large amplitude free vibrations of beams with immovably supported ends have been discussed in the work of Sarma et al. [11].

Most models dealing with non-linear dynamics of flexible beams include cubic non-linear terms in the equations of motion. The literature considering high order of non-linearities is very limited. To extend the study and our understanding of the effects of high order nonlinearities, this paper investigates the non-linear equation of motion of beams with quintic non-linearities. Recently, Sedighi et al. [12] used Homotopy Analysis Method (HAM) to find an analytical solution of transversal oscillation of quintic non-linear beams. Sedighi and Reza [13] used He's Max-Min Approach (MMA) and Amplitude-Frequency Formulation (AFF) to obtain the frequency-amplitude relationship of beam vibrations with quintic nonlinearity. It was clearly shown that the first term in the series expansions was sufficient to produce a highly accurate approximation of the nonlinear system. The Parameter Expansion Method (PEM) was applied by Sedighi and Reza to analyze the lateral vibration of quintic beams [14]. Bakhtiari-Nejad et al. used the coupled Homotopy-Variational method to analyze nonlinear vibrations of beams subjected to axial loads for Clamped-Clamped and Hinged-Hinged boundary conditions [32]. Chebyshev polynomials were used by Mohammadi et al. to analyze the buckling of micro- and nano- beams based on nonlocal Euler beam theory [33].

It is difficult to obtain an exact solution for these problems. As a result, some approximation techniques have been developed. Indeed, these techniques are useful tools in analyzing nonlinear oscillation problems. Traditional analytical methods, which have been widely used for non-linear equations, include the Parameterized Perturbation Method (PPM) [15], the Variational Iteration Method (VIM) [16], the Homotopy Perturbation Method (HPM) [17], the Energy Balance Method (HBM) [18], the Parameter Expansion Method (PEM) [19], and the Min–Max Method [20]. In order to overcome the limitations of traditional analytical methods, the Homotopy Analysis Method (HAM) was developed by Liao [21].

The Equivalent Linearization Method (EML) was first introduced by Krylov and Bugoliubov [22]. The essence of this method is the determination of the equivalent parameters of the nonlinear equation and the linear one. This classical Equivalent Linearization method plays an important role in the quasi-linear theory and leads to results consistent with experimental data. However, the introduction of equivalent parameters by postulating the work done per one cycle (per one period) often gives the inaccurate results. The averaging values of the functions such as sin(t) and cos(t) over one period will be equal to zero thus the information related to these quantities will be lost after the normal averaging process. In 1959, an Equivalent Linearization technique was developed by Caughey to analyze stochastic systems [23]. The Caughey's technique is a useful tool widely used for analyzing nonlinear stochastic problems. Nevertheless, the accuracy of the Equivalent Linearization Method with conventional averaging is normally reduced for middle or strong nonlinear systems [24, 25]. Recently, using the Equivalent Linearization Method proposed by Caughey and introducing a new way for determining averaging values, Anh has suggested a method to analyze the nonlinear oscillators of deterministic systems [26]. Averaging value is calculated in a new way called the weighted averaging value. The effectiveness of this method has been verified through the analysis of some complex nonlinear oscillations [27, 28].

In this paper, Anh's method has been applied to the analysis of free vibration of quintic nonlinear beams compressed by the axial loads. The solutions were compared with the ones given by He's Min-Max Approach (MMA) [13].

2. Methodology

Consider the Euler-Bernoulli beam of length l, moment of inertia I, mass per unit length m and modulus of elasticity E, which is axially compressed by loading P as shown in Fig. 1 [13]:

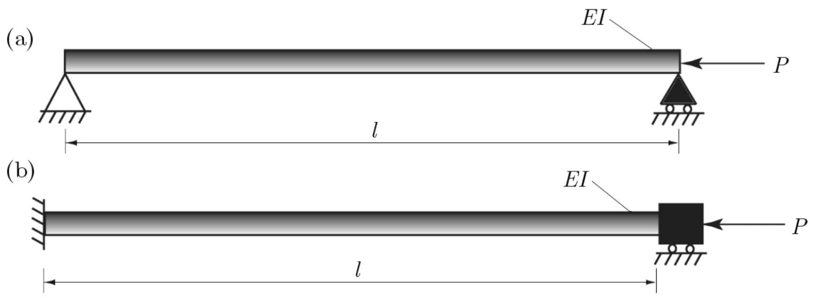


Fig. 1. Schematic of an Euler-Bernoulli beam subjected to an axial load (a) simply-supported beam, (b) clamped-clamped beam.

Denoting the transverse deflection by w, the differential equation governing the equilibrium in the deformed situation is derived as [13]:

$$\frac{d^2}{dx^2}\left\{\frac{EIw''(x,t)}{\sqrt{[1+w'^2(x,t)]^3}}\right\}+Pw''(x,t)\left(1+\frac{3}{2}w'^2\right)+m\ddot{w}(x,t)=0 \tag{1}$$

where $w''(x,t)/\sqrt{[1+w'^2(x,t)]^3}$ is the "exact" expression for the curvature, and using the approximation:

$$\frac{w''(x,t)}{\sqrt{[1+w'^2(x,t)]^3}} \cong w''(x,t)\left[1-\frac{3}{2}w'^2(x,t)+\frac{15}{8}w'^4(x,t)\right] \tag{2}$$

the nonlinear equation (1) can be expressed as:

$$\begin{aligned} &EIw^{(4)}\left(1-\frac{3}{2}w'^2+\frac{15}{8}w'^4\right)-9EIw'''w'w''+\frac{45}{2}EIw'''w'^3w'' \\ &-3EIw''^3+\frac{45}{2}EIw'^2w''^3+Pw''\left(1+\frac{3}{2}w'^2\right)+m\ddot{w}=0 \end{aligned} \tag{3}$$

which is subjected to the following boundary conditions:

For simply-supported (S-S) beam:

$$w(0,t)=\frac{\partial^2 w}{\partial x^2}(0,t)=0, \qquad w(l,t)=\frac{\partial^2 w}{\partial x^2}(l,t)=0 \tag{4}$$

For clamped-clamped (C-C) beam:

$$w(0,t)=\frac{\partial w}{\partial x}(0,t)=0, \qquad w(l,t)=\frac{\partial w}{\partial x}(l,t)=0 \tag{5}$$

Assuming $w(x,t)=q(t)\phi(x)$, where $\phi(x)$ is the first eigenmode of the beam vibration, it can be expressed as:

For S-S beam:

$$\phi(x)=\sin\left(\frac{\pi x}{l}\right) \tag{6}$$

For C-C beam:

$$\phi(x)=\frac{1}{2}\left[1-\cos\left(\frac{2\pi x}{l}\right)\right] \tag{7}$$

Applying the Bubnov-Galerkin method for Eq. (3) yields:

$$\int_0^l \left[\begin{aligned} &EIw^{(4)}\left(1-\frac{3}{2}w'^2+\frac{15}{8}w'^4\right)-9EIw'''w'w''+\frac{45}{2}EIw'''w'^3w'' \\ &-3EIw''^3+\frac{45}{2}EIw'^2w''^3+Pw''\left(1+\frac{3}{2}w'^2\right)+m\ddot{w} \end{aligned}\right]\phi(x)dx=0 \tag{8}$$

By introducing the following non-dimensional variables:

$$\tau=\sqrt{\frac{EI}{ml^4}}t, \qquad \bar{q}=\frac{q}{l} \tag{9}$$

the non-dimensional nonlinear equation of motion about its first mode can be written as:

$$\frac{d^2\bar{q}(\tau)}{d\tau^2}+\gamma_1\bar{q}(\tau)+\gamma_2[\bar{q}(\tau)]^3+\gamma_3[\bar{q}(\tau)]^5=0 \tag{10}$$

where:

For S-S beam:

$$\gamma_1=\pi^4-\frac{Pl^2\pi^2}{EI}, \ \gamma_2=-\frac{3}{8}\pi^6-\frac{3}{8}\frac{Pl^2\pi^4}{EI}, \ \gamma_3=\frac{15}{64}\pi^8 \tag{11}$$

For C-C beam:

$$\gamma_1=\frac{16\pi^4}{3}-\frac{4\pi^2Pl^2}{3EI}, \ \gamma_2=-2\pi^6-\frac{\pi^4Pl^2}{2EI}, \ \gamma_3=\frac{5\pi^8}{4} \tag{12}$$

Eq. (10) is the governing non-linear vibration equation of Euler-Bernoulli beams. The center of the beam subjected to the following initial conditions will be as follows:

$$\bar{q}(0)=A, \quad \dot{\bar{q}}(0)=0 \tag{13}$$

where A denotes the non-dimensional maximum amplitude of oscillation.

3. Basics Idea of the Studied Method

To introduce an overview of the Equivalent Linearization Method, we consider a single degree of freedom system with the nonlinear function depending on displacement which is described by the following nonlinear differential equation:

$$\ddot{X} + g(X) = 0, \quad X(0) = A, \dot{X}(0) = 0 \tag{14}$$

where $g(X)$ is a nonlinear function of X and A is the initial amplitude.

The idea of the Equivalent Linearization Method is to replace the nonlinear term g(X) in Eq. (14) by the linear term as follows [23]:

$$g(X) \to \alpha X \tag{15}$$

By this replacement, the linearized equation of Eq. (14) can be achieved as:

$$\ddot{X} + \alpha X = 0 \tag{16}$$

It can be seen that Eq. (16) is much simpler than Eq. (14). For the solution of the linear equation (16) approaches to the solution of the nonlinear equation (14), one assumes that this can be achieved if we find the coefficient α so that the error between the two equations is the smallest. There are some optimization criteria to find α. In this work, we use the mean-square error criterion, which requires that the error between the two Eqs. (14) and (16) be minimum [24, 25, 31]:

$$e^2(X) = \left[g(X) - \alpha X\right]^2 \to \underset{\alpha}{Min} \tag{17}$$

Performing derivative operation:

$$\frac{\partial \left\langle e^2(X) \right\rangle}{\partial \alpha} = 0$$

yields:

$$\alpha = \frac{\left\langle g(X)X \right\rangle}{\left\langle X^2 \right\rangle} \tag{18}$$

In Eq. (18), the symbol $\langle g \rangle$ denotes the time-averaging operator in classical meaning [22]:

$$\left\langle f(t) \right\rangle = \lim_{T\to+\infty} \frac{1}{T}\int_0^T f(t)dt \tag{19}$$

For a ω-frequency function $f(\omega t)$, the averaging process is taken during one period T, thus the averaging value in Eq. (19) can be calculated as follows:

$$\left\langle f(\omega t) \right\rangle = \frac{1}{T}\int_0^T f(\omega t)dt = \frac{1}{2\pi}\int_0^{2\pi} f(\tau)d\tau, \quad \tau = \omega t \tag{20}$$

The averaging values in Eqs. (19) and (20) are called the classical or conventional averaging values. They often give incorrect results, especially for some periodic functions such as sine or cosine ones because the averaging values of sine and cosine functions per one period will be zero. To improve this issue, instead of using the conventional averaging values in Eqs. (19) and (20), Anh [26] introduced a new method for calculating averaging values of the periodic functions. The idea of the proposed method is as follows: replacing the constant coefficient $1/T$ in Eq. (19) by a weighted coefficient function $h(t)$. Accordingly, the averaging value is calculated in a new way called the weighted averaging value:

$$\left\langle x(t) \right\rangle_w = \int_0^T h(t)x(t)dt \tag{21}$$

where the weighted coefficient function $h(t)$ satisfies the condition:

$$\int_0^T h(t)dt = 1 \tag{22}$$

In Ref. [26], Anh has proposed a weighted function as follows:

$$h(t) = s^2\omega^2 t e^{-s\omega t} \tag{23}$$

where s is a positive constant. Note that the weighted function $h(t)$ given in Eq. (23) satisfies the condition (22).

The weighted coefficient function (23) obtained as a product of the optimistic weighted coefficient t and the pessimistic weighted coefficient $e^{-s\omega t}$, as the two basic weighted coefficient functions, has one maximal value at $t_* = 1/(\omega s)$ and then decreases to zero as $t \to +\infty$. The graphs of basic weighted coefficient functions and the weighted coefficient function in Eq. (23) are shown in Fig. 2 for some values of s parameter (s=1, 2, 3 and 4). The s parameter is considered as an adjustment parameter which can be chosen in the following way: if one requires that the time t_* be equal to $T/n=2\pi/(n\omega)$, where n is a natural number or zero, we get $s=n/(2\pi)$. Therefore, the meaning of s can be specified as follows: for n=1, s=1/(2π) the weighted coefficient (23) has the maximal value after one period, and for n=2, s=2/(2π)=1/π the weighted coefficient (23) has the maximal value after half period. For n=0, s=0 the weighted coefficient (23) has the maximal value at infinity and this case corresponds to the conventional averaging value. The detailed properties of the weighted function *h(t)* in Eq. (23) can be viewed in Refs. [26, 27].

With the periodic solution of linearized equation (16), the averaging values in Eq. (18) can be calculated using Eq. (21):

$$\left\langle x(\omega t)\right\rangle_w = \int_0^{+\infty} s^2\omega^2 t e^{-s\omega t} x(\omega t)dt = \int_0^{+\infty} s^2 \tau e^{-s\tau} x(\tau)d\tau,\ \tau = \omega t \tag{24}$$

We can see that the averaging value in Eq. (24) is an image of function $s^2\tau x(\tau)$ through the Laplace transform. As ω – periodic function $x(\omega t)$ can be expanded into Fourier series, then we can easily calculate the integral in Eq. (24) using Laplace transform. In the next section, the proposed method will be applied to analyze free vibration of quintic nonlinear beam compressed by the axial loads given by Eq. (10).

4. SOLUTION

We will find the approximate analytical solution of Eq. (10). First, the equivalent linear equation of Eq. (10) can be given as follows:

$$\ddot{\overline{q}}(\tau) + \omega^2\overline{q}(\tau) = 0 \tag{25-a}$$

The equation error between the two oscillators given in Eq. (25) and Eq. (10) is:

$$e(\overline{q}) = \gamma_1\overline{q} + \gamma_2\overline{q}^3 + \gamma_3\overline{q}^5 - \omega^2\overline{q} \tag{25-b}$$

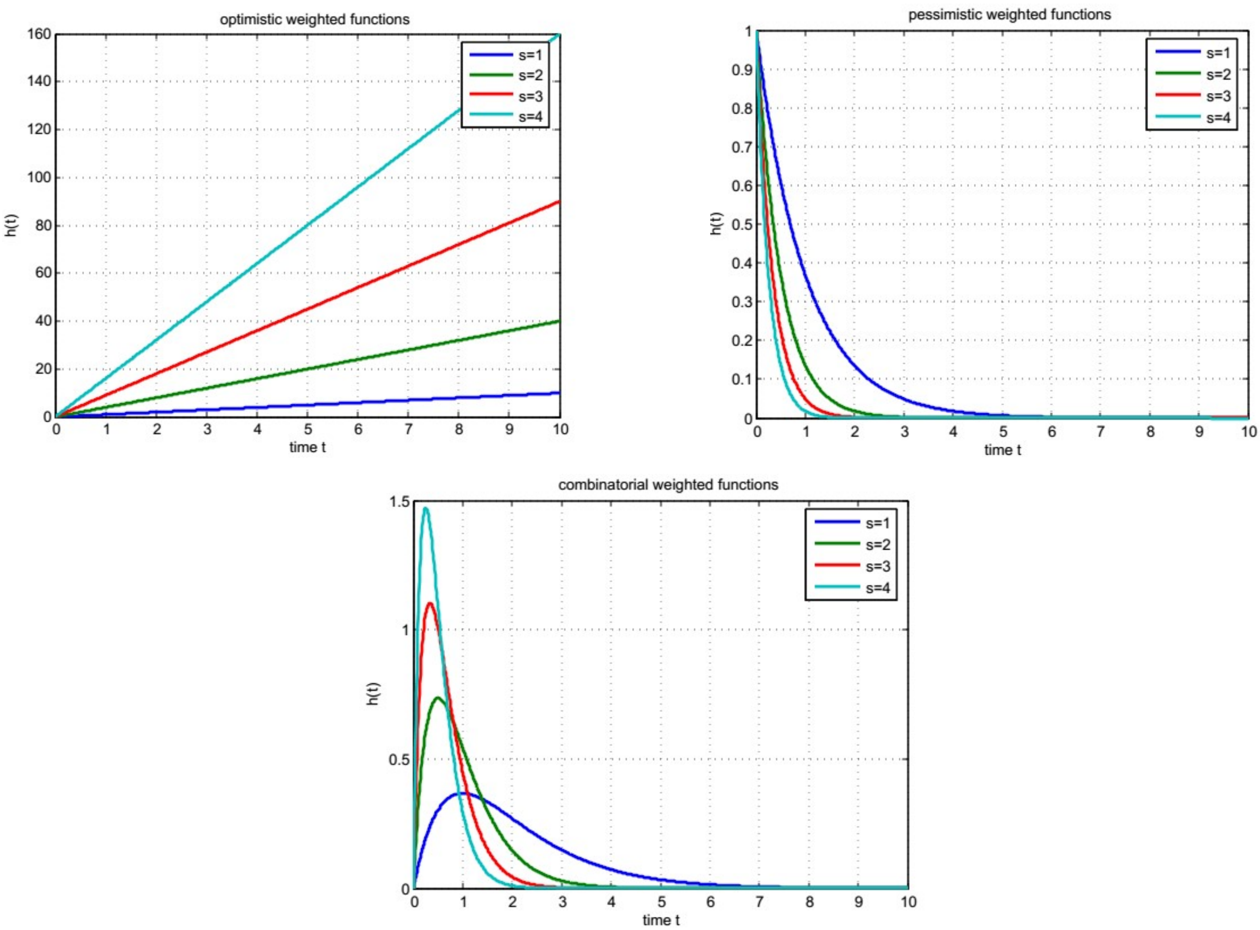


Fig. 2. Graphs of weighted functions

where ω^2 is determined by using the mean-square error criterion as follows:

$$\omega^2 = \frac{\gamma_1\left\langle\overline{q}^2\right\rangle + \gamma_2\left\langle\overline{q}^4\right\rangle + \gamma_3\left\langle\overline{q}^6\right\rangle}{\left\langle\overline{q}^2\right\rangle} \tag{26}$$

The periodic solution of the equivalent linear equation (25) is:

$$\overline{q}(\tau) = A\cos(\omega\tau) \tag{27}$$

With the periodic solution in Eq. (27) and the weighted coefficient function in Eq. (23), we can calculate the averaging operators $\left\langle\overline{q}^2\right\rangle$, $\left\langle\overline{q}^4\right\rangle$ and $\left\langle\overline{q}^6\right\rangle$ in Eq. (26) by using Eq. (24), we get:

$$\left\langle \bar{q}^2 \right\rangle = \left\langle A^2 \cos^2(\omega\tau) \right\rangle_w = A^2 \frac{s^4 + 2s^2 + 8}{(s^2+4)^2} \tag{28}$$

$$\left\langle \bar{q}^4 \right\rangle = \left\langle A^4 \cos^4(\omega\tau) \right\rangle_w = A^4 \frac{248s^4 + 416s^2 + 1536 + 28s^6 + s^8}{(s^2+4)^2(s^2+16)^2} \tag{29}$$

$$\left\langle \bar{q}^6 \right\rangle = \left\langle A^6 \cos^6(\omega\tau) \right\rangle_w = A^6 \frac{1658880 + 440064s^2 + 282496s^4 + 45712s^6 + 3168s^8 + 94s^{10} + s^{12}}{(s^2+4)^2(s^2+16)^2(s^2+36)^2} \tag{30}$$

Substituting Eqs. (28), (29) and (30) into Eq. (26), and with the parameter s chosen equal to 2, we obtain the approximate frequency:

$$\omega = \sqrt{\gamma_1 + 0.72\gamma_2 A^2 + 0.575\gamma_3 A^4} \tag{31}$$

Therefore, the approximate solution for $\bar{q}(\tau)$ is

$$\bar{q}(\tau) = A\cos\left(\sqrt{\gamma_1 + 0.72\gamma_2 A^2 + 0.575\gamma_3 A^4}\ \tau\right) \tag{32}$$

5. Results and Discussions

The approximate frequency $\omega_{present}$ in Eq. (30) obtained by the proposed method, the one obtained by He's Max-Min Approach (MMA) ω_{MMA} achieved by Sedighi and Reza [13], and the exact frequency ω_{exact} were compared. For this nonlinear problem governed by Eq. (10), the exact frequency was stated by Younesian et al. [30]. The comparison results are presented in Tables 1-2 for different values of the parameters γ_1, γ_2, γ_3 and A. From Tables 1-2, we see that the approximate frequency in this paper is better than the one obtained by He's Max-Min Approach. The maximum relative error of Min-Max Approach approaches to 5.86%, while the maximum relative error of the present method is only 1.53%.

The approximate frequency obtained by He's Max-Min Approach is as follows [13]:

$$\omega_{MMA} = \sqrt{\gamma_1 + \frac{3}{4}\gamma_2 A^2 + \frac{5}{8}\gamma_3 A^4} \tag{33}$$

The exact frequency is stated by Younesian et al. [30]:

$$\omega_{exact} = 2\pi \left[4\int_0^{\pi/2} \frac{d\theta}{\sqrt{\gamma_1 + \frac{1}{2}\left(1+\sin^2\theta\right)\gamma_2 A^2 + \frac{1}{3}\left(1+\sin^2\theta+\sin^4\theta\right)\gamma_3 A^4}} \right]^{-1} \tag{34}$$

Table 1. Percentage errors of approximate frequency with $\gamma_1=\gamma_2=\gamma_3=1$

A	ω_{exact}	ω_{MMA}	R. Error (%)	$\omega_{present}$	R. Error (%)
0.1	1.003770000	1.003774128	0.0004	1.003622189	0.015
0.3	1.035540000	1.035645934	0.01	1.034145783	0.13
0.5	1.106540000	1.107502822	0.09	1.102695561	0.35
1	1.523590000	1.541103501	1.15	1.514925741	0.57
3	7.268630000	7.640353395	5.11	7.352210552	1.15
5	19.18150000	20.25771458	5.61	19.45186366	1.41
8	48.294 60000	51.07837116	5.76	49.01305948	1.49
10	75.17740000	79.53615530	5.79	76.30858405	1.50
20	299.2230000	316.7033312	5.84	303.7910466	1.53
50	1867.570000	1976.898075	5.85	1896.193819	1.53
70	3659.980000	3874.264575	5.85	3716.083826	1.53
100	7468.830000	7906.168541	5.86	7583.350249	1.53

Comparison of the relative error of the two frequencies is also presented in Figs. 3-6 for some values of s parameter (s=1, 1.8, 1.9 and 2). The relative error of the approximate solutions depends linearly on γ_1, γ_2 and γ_3, and are nonlinear functions of the initial amplitude A. From Fig. 3, it can be seen that the larger the initial amplitude A is, the larger the relative error becomes. From Figs. 4-6, we find that the variation of the parameters γ_1, γ_2 and γ_3 does not significantly affect the graph of the relative error. The relative error of the present frequency is smaller than the relative error of the MMA frequency. Again, the accuracy of the present solution can be observed.

Table 2. Percentage errors of approximate frequency with $\gamma_1=1$, $\gamma_2=10$, $\gamma_3=100$

A	ω_{exact}	ω_{MMA}	R. Error (%)	$\omega_{present}$	R. Error (%)
0.1	1.039700000	1.039831717	0.01	1.038147388	0.15
0.3	1.462590000	1.476905549	0.98	1.453874135	0.59
0.5	2.524690000	2.604083332	3.14	2.528586562	0.15
1	8.010050000	8.426149773	5.19	8.105553652	1.19
3	67.70970000	71.63099887	5.79	68.72626860	1.50
5	187.1990000	198.1186513	5.83	190.0486779	1.52
8	478.4630000	506.4395324	5.85	485.7795796	1.53
10	747.3230000	791.0442465	5.85	758.7628088	1.53
20	2987.830000	3162.752124	5.85	3033.625059	1.53
50	18671.34000	19764.709737	5.86	18957.66338	1.53
70	36595.36000	38738.375688	5.86	37156.56444	1.53
100	74683.91000	79057.415850	5.86	75829.22919	1.53

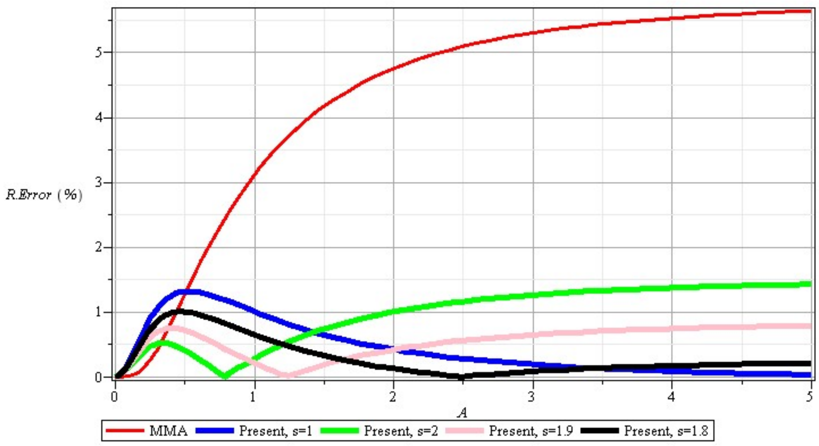


Fig. 3. Comparison of the relative error of two frequencies based on initial amplitude with $\gamma_1 = 1, \gamma_2 = 10, \gamma_3 = 10$

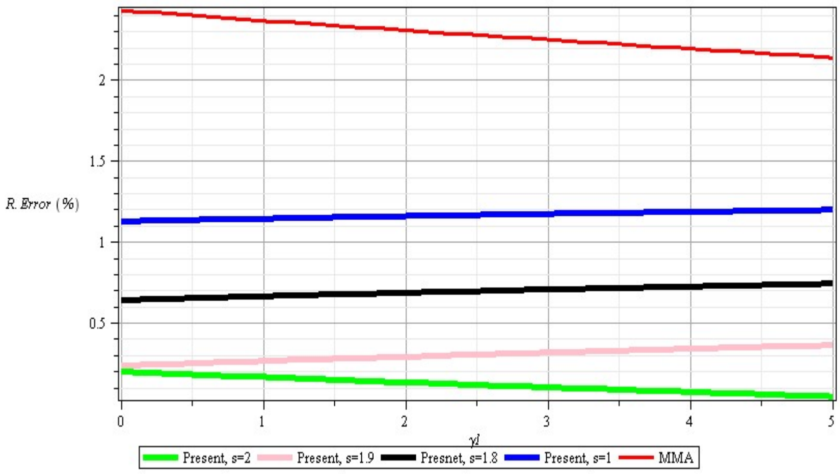


Fig. 4. Comparison of the relative error of two frequencies based on γ_1 with $A = 1, \gamma_2 = 100, \gamma_3 = 10$

In following section, an Euler-Bernoulli beam with the geometrical properties listed in Table 3 is examined [29]. Comparisons of the responses between the present solution and the MMA solution are given in Figs. 7 and 8 for the S-S beam and the C-C beam, respectively.

Table 3. Geometrical properties of the Euler – Bernoulli beam [29]

Item	Notation	Value
Young's modulus (steel)	E	210 GPa
Mass density	m	7850 kg/m^3
Cross sectional area	S	7.69 10^{-3} m^2

Second moment of area	I	30.55 10^{-6} m^4
Beam length	l	18 m
Axial load	P	100 N

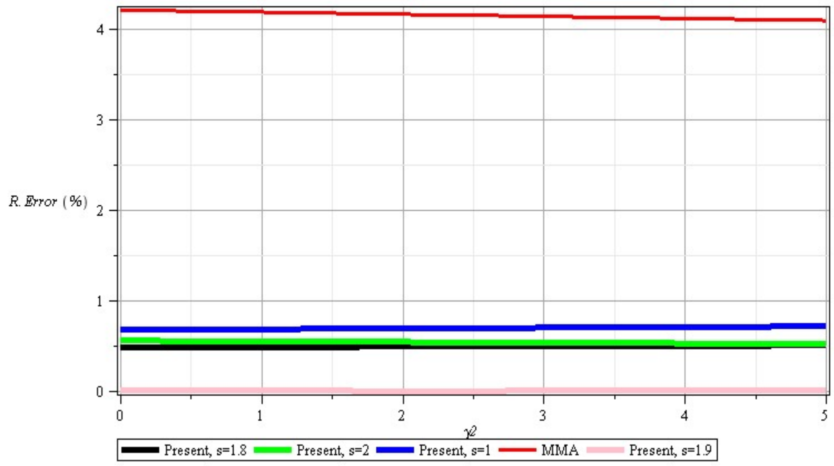


Fig. 5. Comparison of the relative error of two frequencies based on γ_2 with $A=1, \gamma_1=10, \gamma_3=100$

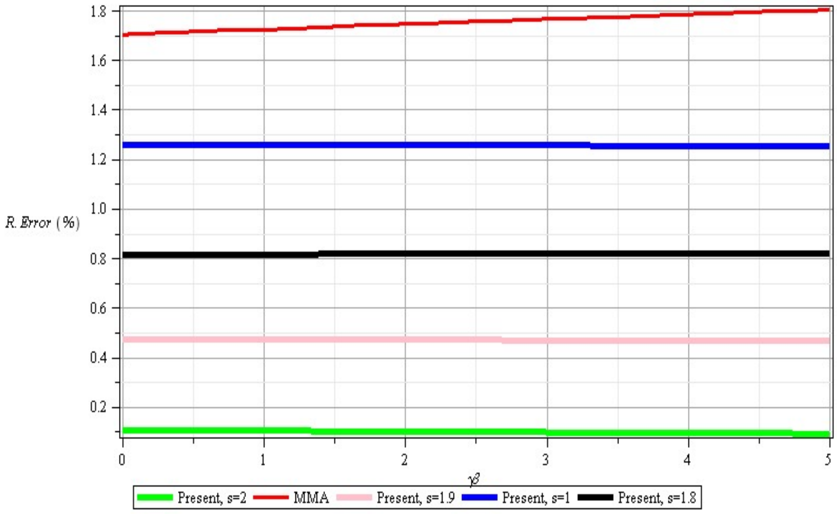


Fig. 6. Comparison of the relative error of two frequencies based on γ_3 with $A=1, \gamma_1=10, \gamma_2=100$

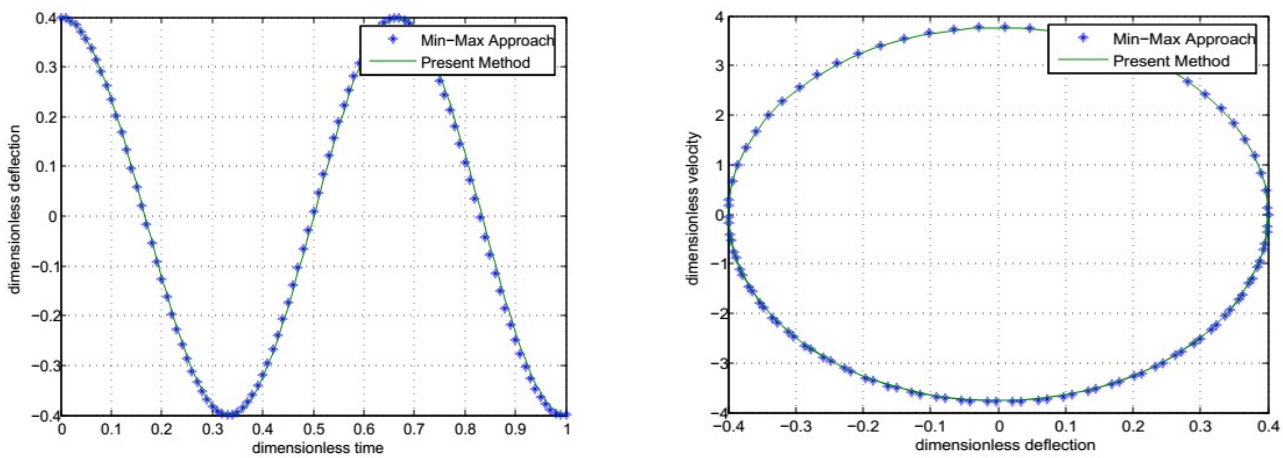


Fig. 7. Comparison of the responses between the present solution and the MMA solution for S-S beam

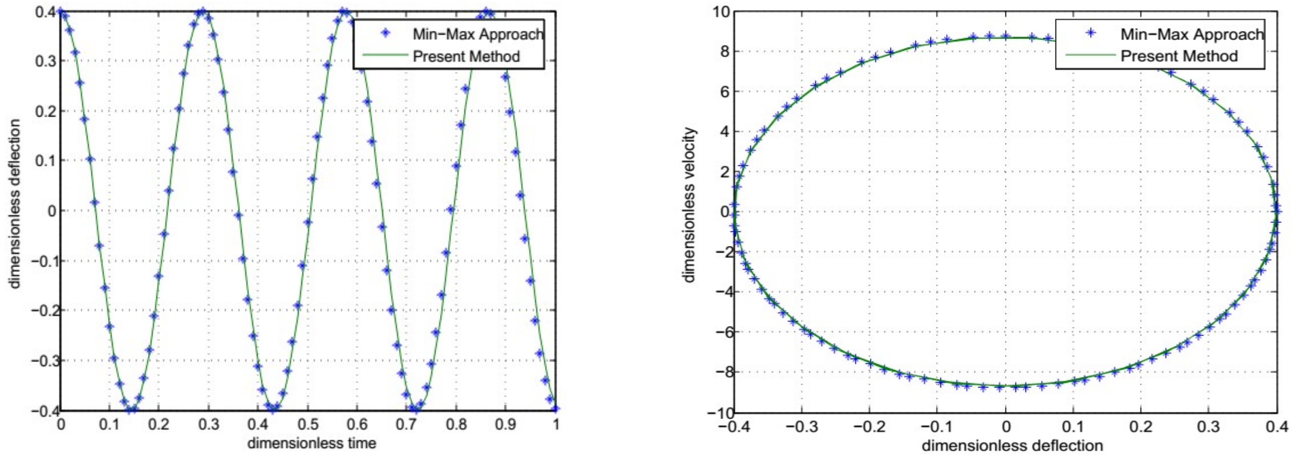
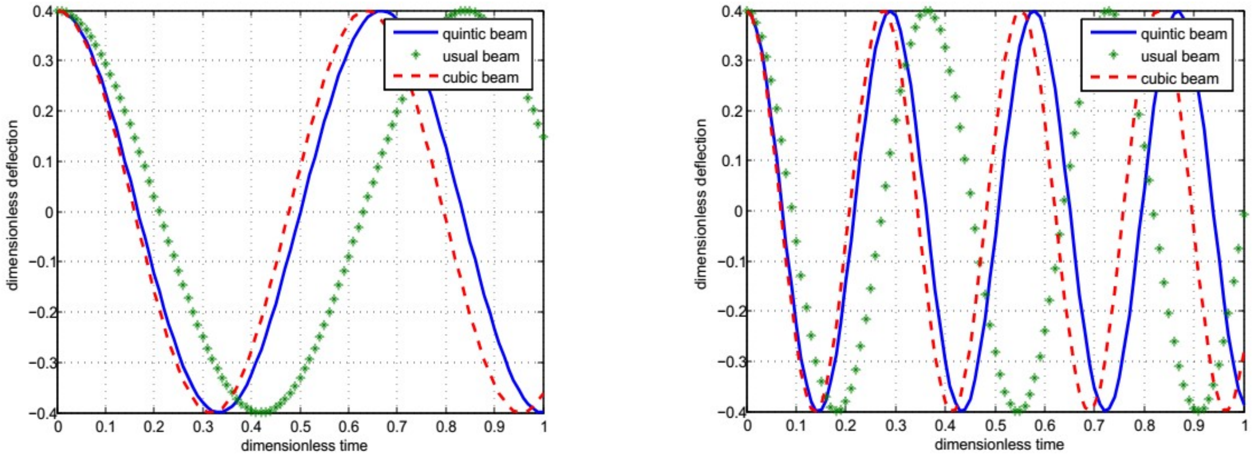


Fig. 8. Comparison of the responses between the present solution and the MMA solution for C-C beam

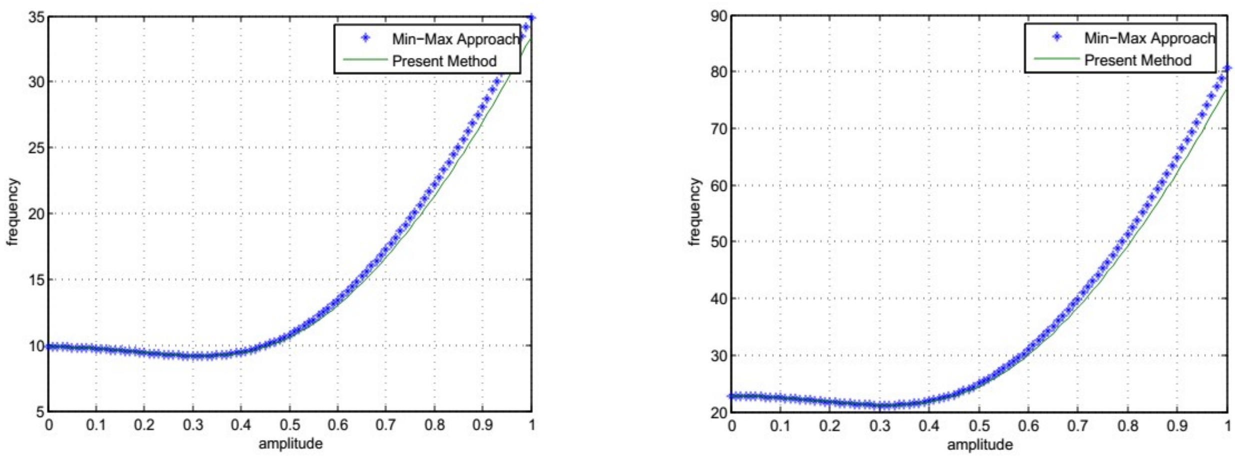
The impact of nonlinear terms on dynamical behavior of the beams is presented in Fig. 9. Note that in this figure, the usual beam (linear beam) corresponds to $\gamma_2 = \gamma_3 = 0$ and the cubic beam corresponds to $\gamma_3 = 0$, respectively. From Fig. 9, we can see that the nonlinear terms have a significantly effect on the response of the S-S beam and the C-C beam.



(a) (b)

Fig. 9. The impact of nonlinear terms on dynamical behavior of the beam: (a) S-S beam, (b) C-C beam.

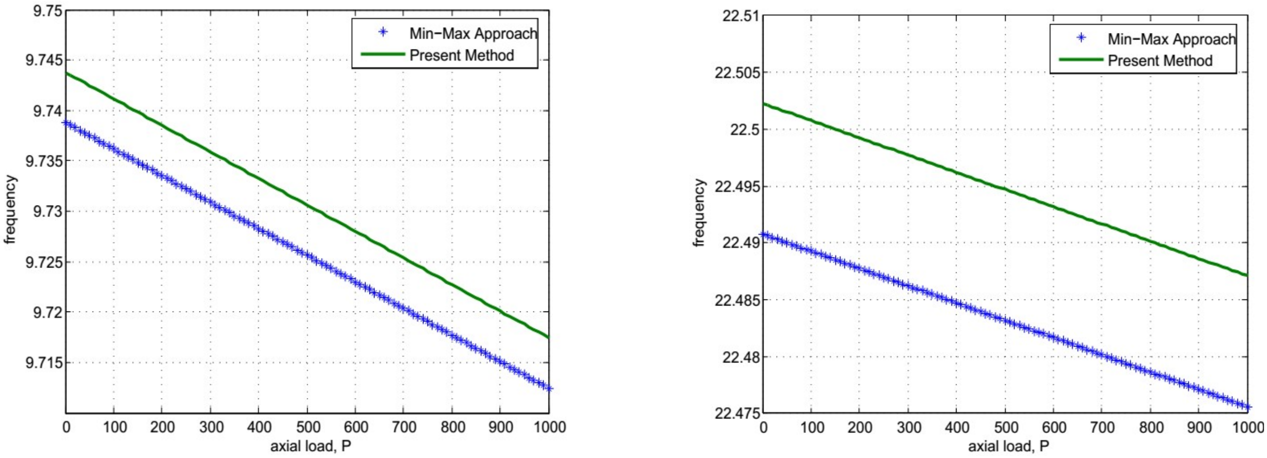
Fig. 10 shows the effect of the initial amplitude on frequency A to the nonlinear frequencies of beams. The approximate frequencies are the quadratic functions of the initial amplitude. When the initial amplitude increases, it will greatly affect the change in the nonlinear frequencies of the systems. In Fig. 9, we can see that when the initial amplitude is small ($A \leq 0.5$), there is no significant change in the nonlinear frequencies of beams, but if the initial amplitude A is lager ($A > 0.5$), the frequencies of beams increase rapidly when the initial amplitude increases.



(a) (b)

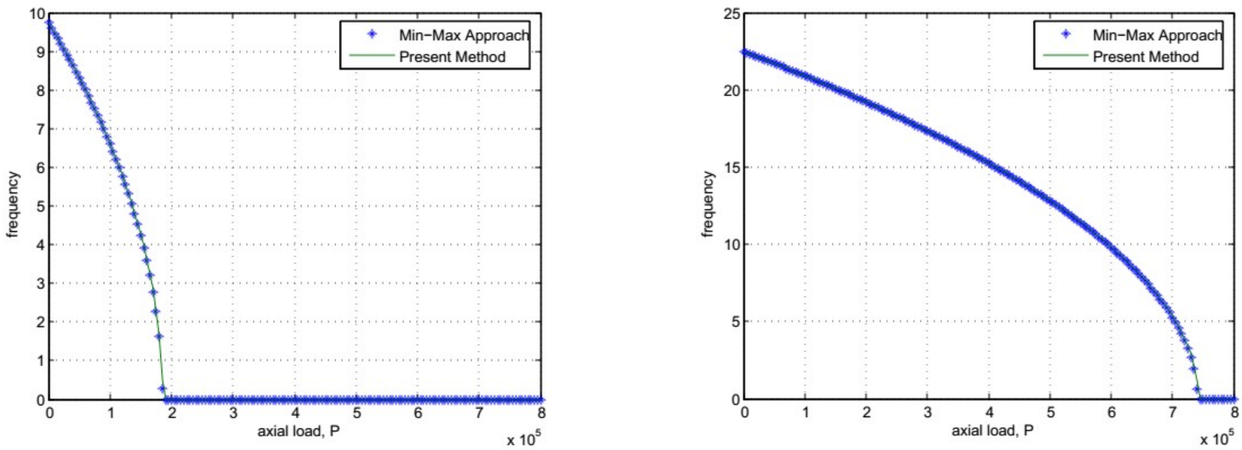
Fig. 10. The effect of the initial amplitude on frequency of the beam: (a) S-S beam, (b) C-C beam.

The effect of the axial compressive force P on frequency is shown in Fig. 11. The nonlinear frequencies of the systems depend linearly on the axial force P. The frequencies decrease when the axial load P increases, and this shows the suitability with reality. The error between the two methods is only 0.0513% for the S-S beam and 0.04% for the C-C beam, respectively. However, if the axial compressive load is extremely large, the beams will be buckled. The frequencies of beams will tend to zero when the axial compressive load increases, and the plots of nonlinear frequencies of beams versus axial force can be seen in Fig. 12.



(a) (b)

Fig. 11. The effect of the axial force P on frequency of the beam with $A=0.1$: (a) S-S beam, (b) C-C beam.



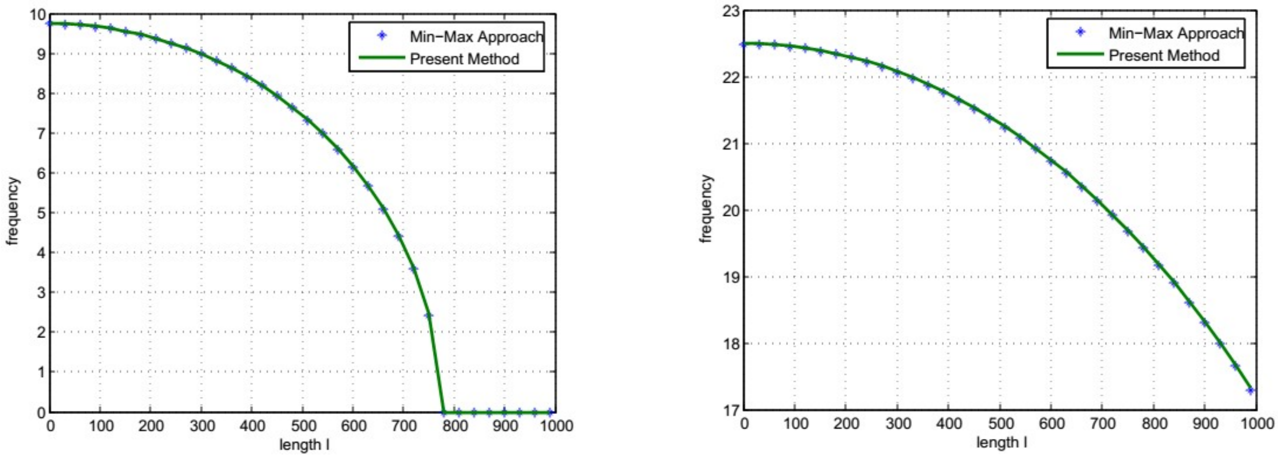
(a) (b)

Fig. 12. The buckled effect of the axial force P on frequency of the beam with $A=0.1$: (a) S-S beam, (b) C-C beam.

Finally, the effect of the length l on nonlinear frequencies are presented in Fig. 13. The nonlinear frequencies decrease as the length of beams increases. We can see that with $l \le 100m$ the length of beams has a small effect on the nonlinear frequency of the beam, but with $l > 100m$ the nonlinear frequencies of beams decrease rapidly when the length increases. With the data in Table 3, the buckling phenomenon occurs with the S-S beam when the length of the beam is approximately 800m.

6. Conclusion

Analytical approximation solution of quintic nonlinear beam is obtained by using the Equivalent Linearization Method with a weighted averaging. Utilizing the advantage in performing of the Equivalent Linearization method and the reliability of the weighted averaging value, the present method has shown effectiveness in the resolution procedure. The approximate solutions are the harmonic oscillations, which are compared with the previous analytical solutions and the exact solutions. The relative error of the two analytical results was investigated with the variation of the system parameters and the initial amplitude. Comparisons showed the accuracy of the present solutions. The impact of nonlinear terms on dynamical behavior of beams and the effect of the initial amplitude on frequencies of beams were investigated in this paper. Furthermore, the effect of the axial force and the length of beams on frequencies were also studied. This method can be further developed for strongly nonlinear systems and multi-degree of freedom vibrations.



(a) (b)

Fig. 13. The effect of length *l* on frequency of the beam with $A=0.1$: (a) S-S beam, (b) C-C beam.

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Conflict of Interest

The authors declare no conflict of interest.

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