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Coherent Frames

Ataollah Askari Hemmat[1], Ahmad Safapour[2], and Zohreh Yazdani Fard[3*]

Abstract. Frames which can be generated by the action of some operators (e.g. translation, dilation, modulation, ...) on a single element f in a Hilbert space, called coherent frames. In this paper, we introduce a class of continuous frames in a Hilbert space $\mathcal{H}$ which is indexed by some locally compact group G, equipped with its left Haar measure. These frames are obtained as the orbits of a single element of Hilbert space $\mathcal{H}$ under some unitary representation π of G on $\mathcal{H}$. It is interesting that most of important frames are coherent. We also investigate canonical dual and combinations of this frames.

1. Introduction

In 1946 Gabor [12] introduced a method for reconstructing signals which led eventually to the theory of wavelets. Later in 1952 Duffin and Schaeffer [10] introduced frame theory for Hilbert spaces to study some problems in nonharmonic Fourier series. Frames reintroduced in 1986 by Daubechies, Grossmann and Meyer [9]. Nowadays frames have become an alternative to orthonormal basis for reconstructing elements of a Hilbert space. Frames have been used in characterization of function spaces and other fields such as signal and image processing [6], filter bank theory [5] and wireless communications [15].

The concept of generalization of frames to a family indexed by some measure space was proposed by Kaiser [16] and independently by Ali, Antoine and Gazeau [1]. Kaiser used the terminology generalized frames. Also, in mathematical physics these frames are referred to as coherent states.

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* Corresponding author.

A frame $\{f_i\}_i$ such that all elements f_i appear by the action of some operators (e.g. translation, dilation, modulation, ...) on a single element f in the Hilbert space, called *coherent frame*. Grächening [13] investigated coherent frames of the form $\{\pi(x_j)g\}_{j\in J}$, where π is a unitary square integrable representation of a locally compact group G on a Hilbert space $\mathcal{H}$. Balazs and Stoven [4] introduced θ-pseudo-coherent frames of the form $\{\theta(\lambda)h\}_{\lambda\in\Lambda}$ for Hilbert Space $\mathcal{H}$, where Λ is a discrete set, $\theta(\lambda)$ is a bounded operator from $\mathcal{H}$ into $\mathcal{H}$ and there exist $\phi : \Lambda\times\Lambda \to \mathbb{C}$ and $\mu : \Lambda\times\Lambda\to\Lambda$ which satisfy:

(i) for all $\lambda\in\Lambda$ the mapping $\lambda' \mapsto \mu(\lambda,\lambda')$ is a bijection from Λ onto Λ,
(ii) $\theta(\lambda)^*\theta(\lambda') = \phi(\lambda,\lambda')\theta(\mu(\lambda,\lambda'))$,
(iii) $\theta(\lambda)\theta(\mu(\lambda,\lambda')) = \overline{\phi(\lambda,\lambda')}\theta(\lambda')$.

In this paper, we consider a continuous frame $\{f_g\}_{g\in G}$ indexed by a locally compact group G, equipped with the left Haar measure μ for which all the elements f_g appear by the action of G on a single element $f\in\mathcal{H}$ via a unitary representation of G on $\mathcal{H}$ and study canonical dual and combinations of this frames.

2. Basic Frame Theory

In this section we introduce some definitions and basic facts about frames theory that are borrowed and adopted from [8].

A continuous frame for a Hilbert space $\mathcal{H}$ is a family $\{f_m\}_{m\in\mathcal{M}}$ indexed by a measure space $(\mathcal{M},\mu)$ such that

- for all $f\in\mathcal{H}$, $m\mapsto\langle f, f_m\rangle$ is a measurable function on $\mathcal{M}$;
- there exist constants $A, B > 0$ such that for each $f\in\mathcal{H}$

$$A\|f\|^2 \le \int_{\mathcal{M}} |\langle f, f_m\rangle|^2 \,\mathrm{d}\mu(m) \le B\|f\|^2 . \tag{2.1}$$

A and B are lower and upper frame bounds. The frame $\{f_m\}_{m\in\mathcal{M}}$ is called a tight frame if (2.1) holds for some $A = B$ and a normalized tight frame if (2.1) holds for some $A = B = 1$. A uniform frame is a frame in which all the elements have equal norms.

The discrete frames correspond to a counting measure space on a countable space.

If there is another frame $\{g_m\}_{m\in\mathcal{M}}\subset\mathcal{H}$ satisfying

$$f = \int_{\mathcal{M}} \langle f, f_m\rangle g_m, \quad \forall f\in\mathcal{H},$$

then $\{g_m\}_{m\in\mathcal{M}}$ is said to be a dual of $\{f_m\}_{m\in\mathcal{M}}$.

The continuous frame operator $S : \mathcal{H} \to \mathcal{H}$ is weakly defined by

$$\langle Sf, g\rangle = \int_{\mathcal{M}} \langle f, f_m\rangle\langle f_m, g\rangle \,\mathrm{d}\mu(m), \quad \forall f, g \in \mathcal{H}.$$

The operator S is bounded, positive and invertible.
If $\{f_m\}_{m\in\mathcal{M}}$ is a continuous frame, then $\left\{S^{-1}f_m\right\}_{m\in\mathcal{M}}$ is the canonical dual frame and every $f \in \mathcal{H}$ can be reconstructed as

$$f = S^{-1}Sf = \int_{\mathcal{M}} \langle f, f_m\rangle S^{-1}f_m \,\mathrm{d}\mu(m).$$

The operator S is a multiple of the identity if and only if $\{f_m\}_{m\in\mathcal{M}}$ is a tight frame.

3. Coherent Frames

Let G be a locally compact group with the left Haar measure μ. A unitary representation π of G on a Hilbert space $\mathcal{H}$ is a strongly continuous group homomorphism from G into the group of unitary operators on $\mathcal{H}$, which means that π satisfies the properties $\pi(gg') = \pi(g)\pi(g'), \pi(g)^* = \pi\left(g^{-1}\right) = \pi(g)^{-1}$, and $\lim\limits_{n\to\infty} g_n = g$ andfor each $x \in \mathcal{H}$ we have $\lim\limits_{n\to\infty} \pi(g_n)x = \pi(g)x$ (see e.g. [11], Section 3.1).

Definition 3.1. Let G be a locally compact group. A *coherent frame* for a Hilbert space $\mathcal{H}$ is a continuous frame $\{\pi(g)\phi\}_{g\in G}$, where π is a unitary representation of G on $\mathcal{H}$ and $\phi \in \mathcal{H}$.

Obviously, coherent frames are uniform. Before we develop the theory for coherent frames, we mention some examples of coherent frames.

Example 3.2. Let $G_{\text{aff}} = \{(b, a) \mid b, a \in \mathbb{R}, a \neq 0\} = \mathbb{R} \rtimes \mathbb{R}_*$ be the group of affine transformations on $\mathbb{R}$, $ax + b$ group, with the natural action $x \mapsto ax + b$ and group low $(b, a)(b', a') = (b + ab', aa')$. The unit element is $(0, 1)$ and the inverse of (b, a) is $\left(-a^{-1}b, a^{-1}\right)$. On G_{aff}, $\frac{1}{a^2}\,\mathrm{d}a\,\mathrm{d}b$ is a left Haar measure where $\mathrm{d}a\,\mathrm{d}b$ is the Lebesgue measure. If $\psi \in L^2(\mathbb{R})$ is admissible, i.e.,

$$C_\psi := \int_{-\infty}^{+\infty} \frac{\left|\hat{\psi}(\gamma)\right|^2}{|\gamma|} \,\mathrm{d}\gamma < \infty,$$

then the family $\{\psi^{b,a}\}_{(b,a)\in G_{\text{aff}}} = \{\pi(b,a)\psi\}_{(b,a)\in G_{\text{aff}}}$ is a tight coherent frame with frame bound C_ψ for $L^2(\mathbb{R})$, where π is a unitary representation of G_{aff} on $L^2(\mathbb{R})$ defined by

$$\begin{aligned}(\pi(b,a)f)(x) &= (T_bD_af)(x)\\ &= \frac{1}{\sqrt{|a|}}f\left(\frac{x-b}{a}\right), \quad f\in L^2(\mathbb{R}), \quad x\in\mathbb{R}.\end{aligned}$$

Actually, the *continuous wavelet transform* of a function $f\in L^2(\mathbb{R})$ with respect to the admissible function $\psi\in L^2(\mathbb{R})$ is defined by

$$\begin{aligned}W_\psi(f)(b,a) &= \left\langle f,\psi^{b,a}\right\rangle\\ &= \int_{\mathbb{R}} f(x)\frac{1}{\sqrt{|a|}}\overline{\psi\left(\frac{x-b}{a}\right)}\,\mathrm{d}x.\end{aligned}$$

Let $f,g\in L^2(\mathbb{R})$. Then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} W_\psi(f)(b,a)\overline{W_\psi(g)(b,a)}\frac{1}{a^2}\,\mathrm{d}a\,\mathrm{d}b = C_\psi\langle f,g\rangle.$$

For details, see the Proposition 11.1.1 and Corollary 11.1.2 in [8].

Example 3.3. Let $G=\mathbb{R}^2$ equipped with the Lebesgue measure $\mathrm{d}a\,\mathrm{d}b$. If $g\in L^2(\mathbb{R})-\{0\}$, then the family $\{g^{a,b}\}_{(a,b)\in G} = \{\pi(a,b)g\}_{(a,b)\in G}$ is a tight coherent frame, where π is a unitary representation of G on $L^2(\mathbb{R})$ defined by

$$\begin{aligned}(\pi(a,b)g)(x) &= (E_bT_ag)(x)\\ &= g(x-a)e^{2\pi ixb}, \quad x\in\mathbb{R}.\end{aligned}$$

Actually, the *short-time Fourier transform* of a function $f\in L^2(\mathbb{R})$ with respect to the window function $g\in L^2(\mathbb{R})-\{0\}$ is given by

$$\begin{aligned}\Psi_g(f)(y,\gamma) &= \langle f, E_\gamma T_y g\rangle\\ &= \int_{\mathbb{R}} f(x)\overline{g(x-y)}e^{-2\pi ix\gamma}\,\mathrm{d}x, \quad y,\gamma\in\mathbb{R}.\end{aligned}$$

Let $f_1,f_2,g_1,g_2\in L^2(\mathbb{R})$. Then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \Psi_{g_1}(f_1)(a,b)\overline{\Psi_{g_2}(f_2)(a,b)}\,\mathrm{d}b\,\mathrm{d}a = \langle f_1,f_2\rangle\langle g_1,g_2\rangle.$$

For details, see the Proposition 8.1.2 in [8].

Example 3.4. The unit 2-sphere, S^2, centered at the origin is defined as the following subset of $\mathbb{R}^3$: $S^2 = \{\omega=(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$. In polar coordinates, we write $\omega=(\theta,\varphi)$, where $\theta\in[0,\pi]$ and $\varphi\in[0,2\pi)$ are azimuthal and polar angles, respectively and $x=\sin\theta\cos\varphi, y=$

$\sin\theta\cos\varphi, z = \cos\theta$. $\mathrm{d}\mu(\omega) = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ is the usual rotation invariant measure on S^2. Affine transformations on S^2 are of two types:

(1) Motions are given by elements of the rotation group $SO(3)$,
(2) Dilations by a scale factor $a \in \mathbb{R}_*^+$ which acts on a point $\omega = (\theta, \varphi)$ by $D_a(\theta, \varphi) = (\theta_a, \varphi)$ with $\tan\frac{\theta_a}{2} = a\tan\frac{\theta}{2}$.

Let X be the group of affine transformations on S^2. X embeds into the Lorentz group $SO_0(3,1)$ via the Iwasawa decomposition $SO_0(3,1) = SO(3).\mathbb{R}_*^+.\mathbb{C}$. Thus $X \simeq \frac{SO_0(3,1)}{\mathbb{C}}$. For any $\eta \in L^2(S^2, \mathrm{d}\mu)$ such that

$$\int_0^{2\pi} \eta(\theta,\varphi)\,\mathrm{d}\varphi \neq 0, \qquad \int_{S^2} \frac{\eta(\theta,\varphi)}{1+cos\theta}\,\mathrm{d}\mu(\theta,\varphi) = 0,$$

the family $\{\rho(x)\eta\}_{x\in X}$ is a coherent frame for $L^2(S^2, \mathrm{d}\mu)$, where ρ is a unitary representation of X on $L^2(S^2, d\mu)$ defined by

$$[\rho(x)f](\omega) = \frac{2af\left((\gamma a)^{-1}\omega\right)}{(a^2-1)\cos\theta + (a^2+1)}, \quad x = (\gamma, a) \in X,\ \omega = (\theta,\varphi) \in S^2.$$

For details, see Proposition 3.4 in [2].

Now, we show that the canonical dual of a coherent frame is also a coherent frame. Balazs and Stoven showed the same result for discrete coherent frames(see Proposition 5.2 in [4]).

Lemma 3.5. *Let $\{\pi(g)\phi\}_{g\in G}$ be a coherent frame with frame operator S. Then S commutes with $\pi(g)$ for every $g \in G$.*

Proof. Let $x, y \in \mathcal{H}$ and $g' \in G$.

$$\begin{aligned}
\left\langle S\pi\left(g'\right)x, y\right\rangle &= \int_G \left\langle \pi\left(g'\right)x, \pi(g)\phi\right\rangle \langle \pi(g)\phi, y\rangle\,\mathrm{d}\mu(g)\\
&= \int_G \left\langle x, \pi\left(g'^{-1}g\right)\phi\right\rangle \left\langle \pi\left(g'^{-1}g\right)\phi, \pi\left(g'^{-1}\right)y\right\rangle\,\mathrm{d}\mu(g)\\
&= \int_G \langle x, \pi(g)\phi\rangle \left\langle \pi(g)\phi, \pi\left(g'^{-1}\right)y\right\rangle\,\mathrm{d}\mu(g)\\
&= \left\langle Sx, \pi\left(g'^{-1}\right)y\right\rangle\\
&= \left\langle \pi\left(g'\right)Sx, y\right\rangle.
\end{aligned}$$

Then $S\pi(g') = \pi(g')S$, for every $g' \in G$. □

Proposition 3.6. *Let $\{\pi(g)\phi\}_{g\in G}$ be a coherent frame for $\mathcal{H}$. Then the canonical dual has the form $\{\pi(g)\psi\}_{g\in G}$ for some $\psi \in \mathcal{H}$.*

Proof. Let S be the frame operator of $\{\pi(g)\phi\}_{g\in G}$. Then $S\pi(g) = \pi(g)S$ for all $g \in G$ which implies that $S^{-1}\pi(g) = \pi(g)S^{-1}$ for all $g \in G$. Putting $\psi = S^{-1}\phi$ implies $S^{-1}\pi(g) = \pi(g)\psi$ for all $g \in G$. Therefore, the canonical dual frame of $\{\pi(g)\phi\}_{g\in G}$ is $\left\{S^{-1}\pi(g)\phi\right\}_{g\in G} = \{\pi(g)\psi\}_{g\in G}$. ∎

Corollary 3.7. *The canonical dual of a coherent frame is also a coherent frame.*

Recall that the frame operator S for a coherent frame is a positive invertible operator, and therefore has a positive and invertible square root operator $S^{\frac{1}{2}}$. The following proposition shows that every coherent frame can be associated to a normalized tight coherent frame.

Proposition 3.8. *Let $\{\pi(g)\phi\}_{g\in G}$ be a coherent frame for $\mathcal{H}$ with frame operator S. Then $\left\{S^{-\frac{1}{2}}\pi(g)\phi\right\}_{g\in G}$ is a coherent normalized tight frame for $\mathcal{H}$.*

Proof. The family $\left\{S^{-\frac{1}{2}}\pi(g)\phi\right\}_{g\in G}$ is a normalized tight continuous frame for $\mathcal{H}$ by Theorem 2.2 in [3]. For all $g \in G$, S commutes with $\pi(g)$ and so $S^{-\frac{1}{2}}$ commutes with each $\pi(g)$. Let $\psi = S^{-\frac{1}{2}}\phi$. Then

$$\left\{S^{-\frac{1}{2}}\pi(g)\phi\right\}_{g\in G} = \{\pi(g)\psi\}_{g\in G},$$

is a normalized tight coherent frame. ∎

In the rest of this paper, we show that coherent frames can be combined as follows:

- the direct sum of disjoint coherent frames is a coherent frame.
- the tensor product of coherent frames is a coherent frame.

For each $i = 1, \ldots, n$, let $\mathcal{H}_i$ be a Hilbert space. The direct sum of Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ is the Hilbert space $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ with inner product

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \sum_{i=1}^{n} \langle x_i, y_i \rangle, \quad x_i, y_i \in \mathcal{H}_i, i = 1, \ldots, n.$$

If $T_i : \mathcal{H}_i \to \mathcal{H}_i, i = 1, \ldots, n$, are linear operators on $\mathcal{H}_i$, the direct sum of operators T_i is a linear operator $T_1 \oplus \cdots \oplus T_n$ on $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ defined by

$$(T_1 \oplus \cdots \oplus T_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} T_1 x_1 \\ \vdots \\ T_n x_n \end{pmatrix}, \quad x_i \in \mathcal{H}_i, i = 1, \ldots, n.$$

Definition 3.9. Let $\{\phi_m\}_{m\in\mathcal{M}}$ and $\{\psi_m\}_{m\in\mathcal{M}}$ be continuous frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Two continuous frames $\{\phi_m\}_{m\in\mathcal{M}}$ and $\{\psi_m\}_{m\in\mathcal{M}}$ are called *disjoint* if for all $x \in \mathcal{H}$ and for all $y \in \mathcal{K}$ we have

$$\int_{\mathcal{M}} \langle x, \phi_m\rangle_{\mathcal{H}} \langle \psi_m, y\rangle_{\mathcal{K}} \, \mathrm{d}\mu(m) = 0.$$

Theorem 3.10. *Let $\Phi = \{\pi(g)\phi\}_{g\in G}$ and $\Psi = \{\rho(g)\psi\}_{g\in G}$ be disjoint coherent frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Then the direct sum*

$$\Phi \oplus \Psi = \left\{ \begin{pmatrix} \pi(g)\phi \\ \rho(g)\psi \end{pmatrix} \right\}_{g\in G},$$

is a coherent frame for direct sum Hilbert space $\mathcal{H} \oplus \mathcal{K}$ with frame operator

$$S_{\Phi\oplus\Psi} = \begin{pmatrix} S_\Phi & \\ & S_\Psi \end{pmatrix}.$$

Proof. Let $x \in \mathcal{H}$ and $t \in \mathcal{K}$;

$$\begin{aligned}
I &= \int_G \left| \left\langle \begin{pmatrix} x \\ t \end{pmatrix}, \begin{pmatrix} \pi(g)\phi \\ \rho(g)\psi \end{pmatrix} \right\rangle \right|^2 \mathrm{d}\mu(g) \\
&= \int_G \left\langle \begin{pmatrix} x \\ t \end{pmatrix}, \begin{pmatrix} \pi(g)\phi \\ \rho(g)\psi \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} \pi(g)\phi \\ \rho(g)\psi \end{pmatrix}, \begin{pmatrix} x \\ t \end{pmatrix} \right\rangle \mathrm{d}\mu(g) \\
&= \int_G \langle x, \pi(g)\phi\rangle \langle \pi(g)\phi, x\rangle \, \mathrm{d}\mu(g) + \int_G \langle x, \pi(g)\phi\rangle \langle \rho(g)\psi, t\rangle \, \mathrm{d}\mu(g) \\
&\quad + \int_G \langle t, \rho(g)\psi\rangle \langle \pi(g)\phi, x\rangle \, \mathrm{d}\mu(g) + \int_G \langle t, \rho(g)\psi\rangle \langle \rho(g)\psi, t\rangle \, \mathrm{d}\mu(g).
\end{aligned}$$

Since Φ and Ψ are disjoint, then

$$I = \int_G |\langle x, \pi(g)\phi\rangle|^2 \, \mathrm{d}\mu(g) + \int_G |\langle t, \rho(g)\psi\rangle|^2 \, \mathrm{d}\mu(g),$$

and therefore $\Phi \oplus \Psi$ is a continuous frame. On the other hand,

$$\Phi \oplus \Psi = \left\{ \begin{pmatrix} \pi(g)\phi \\ \rho(g)\psi \end{pmatrix} \right\}_{g\in G} = \left\{ (\pi \oplus \rho)(g) \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\}_{g\in G},$$

where $\pi \oplus \rho$, is a unitary representation of G on $\mathcal{H} \oplus \mathcal{K}$.

Now, let $x, y \in \mathcal{H}$ and $t, u \in \mathcal{K}$. Then, we have

$$
\begin{aligned}
&\left\langle S_{\Phi\oplus\Psi}\begin{pmatrix} x \\ t \end{pmatrix}, \begin{pmatrix} y \\ u \end{pmatrix}\right\rangle \\
&\quad = \int_G \left\langle \begin{pmatrix} x \\ t \end{pmatrix}, \begin{pmatrix} \pi(g)\phi \\ \rho(g)\psi \end{pmatrix}\right\rangle \left\langle \begin{pmatrix} \pi(g)\phi \\ \rho(g)\psi \end{pmatrix}, \begin{pmatrix} y \\ u \end{pmatrix}\right\rangle \mathrm{d}\mu(g) \\
&\quad = \int_G \left\langle \left\langle \begin{pmatrix} x \\ t \end{pmatrix}, \begin{pmatrix} \pi(g)\phi \\ \rho(g)\psi \end{pmatrix}\right\rangle \begin{pmatrix} \pi(g)\phi \\ \rho(g)\psi \end{pmatrix}, \begin{pmatrix} y \\ u \end{pmatrix}\right\rangle \mathrm{d}\mu(g) \\
&\quad = \int_G \left\langle \begin{pmatrix} (\langle x, \pi(g)\phi\rangle + \langle t, \rho(g)\psi\rangle)\pi(g)\phi \\ (\langle x, \pi(g)\phi\rangle + \langle t, \rho(g)\psi\rangle)\rho(g)\psi \end{pmatrix}, \begin{pmatrix} y \\ u \end{pmatrix}\right\rangle \mathrm{d}\mu(g) \\
&\quad = \int_G \langle x, \pi(g)\phi\rangle\langle \pi(g)\phi, y\rangle \,\mathrm{d}\mu(g) + \int_G \langle t, \rho(g)\psi\rangle\langle \pi(g)\phi, y\rangle \,\mathrm{d}\mu(g) \\
&\qquad + \int_G \langle x, \pi(g)\phi\rangle\langle \rho(g)\psi, u\rangle \,\mathrm{d}\mu(g) + \int_G \langle t, \rho(g)\psi\rangle\langle \rho(g)\psi, u\rangle \,\mathrm{d}\mu(g) \\
&\quad = \langle S_\Phi x, y\rangle + \langle S_\Psi t, u\rangle \\
&\quad = \left\langle \begin{pmatrix} S_\Phi \\ S_\Psi \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}, \begin{pmatrix} y \\ u \end{pmatrix}\right\rangle.
\end{aligned}
$$

□

Corollary 3.11. *Let $\Phi_i = \{\pi_i(g)\phi\}_{g\in G}, i = 1, \dots, n$, be mutually disjoint coherent frames for Hilbert spaces $\mathcal{H}_i, i = 1, \dots, n$, respectively. Then the direct sum $\Phi_1 \oplus \cdots \oplus \Phi_n$ is a coherent frame for direct sum Hilbert space $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ with frame operator*

$$S_{\Phi_1\oplus\cdots\oplus\Phi_n} = S_{\Phi_1} \oplus \cdots \oplus S_{\Phi_n}.$$

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. The tensor product of $\mathcal{H}$ and $\mathcal{K}$ is the set $\mathcal{H} \otimes \mathcal{K}$ of all antilinear maps $T : \mathcal{K} \to \mathcal{H}$ such that

$$\sum_j \|Tu_j\|^2 < \infty,$$

for some, and hence every orthonormal basis $\{u_j\}_j$ of $\mathcal{K}$.

Moreover, for every $T \in \mathcal{H} \otimes \mathcal{K}$, we set

$$|||T|||^2 = \sum_j \|Tu_j\|^2.$$

By Theorem 7.12 in [11], $\mathcal{H} \otimes \mathcal{K}$ is a Hilbert space with the norm $|||.|||$ and associated inner product

$$\langle Q, T\rangle = \sum_j \langle Qu_j, Tu_j\rangle,$$

where $\{u_j\}_j$ is any orthonormal basis of $\mathcal{K}$.

If $x \in \mathcal{H}$ and $t \in \mathcal{K}$, the map $u \mapsto \langle t, u\rangle x \quad (u \in \mathcal{K})$ belongs to $\mathcal{H} \otimes \mathcal{K}$; we denote it by $x \otimes t$.

If $x, y \in \mathcal{H}$ and $t, u \in \mathcal{K}$, then by [11]

$$\||x \otimes t\|| = \|x\|\|t\|,$$
$$\langle x \otimes t, y \otimes u\rangle = \langle x, y\rangle\langle t, u\rangle.$$

Theorem 3.12. *Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be Hilbert spaces and $F_1 = \{f_m^1\}_{m \in \mathcal{M}_1}$; $\ldots$; $F_n = \{f_m^n\}_{m \in \mathcal{M}_n}$ be continuous frames for Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ with frame bounds $A_1, B_1; \ldots; A_n, B_n$, respectively. Then*

$$F_1 \otimes \cdots \otimes F_n = \left\{ f_{m_1}^1 \otimes \cdots \otimes f_{m_n}^n : f_{m_j}^j \in F_j, 1 \leq j \leq n \right\},$$

is a continuous frame for Hilbert tensor product space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ with frame bounds $A_1 A_2 \cdots A_n, B_1 B_2 \cdots B_n$. In particular, if $F_1, \ldots, F_n$ are normalized tight continuous frames, then it is a normalized tight continuous frame.

By using the associativity of tensor product (Proposition 2.6.5 in [17]) and by induction it is enough to prove the theorem for $n = 2$.

Theorem 3.13. *Let $F = \{f_m\}_{m \in \mathcal{M}}$ and $G = \{g_{m'}\}_{m' \in \mathcal{M}'}$ be continuous frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ with frame bounds $A, B; C, D$, respectively. Then the tensor product $F \otimes G = \{f_m \otimes g_{m'}\}_{m \in \mathcal{M}, m' \in \mathcal{M}'}$ is a continuous frame for Hilbert tensor product space $\mathcal{H} \otimes \mathcal{K}$ with frame bounds AC, BD. Moreover $F \otimes G$ is a normalized tight continuous frame if F and G are.*

Proof. Let $x \in \mathcal{H}$ and $t \in \mathcal{K}$;

$$\begin{aligned}
&\int_{\mathcal{M}} \int_{\mathcal{M}'} |\langle x \otimes t, f_m \otimes g_{m'}\rangle|^2 \, \mathrm{d}\mu \, \mathrm{d}\mu' \\
&\qquad = \int_{\mathcal{M}} \int_{\mathcal{M}'} \langle x \otimes t, f_m \otimes g_{m'}\rangle \langle f_m \otimes g_{m'}, x \otimes t\rangle \, \mathrm{d}\mu \, \mathrm{d}\mu' \\
&\qquad = \int_{\mathcal{M}} \int_{\mathcal{M}'} \langle x, f_m\rangle\langle t, g_{m'}\rangle\langle f_m, x\rangle\langle g_{m'}, t\rangle \, \mathrm{d}\mu \, \mathrm{d}\mu' \\
&\qquad = \int_{\mathcal{M}} |\langle x, f_m\rangle|^2 \, \mathrm{d}\mu \int_{\mathcal{M}'} |\langle t, g_{m'}\rangle|^2 \, \mathrm{d}\mu'.
\end{aligned}$$

Then $F \otimes G$ is a continuous frame for Hilbert tensor product space $\mathcal{H} \otimes \mathcal{K}$. □

Corollary 3.14. *Let $\Phi = \{\pi(g)\phi\}_{g \in G}$ and $\Psi = \{\rho(g')\psi\}_{g' \in G'}$ be coherent frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. The tensor product $\Phi \otimes \Psi = \{\pi(g)\phi \otimes \rho(g')\psi\}_{g \in G, g' \in G'}$ is a coherent frame for Hilbert tensor product space $\mathcal{H} \otimes \mathcal{K}$.*

Proof. By Theorem 3.13, $\Phi \otimes \Psi$ is a continuous frame. On the other hand,

$$\begin{aligned}\Phi \otimes \Psi &= \{\pi(g)\phi \otimes \rho(g')\psi\}_{g\in G, g'\in G'} \\ &= \left\{ \left((\pi \otimes \rho)(g,g') \right) (\phi \otimes \psi) \right\}_{(g,g')\in G\times G'},\end{aligned}$$

where $\pi \otimes \rho$ is a unitary representation of $G \times G'$ on $\mathcal{H} \otimes \mathcal{K}$. □

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[1] Department of Mathematics, Faculty of Mathematics and Computer Sciences, Shahid Bahonar University of Kerman, P.O.Box 76169-133, Kerman, Iran.

E-mail address: askari@mail.uk.ac.ir

[2] Department of Mathematics, Faculty of Mathematical Sciences, Vali-e-Asr University of Rafsanjan, P.O.Box 518, Rafsanjan, Iran.

E-mail address: safapour@vru.ac.ir

[3] Department of Mathematics, Faculty of Mathematical Sciences, Vali-e-Asr University of Rafsanjan, P.O.Box 518, Rafsanjan, Iran.

E-mail address: z.yazdanifard@stu.vru.ac.ir