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The Norm Estimates of Pre-Schwarzian Derivatives of Spirallike Functions and Uniformly Convex α-spirallike Functions

Zahra Orouji[1*] and Rasul Aghalary[2]

Abstract. For a constant $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we define a subclass of the spirallike functions, $SP_p(\alpha)$, the set of all functions $f \in \mathcal{A}$

$$\operatorname{Re}\left\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|.$$

In the present paper, we shall give the estimate of the norm of the pre-Schwarzian derivative $\mathrm{T}_f = f''/f'$ where $\|\mathrm{T}_f\| = \sup_{z\in\Delta}(1 - |z|^2)|\mathrm{T}_f(z)|$ for the functions in $SP_p(\alpha)$.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f on the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

and let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions.

Let f and g be analytic in Δ. The function f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$, and $f(z) = g(w(z))$ on Δ.

For a real number α with $0 \leq \alpha < 1$, a function $f \in \mathcal{A}$ is called starlike of order α if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \Delta,$$

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* Corresponding author.

and f is called convex of order α if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha, \quad z\in\Delta.$$

We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the classes of starlike and convex functions of order α, respectively. It follows that $f(z)\in\mathcal{K}(\alpha)$ if and only if $zf'(z)\in\mathcal{S}^*(\alpha)$. The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were studied by many authors (see for example [2, 11]).

Let $T_f=f''/f'$ denote pre-Schwarzian derivative of f. Pre-Schwarzian derivative has several applications in the theory of Teichmuller spaces as well as in the theory of locally univalent functions. For a locally univalent holomorphic function f in Δ, a norm of T_f is defined by

$$\|T_f\|=\sup_{z\in\Delta}\left(1-|z|^2\right)|T_f(z)|.$$

It is known that $\|T_f\|\le 6$ for $f\in\mathcal{S}$ and conversely, for $f\in\mathcal{A}$, $\|T_f\|\le 1$ implies $f\in\mathcal{S}$, and these bounds are sharp (see [1]). The norm estimates for typical subclasses of univalent functions are investigated by many authors such as [5, 6, 8]. The next result was improved by Yamashita [11].

Theorem 1.1. *Let $0\le\alpha<1$ and $f\in\mathcal{A}$.*

(i) If $f\in\mathcal{S}^*(\alpha)$, then $\|\mathrm{T}_f\|\le 6-4\alpha$.
(ii) If $f\in\mathcal{K}(\alpha)$, then $\|\mathrm{T}_f\|\le 4(1-\alpha)$.
(iii) If $|Arg\,(zf'(z)/f(z))|<\alpha\pi/2$, then $\|\mathrm{T}_f\|\le M(\alpha)+2\alpha$, where $M(\alpha)$ is given by

$$M(\alpha)=\frac{4\alpha c(\alpha)}{(1-\alpha)c^2(\alpha)+1+\alpha},$$

and $c(\alpha)$ is the unique solution of the following equation in the interval $(1,\infty)$:

$$(1-\alpha)c^{\alpha+2}+(1+\alpha)c^{\alpha}-c^2-1=0.$$

Remark 1.2. If $\|\mathrm{T}_f\|<2$ then f is bounded (see [5]).

Definition 1.3. The function f is uniformly convex (starlike) if for every circular arc γ contained in Δ with center $\xi\in\Delta$ the image arc $f(\gamma)$ is convex (starlike with respect to $f(\xi)$). The class of all uniformly convex (starlike) functions is denoted by $UCV(UST)$.

These classes were studied by A.W. Goodman [3, 4]. In [3, 4] it is shown that

$$f\in UCV\Longleftrightarrow \operatorname{Re}\left\{1+(z-\xi)\frac{f''(z)}{f'(z)}\right\}\ge 0, \quad z,\xi\in\Delta$$

and

$$f \in UST \Longleftrightarrow \operatorname{Re}\left\{\frac{(z-\xi)f'(z)}{f(z)-f(\xi)}\right\} \geq 0, \quad z, \xi \in \Delta.$$

Rϕnning [10] and Ma and Minda [7] have proved the following characterization for the functions in UCV.

Theorem 1.4. *Let $f \in \mathcal{A}$. Then $f \in UCV$ if and only if*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in \Delta.$$

Corollary 1.5 ([10])**.** *A function $f \in \mathcal{A}$ is uniformly convex if and only if $z\mathrm{T}_f(z) \in W$ for any $z \in \Delta$, where W is the domain*

$$\Big\{\omega = u+iv; v^2 < 2u+1\Big\}.$$

The conformal map $g : \Delta \to \mathbb{C}$ is given by $g(0) = 0$ and

$$g(z) = \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 = \frac{8z}{\pi^2}\left(1+\frac{z}{3}+\frac{z^2}{5}+\frac{z^3}{7}+\cdots\right)^2, \tag{1.1}$$

which maps the unit disc Δ onto W (see [7], Example 1).

Therefore, $f \in \mathcal{A}$ is uniformly convex if and only if $z\mathrm{T}_f(z)$ is subordinate to the function $g(z)$. Kim and Sugawa [5] give the sharp estimate of the norm of the pre-Schwarzian derivative for the functions in UCV as follow.

Theorem 1.6 ([5], Theorem 4.5)**.** *If $f \in \mathcal{A}$ is uniformly convex, then we have*

$$\|\mathrm{T}_f\| \leq h(t_2) = 0.94779..., \tag{1.2}$$

where

$$h(t) = \frac{8t^2}{\pi^2}\frac{\cosh t}{\sinh^2 t}, \quad 0 < t < \infty, \tag{1.3}$$

assumes its maximum at the point $t = t_2 = 1.6061152...$, and equality occurs only when f is a rotation of the function $F \in \mathcal{A}$ determined by $T_F(z) = g(z)/z$, where $g(z)$ is given by (1.1).

Let Γ_ω be the image of an arc $\Gamma_z : z = z(t), (a \leq t \leq b)$ under the function $w = f(z)$. We say that the arc Γ_ω is convex α-spirallike if

$$\arg\left(\frac{z''(t)}{z'(t)}+\frac{z'(t)f''(z)}{f'(z)}\right),$$

lies between α and $\alpha+\pi$.

Definition 1.7. For a constant α with $|\alpha| < \pi/2$, the function f is uniformly convex α-spiral function if the image of every circular arc Γ_z with center at ξ lying in Δ is convex α-spirallike (see [9]).

The class of all uniformly convex α-spiral functions is denoted by $UCSP(\alpha)$. In particular, $UCSP(0) = UCV$. The next results were proved by Ravichandran and Selvaraj [9].

Lemma 1.8 ([9], Theorem 6)**.** *A function $f(z) \in \mathcal{A}$ is in $UCSP(\alpha)$ if and only if*

$$\operatorname{Re}\left\{e^{-i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in \Delta.$$

Lemma 1.9 ([9],Theorem 9)**.** *Let $f(z) \in \mathcal{A}$ and $s(z)$ be defined by*

$$f'(z) = (s'(z))^{e^{i\alpha}\cos\alpha}, \quad z \in \Delta.$$

Then $f(z) \in UCSP(\alpha)$ if and only if $s(z) \in UCV$.

The main object of this paper, is investigating of the norm estimates of pre-Schwarzian derivative of the classes $UCSP(\alpha)$ and $SP_p(\alpha)$. Our results extend the result obtained by [5].

In the rest of the paper, we denote by K the value

$$0.94774... \tag{1.4}$$

which is the maximum of the function h defined by (1.3) at the point $t_2 = 1.6061152....$

2. Main Results

Now we can prove our first result.

Theorem 2.1. *Let $f \in \mathcal{A}$ be in $UCSP(\alpha)$. Then $\|\mathrm{T}_f\| \leq K\cos\alpha$ where $K = 0.94774...$ is given by* (1.4)*.*

Proof. Let $f \in \mathcal{A}$ be in $UCSP(\alpha)$ and $s(z)$ be defined by

$$f'(z) = (s'(z))^{e^{i\alpha}\cos\alpha}, \quad z \in \Delta. \tag{2.1}$$

By Lemma 1.9, $s(z) \in UCV$ and therefore by Theorem 1.6, $\|\mathrm{T}_s\| \leq K$. Now, in view of (2.1) we have

$$\frac{f''(z)}{f'(z)} = e^{i\alpha}\cos\alpha\frac{s''(z)}{s'(z)}, \quad z \in \Delta,$$

and so,

$$\|\mathrm{T}_f\| = |e^{i\alpha}\cos\alpha|\|\mathrm{T}_s\| \leq K|\cos\alpha|.$$

□

The class of functions $F(z) = zf'(z), f(z) \in UCSP(\alpha)$ is a subclass of the spirallike functions and we denote it by $SP_p(\alpha)$. By Lemma 1.8, the function $f \in \mathcal{A}$ is in $SP_p(\alpha)$ if and only if

$$\mathrm{Re}\left\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \Delta.$$

Geometrically it means that $zf'(z)/f(z)$ lies in the parabolic region

$$\Omega_\alpha = \Big\{\omega : \mathrm{Re}\{e^{-i\alpha}\omega\} > |\omega - 1|\Big\}.$$

In the next theorem, we shall give the estimate for the norm of pre-Schwarzian derivative of the class $SP_p(\alpha)$.

Theorem 2.2 ([9], Theorem 7)**.** *A function $f \in \mathcal{A}$ is in $SP_p(\alpha)$ if and only if*

$$\frac{zf'(z)}{f(z)} \prec e^{i\alpha}(\cos\alpha P(z) - i\sin\alpha)$$

where

$$P(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$$

is the function which maps Δ onto $\Omega_0 = \{u + iv, v^2 < 2u - 1, u > 0\}$.

Note that Ω_0 is the interior of a parabola in the right half-plane which is symmetric about the real axis and has vertex at $(1/2, 0)$.

Theorem 2.3. *Let $f \in \mathcal{A}$ be in $SP_p(\alpha)$. Then we have*

$$\begin{aligned}\|\mathrm{T}_f\| &\leq \max_{y\in\mathbb{R}} \frac{8}{\pi}\sqrt{\frac{1+y^2}{y^4+6y^2-8y\tan\alpha+1+4\tan^2\alpha}} + K\cos\alpha \\ &\leq \frac{8}{\pi} + K\cos\alpha,\end{aligned}$$

where K is given by (1.4).

Proof. Let $f \in \mathcal{A}$ be in $SP_p(\alpha)$. By setting $p(z) = zf'(z)/f(z)$ we have

$$z\mathrm{T}_f(z) = \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} + p(z) - 1, \quad z \in \Delta. \tag{2.2}$$

By Theorem 2.2, we have $p(z) \prec q(z)$ where

$$q(z) = 1 + \frac{2e^{i\alpha}\cos\alpha}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad z \in \Delta.$$

Therefore there exists an analytic function $w : \Delta \to \Delta$ with $w(0) = 0$ such that $p(z) = q(w^2(z))$ and so

$$p(z) = 1 + \frac{2e^{i\alpha}\cos\alpha}{\pi^2}\left(\log\frac{1+w(z)}{1-w(z)}\right)^2, \quad z \in \Delta. \tag{2.3}$$

Differentiating logarithmically, we obtain

$$\frac{p'(z)}{p(z)} = \frac{\frac{8e^{i\alpha}\cos\alpha}{\pi^2}\log\left(\frac{1+w(z)}{1-w(z)}\right)\frac{w'(z)}{1-w^2(z)}}{1+\frac{2e^{i\alpha}\cos\alpha}{\pi^2}\left(\log\frac{1+w(z)}{1-w(z)}\right)^2}. \tag{2.4}$$

Upon using Schwarz-Pick Lemma, we have

$$|w'(z)|/|1-w^2(z)| \le 1/(1-|z|^2),$$

and so by using (2.2) to (2.4), for $z \in \Delta$ yields

$$\begin{aligned}(1-|z|^2)|\mathrm{T}_f(z)| \le & \frac{\frac{8\cos\alpha}{\pi^2}\left|\log\frac{1+w(z)}{1-w(z)}\right|}{\left|1+\frac{2e^{i\alpha}\cos\alpha}{\pi^2}\left(\log\frac{1+w(z)}{1-w(z)}\right)^2\right|} \\ & + \frac{1-|z|^2}{|z|}\frac{2\cos\alpha}{\pi^2}\left|\log\frac{1+w(z)}{1-w(z)}\right|^2.\end{aligned} \tag{2.5}$$

Since by (1.1),

$$g(z) = \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2,$$

has positive Taylor coefficient, we see that

$$|g(w^2(z))| \le g\left(|w^2(z)|\right) \le g(|z|).$$

Kim and Sugawa [5], proved that

$$\sup_{z\in\Delta}\frac{1-|z|^2}{|z|}\frac{2}{\pi^2}\left|\log\frac{1+w(z)}{1-w(z)}\right|^2 \le \sup_{0<x<1}(1-x^2)\frac{g(x)}{x} = K, \tag{2.6}$$

where $K = 0.94774...$ is given by (1.4). Set

$$\begin{aligned} I := \sup_{z\in\Delta} & \frac{\frac{8\cos\alpha}{\pi^2}\left|\log\frac{1+w(z)}{1-w(z)}\right|}{\left|1+\frac{2}{\pi^2}e^{i\alpha}\cos\alpha\left(\log\frac{1+w(z)}{1-w(z)}\right)^2\right|} \\ &= \frac{8}{\pi\sqrt{2}}\sup_{(x,y)\in\Omega}\left(\frac{\cos^2\alpha|x+iy|}{|1+e^{i\alpha}\cos\alpha(x+iy)|^2}\right)^{\frac{1}{2}},\end{aligned} \tag{2.7}$$

where

$$x+iy := \frac{2}{\pi^2}\left(\log\frac{1+w(z)}{1-w(z)}\right)^2,$$

belongs to $\Omega = \{x+iy, y^2 < 2x+1\}$ and so

$$I = \frac{8}{\pi\sqrt{2}}\sup_{(x,y)\in\Omega}\left(\frac{\cos^2\alpha\sqrt{x^2+y^2}}{1+\cos^2\alpha(x^2+y^2)+2x\cos^2\alpha-2y\sin\alpha\cos\alpha}\right)^{\frac{1}{2}}.$$

By using the maxizem principle of subharmonic functions and setting $x = \frac{y^2-1}{2}$ we obtain

(2.8)

$$I = \frac{8}{\pi\sqrt{2}} \sup_{y\in\mathbb{R}} \left(\frac{\frac{1}{2}\cos^2\alpha(1+y^2)}{1 + \left(\frac{1}{4}\cos^2\alpha\right) y^4 + \left(\frac{3}{2}\cos^2\alpha\right) y^2 - 2y\sin\alpha\cos\alpha - \frac{3}{4}\cos^2\alpha} \right)^{\frac{1}{2}}$$

$$= \frac{8}{\pi} \sup_{y\in\mathbb{R}} \left(\frac{1+y^2}{y^4 + 6y^2 - 8y\tan\alpha + 1 + 4\tan^2\alpha} \right)^{\frac{1}{2}}.$$

Therefore by relations (2.5)-(2.8) we have

$$\|\mathrm{T}_f\| \leq \sup_{y\in\mathbb{R}} \frac{8}{\pi} \sqrt{\frac{1+y^2}{y^4 + 6y^2 - 8y\tan\alpha + 1 + 4\tan^2\alpha}} + K\cos\alpha. \tag{2.9}$$

We claim that the right side of (2.9) is bounded. Let

$$h(y,\alpha) = \frac{1+y^2}{y^4 + 6y^2 - 8y\tan\alpha + 1 + 4\tan^2\alpha}, \quad y \in \mathbb{R}, |\alpha| < \frac{\pi}{2}. \tag{2.10}$$

Then $\frac{\partial h}{\partial \alpha} = 0$ if and only if

$$8(y^2+1)(1+\tan^2\alpha)(-y+\tan\alpha) = 0,$$

or if and only if $y = \tan\alpha$ and also $\frac{\partial h}{\partial y} = 0$ if and only if

$$2y(y^4 + 6y^2 - 8y\tan\alpha + 1 + 4\tan^2\alpha) = (y^2+1)(4y^3 + 12y - 8\tan\alpha).$$

Hence $\partial h/\partial\alpha = \partial h/\partial y = 0$ if and only if $y = \tan\alpha = 0$. Also it is easy to see that $h_{\alpha\alpha}(0,0) < 0$ and $h_{\alpha\alpha}(0,0)h_{yy}(0,0) - h^2_{\alpha y}(0,0)$ is positive. So the function $h(y,\alpha)$ takes its maximum value at the point $y = \alpha = 0$. But in view of (2.10), we have $h(0,0) = 1$ and so its maximum is 1.

Now, the relation (2.9) yields

$$\|\mathrm{T}_f\| \leq \frac{8}{\pi} + K\cos\alpha$$

and the proof is complete. □

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[1] Department of Mathematics, Faculty of Science, Urmia University, Urmia, Iran.

E-mail address: z.orouji@urmia.ac.ir

[2] Department of Mathematics, Faculty of Science, Urmia University, Urmia, Iran.

E-mail address: raghalary@yahoo.com