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Some Fixed Point Results for the Generalized F-suzuki Type Contractions in b-metric Spaces

Sumit Chandok[1], Huaping Huang[2*], and Stojan Radenović[3]

Abstract. Compared with the previous work, the aim of this paper is to introduce the more general concept of the generalized F-Suzuki type contraction mappings in b-metric spaces, and to establish some fixed point theorems in the setting of b-metric spaces. Our main results unify, complement and generalize the previous works in the existing literature.

1. Introduction and Preliminaries

Before the start of this article, let us denote the letters $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ as the set of all natural numbers, the set of all real numbers and the set of all nonnegative real numbers, respectively. We denote Ψ as the set of all functions $\psi : [0,\infty) \to [0,\infty)$ such that ψ is continuous and $\psi^{-1}(0) = 0$.

Recently, Wardowski [13] introduced a new type of contractive mappings called F-contraction and proved a fixed point theorem as a generalization of the Banach contraction principle as follows:

Definition 1.1 ([13])**.** Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a mapping satisfying

- (F1) F is strictly increasing, i.e. for all $\alpha, \beta \in \mathbb{R}_+$, $\alpha < \beta$ implies that $F(\alpha) < F(\beta)$;
- (F2) For any sequence $\{\alpha_n\}$ of positive numbers, $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$ are equivalent;
- (F3) There exists $k \in (0,1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

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* Corresponding author.

For some examples satisfying Definition 1.1, the reader may refer to [14] and [13].

Definition 1.2 ([13])**.** Let (X, d) be a metric space. A map $T : X \to X$ is said to be F-contraction on X if there exist F in Definition 1.1 and $\tau > 0$ such that

$$x, y \in X \text{ with } d(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F(d(Tx, Ty)) \le F(d(x, y)).$$

Theorem 1.3. *Let (X, d) be a complete metric space and $T : X \to X$ an F-contraction. Then T has a unique fixed point $x^* \in X$ and, for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.*

Many researchers have focused on F-contractions and obtained some results in the field (see [1–3, 5, 8, 10–12] and their references cited therein). Motivated by [13], throughout this paper, we introduce the concept of the generalized F-Suzuki contractions and obtain some relevant fixed point results in the setting of b-metric spaces.

To begin with, we shall give a definition and a lemma which will be used in the sequel.

Definition 1.4 ([4, 6])**.** Let X be a (nonempty) set and $s \ge 1$ a given real number. A function $d : X \times X \to [0, +\infty)$ is called a b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_1) $d(x, y) = 0$ if and only if $x = y$;
(b_2) $d(x, y) = d(y, x)$;
(b_3) $d(x, z) \le s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric when $s = 1$. The following example shows that in general a b-metric need not necessarily be a metric (see also [7, 9]).

Example 1.5. Let (X, ρ) be a metric space, and $d(x, y) = (\rho(x, y))^p$, where $p > 1$ is a real number. Then d is a b-metric with coefficient $s = 2^{p-1}$, but d is not a metric on X.

Otherwise, for more concepts such as b-convergence, b-completeness and b-Cauchy sequence in b-metric spaces, we refer the reader to [7, 9] and the references mentioned therein.

Lemma 1.6 ([9], Lemma 3.1)**.** *Let $\{y_n\}$ be a sequence in the b-metric space (X, d) with $s \ge 1$, such that*

$$d(y_{n+1}, y_{n+2}) \le \lambda d(y_n, y_{n+1}), \tag{1.1}$$

for some $\lambda \in [0, \frac{1}{s})$ and each $n = 1, 2, \ldots$. Then $\{y_n\}$ is a b-Cauchy sequence in (X, d).

2. Main Results

In the following, let us denote $\mathfrak{F}$ as the collection of all mappings $F : \mathbb{R}_+ \to \mathbb{R}$ satisfying

(F1) F is strictly increasing;
(F2) F is continuous on $(0, \infty)$.

Definition 2.1. Let (X, d) be a b-metric space with coefficient $s \geq 1$. A map $T : X \to X$ is said to be a generalized F-Suzuki contraction if there exists $F \in \mathfrak{F}$ such that, for all $x, y \in X$ with $x \neq y$,

$$\frac{1}{2s}d(x,Tx) < d(x,y) \Rightarrow F(s^{\varepsilon}d(Tx,Ty)) \leq F(M_T(x,y)) - \psi(M_T(x,y)), \tag{2.1}$$

where $\psi \in \Psi$ and $\varepsilon > 1$ is a constant and

$$\begin{aligned} M_T(x,y) = \max\Bigg\{ & d(x,y), d(T^2x,y), \frac{d(Tx,y)+d(x,Ty)}{2s}, \\ & \frac{d(T^2x,x)+d(T^2x,Ty)}{2s}, d(T^2x,Ty)+d(T^2x,Tx), \\ & d(T^2x,Ty)+d(Tx,x), d(Tx,y)+d(y,Ty)\Bigg\}. \end{aligned}$$

Remark 2.2. Compared with Definition 2.1 of [11], Definition 2.1 has more general character, since if $\varepsilon = 5$, then Definition 2.1 is reduced to Definition 2.1 of [11]. That is to say, Definition 2.1 is a large generalization of Definition 2.1 of [11].

Theorem 2.3. *Let (X, d) be a b-complete b-metric space with coefficient $s > 1$ and $T : X \to X$ be a generalized F-Suzuki contraction. Then T has a unique fixed point $x^* \in X$ and, for every $x_0 \in X$ the iterative sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ b-converges to x^*.*

Proof. Take $x_0 \in X$ and let $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, Tx_{n_0}) = 0$, then $x = x_{n_0}$ becomes a fixed point of T, which completes the proof. So, in the rest of the proof, we always assume that $0 < d(x_n, Tx_n)$, for all $n \in \mathbb{N}$. Hence, we have

$$\frac{1}{2s}d(x_n,Tx_n) < d(x_n,Tx_n) = d(x_n,x_{n+1}), \quad \text{for all } n \in \mathbb{N}.$$

Then by (2.1) we have

$$F(s^{\varepsilon}d(Tx_n,Tx_{n+1})) \leq F(M_T(x_n,x_{n+1})) - \psi(M_T(x_n,x_{n+1})).$$

Since

$$\begin{aligned}
&\max\{d(x_n,x_{n+1}),d(x_{n+2},x_{n+1})\}\\
&\quad\le M_T(x_n,x_{n+1})\\
&\quad=\max\Bigg\{d(x_n,x_{n+1}),d(x_{n+2},x_{n+1}),\frac{d(x_n,x_{n+2})}{2s},\frac{d(x_{n+2},x_n)}{2s},\\
&\qquad d(x_{n+2},x_{n+1}),d(x_{n+1},x_n),d(x_{n+1},x_{n+2})\Bigg\}\\
&\quad\le\max\Bigg\{d(x_n,x_{n+1}),d(x_{n+2},x_{n+1}),\\
&\qquad\frac{s[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})]}{2s}\Bigg\}\\
&\quad\le\max\{d(x_n,x_{n+1}),d(x_{n+2},x_{n+1})\},
\end{aligned}$$

we get

$$M_T(x_n,x_{n+1})=\max\{d(x_n,x_{n+1}),d(x_{n+2},x_{n+1})\}.$$

Then

$$\begin{aligned}
F(s^\varepsilon d(x_{n+1},x_{n+2}))&=F(s^\varepsilon d(Tx_n,Tx_{n+1}))\\
&\le F(\max\{d(x_n,x_{n+1}),d(x_{n+2},x_{n+1})\})\\
&-\psi(\max\{d(x_n,x_{n+1}),d(x_{n+2},x_{n+1})\})\\
&\le F(\max\{d(x_n,x_{n+1}),d(x_{n+2},x_{n+1})\}).
\end{aligned}\tag{2.2}$$

By the monotonicity of F, it follows immediately from (2.2) that

$$s^\varepsilon d(x_{n+1},x_{n+2})\le\max\{d(x_n,x_{n+1}),d(x_{n+2},x_{n+1})\}.\tag{2.3}$$

If

$$d(x_n,x_{n+1})<d(x_{n+2},x_{n+1}),$$

then (2.3) gives

$$s^\varepsilon d(x_{n+1},x_{n+2})\le d(x_{n+2},x_{n+1}),$$

which implies that $d(x_{n+1},x_{n+2})=0$, a contradiction. If

$$d(x_n,x_{n+1})\ge d(x_{n+2},x_{n+1}),$$

then (2.3) gives that

$$d(x_{n+1},x_{n+2})\le\frac{1}{s^\varepsilon}d(x_n,x_{n+1}).$$

Hence, by using Lemma 1.6, $\{x_n\}$ is a b-Cauchy sequence. As (X,d) is b-complete, then $\{x_n\}$ b-converges to some point $x^*\in X$. Therefore, $\lim_{n\to\infty}d(x_n,x^*)=0$.

We claim that, for every $n \in \mathbb{N}$,

$$\frac{1}{2s}d(x_n, Tx_n) < d(x_n, x^*), \tag{2.4}$$

or

$$\frac{1}{2s}d(Tx_n, T^2x_n) < d(Tx_n, x^*). \tag{2.5}$$

Indeed, suppose, on the contrary, that there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2s}d(x_m, Tx_m) \geq d(x_m, x^*), \qquad \frac{1}{2s}d(Tx_m, T^2x_m) \geq d(Tx_m, x^*). \tag{2.6}$$

Then

$$2sd(x_m, x^*) \leq d(x_m, Tx_m) \leq s\left[d(x_m, x^*) + d(x^*, Tx_m)\right],$$

which implies that $d(x_m, x^*) \leq d(x^*, Tx_m)$. As a result, we have

$$\begin{aligned} d(Tx_m, T^2x_m) &\leq \frac{1}{s^\varepsilon}d(x_m, Tx_m) \\ &\leq \frac{1}{s^\varepsilon} \cdot [sd(x_m, x^*) + sd(x^*, Tx_m)] \\ &\leq \frac{1}{s^\varepsilon} \cdot 2sd(x^*, Tx_m). \end{aligned} \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$d(Tx_m, T^2x_m) \leq \frac{1}{s^\varepsilon}d(Tx_m, T^2x_m).$$

This is a contradiction unless $d(Tx_m, T^2x_m) = 0$. Consequently, (2.4) or (2.5) hold.

If (2.4) is true, then

$$\begin{aligned} F\left(s^\varepsilon d(x_{n+1}, Tx^*)\right) &= F\left(s^\varepsilon d(Tx_n, Tx^*)\right) \\ &\leq F(M_T(x_n, x^*)) - \psi(M_T(x_n, x)). \end{aligned} \tag{2.2}$$

Since

$$\begin{aligned} d(x^*, Tx^*) &\leq M_T(x_n, x^*) \\ &= \max\Bigg\{d(x_n, x^*), d(x_{n+2}, x^*), \frac{d(x_{n+1}, x^*) + d(x_n, Tx^*)}{2s}, \\ &\quad \frac{d(x_{n+2}, x_n) + d(x_{n+2}, Tx^*)}{2s}, d(x_{n+2}, Tx^*) + d(x_{n+2}, x_{n+1}), \\ &\quad d(x_{n+2}, Tx^*) + d(x_{n+1}, x_n), d(x_{n+1}, x^*) + d(x^*, Tx^*)\Bigg\} \\ &\leq \max\Bigg\{d(x_n, x^*), d(x_{n+2}, x^*), \frac{d(x_{n+1}, x^*) + d(x_n, Tx^*)}{2s}, \\ &\quad \frac{s[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)] + d(x_{n+2}, Tx^*)}{2s}, \\ &\quad d(x_{n+2}, Tx^*) + d(x_{n+2}, x_{n+1}), d(x_{n+2}, Tx^*) + d(x_{n+1}, x_n), \\ &\quad d(x_{n+1}, x^*) + d(x^*, Tx^*)\Bigg\}, \end{aligned}$$

taking limit from both sides of the above inequality, we get

$$\lim_{n\to\infty} M_T(x_n, x^*) = d(x^*, Tx^*).$$

Thus by (2.2) and the continuity of ψ, it is easy to see that

$$F(s^\varepsilon d(x^*, Tx^*)) \leq F(d(x^*, Tx^*)) - \psi(d(x^*, Tx^*)) \leq F(d(x^*, Tx^*)),$$

which yields that $x^* = Tx^*$.

If (2.5) is true, using a similar method as the above, we have $x^* = Tx^*$.

Now we show that T has only one fixed point. Indeed, if $y^* \in X$ is another fixed point of T, then

$$0 = \frac{1}{2s} d(x^*, Tx^*) < d(x^*, y^*),$$

and from (2.2), we obtain

$$\begin{aligned} F(s^\varepsilon d(x^*, y^*)) &= F(s^\varepsilon d(Tx^*, Ty^*)) \\ &\leq F(M_T(x^*, y^*)) - \psi(M_T(x^*, y^*)) \\ &= F(d(y^*, x^*)) - \psi(d(y^*, x^*)) \\ &\leq F(d(y^*, x^*)). \end{aligned}$$

This gives $d(y^*, x^*) = 0$. Hence $y^* = x^*$. This completes the proof. □

Remark 2.4. Compared with Theorem 2.2 of [11], Theorem 2.3 has a sharp superiority. Indeed, because of the arbitrary value of the constant $\varepsilon > 1$, it evidently contains the special case $\varepsilon = 5$. Accordingly, our

Theorem 2.3 greatly generalizes Theorem 2.2 from [11]. Otherwise, our proof of Theorem 2.3 is simpler than the one of Theorem 2.2 from [11] since the former avoids the discontinuity problems of b-metric while the latter deals with it too complicate.

The following three theorems can be obtained easily by repeating the steps in the proof of Theorem 2.3 and therefore we omit their proofs.

Theorem 2.5. *Let (X,d) be a b-complete b-metric space with coefficient $s>1$ and $T: X \to X$ be a map such that, for every $x, y \in X$,*

$$\frac{1}{2s} d(x,Tx) < d(x,y) \quad \Rightarrow \quad F(s^{\varepsilon} d(Tx,Ty)) \le F(M_T(x,y)),$$

where $F \in \mathfrak{F}$ and $\varepsilon > 1$ is a constant. Then T has a unique fixed point $x^* \in X$, and for every $x_0 \in X$ the iterative sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ b-converges to x^*.

Theorem 2.6. *Let (X,d) be a b-complete b-metric space with coefficient $s>1$ and $T: X \to X$ be a map such that, for every $x, y \in X$*

$$\frac{1}{2s} d(x,Tx) < d(x,y) \quad \Rightarrow \quad F(d(Tx,Ty)) \le F(M_T(x,y)) - \psi(N(x,y)),$$

where $F \in \mathfrak{F}$, $\psi \in \Psi$ and

$$N(x,y) = \max\Big\{ d(x,y), d(x,Tx), d(y,Ty), d(T^2x,y), d(T^2x,Ty), \\ d(T^2x,Tx), \frac{d(Tx,y)+d(x,Ty)}{2}, \frac{d(T^2x,x)+d(T^2x,Ty)}{2} \Big\}.$$

Then T has a unique fixed point $x^* \in X$, and for every $x_0 \in X$ the iterative sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ b-converges to x^*.

Theorem 2.7. *Let (X,d) be a b-complete b-metric space with coefficient $s>1$ and $T: X \to X$ be a map such that, for every $x, y \in X$*

$$\frac{1}{2s} d(x,Tx) < d(x,y) \quad \Rightarrow \quad F(d(Tx,Ty)) \le F(M_T(x,y)) - \psi(d(x,y)),$$

where $F \in \mathfrak{F}$ and $\psi \in \Psi$. Then T has a unique fixed point $x^* \in X$, and for every $x_0 \in X$, the iterative sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ b-converges to x^*.

Remark 2.8. Since a b-metric space is a metric space when $s=1$, then we easily obtain the corresponding results in the setting of metric spaces.

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[1] School of Mathematics, Thapar University, Patiala-147004, India.
E-mail address: sumit.chandok@thapar.edu

[2] School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, PR China.
E-mail address: mathhhp@163.com

[3] Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120, Beograd, Serbia.
E-mail address: radens@beotel.net