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Optimization of the break-even point for non-homogeneous products sales

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Abstract

Break-even point analysis is a classic management accounting tool. In the case of the sale of one product, the notion of the break-even point is well described in the literature, conceptually simple, and relatively easy to apply in business practice. However, when it comes to heterogeneous sales, consisting of various products, this problem is presented less frequently, and the methods used in this case exploit rather arbitrary and often ambiguous criteria. The aim of the article is to present and analyze alternative ways of determining break-even points for non-homogeneous sales based on econometric modeling methods. Production levels determined by the proposed methods meet the classical condition set for the break-even point, and in addition are optimal from the point of view of criteria used in the economic analysis. Three methods were presented: first – based on the classic criterion of profit maximization and linear programming, second – minimizing variable production costs and taking into account the scale effects on the production costs, and third – taking into account the random aspect of the business operations and maximizing the probability of profitability. According to the authors' knowledge, the proposed methods are original and are not known in the existing literature on the subject.

Keywords: break-even point, heterogeneous production, production costs.

Streszczenie
Analiza progu rentowności dla produktów niejednorodnych

Analiza progu rentowności jest klasycznym narzędziem rachunkowości zarządczej. W przypadku sprzedaży jednego produktu pojęcie progu rentowności jest dobrze opisane w literaturze, łatwe konceptualnie i stosunkowo proste do zastosowania w praktyce gospodarczej. Jeśli chodzi o sprzedaż niejednorodną, składającą się z różnorodnych produktów, to problem ten jest prezentowany znacznie rzadziej, a stosowane w tym przypadku metody wykorzystują dość arbitralne i często niejednoznaczne kryteria. Celem artykułu jest przedstawienie i analiza alternatywnych sposobów ustalania progów rentowności dla sprzedaży niejednorodnej, według metod modelowania ekonometrycznego. Poziomy produkcji wyznaczane proponowanymi przez nas metodami spełniają klasyczny warunek stawiany progowi rentowności, a dodatkowo są również optymalne z punktu widzenia stosowanych w analizie ekonomicznej kryteriów. Przedstawione zostały trzy metody: pierwsza – oparta na klasycznym kryterium maksymalizacji przychodu oraz programowaniu liniowym, druga – minimalizująca zmienne koszty produkcji i uwzględniająca efekty skali po stronie kosztów produkcji oraz trzecia – uwzględniająca losowy aspekt działalności przedsiębiorstwa i maksymalizująca prawdopodobieństwo osiągnięcia rentowności. Według wiedzy autorów, zaproponowane metody są oryginalne i nie są znane w dotychczasowej literaturze przedmiotu.

Słowa kluczowe: próg rentowności, produkcja niejednorodna, koszty produkcji.

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1. Introduction and basic definitions

Break-even point analysis, as part of Cost-Volume-Profit (CVP) analysis, is a standard tool in management accounting. In the case of selling uniform goods, which usually applies to raw materials or half-finished products (e.g., cement, coal), the problem of setting the break-even point comes down to determining what amount of the product will cover the total fixed costs (see: e.g., Nowak, 1996).

Let F denote total fixed costs, c – the unit cost of production, p – unit selling price, and therefore $m = p - c$ is the margin per unit. The break-even point q determines the quantity of the product that needs to be sold to cover the fixed costs of production. In the case of homogeneous products sales, the calculations are easy. With the above notation, the break-even point is determined by the following formula:

$$q = \frac{F}{p-c} = \frac{F}{m}.$$

Things get more complicated when selling non-homogeneous products to many recipients. Research conducted in this scope is quite complex (see: Ćwiąkała-Małys, Nowak, 2005; Correa, 1984). Most textbooks state very demanding assumptions and narrow boundaries of applicability of the break-even analysis in this case. For example, Dutta (2004, p. 16.11) writes, "Where multiple products [...] are involved, break-even point cannot be stated in units." Heitger et al. (2007, p. 113) state, "Multiple-product breakeven analysis requires a constant sales mix. However, it is virtually impossible to predict with certainty the sales mix." The most popular approach is to determine the overall break-even point in sales revenue as a ratio of the total fixed costs divided by the weighted average contribution margin ratio, and then to distribute the break-even sales proportionally to the sales mix, which is assumed to be constant (Garrison et al., 2012). We are also dealing with many concepts of determining the break-even point in the case of heterogeneous production. In particular, A. Żwirbla (2015) presents a literature review of the current break-even analysis and new proposals in this scope of research citing, among others, the results of J. Czekaj and Z. Dresler (1999), M. Dobija (2001), E. Nowak (2001), Z. Krokosz-Krynke (2007), J. Mielcarek (2005), B. Pomykalska (2007) and A. Ćwiąkała-Małys and W. Nowak (2005, 2009). In a simplified way, these considerations concern a multi-aspect analysis of the relationships between the quantitative side of multi-assortment production and production costs, and the margins obtained from the sale of goods. In particular, we are looking for such a quantitative structure (vector) of production, for which total margin will cover fixed costs. It is not always the case that such a problem leads to an unambiguous solution, that is, to establishing a unique structure of production (cf. Żwirbla, 2014; 2015, p. 141). Therefore, it is desirable to formulate the problem, by introducing additional assumptions in order to obtain an unambiguous structure of production, in such a way that total margin will balance fixed production costs.

Żwirbla (2015) focuses mainly on the concept of the break-even point in sales revenue, while the solutions proposed in this paper set the goal of establishing a break-even point in units, which does not play down any of the approaches since they are equivalent, to some extent. Secondly, researchers (cf. Potkanya, Krajcirova, 2015; Żwirbla, 2015) most often treat the sales mix as fixed, known in advance, or based on the sales results from previous periods, which is a well known and common accountant practice. However, this approach assumes this structure will not change in the future periods. Moreover, if this structure is set on the basis of predictions, it may be a source of additional errors. In contrast to this, and what we think is quite new, is that we treat the sales mix as unknown, since it is an element of the sought-after solution, which is a multi-product break-even point. Therefore, this study is intended to be a contribution to the determination of such a threshold which has some optimal properties. Moreover, ideas of the construction of the proposed solutions appear to be clear and simple. To this end, we use classical methods of econometric modeling.

Taking into account the uncertainty of business operation leads to new methods and results. Although the probabilistic approach in CVP analysis is not new (Bierman 1963; Jaedicke, Robichek, 1964; Johnson, Simik, 1971; Dickinson, 1974; Yunker, Yunker, 1982; Phillips, 1994), statistical and stochastic methods are not widely popular among practitioners (Drury et al., 1993). One of our proposals is part of this trend of research.

The paper is organized as follows. The remainder of this section presents the necessary notations and states a few basic observations about costs, margin, and profit. Section 2 is divided into three parts, each presenting one method of determining the break-even point for heterogeneous sales with a short analysis and an example. Section 3 concludes our considerations.

Now we introduce the necessary notations. Let:

q_{ij} denote the size of sales of the i-th product to the j-th recipient,

c_i be the unit cost of production of the i-th product,

p_{ij} be the unit selling price of the i-th product to the j-th recipient, hence

$m_{ij} = p_{ij} - c_i$ is the unit margin achieved from the sale of the i-th product to the j-th recipient,

for $i = 1,\ldots,n$, $j = 1,\ldots,z$, $n \geq 1$, $z \geq 1$. Let us note that some margins may be negative. We assume that the unit sales margins of individual products sold to respective customers are known in advance. The total margin from the sale of all goods is determined by the function:

$$d\left(q_{11},q_{12},\ldots,q_{nz}\right) = \sum_{i=1}^{n}\sum_{j=1}^{z} m_{ij}q_{ij} \tag{1}$$

Where we assume that $q_{ij} \geq 0$ for $i = 1,\ldots,n$, $j = 1,\ldots,z$ and $m_{ij} > 0$ for at least one pair (i, j), $i = 1,\ldots,n$, $j = 1,\ldots,z$. In this case, determining the break-even point will proceed by establishing the levels for the number of goods sold, that is, the numbers q_{ij}, $i = 1,\dots,n$, $j = 1,\dots,z$, so they fulfill the criterion:

$$d(q_{11},q_{12},\dots,q_{nz}) = F. \tag{2}$$

From the algebraic point of view we are dealing with solving one equation of nz unknowns, hence, as is well-known, there will be many solutions. To obtain a unique solution we need to introduce some additional conditions. Let $q_i = \sum_{j=1}^{z} q_{ij}$ be the number of produced goods of the i-th kind. We can transform the equation (1) as follows:

$$\begin{aligned} d\left(q_{11},q_{12},\dots,q_{nz}\right) &= \sum_{i=1}^{n}\sum_{j=1}^{z}\left(p_{ij}-c_i\right)q_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{z} p_{ij}q_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{z} c_i q_{ij} = \\ &= \sum_{i=1}^{n}\sum_{j=1}^{z} p_{ij}q_{ij} - \sum_{i=1}^{n} c_i \sum_{j=1}^{z} q_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{z} p_{ij}q_{ij} - \sum_{i=1}^{n} c_i q_i . \end{aligned} \tag{3}$$

The average selling price of i-th goods is described by the formula

$$\bar{p}_i = \frac{\sum_{j=1}^{z} p_{ij}q_{ij}}{\sum_{j=1}^{z} q_{ij}} = \frac{\sum_{j=1}^{z} p_{ij}q_{ij}}{q_i}.$$

From this equality we obtain

$$\bar{p}_i q_i = \sum_{j=1}^{z} p_{ij}q_{ij},$$

which allows us to write (3) in the form:

$$d\left(q_{11},q_{12},\dots,q_{nz}\right) = \sum_{i=1}^{n}\bar{p}_i q_i - \sum_{i=1}^{n} c_i q_i = \sum_{i=1}^{n}\left(\bar{p}_i - c_i\right)q_i = \sum_{i=1}^{n}\bar{m}_i q_i, \tag{4}$$

where: $\bar{m}_i = \bar{p}_i - c_i$ is the average unit margin obtained from the sale of i-th goods to all the recipients. Thus, we have reduced the number of variables to n instead of nz. If i-th goods are sold to all recipients at the same price p_i, which may occur, for example, in the case of wholesale prices, then $\bar{p}_i = p_i$ and $\bar{m}_i = m_i$, where m_i is the unit margin of i-th goods. Therefore, a special case of the function defined by formula (4) is

$$d\left(q_1,q_2,\dots,q_z\right) = \sum_{i=1}^{n} m_i q_i.$$

2. Proposals of criteria for the Break-Even Point

We will now present our proposals of methods for determining the break-even point for heterogeneous sales.

2.1. Maximizing sale profit at fixed means of production

Let $A = [a_{ji}]$, where a_{ji} is the amount of j-th mean of production needed to produce one unit of i-th product, $i = 1,\ldots,n$, $j = 1,\ldots,k$. By $s = [s_1,\ldots,s_k]^T$ we denote the admissible levels of resources. The equation describing acceptable variants of the structure of production (and sales) is:

$$Aq \leq s,$$

Where $q = \left[q_1,\ldots,q_n\right]^T$. Thus, the optimization problem is:

$$\begin{cases} d(q_1,q_2,\ldots,q_n) & \to & \max, \\ Aq & \leq & s, \\ q & \geq & 0. \end{cases} \tag{5}$$

Due to the linear form of the objective function and constraints, the problem can be effectively solved by standard algorithms of linear programming such as the simplex method. Moreover, in simple cases when $n = 2$ or $k = 2$, it can be solved and visualized (after possible translation to the dual problem) by a graphical method, see (Trzaskalik, 2008).

The solution to problem (5) will usually not satisfy condition (2). However, as is done in the TOPSIS multi-criteria decision-making method (Hwang, Yoon, 1981), we can look for a production basket from the set of solutions of (2) which will be the closest, in the sense of some metric, to the optimal solution of (5), treated here as an ideal solution. A natural metric suitable for this problem seems to be the weighted Manhattan distance, where the weights are product margins (one can also consider prices or variable costs). More specifically, the distance between the production baskets $q^1 = \left(q_1^1,\ldots,q_n^1\right)$ and $q^2 = (q_1^2,\ldots,q_n^2)$ can be calculated as:

$$dist\left(q^1,q^2\right) = \sum_{i=1}^{n} m_i |q_i^1 - q_i^2| .$$

Example 2.1. Consider the case of two products ($n = 2$) and three resources ($k = 3$). We assume that matrix A resulting from the production technology has the form:

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

The vector of resources available to the company is equal to $s = [45\ \ 20\ \ 36]$ and the margins of the products are equal to $m_1 = 10$, $m_2 = 8$, respectively. Therefore, the optimization problem has the following form:

$$\begin{cases} 10q_1 + 8q_2 & \to & \max, \\ q_1 + 3q_2 & \leq & 45, \\ q_1 + q_2 & \leq & 20, \\ 2q_1 + q_2 & \leq & 36, \\ q_1, q_2 & \geq & 0. \end{cases}$$

The graphical solution is shown in Figure 1. The dashed lines correspond to the technological limitations associated with consecutive resources, the shaded polygon is the set of admissible production plans, and the bold line is the isoline of the total margin (a line containing production plans of equal total margin) tangent to the set of admissible production plans. The point of tangency of the isoline to the set of feasible solutions $q^* = (16, 4)$ determines the ideal production plan. Now, assuming fixed costs $F = 150$, we proceed to look for the production plan satisfying the break-even condition: $10q_1 + 8q_2 = 150$ and the nearest to q^* in the following distance:

$$\min_{q_1, q_2}\left(10\left|16 - q_1\right| + 8\left|4 - q_2\right|\right).$$

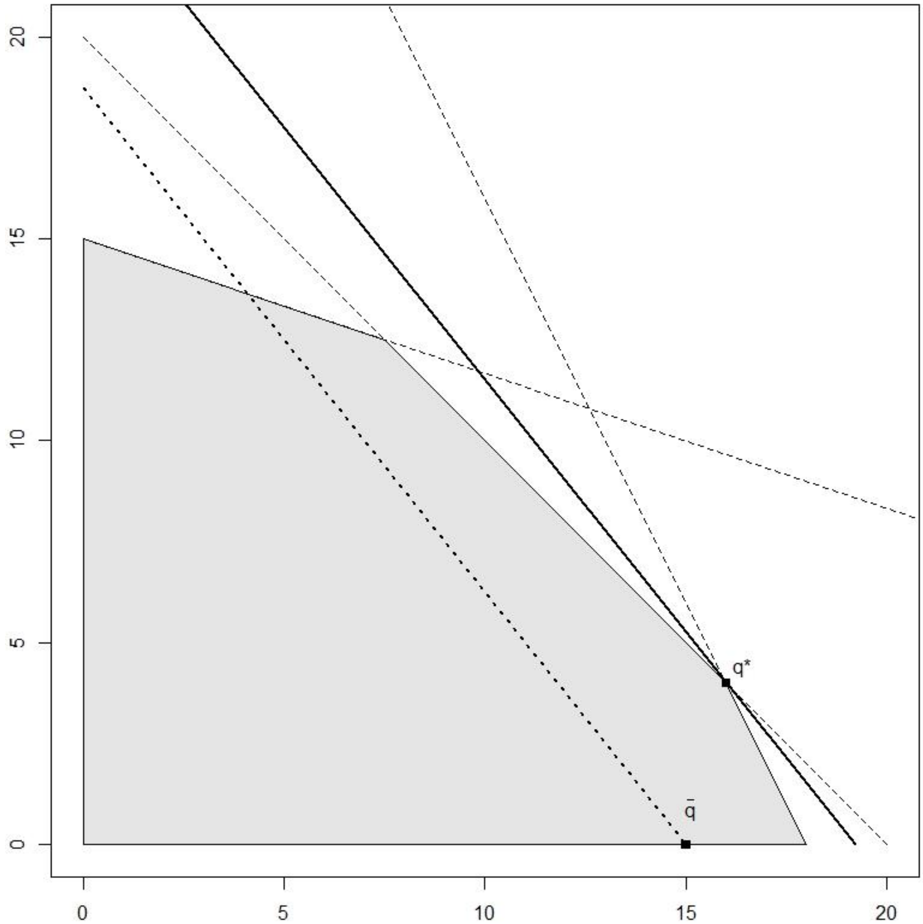
We obtain exactly one integer solution to this problem: $\bar{q} = (15, 0)$, which means that our break-point is obtained producing 15 units of the first product and 0 units of the second. However in the case of divisible products, we would obtain an infinite number of solutions – this is the weakness of the proposed Manhattan distance.

2.2. Minimization of the variable cost of production

Another approach to the problem of determining the level of break-even production is to establish production volumes of individual products to the point for which the function of the overall variable costs $f(q_1, q_2, \ldots, q_n)$ attains the minimum, provided a certain level of total margin covering fixed costs is achieved:

$$d\left(q_1, q_2, \ldots, q_n\right) = \sum_{i=1}^{n} m_i q_i = F.$$

Figure 1. Maximization of total margin under fixed resources of means of production. Solution with the graphical method.
Dashed lines – boundaries, bold line – optimal total margin isoline,
shaded set – feasible production plans, dotted line – set of all break-even points,
q^* – the optimal production plan, $\bar{q}$ – break-even point closest to q^*.
Axes: horizontal – q_1, vertical – q_2



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Synthetically, we get the following optimization problem:

$$\begin{cases} f(q_1,q_2,\ldots,q_n) & \to & \min, \\ \sum_{i=1}^{n} m_i q_i & = & F, \\ q_i & \geq & 0, \qquad i=1,\ldots,n. \end{cases} \tag{6}$$

Let us denote the solution of the problem formulated above by $\bar{q} = \left(\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n\right)$. The resulting sequence of sell sizes for individual products can be interpreted as the optimal break-even points, because it ensures that the total margin from sales will reach at least the level of fixed costs of production.

Total costs of production can be modeled with linear or polynomial functions, (see: Pawłowski, 1970; Pawłowski, 1975). Our approach leads to an interesting solution when the total costs of production are described by the function of the form:

$$f\left(q_1, q_2, \ldots, q_n\right) = \sum_{i=1}^{n} \beta_i q_i^{\alpha},$$

where $\beta_i > 0$ for $i = 1, \ldots, n$ and $\alpha > 1$. The assumption $\alpha > 1$ indicates negative economies of scale associated with the disproportionate increase in production costs in relation to its volume. With the above assumptions, the total cost function is strictly convex and attains the minimum within the set of feasible solutions. The Lagrange functional for problem (6) now has the form:

$$L\left(q_1, q_2, \ldots, q_n, \lambda\right) = \sum_{i=1}^{n} \beta_{\mathrm{i}} q_i^{\alpha} - \lambda\left(\sum_{i=1}^{n} m_i q_i - F\right).$$

Equating its derivatives to zero, we get the first-order conditions:

$$\frac{\partial L}{\partial q_i} = \alpha \beta_i q_i^{\alpha-1} - \lambda m_i = 0, \qquad i = 1, \ldots, n,$$

whence

$$\bar{q}_i = \left(\frac{\lambda m_i}{\alpha \beta_i}\right)^{1/(\alpha-1)} = \left(\frac{\lambda}{\alpha}\right)^{1/(\alpha-1)} \left(\frac{m_i}{\beta_i}\right)^{1/(\alpha-1)}, \qquad i=1, \ldots, n.$$

Substituting these quantities into the boundary condition, we get

$$\sum_{i=1}^{n} m_i \left(\frac{\lambda}{\alpha}\right)^{1/(\alpha-1)} \left(\frac{m_i}{\beta_i}\right)^{1/(\alpha-1)} = F,$$

which, after transformations, leads to (we change the indices to avoid notational collision in the final formula)

$$\left(\frac{\lambda}{\alpha}\right)^{1/(\alpha-1)} = \frac{F}{\sum_{j=1}^{n} m_j \left(\frac{m_j}{\beta_j}\right)^{1/(\alpha-1)}},$$

and thus finally

$$\bar{q}_i = F \times \frac{\left(\frac{m_i}{\beta_i}\right)^{\frac{1}{\alpha-1}}}{\sum_{j=1}^{n} m_j \left(\frac{m_j}{\beta_j}\right)^{\frac{1}{\alpha-1}}}, \qquad i=1,\ldots,n.$$

For $\alpha \to \infty$ we have $(m_i/\beta_i)^{1/(\alpha-1)} \to 1$, hence the production profile obtained by this method will be more and more uniform, that is $\lim_{\alpha\to\infty} \bar{q}_i = F/\sum_{j=1}^{n} m_j$ for $i = 1,\ldots,n$. For $\alpha \to 1$, the presented method will recommend focusing solely on the production of the product (or products) characterized by the greatest ratio of margin to the cost m_i/β_i. For $\alpha \le 1$ we can expect a similar decision, with the stipulation that the criterion deciding which product should be produced will be the maximal m_i^{α}/β_i ratio.

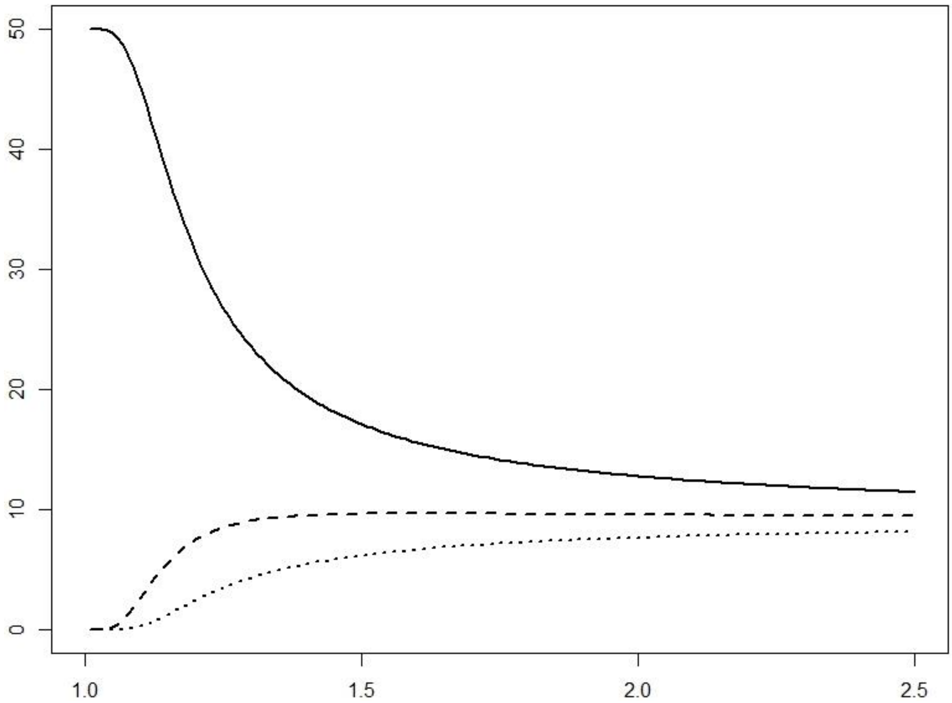
Example 2.2. Let $n = 3$, $F = 100$, $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 5$, $m_1 = 2$, $m_2 = 3$, $m_3 = 6$. The optimal production quantities for different values of the coefficient α are shown in Table 1 and Figure 2.

Table 1. Optimal production in Example 2.2 for selected values of economies of scale α.

α	$\underline{q}_1$	$\underline{q}_2$	$\underline{q}_3$
1.1	45.35	2.55	0.27
1.5	17.10	9.62	6.16
2.0	12.74	9.55	7.64

Source: own preparation.

Figure 2. Optimal size of production to minimize total costs and to cover fixed costs by total margin: $\underline{q}_1$ – continuous line, $\underline{q}_2$ – dashed line, $\underline{q}_3$ – dotted line.
Axes: horizontal – α, vertical – production levels



Source: own preparation.

Where economies of scale are different for each of the produced goods, i.e., a function of the total variable costs, it takes the form:

$$f(q_1,\ldots,q_n)=\sum_{i=1}^{n}\beta_i q_i^{\alpha_i},$$

where $\beta_i > 0$, $\alpha_i > 0$ for $i = 1,\ldots,$n, an optimization problem will not usually have an analytical solution. One can seek approximate solutions with numerical methods.

Example 2.3. Let $n = 3$, $F = 100$, $\beta_1 = 1$, $\beta_2 = 2$, $m_1 = 2$, $m_2 = 3$, $\alpha_1 = 2$, $\alpha_2 = 1.5$. We temporarily allow the parameters β_3, m_3 and α_3 to be arbitrary. The Lagrangian has the form:

$$L\left(q_1,q_2,q_3,\lambda\right) = q_1^2 + 2q_2^{3/2} + \beta_3 q_3^{\alpha_3} - \lambda(2q_1 + 3q_2 + m_3 q_3 - 100)$$

Equating its derivatives with respect to consecutive variables to zero, for $\alpha_3 \neq 1$ we obtain:

$$\frac{\partial L}{\partial q_1} = 2q_1 - 2\lambda = 0, \qquad \Rightarrow \underline{q}_1 = \lambda,$$
$$\frac{\partial L}{\partial q_2} = 3q_2^{1/2} - 3\lambda = 0, \qquad \Rightarrow \underline{q}_2 = \lambda^2,$$
$$\frac{\partial L}{\partial q_3} = \alpha_3\beta_3 q_3^{\alpha_3 - 1} - m_3\lambda = 0, \quad \Rightarrow \underline{q}_3 = \left(\frac{\lambda m_3}{\alpha_3\beta_3}\right)^{1/(\alpha_3-1)}.$$

The value of $\overline{q}_3$ is determined from the restrictions:

$$m_1\underline{q}_1 + m_2\underline{q}_2 + m_3\underline{q}_3 = F \quad \Leftrightarrow \quad 2\lambda + 3\lambda^2 + m_3\left(\frac{\lambda m_3}{\alpha_3\beta_3}\right)^{1/(\alpha_3-1)} = 100.$$

For $\alpha_3 = 1$ we directly obtain $\lambda = \frac{\beta_3}{m_3}$, and thus

$$\underline{q}_1 = \frac{\beta_3}{m_3}, \quad \underline{q}_2 = \left(\frac{\beta_3}{m_3}\right)^2, \quad \underline{q}_3 = \frac{100 - 2\frac{\beta_3}{m_3} - 3\left(\frac{\beta_3}{m_3}\right)^2}{m_3},$$

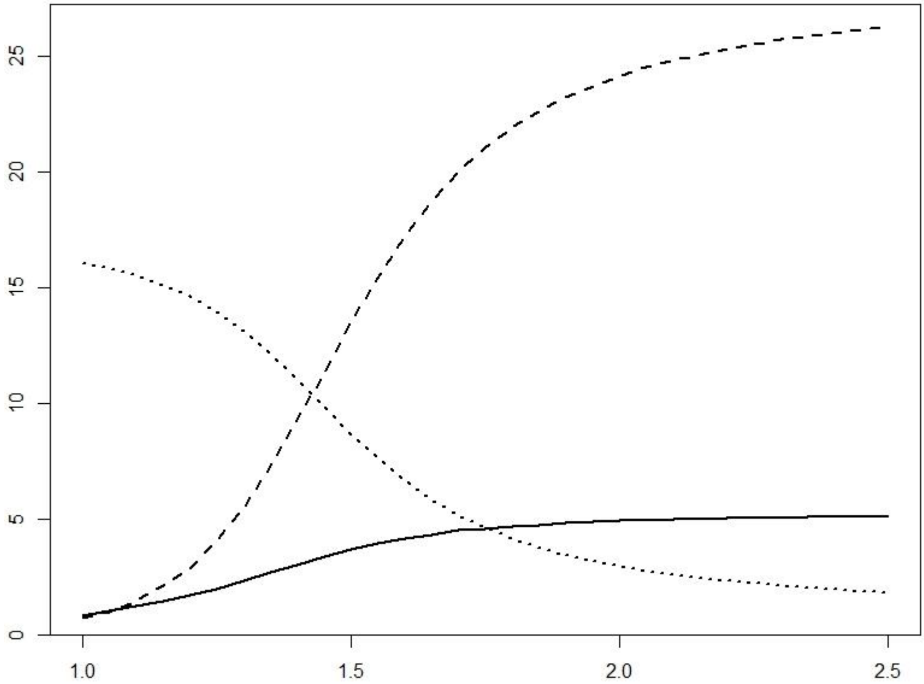
as long as the expression on the right side of the last equality is non-negative.

Figure 3 shows the effect of changes in the economies of scale parameter. With fixed values of $\beta_3 = 5$ and $m_3 = 6$, we change the value of the parameter α_3. One can see a decrease in the amount of the 3rd product, accompanied by an increase in the corresponding rate of economies of scale. It is replaced by two other products, where the share of a second product, with a lower coefficient of economies of scale, is growing more rapidly.

The effect of changes in the cost-factor β_3 is quite simple: with the fixed values of m_3 and α_3, with a decrease of β_3 one can observe a slow increase of the 3rd product and a similar slow decrease in the amount of the other two products converging to zero.

The impact of margin changes on the scale of the optimal output is shown in Figure 4. With fixed values α_3=1 and β_3=5, we change the value of parameter m_3 within the interval [0.92, 8] (for $m_3 < \frac{1+\sqrt{301}}{20} \approx 0.9175$, our solution gives $\underline{q}_3 < 0$). For small values of margin m_3, production of the 3rd product is not advisable. When the margin exceeds a certain threshold, the share of those products rapidly grows at the expense of other products. With a further increase in the margin, lower production is needed to cover fixed costs, hence the volume of the optimal level of production of the 3rd product falls as well.

Figure 3. The optimum output at different levels of scale effects for various products, with the changing value of α_3: $\underline{q}_1$ – continuous line, $\underline{q}_2$ – dashed line, $\underline{q}_3$ – dotted line. Axes: horizontal – α_3, vertical – production levels



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2.3. Minimizing the probability of achieving profitability

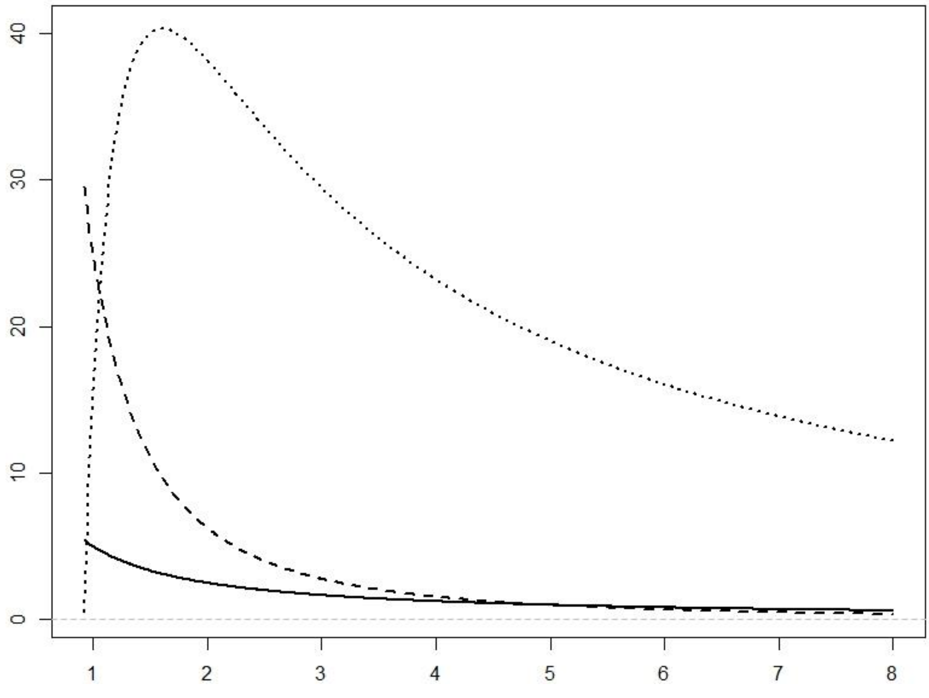
The margins achieved by an enterprise may, in fact, be different from the expected ones, due to price fluctuations, both in terms of costs and revenues. Therefore, it is justified to treat those quantities as random variables and, as a consequence, exceeding the break-even point is a random event, the occurrence of which is desired by the company. As a necessary prerequisite for determining the break-even point, we suggest a sufficiently large probability of the total margin exceeding the fixed costs. This approach may be seen as a competitive model to Johnson and Simik (1971), where the number of units sold is modeled as a random variable and a simplification of Yunker and Yunker (1982), where a much more general model was proposed.

We assume that the individual margins have a joint normal distribution, where m_i has the expected value $\overline{m}_i$ and variance σ_i^2. Furthermore, we assume that margins are correlated, assuming ρ_{ij} as the correlation coefficient between the margin m_i and m_j, $i,j = 1,\ldots,n$. The total margin of the company, under production of $\left(q_1,\ldots,q_n\right)$ is the random variable w ith a normal distribution with the following parameters:

$$\overline{m} = \boldsymbol{E}\left(d(\mathrm{q}_1,\ldots,q_n)\right) = \boldsymbol{E}\left(\sum_{i=1}^{n} m_i q_i\right) = \sum_{i=1}^{\mathrm{n}} \overline{m}_i q_i,$$

$$\sigma^2 = \boldsymbol{Var}\left(d(q_1,\ldots,q_n)\right) = \boldsymbol{Var}\left(\sum_{i=1}^{n} m_i q_i\right) = \sum_{i=1}^{n} \sigma_i^2 q_i + 2 \sum_{i,j=1,\, i<j}^{n} q_i q_j \sigma_i \sigma_j \rho_{ij}\,.$$

Figure 4. The dependence of the optimum production levels on m_3:
$\underline{q}_1$ – continuous line, $\underline{q}_2$ – dashed line, $\underline{q}_3$ – dotted line.
Axes: horizontal – m_3, vertical – production levels



Source: own preparation.

With a fixed confidence level $\alpha \in (0,1)$, usually close to 1, we now define the break-even point as the level of production for which

$$\boldsymbol{P}\left(d(q_1,\ldots,q_n) \geq F\right) = \alpha.$$

Denoting by Φ^{-1} the inverse of the cumulative distribution function of the standard normal distribution, this condition can be written in the form

$$F = \overline{m} + \sigma \Phi^{-1}(1-\alpha), \tag{7}$$

bearing in mind that $\overline{m}$ and σ are functions of production levels $q_1,\ldots,q_n$. This condition will allow us to determine the set of acceptable levels of production. To indicate a specific level of production, we use the criterion of minimizing the total costs:

$$f\left(q_1,\ldots,q_n\right) \rightarrow \min$$

Example 2.4. Let $n = 2$, $F = 100$, $m_1 = 2$, $m_2 = 1$, $\sigma_1^2 = 0.5$, $\sigma_2^2 = 0.2$, $\rho_{12} = -0.4$ and $\alpha = 0.95$. Equation (7) has the form

$$100 = 2q_1 + q_2 + \sqrt{0.5q_1^2 + 0.2q_2^2 - 0.4\sqrt{0.5 \times 0.2}q_1q_2} \times \Phi^{-1}(1 - 0.05),$$

and we can solve it for q_2 as a function of $q_1 \in \left[0,q_{1,\max}\right]$, where $q_{1,\max} \approx 119.5$:

$$q_2 = 125 - 2.5791q_1 + \sqrt{3.3425q_2^2 - 144.7642q_2 + 3125}.$$

Assuming a linear function of total variable costs $f\left(q_1,q_2\right) = 3q_1 + 2q_2$, we obtain the value of the break-even point: $(\underline{q}_1,\underline{q}_2) = (37.4380, 77.3347)$.

Conclusions

We have shown how one can establish the break-even points for non-homogenous product sales. In Section 2.1., we maximized the total margin, taking into account the limitations of resources. In Section 2.2 cost effects of scale of production were considered. In Section 2.3 we considered the uncertainty of economic events, taking a sufficiently high probability of exceeding the break-even point for the criterion of the admissibility of the production plan.

Our considerations have led to new and usually unambiguous methods for determining the break-even point for a set of non-homogenous products, optimal with regard to some reasonably determined conditions. The presented solutions are quite simple and easy to motivate in terms of well known microeconomic criteria. Different approaches may be utilized by particular business units suitably to their goals. Two of the presented methods are deterministic, and the third manifests a stochastic approach to financial modeling. For large, real-life problems, the proposed methods typically necessitate the use of numerical methods, although in the era of modern computers this should not constitute an obstacle to the application of the presented methods. On the other hand, however, optimization problems suffer from the curse of dimensionality, which in the case of a large variety of products can be a limitation. Also, the reliable estimation of the parameters used in the presented methods, like the function of variable costs or parameters of the distribution of future prices, may prove a definitely more serious problem in practice. The assumption of normality in the third proposed method probably is not true for most companies and products, and should be replaced by more appropriate distribution. Another weakness of our proposals is that we assume equality of production and sales, that is, all manufactured products will be sold.

The considerations may be further enriched with other various aspects of optimizing production sizes. One of the possibilities is to include in the formulation of optimization tasks the degree of utilization of production capacities, understood as the maximum number of individual products that it is possible to manufacture at a given time with the help of the machines (generally – fixed assets) installed in the company.

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