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G-dual Frames in Hilbert C^*-module Spaces

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Abstract. In this paper, we introduce the concept of g-dual frames for Hilbert C^*-modules, and then the properties and stability results of g-dual frames are given. A characterization of g-dual frames, approximately dual frames and dual frames of a given frame is established. We also give some examples to show that the characterization of g-dual frames for Riesz bases in Hilbert spaces is not satisfied in general Hilbert C^*-modules.

1. Introduction

Let $\mathcal{A}$ be a C^*-algebra. A left pre Hilbert C^*-module $\mathcal{H}$ over $\mathcal{A}$ (or a pre Hilbert $\mathcal{A}$-module) is a linear space which is a left $\mathcal{A}$-module together with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ with following properties:

(i) $\langle x, x\rangle \geq 0$, $x \in \mathcal{H}$;
(ii) $\langle x, x\rangle = 0$ implies that $x = 0$;
(iii) $\langle \alpha x + y, z\rangle = \alpha \langle x, z\rangle + \langle y, z\rangle$, $\alpha \in \mathbb{C}$ and $x, y, z \in \mathcal{H}$;
(iv) $\langle ax, y\rangle = a \langle x, y\rangle$, $x, y \in \mathcal{H}$ and $a \in \mathcal{A}$;
(v) $\langle x, y\rangle = \langle y, x\rangle^*$, $x, y \in \mathcal{H}$.

We set $\|x\|^2_{\mathcal{H}} = \| \langle x, x\rangle \|_{\mathcal{A}}$ for each $x \in \mathcal{H}$. Then $\|.\|_{\mathcal{H}}$ is a norm on $\mathcal{H}$ and satisfies the following properties:

(*i*) $\|ax\|_{\mathcal{H}} \leq \|a\| \|x\|_{\mathcal{H}}$, $a \in \mathcal{A}$ and $x \in \mathcal{H}$;
(*ii*) $\langle x, y\rangle \langle y, x\rangle \leq \|y\|^2_{\mathcal{H}} \langle x, x\rangle$, $x, y \in \mathcal{H}$;
(*iii*) $\| \langle x, y\rangle \| \leq \|x\|_{\mathcal{H}} \|y\|_{\mathcal{H}}$, $x, y \in \mathcal{H}$;

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(see [13]). A pre-Hilbert A-module $\mathcal{H}$ is called a Hilbert $\mathcal{A}$-module (or a Hilbert C^*-module over $\mathcal{A}$) if it is complete with respect to the norm $\|.\|_{\mathcal{H}}$. For example, the C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with the $\mathcal{A}$-valued inner product of elements $a, b \in \mathcal{A}$ defined by $\langle a, b\rangle := ab^*$. In this paper, we deal with finitely or countably generated Hilbert C^*-modules. A Hilbert $\mathcal{A}$-module $\mathcal{H}$ is called finitely generated if there exists a finite set $\mathcal{F} \subseteq \mathcal{H}$ such that $\mathcal{H}$ equals the linear span (over $\mathbb{C}$ and $\mathcal{A}$) of this set. A Hilbert $\mathcal{A}$-module $\mathcal{H}$ is called countably generated if there exists a countable set $\mathcal{F} \subseteq \mathcal{H}$ such that $\mathcal{H}$ equals the norm-closure of the linear span (over $\mathbb{C}$ and $\mathcal{A}$) of this set. For a unital C^*-algebra $\mathcal{A}$ and a countable set I of indices,

$$\ell^2(\mathcal{A}) = \left\{ \{a_i\}_{i\in I} \subseteq \mathcal{A} : \sum_{i=1}^{\infty} a_i a_i^* \text{converges in norm} \right\},$$

is a Hilbert $\mathcal{A}$-module with the inner product

$$\langle \{a_i\}_i, \{b_i\}_i\rangle = \sum_{i=1}^{\infty} a_i b_i^*.$$

The set $\{e_i : i \in I\}$ that each e_i takes $1_{\mathcal{A}}$ in i and $0_{\mathcal{A}}$ everywhere else, is a generating set for $\ell^2(\mathcal{A})$ and it is called the standard orthonormal basis of $\ell^2(\mathcal{A})$.

For Hilbert C^*-modules V and W, a map $T : V \to W$ is called adjointable if there is a map $T^* : W \to V$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad x \in V, y \in W.$$

It is easy to see that every adjointable operator is $\mathcal{A}$-linear and bounded. The converse is true in Hilbert spaces: every bounded operator is adjointable. But this is no longer true in Hilbert C^*-modules. We denote by $\mathbf{L}(V, W)$ the set of all adjointable maps from V to W. In fact, $\mathbf{L}(V, W)$ is a Banach space with respect to the operator norm. Moreover, $\mathbf{L}(V, V)$ is a C^*-algebra and we will denote it by $\mathbf{L}(V)$. Note that the theory of Hilbert C^*-modules is quite different from that of Hilbert spaces. For more details about Hilbert C^*-modules we refer the reader to [13].

Proposition 1.1 ([16])**.** *Let $\mathcal{A}$ be a C^*-algebra. If $a, b \in \mathcal{A}$ are self-adjoint and $c \in \mathcal{A}$, then $a \leq b$ implies $c^*ac \leq c^*bc$.*

Frames in a Hilbert space can be viewed as redundant bases which are generalization of orthonormal bases. Indeed, frames are a tool for the construction of series expansions in Hilbert spaces. Frames were introduced by Duffin and Schaeffer [5] in 1952 for separable Hilbert spaces to deal with some problems in nonharmonic Fourier analysis. Hilbert C^*-module frames are generalization of Hilbert space frames. Frank and Larson [6, 7] extended theory of frames known for (separable)

Hilbert spaces to similar sets in C^*-algebras and (finitely and countably generated) Hilbert C^*-modules. However, some properties of frames in Hilbert spaces hold also for Hilbert C^*-modules and often require different proofs. Moreover, there are many essential differences between Hilbert space frames and Hilbert C^*-module frames. It is known that every Hilbert space admits a frame while it has shown in [14] that not every Hilbert C*-module admits a frame. By Kasparov Stabilization Theorem, we infer that every finitely or countably generated Hilbert C^*-module has a frame (see [6]), so in this paper, we consider Hilbert C^*-modules which are finitely or countably generated. For more details on these topics we refer to [8, 10, 14, 17–19].

Throughout the paper $\mathcal{H}$ denotes a Hilbert C^*-module, $\mathcal{A}$ denotes a unital C^*-algebra, and I is a finite or countable index set. The notations Φ, Ψ and Γ are used to denote the sequences $\{\varphi_i\}_{i\in I}, \{\psi_i\}_{i\in I}$ and $\{\gamma_i\}_{i\in I}$ in $\mathcal{H}$, respectively. We now introduce the definition of frames in Hilbert C^*-modules.

Definition 1.2. A sequence Φ is called a (standard) *frame* for $\mathcal{H}$ if there exist constants $0 < A \leq B$ such that

$$A \langle h, h\rangle \leq \sum_{i\in I} \langle h, \varphi_i\rangle \langle \varphi_i, h\rangle \leq B \langle h, h\rangle, \quad h \in \mathcal{H}, \tag{1.1}$$

where the sum in the middle of the inequality is convergent in norm.

The constants A, B are called the *lower* and *upper frame bounds*, respectively. If $A = B$, the frame Φ is called a *tight frame* and if $A = B = 1$, it is called a *normalized tight frame* or *Parseval frame*. A sequence Φ is called a (standard) *Bessel sequence* for $\mathcal{H}$ if the right inequality in (1.1) is required.

If Φ is a Bessel sequence for a Hilbert $\mathcal{A}$-module $\mathcal{H}$, then the operator

$$T_\Phi : \ell^2(\mathcal{A}) \to \mathcal{H}, \qquad T_\Phi\left(\{a_i\}_{i\in I}\right) = \sum_{i\in I} a_i\varphi_i,$$

is well defined, adjointable and bounded. The operator T_Φ is called the *synthesis operator*. The adjoint operator of T_Φ is given by

$$U_\Phi = T_\Phi^* : \mathcal{H} \to \ell^2(\mathcal{A}), \qquad U_\Phi(h) = \{\langle h, \varphi_i\rangle\}_{i\in I},$$

and is called the *analysis operator*. By composing T_Φ with its adjoint T_Φ^* we obtain the frame operator

$$S_\Phi : \mathcal{H} \to \mathcal{H}, \qquad S_\Phi(h) = T_\Phi U_\Phi(h) = \sum_{i\in I} \langle h, \varphi_i\rangle \varphi_i.$$

The frame operator S_Φ is a positive operator and will be invertible if the Bessel sequence Φ is a frame for $\mathcal{H}$ [6].

Definition 1.3. A frame Φ of nonzero elements in Hilbert $\mathcal{A}$-module $\mathcal{H}$ is called a (standard) *Riesz basis* if

$$\sum_{i\in J} a_i\varphi_i = 0,$$

for $J \subseteq I$ and $a_i \in \mathcal{A}$ implies $a_i\varphi_i = 0$ for each $i \in J$.

Assume that $\mathcal{H}$ is a Hilbert C^*-module and Φ is a frame for $\mathcal{H}$. A sequence Ψ in $\mathcal{H}$ is said to be a dual sequence of Φ if

$$h = \sum_{i\in I} \langle h, \psi_i\rangle \varphi_i,$$

holds for all $h \in \mathcal{H}$. Since S_Φ is invertible, we have

$$h = S_\Phi S_\Phi^{-1} h = \sum_{i\in I} \left\langle h, S_\Phi^{-1}\varphi_i\right\rangle \varphi_i, \quad h \in \mathcal{H}.$$

Then $\{S_\Phi^{-1}(\varphi_i)\}_{i\in I}$ is a dual of Φ. This dual is called the *canonical dual frame* of Φ and is denoted by $\tilde{\Phi}$. We will use the following results in this paper.

Proposition 1.4 ([10])**.** *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and Φ, Ψ be two Bessel sequences in $\mathcal{H}$. If*

$$h = \sum_{i\in I} \langle h, \psi_i\rangle \varphi_i,$$

holds for all $h \in H$, then both Φ and Ψ are frames of $\mathcal{H}$ and

$$h = \sum_{i\in I} \langle h, \varphi_i\rangle \psi_i,$$

holds for all $h \in \mathcal{H}$.

Theorem 1.5. *[10] Let $\mathcal{H}$ be a Hilbert C^*-module and Φ be a frame for $\mathcal{H}$ with analysis operator U_Φ. Then the following statements are equivalent:*

(i) *Φ has a unique dual frame;*
(ii) *U_Φ is onto and therefore it is invertible;*
(iii) *T_Φ is injective and therefore it is an invertible operator.*

If each of the equivalent conditions is satisfied, Φ will be a Riesz basis for $\mathcal{H}$.

2. g-dual Frames

The concept of g-dual frame for Hilbert spaces was introduced by Dehghan and Hasankhanifard in [3]. They also presented g-duals for $L^2(0,\infty)$ [11]. This concept extended to generalized frames by Dengfeng and Yanting [4]. In this section, we introduce g-dual frames for a given frame in Hilbert C^*-modules and express some results about them.

Definition 2.1. Let Φ be a Bessel sequence for a Hilbert C^*-module $\mathcal{H}$. A Bessel sequence Ψ in $\mathcal{H}$ is called a *generalized dual* of Φ if $T_\Phi U_\Psi$ is invertible.

If we set $G = (T_\Phi U_\Psi)^{-1}$, then we have

$$h = \sum_{i \in I} \langle Gh, \psi_i \rangle \varphi_i,$$

for each $h \in \mathcal{H}$. Since $T_\Phi U_\Psi$ is adjointable, G will be adjointable and we have

$$h = \sum_{i \in I} \langle h, G^* \psi_i \rangle \varphi_i,$$

for every $h \in \mathcal{H}$. Because $\{G^* \psi_i\}_{i \in I}$ and $\{\varphi_i\}_{i \in I}$ are Bessel sequences, by Proposition 1.4, they will be frames. Invertibility of G^* implies that Ψ is a frame for $\mathcal{H}$. From now on, we use g-dual frame for generalized dual sequence. If Ψ is a g-dual frame of Φ, then the operators $T_\Phi U_\Psi$ and $T_\Psi U_\Phi$ are invertible. This implies that Φ is also a g-dual frame of Ψ. Since $T_\Phi U_\Phi = S_\Phi$ is invertible, every frame Φ is a g-dual frame of itself. If Φ^d is a dual frame of Φ, we have $T_\Phi U_{\Phi^d} = Id_{\mathcal{H}}$. So every dual frame of a frame is a g-dual frame of it.

Another concept that is related to this discussion is the approximately dual frames. Approximately dual frames were introduced by Christensen and Laugesen [2] for separable Hilbert spaces and were extended to Hilbert C^*-modules by Mirzaee [15]. Recall that two Bessel sequences Φ and Ψ of Hilbert C^*-module $\mathcal{H}$ are called *approximately dual frames* if $\|T_\Phi U_\Psi - Id_{\mathcal{H}}\| < 1$ or $\|T_\Psi U_\Phi - Id_{\mathcal{H}}\| < 1$. It is clear that approximately dual frames are g-dual frames. Before characterizing g-dual frames we state some results which are similar to results in g-dual frames on Hilbert spaces. In the following proposition $\mathcal{Z}(\mathcal{A}) = \{a \in \mathcal{A} : ab = ba, \forall b \in \mathcal{A}\}$ is the center of $\mathcal{A}$.

Proposition 2.2. *Let Φ be a frame for a Hilbert $\mathcal{A}$-module $\mathcal{H}$ and Ψ be a g-dual frame of Φ with $(T_\Phi U_\Psi)^{-1} = G$. If S_Φ is the frame operator of Φ and $a \in \mathcal{Z}(\mathcal{A})$, then the sequence*

$$\Psi^a = \left\{ a\psi_i + (1_{\mathcal{A}} - a) \left(G^{-1}\right)^* S_\Phi^{-1} \varphi_i \right\}_{i \in I},$$

is a g-dual frame of Φ with $(T_\Phi U_{\Psi^a})^{-1} = G$.

Proof. Since Ψ is a g-dual frame of Φ, we have

$$T_\Phi U_{\Psi^a} G h = \sum_{i \in I} \langle Gh, \psi_i^a \rangle \varphi_i$$

$$
\begin{aligned}
&= \sum_{i\in I} \langle Gh, a\psi_i\rangle \varphi_i + \sum_{i\in I} \left\langle Gh, (1_{\mathcal{A}} - a)\left(G^{-1}\right)^* S_\Phi^{-1}\varphi_i\right\rangle \varphi_i \\
&= \sum_{i\in I} \langle Gh, \psi_i\rangle a^*\varphi_i + \sum_{i\in I} \left\langle Gh, \left(G^{-1}\right)^* S_\Phi^{-1}\varphi_i\right\rangle (1_{\mathcal{A}} - a)^*\varphi_i \\
&= a^* \sum_{i\in I} \langle Gh, \psi_i\rangle \varphi_i + (1_{\mathcal{A}} - a^*) \sum_{i\in I} \left\langle h, S_\Phi^{-1}\varphi_i\right\rangle \varphi_i \\
&= a^*h + (1_{\mathcal{A}} - a^*)h = h, \quad h \in \mathcal{H}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
GT_\Phi U_{\Psi^a} h &= G\left(\sum_{i\in I} \langle h, \psi_i^a\rangle \varphi_i\right) \\
&= G\left(\sum_{i\in I} \langle h, a\psi_i\rangle \varphi_i\right) \\
&\quad + G\left(\sum_{i\in I} \left\langle h, (1_{\mathcal{A}} - a)\left(G^{-1}\right)^* S_\Phi^{-1}\varphi_i\right\rangle \varphi_i\right) \\
&= a^*G\left(\sum_{i\in I} \langle h, \psi_i\rangle \varphi_i\right) \\
&\quad + (1_{\mathcal{A}} - a^*)G\left(\sum_{i\in I} \left\langle G^{-1}h, S_\Phi^{-1}\varphi_i\right\rangle \varphi_i\right) \\
&= a^*h + (1_{\mathcal{A}} - a^*)h = h, \quad h \in \mathcal{H}.
\end{aligned}
$$

Therefore $T_\Phi U_{\Psi^a}$ is invertible and $(T_\Phi U_{\Psi^a})^{-1} = G$. □

The following proposition shows that g-duality is preserved under adjointable invertible operators.

Proposition 2.3. *Let Ψ be a g-dual frame of Φ in a Hilbert C^*-module $\mathcal{H}$ with $(T_\Phi U_\Psi)^{-1} = G$ and E, F be two adjointable invertible operators on $\mathcal{H}$. Then $E\Psi$ is a g-dual frame of $F\Phi$ with $[T_{F\Phi}U_{E\Psi}]^{-1} = (E^*)^{-1}GF^{-1}$.*

Proof. Since $T_{F\Phi} = FT_\Phi$ and $U_{E\Psi} = U_\Psi E^*$, we have

$$
\begin{aligned}
T_{F\Phi}U_{E\Psi}(E^*)^{-1}GF^{-1} &= FT_\Phi U_\Psi E^*(E^*)^{-1}GF^{-1} \\
&= FT_\Phi U_\Psi GF^{-1} \\
&= Id_{\mathcal{H}},
\end{aligned}
$$

and

$$
(E^*)^{-1}GF^{-1}T_{F\Phi}U_{E\Psi} = (E^*)^{-1}GF^{-1}FT_\Phi U_\Psi E^*
$$

$$= (E^*)^{-1} G T_\Phi U_\Psi E^*$$
$$= Id_{\mathcal{H}}.$$

Hence $E\Psi$ and $F\Phi$ are g-dual frames. □

We can weighted Bessel sequences and verify g-duality between them. Consider

$$\ell^\infty(\mathcal{A}) = \left\{ \{a_i\}_{i\in I} \subseteq \mathcal{A} : \sup_{i\in I} \|a_i\| < \infty \right\}.$$

Proposition 2.4. *Let Φ be a Bessel sequence for a Hilbert $\mathcal{A}$-module $\mathcal{H}$ with a Bessel bound B and $m = \{m_i\}_{i\in I} \in \ell^\infty(\mathcal{A})$. Then $\{m_i\varphi_i\}_{i\in I}$ is a Bessel sequence for $\mathcal{H}$.*

Proof. By Proposition 1.1, we have

$$\langle x, \varphi_i\rangle m_i^* m_i \langle \varphi_i, x\rangle \le \langle x, \varphi_i\rangle \|m_i\|^2 \langle \varphi_i, x\rangle \le \langle x, \varphi_i\rangle \|m\|_\infty^2 \langle \varphi_i, x\rangle ,$$

for each $i \in I$ and each $x \in \mathcal{H}$. So we get

$$\begin{aligned}\sum_{i\in I} \langle x, m_i\varphi_i\rangle \langle m_i\varphi_i, x\rangle &= \sum_{i\in I} \langle x, \varphi_i\rangle m_i^* m_i \langle \varphi_i, x\rangle \\ &\le \|m\|_\infty^2 \sum_{i\in I} \langle x, \varphi_i\rangle \langle \varphi_i, x\rangle \\ &\le \|m\|_\infty^2 B \langle x, x\rangle ,\end{aligned}$$

for each $x \in \mathcal{H}$. □

Proposition 2.5. *Let Φ, Ψ and Γ be Bessel sequences in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ and $m, m' \in \ell^\infty(\mathcal{A})$. Then $m\Psi + m'\Gamma$ is a g-dual frame of Φ if and only if $T_\Phi U_{m\Psi} + T_\Phi U_{m'\Gamma}$ is invertible.*

Proof. Since $m, m' \in \ell^\infty(\mathcal{A})$, it follows from Proposition 2.4 that $m\Psi + m'\Gamma$ is a Bessel sequence for $\mathcal{H}$, and we have

$$\begin{aligned}T_\Phi U_{m\Psi+m'\Gamma}(h) &= \sum_{i\in I} \left\langle h, m_i\psi_i + m_i'\gamma_i\right\rangle \varphi_i \\ &= \sum_{i\in I} \langle h, m_i\psi_i\rangle \varphi_i + \sum_{i\in I} \left\langle h, m_i'\gamma_i\right\rangle \varphi_i \\ &= T_\Phi U_{m\Psi} + T_\Phi U_{m'\Gamma}(h), \quad h \in \mathcal{H}.\end{aligned}$$

This completes the proof. □

3. Characterization of g-dual Frames, Approximately Dual Frames and Dual Frames in Hilbert C^*-modules

In this section, we characterize all g-dual frames, approximately dual frames and dual frames of a given frame Φ in a Hilbert C^*-module $\mathcal{H}$.

For this we first introduce the following notation:

$$\operatorname{ran}_{\mathbf{L}(\mathcal{H},\ell^2(\mathcal{A}))}(T_\Phi) := \left\{\Theta \in \mathbf{L}(\mathcal{H},\ell^2(\mathcal{A})) : T_\Phi\Theta = 0\right\}.$$

Proposition 3.1. *Let Φ be a frame for a Hilbert C^*-module $\mathcal{H}$. Then we have*

$$\operatorname{ran}_{\mathbf{L}(\mathcal{H},\ell^2(\mathcal{A}))}(T_\Phi) = \left\{U_F - U_\Phi S_\Phi^{-1}T_\Phi U_F : F \textit{ is a Bessel sequence in } \mathcal{H}\right\}.$$

Proof. First assume that F is a Bessel sequence in $\mathcal{H}$, then we have

$$\begin{aligned} T_\Phi U_F - T_\Phi U_\Phi S_\Phi^{-1}T_\Phi U_F &= T_\Phi U_F - S_\Phi S_\Phi^{-1}T_\Phi U_F \\ &= T_\Phi U_F - T_\Phi U_F \\ &= 0. \end{aligned}$$

So we get

$$U_F - U_\Phi S_\Phi^{-1}T_\Phi U_F \in \operatorname{ran}_{\mathbf{L}(\mathcal{H},\ell^2(\mathcal{A}))}(T_\Phi).$$

Conversely, let $0 \neq \Theta \in \operatorname{ran}_{\mathbf{L}(\mathcal{H},\ell^2(\mathcal{A}))}(T_\Phi)$ and Θ^* be the adjoint operator of Θ. Then we have

$$\Theta(h) = \{\langle\Theta(h), e_i\rangle\}_{i\in I} = \{\langle h, \Theta^* e_i\rangle\}_{i\in I}, \quad h \in \mathcal{H},$$

where $\{e_i\}_{i\in I}$ is the standard orthonormal basis of $\ell^2(\mathcal{A})$. Therefore Θ is the analysis operator of the Bessel sequence $\{\Theta^* e_i\}_{i\in I}$. If we set $F = \{\Theta^* e_i\}_{i\in I}$ then $\Theta = U_F$ and

$$U_F - U_\Phi S_\Phi^{-1}T_\Phi U_F = U_F - 0 = U_F = \Theta.$$

If $\Theta = 0$, we set $F = \Phi$. Then the proof is completed. $\square$

Now we characterize all g-dual frames, approximately dual frames and dual frames of a given frame Φ. We will show that for every adjointable invertible operator G on $\mathcal{H}$ we have a g-dual frame of Φ and for every adjointable invertible operator G on $\mathcal{H}$ with $\|Id_\mathcal{H} - G\| < 1$ we have an approximately dual frame of Φ.

Theorem 3.2. *Let Φ be a frame for a Hilbert C^*-module $\mathcal{H}$ and $\{e_i\}_{i\in I}$ be the standard orthonormal basis of $\ell^2(\mathcal{A})$. Then all g-dual frames of Φ are precisely the sequences Φ^g such that*

$$\varphi_i^g = (S_\Phi G^*)^{-1}\varphi_i + \Theta^*(e_i),$$

where $\Theta \in \operatorname{ran}_{\mathbf{L}(\mathcal{H},\ell^2(\mathcal{A}))}(T_\Phi)$ and G is an invertible adjointable operator on $\mathcal{H}$. In particular, $G^{-1} = T_\Phi U_{\Phi^g}$.

Proof. Suppose Φ^g is a g-dual frame of Φ with $(T_\Phi U_{\Phi^g})^{-1} = G$. If we set $\Theta := U_{\Phi^g} - U_\Phi S_\Phi^{-1}T_\Phi U_{\Phi^g}$, then Proposition 3.1 implies that $\Theta \in \operatorname{ran}_{\mathbf{L}(H,\ell^2(\mathcal{A}))}(T_\Phi)$ and

$$\begin{aligned} (S_\Phi G^*)^{-1}\varphi_i + \Theta^* e_i &= (G^*)^{-1}S_\Phi^{-1}\varphi_i + \left(T_{\Phi^g} - T_{\Phi g}U_\Phi S_\Phi^{-1}T_\Phi\right)e_i \\ &= (G^*)^{-1}S_\Phi^{-1}\varphi_i + \varphi_i^g - T_{\Phi^g}U_\Phi S_\Phi^{-1}\varphi_i \end{aligned}$$

$$= (G^*)^{-1}S_\Phi^{-1}\varphi_i + \varphi_i^g - (G^*)^{-1}S_\Phi^{-1}\varphi_i$$
$$= \varphi_i^g.$$

Conversely, let $\Theta \in \mathrm{ran}_{\mathbf{L}(H,\ell^2(\mathcal{A}))}(T_\Phi)$ and G be an adjointable invertible operator on $\mathcal{H}$. Suppose that Φ^g is a sequence in $\mathcal{H}$ such that $\varphi_i^g = (S_\Phi G^*)^{-1}\varphi_i + \Theta^* e_i$. Then Φ^g is a Bessel sequence in $\mathcal{H}$ and we have

$$\begin{aligned} T_\Phi U_{\Phi^g} Gh &= \sum_{i\in I} \langle Gh, \varphi_i^g\rangle \varphi_i \\ &= \sum_{i\in I} \left\langle Gh, (G^*)^{-1}S_\Phi^{-1}\varphi_i\right\rangle \varphi_i + \sum_{i\in I} \langle Gh, \Theta^* e_i\rangle \varphi_i \\ &= \sum_{i\in I} \left\langle h, S_\Phi^{-1}\varphi_i\right\rangle \varphi_i + T_\Phi \Theta G h = h, \end{aligned}$$

and

$$\begin{aligned} G T_\Phi U_{\Phi^g} h &= G\left(\sum_{i\in I} \langle h, \varphi_i^g\rangle \varphi_i\right) \\ &= G\left(\sum_{i\in I} \langle h, (G^*)^{-1}S_\Phi^{-1}\varphi_i\rangle \varphi_i\right) + G\left(\sum_{i\in I} \langle h, \Theta^* e_i\rangle \varphi_i\right) \\ &= G\left(\sum_{i\in I} \left\langle G^{-1}h, S_\Phi^{-1}\varphi_i\right\rangle \varphi_i\right) + G T_\Phi \Theta h = h, \end{aligned}$$

for every $h \in \mathcal{H}$. Then $(T_\Phi U_{\Phi^g})^{-1} = G$ and Φ^g is a g-dual frame of Φ. $\square$

The proofs of the following theorems are similar to the proof of Theorem 3.2 and we omit them.

Theorem 3.3. *Let Φ be a frame for a Hilbert C^*-module $\mathcal{H}$ and $\{e_i\}_{i\in I}$ be the standard orthonormal basis of $\ell^2(\mathcal{A})$. Then all approximately dual frames of Φ are precisely the sequences of the form*

$$\varphi_i^{ad} = G^* S_\Phi^{-1}\varphi_i + \Theta^* e_i,$$

where $\Theta \in \mathrm{ran}_{\mathbf{L}(\mathcal{H},\ell^2(\mathcal{A}))}(T_\Phi)$ and G is an adjointable invertible operator on $\mathcal{H}$ such that $\|Id_{\mathcal{H}} - G\| < 1$. In this case, $G = T_\Phi U_{\Phi^{ad}}$.

Theorem 3.4. *Let Φ be a frame for a Hilbert C^*-module $\mathcal{H}$ and $\{e_i\}_{i\in I}$ be the standard orthonormal basis of $\ell^2(\mathcal{A})$. Then all dual frames of Φ are precisely the sequences of the form*

$$\varphi_i^{d} = S_\Phi^{-1}\varphi_i + \Theta^* e_i,$$

where $\Theta \in \mathrm{ran}_{\mathbf{L}(\mathcal{H},\ell^2(\mathcal{A}))}(T_\Phi)$.

Remark 3.5. *If G is an adjointable invertible operator on a Hilbert C^*-module $\mathcal{H}$ and Φ is a frame for $\mathcal{H}$, then by Theorem 3.2 we can introduce a g-dual frame Φ^g of Φ with $(T_\Phi U_{\Phi^g})^{-1} = G$. If G has an extra condition $\|G - Id_{\mathcal{H}}\| < 1$, then we have an approximately dual frame of Φ. If G is an adjoinable positive and onto operator on $\mathcal{H}$, we can also introduce a frame that G is the frame operator of it. Indeed, if Φ is a tight frame of $\mathcal{H}$, then G is the frame operator of the frame $G^{\frac{1}{2}}\Phi$.*

All of these characterizations are exactly the same characterization that were presented in Hilbert spaces [3, 12], but the characterization of g-dual frames for Riesz bases in Hilbert spaces is not satisfied in general Hilbert C^*-modules. In Hilbert spaces every g-dual frame for a Riesz basis Φ is of the form $G\Phi$ where G is an invertible operator and of course every g-dual frame of a Riesz basis is a Riesz basis [3]. But in a Hilbert C^*-module there exists a dual frame of a Riesz basis that is not a Riesz basis (see Example 3.6 in [10]). Since every dual frame of a frame in a Hilbert C^*-module is a g-dual frame, we have a g-dual frame that is not a Riesz basis. Because Riesz bases in Hilbert C^*-modules are preserved under invertible adjointable operators, a g-dual frame of a frame Φ is not in general of the form $G\Phi$, where G is an invertible adjointable operator.

In Hilbert spaces every two Riesz basis are g-dual frames of each other, and moreover, if Φ and Ψ are Riesz bases, then $(T_\Phi U_\Psi)^{-1} = T_{\tilde{\Psi}} U_{\tilde{\Phi}}$ [3], but this is no longer true for Hilbert C^*-modules. We consider the following examples.

Example 3.6. Let $\mathbf{M}_{2\times 2}(\mathbb{C})$ denote the C^*-algebra of all 2×2 complex matrices. Then $\mathbf{M}_{2\times 2}(\mathbb{C})$ is a Hilbert C^*-module with the inner product $\langle A, B\rangle = AB^*$ for $A, B \in \mathbf{M}_{2\times 2}(\mathbb{C})$.
Now we set

$$\Phi = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right\}, \qquad \Psi = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then Φ and Ψ are Riesz bases, but $T_\Phi U_\Psi = 0$ and hence it is not invertible. So Ψ is not a g-dual frame of Φ.

Example 3.7. In the Example 3.6 we set

$$\Phi = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \qquad \Psi = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right\}.$$

Then Φ is a Parseval frame and Ψ is a tight frame with bound 2. Also both Φ and Ψ are Riesz Bases. We have that $T_\Phi U_\Psi$ is invertible with $(T_\Phi U_\Psi)^{-1} = T_\Psi U_\Phi$. But $T_\Psi U_\Phi$ is not equal to $T_{\tilde{\Psi}} U_{\tilde{\Phi}} = \frac{1}{2} T_\Psi U_\Phi$.

Proposition 3.8. *Let Φ be a Riesz basis for a Hilbert C^*-module $\mathcal{H}$. If Φ has a unique dual frame, then every g-dual frame Φ^g of Φ is a Riesz* *basis of the form $G\Phi$ where G is an adjointable invertible operator on $\mathcal{H}$ and $(T_\Phi U_{\Phi^g})^{-1} = T_{\tilde{\Phi}^g} U_{\tilde{\Phi}}$.*

Proof. Since Φ has a unique dual frame, by Theorem 1.5, T_Φ is invertible. If Φ^g is a g-dual frame of Φ, $T_\Phi U_{\Phi^g}$ is invertible and hence U_{Φ^g} will be invertible. By Theorem 1.5, Φ^g is a Riesz basis and we have

$$\begin{aligned} T_\Phi U_{\Phi^g} T_{\tilde{\Phi}^g} U_{\tilde{\Phi}} &= T_\Phi U_{\Phi^g} S_{\Phi^g}^{-1} T_{\Phi^g} U_\Phi S_\Phi^{-1} \\ &= T_\Phi U_{\Phi^g} U_{\Phi^g}^{-1} T_{\Phi^g}^{-1} T_{\Phi^g} U_\Phi S_\Phi^{-1} \\ &= Id_{\mathcal{H}}. \end{aligned}$$

If $\{e_i\}_{i\in I}$ is the standard orthonormal basis of $\ell^2(\mathcal{A})$, then by invertibility of U_Φ and U_{Φ^g} we have

$$U_\Phi^{-1} e_i = \varphi_i, \qquad U_{\Phi^g}^{-1} e_i = \varphi_i^g,$$

so

$$\varphi_i^g = U_{\Phi^g}^{-1} U_\Phi \varphi_i.$$

Now we set $G = U_{\Phi^g}^{-1} U_\Phi$ and the proof is completed. □

4. Stability of g-dual Frames

Let Φ and Ψ be Bessel sequences in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ and let $m = \{m_i\}_{i\in I} \in \ell^\infty(\mathcal{A})$. The operator

$$M_{m,\Phi,\Psi} : \mathcal{H} \to \mathcal{H}, \qquad M_{m,\Phi,\Psi} h = \sum_{i\in I} m_i \langle h, \psi_i \rangle \varphi_i,$$

is called a *Bessel multiplier*. If we set $m = \{1_\mathcal{A}\}$, then $M_{\{1_\mathcal{A}\},\Phi,\Psi} = T_\Phi U_\Psi$. The invertibility of $M_{m,\Phi,\Psi}$ and representation of the inverse were verified in [1, 9, 20] for Hilbert spaces and for Hilbert C^*-modules. We explain some of these results for g-dual frames in Hilbert C^*-modules. We will use the following proposition.

Proposition 4.1 ([9])**.** *Let $\mathbf{B}$ be a Banach space and $F : \mathbf{B} \to \mathbf{B}$ be invertible on $\mathbf{B}$. Suppose that $G : \mathbf{B} \to \mathbf{B}$ is a bounded operator such that $\|Gb - Fb\| \le v\|b\|$ for all b in $\mathbf{B}$, where $v \in [0, \frac{1}{\|F^{-1}\|})$. Then*

(i) *G is invertible on $\mathbf{B}$,*

$$G^{-1} = \sum_{k=0}^{\infty} \left[F^{-1}(F-G)\right]^k F^{-1},$$

and

$$\left\| G^{-1} - \sum_{k=0}^{n} [F^{-1}(F-G)]^k F^{-1} \right\| \le \|F^{-1}\| \sum_{k=n+1}^{\infty} \|F^{-1}(F-G)\|^k.$$

(ii)

$$\frac{1}{v+\|F\|}\|b\| \leq \|G^{-1}b\| \leq \frac{1}{(\frac{1}{\|F^{-1}\|}-v)}\|b\|, \quad b \in \mathbf{B}.$$

Theorem 4.2. *Let Φ be a frame for a Hilbert C^*-module $\mathcal{H}$ and Ψ be a sequence in $\mathcal{H}$. If there exists $\mu \in [0, \frac{A_\Phi^2}{B_\Phi})$ such that*

$$\sum_{i\in I} \langle h, \psi_i - \varphi_i\rangle \langle \psi_i - \varphi_i, h\rangle \leq \mu \langle h, h\rangle, \quad h \in \mathcal{H},$$

then Ψ is a frame for $\mathcal{H}$, $T_\Phi U_\Psi$ is invertible on $\mathcal{H}$ and

$$\frac{1}{B_\Phi + \sqrt{\mu B_\Phi}}\|h\| \leq \|(T_\Phi U_\Psi)^{-1}h\| \leq \frac{1}{A_\Phi - \sqrt{\mu B_\Phi}}\|h\|, \tag{4.1}$$

$$(T_\Phi U_\Psi)^{-1} = \sum_{i\in I} \left[S_\Phi^{-1}(S_\Phi - T_\Phi U_\Psi)\right]^k S_\Phi^{-1}.$$

By invertibility of $T_\Phi U_\Psi$, the sequence Ψ is a g-dual frame of Φ.

Proof. For $\mu = 0$, we have $\Phi = \Psi$ and therefore $T_\Phi U_\Psi = S_\Phi$ which is invertible. Let $\mu > 0$. Since $\mu < \frac{A_\Phi^2}{B_\Phi} < A_\Phi$, we infer that Ψ is a frame for $\mathcal{H}$ (see Corollary 3.5 in [8]). We also have

$$\begin{aligned}\|T_\Phi U_\Psi h - S_\Phi h\| &= \Big\|\sum_{i\in I} \langle h, \psi_i\rangle \varphi_i - \sum_{i\in I} \langle h, \varphi_i\rangle \varphi_i\Big\| \\ &= \Big\|\sum_{i\in I} \langle h, \psi_i - \varphi_i\rangle \varphi_i\Big\| \\ &= \|T_\Phi U_{\Psi-\Phi} h\| \leq \sqrt{\mu B_\Phi}\|h\|, \quad h \in \mathcal{H}.\end{aligned}$$

Since $\sqrt{\mu B_\Phi} < A_\Phi \leq \frac{1}{\|S_\Phi^{-1}\|}$, by Proposition 4.1, we infer $T_\Phi U_\Psi$ is invertible and satisfies (4.1). □

Proposition 4.3. *Let Φ be a frame for a Hilbert C^*-module $\mathcal{H}$ and Ψ be a sequence in $\mathcal{H}$. Assume that there exists $\mu \in [0, \frac{1}{B_\Phi})$ such that*

$$\sum_{i\in I} \left\langle h, \psi_i - \varphi_i^d\right\rangle \left\langle \psi_i - \varphi_i^d, h\right\rangle \leq \mu \langle h, h\rangle, \quad h \in \mathcal{H}$$

for some dual frame Φ^d of Φ. Then Ψ is a g-dual frame for $\mathcal{H}$ and $T_\Phi U_\Psi$ is invertible on $\mathcal{H}$ with

$$\frac{1}{1 + \sqrt{\mu B_\Phi}}\|h\| \leq \|(T_\Phi U_\Psi)^{-1}h\| \leq \frac{1}{1 - \sqrt{\mu B_\Phi}}\|h\|, \quad \forall h \in \mathcal{H}.$$

Furthermore, $(T_\Phi U_\Psi)^{-1} = \sum_{k=0}^{\infty}(Id_\mathcal{H} - T_\Phi U_\Psi)^k$.

Proof. If $\mu = 0$, then $\Phi^d = \Psi$ and $T_\Phi U_\Psi = Id_{\mathcal{H}}$ is invertible. Assume that $\mu > 0$. Since Φ^d is a frame with frame bounds $\frac{1}{B_\Phi}$, $\frac{1}{A_\Phi}$ and $\mu < \frac{1}{B_\Phi}$, we get Ψ is a frame for $\mathcal{H}$ (see Corollary 3.5 in [8]). Moreover, we have

$$\begin{aligned}\|T_\Phi U_\Psi h - h\| &= \left\| \sum_{i\in I} \langle h, \psi_i\rangle \varphi_i - \sum_{i\in I} \left\langle h, \varphi_i^d \right\rangle \varphi_i \right\| \\ &= \left\| \sum_{i\in I} \left\langle h, \psi_i - \varphi_i^d \right\rangle \varphi_i \right\| \leq \sqrt{\mu B_\Phi}\|h\|,\end{aligned}$$

for each $h \in \mathcal{H}$. Now we can apply Proposition 4.1 to complete the proof. □

We recall that two frames Φ and Ψ in a Hilbert C^*-module $\mathcal{H}$ are called *equivalent* if there exists an adjointable invertible operator F on $\mathcal{H}$ such that $\psi_i = F\varphi_i$ for each $i \in I$.

Proposition 4.4. *Let Φ and Ψ be equivalent frames in a Hilbert C^*-module $\mathcal{H}$. Then Ψ is a g-dual frame of Φ and $(T_\Phi U_\Psi)^{-1} = T_{\tilde{\Psi}} U_{\tilde{\Phi}}$.*

Proof. By the assumption, there exists an adjointable invertible operator F on $\mathcal{H}$ that $\psi_i = F\varphi_i$ for every $i \in I$. Then $T_\Psi = FT_\Phi$ and $U_\Psi = U_\Phi F^*$. Therefore

$$\begin{aligned}T_\Phi U_\Psi T_{\tilde{\Psi}} U_{\tilde{\Phi}} &= T_\Phi U_\Phi F^* S_\Psi^{-1} F T_\Phi U_\Phi S_\Phi^{-1} \\ &= T_\Phi U_\Phi F^* (F^*)^{-1} S_\Phi^{-1} F^{-1} F \\ &= Id_{\mathcal{H}},\end{aligned}$$

and

$$\begin{aligned}T_{\tilde{\Psi}} U_{\tilde{\Phi}} T_\Phi U_\Psi &= S_\Psi^{-1} F T_\Phi U_\Phi S_\Phi^{-1} T_\Phi U_\Phi F^* \\ &= S_\Psi^{-1} F T_\Phi U_\Phi F^* = S_\Psi^{-1} T_\Psi U_\Psi \\ &= Id_{\mathcal{H}}.\end{aligned}$$

□

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