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The Solvability of Concave-Convex Quasilinear Elliptic Systems Involving p-Laplacian and Critical Sobolev Exponent

Somayeh Khademloo[1]* and Saeed Khanjany Ghazi[2]

Abstract. In this work, we study the existence of non-trivial multiple solutions for a class of quasilinear elliptic systems equipped with concave-convex nonlinearities and critical growth terms in bounded domains. By using the variational method, especially Nehari manifold and Palais-Smale condition, we prove the existence and multiplicity results of positive solutions.

1. Introduction and Main Results

The aim of this paper is to establish the existence and multiplicity of nontrivial positive solutions of the following quasilinear elliptic system:

(1.1)
$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda |u|^{q-2}u + \frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & x\in\Omega, \\ -\Delta_p v + a(x)|v|^{p-2}v = \mu |v|^{q-2}v + \frac{\beta}{\alpha+\beta}|v|^{\beta-2}v|u|^{\alpha}, & x\in\Omega, \\ u=v=0, & x\in\partial\Omega. \end{cases}$$

Here $\Omega\subset\mathbb{R}^N (N\geq 2)$ is a smooth bounded domain, $0\in\Omega, \alpha>1, \beta>1$, $\alpha+\beta=p^*$, and $p^*=\dfrac{pN}{N-p}$ is the critical Sobolev exponent. Also $\lambda,\mu>0$, $1<q<p<N$, $\Delta_p u = \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big)$ is the p-Laplacian, and $a(x)$ is the weight function that is also positive and bounded.

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* Corresponding author.

Many problems in science and engineering are described by semilinear and quasilinear elliptic equations and systems which usually difficult to solve, we refer the reader to Ambrosetti et al. [3, 2], Hsu [13, 14], Brown et al. [10, 11] and Wu [16, 17], and so forth. Although by methods of nonlinear analysis like variational method, we are able to tackle such problems. Afrouzi and Rasouli [1] have investigated the following system with subcritical nonlinearity:

(1.2)
$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda|u|^{q-2}u + \frac{\alpha}{\alpha+\beta}c(x)|u|^{\alpha-2}u|v|^{\beta}, & x \in \Omega, \\ -\Delta_p v + a(x)|v|^{p-2}v = \mu|v|^{q-2}v + \frac{\beta}{\alpha+\beta}c(x)|v|^{\beta-2}v|u|^{\alpha}, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases}$$

where $p > 2$ and $2 < \alpha + \beta < p < q < p^*$, and the weight c satisfy some suitable conditions. They have proved that, there exists $\delta^* > 0$ such that if the parameters $\lambda, \mu > 0$, satisfy

$$0 < \lambda^{\frac{p}{q-p}} + \mu^{\frac{p}{q-p}} < \delta^*,$$

then the problem (1.2) has at least two nontrivial positive solutions (u_0^+, v_0^+) and (u_0^-, v_0^-).

Set $p = 2, a(x) = 0$ and $1 < q < 2 < \alpha + \beta < 2^*$ and also by adding sign changing functions f, g and h that $h \in C(\overline{\Omega})$, the problem (1.1) changes to

$$\begin{cases} -\Delta u = \lambda f(x)|u|^{q-2}u + \frac{\alpha}{\alpha+\beta}h(x)|u|^{\alpha-2}u|v|^{\beta}, & x \in \Omega, \\ -\Delta v = \mu g(x)|v|^{q-2}v + \frac{\beta}{\alpha+\beta}h(x)|v|^{\beta-2}v|u|^{\alpha}, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Wu in [16] proved that the system (1.3) has two positive solutions with same conditions as in [1]. Our main results are as follows.

Theorem 1.1. *If λ, μ satisfy*

$$0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_1,$$

then (1.1) *has at least one positive solution.*

Theorem 1.2. *If λ, μ satisfy*

$$0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_2,$$

then (1.1) *has at least two positive solutions.*

We divide this paper into four sections. In the next section, we give properties of Nehari manifold and set up the variational method. The proof of Theorem 1.1 is in the third section and in the last section, by the Palais-Smale condition, we prove Theorem 1.2.

2. Nehari Manifold

Problem (1.1) is posed in the framework of the Sobolev space $W = W_0^{1,p} \times W_0^{1,p}$ with the standard norm

$$\parallel z \parallel = \left(\int_\Omega \left(|\nabla u|^p + a(x)|u|^p \right) dx + \int_\Omega \left(|\nabla v|^p + a(x)|v|^p \right) dx \right)^{1/p},$$

for any $z = (u, v) \in W$. An element z is said to be a weak solution of (1.1) if

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^{p-2} \nabla u \nabla \varphi_1 + a(x) u \varphi_1 \right) dx + \int_\Omega \left(|\nabla v|^{p-2} \nabla v \nabla \varphi_2 + a(x) v \varphi_2 \right) dx \\ &\quad - \lambda \int_\Omega |u|^{q-2} u \varphi_1 - \mu \int_\Omega |v|^{q-2} v \varphi_2 dx - \frac{\alpha}{\alpha+\beta} \int_\Omega |u|^{\alpha-2} u |v|^\beta \varphi_1 dx \\ &\quad - \frac{\beta}{\alpha+\beta} \int_\Omega |v|^{\beta-2} v |u|^\alpha \varphi_1 dx = 0, \end{aligned}$$

for all $(\varphi_1, \varphi_2) \in W$.

Thus the corresponding energy functional of (1.1) is defined by

$$J_{\lambda,\mu}(z) = \frac{1}{p} \parallel z \parallel^p - \frac{1}{q} K_{\lambda,\mu}(z) - \frac{1}{\alpha+\beta} L(z),$$

where $K_{\lambda,\mu}, L : W \to \mathbb{R}$ are the functionals defined by

$$K_{\lambda,\mu}(z) = \int_\Omega (\lambda |u|^q + \mu |v|^q) d(x), \qquad L(z) = \int_\Omega |u|^\alpha |v|^\beta dx.$$

As the energy functional $J_{\lambda,\mu}$ is not bounded below on W, we consider the Nehari manifold $N_{\lambda,\mu} = \left\{ z \in W \setminus \{0\} \, | \langle J'_{\lambda,\mu}(z), z \rangle = 0 \right\}$. Thus $z \in N_{\lambda,\mu}$ if and only if

$$\langle J'_{\lambda,\mu}(z), z \rangle = \parallel z \parallel^p - K_{\lambda,\mu}(z) - L(z) = 0. \tag{2.1}$$

Note that $N_{\lambda,\mu}$ contains every nonzero solution of (1.1). Define

$$\phi_{\lambda,\mu}(z) = \langle J'_{\lambda,\mu}(z), z \rangle.$$

Then for $z \in N_{\lambda,\mu}$,

$$\begin{aligned} \langle \phi'_{\lambda,\mu}(z), z\rangle &= p \parallel z \parallel^p - qK_{\lambda,\mu}(z) - p^* L(z) \\ &= (p-q) \parallel z \parallel^p - (p^*-q)L(z) \\ &= (p^*-q)K_{\lambda,\mu}(z) - (p^*-p) \parallel z \parallel^p . \end{aligned} \tag{2.2}$$

Now, similar to the method used in [15], we split $N_{\lambda,\mu}$ into three disjoint parts:

$$\begin{aligned} N^+_{\lambda,\mu} &= \left\{ z \in N_{\lambda,\mu} : \langle \phi'_{\lambda,\mu}(z), z\rangle > 0 \right\}, \\ N^0_{\lambda,\mu} &= \left\{ z \in N_{\lambda,\mu} : \langle \phi'_{\lambda,\mu}(z), z\rangle = 0 \right\}, \\ N^-_{\lambda,\mu} &= \left\{ z \in N_{\lambda,\mu} : \langle \phi'_{\lambda,\mu}(z), z\rangle < 0 \right\}. \end{aligned}$$

To state our main result, we present some important properties of $N^+_{\lambda,\mu}$, $N^0_{\lambda,\mu}$ and $N^-_{\lambda,\mu}$.

Lemma 2.1. *$J_{\lambda,\mu}$ is coercive and bounded from below on $N_{\lambda,\mu}$.*

Proof. If $z \in N_{\lambda,\mu}$, then by (2.1), the Hölder inequality and the Sobolev embedding theorem we have

(2.3)
$$\begin{aligned} J_{\lambda,\mu}(z) &= \frac{p^*-p}{pp^*} \parallel z \parallel^p - \frac{p^*-q}{qp^*} K_{\lambda,\mu}(z) \\ &\geq \frac{1}{N} \parallel z \parallel^p - \frac{p^*-q}{qp^*} S^{-\frac{q}{p}} |\Omega|^{\frac{p^*-q}{p^*}} \left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}} \parallel z \parallel^q, \end{aligned}$$

where S is the best Sobolev constant for the embedding of $W_0^{1,p}$ in $L^{p^*}(\Omega)$ defined by

$$S = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega (|\nabla u|^p + a(x)|u|^p)\, dx}{\left(\displaystyle\int_\Omega | u |^{p^*} dx \right)^{\frac{p}{p^*}}}. \tag{2.4}$$

Since $1 < q < p$, we see that $J_{\lambda,\mu}$ is coercive and bounded below on $N_{\lambda,\mu}$. □

By modifying the proof of Alves et al. [4] (Theorem 5), we have

$$S_{\alpha,\beta} = \left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \right) S, \tag{2.5}$$

where S is the best Sobolev constant defined in (2.3) and

$$S_{\alpha,\beta} = \inf_{u,v\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\| z \|^p}{\left(\int_\Omega |u|^\alpha |v|^\beta dx\right)^{\frac{p}{p^*}}}. \tag{2.6}$$

This is achieved if and only if $\Omega = \mathbb{R}^N$ by the function

$$U_\varepsilon(x) = C_N \left(\frac{\varepsilon^{\frac{1}{p}}}{\varepsilon + |x|^{\frac{p}{p-1}}} \right)^{(N-p)/p}, \quad \varepsilon > 0.$$

Lemma 2.2. *Suppose that* z_0 *is a local minimizer for* $J_{\lambda,\mu}$ *on* $N_{\lambda,\mu}$ *and that* $z_0 \notin N_{\lambda,\mu}$*, then* z_0 *is a critical point of the* $J_{\lambda,\mu}$.

Proof. If z_0 is a local minimizer for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$. Then z_0 is a solution of optimization problem. Since $\phi_{\lambda,\mu}(z) = \langle J'_{\lambda,\mu}(z), z\rangle$, then by the theory of Lagrange multipliers, there exists $\gamma \in \mathbb{R}$ such that

$$\langle J'_{\lambda,\mu}(z_0), z_0\rangle = \gamma \langle \phi'_{\lambda,\mu}(z_0), z_0\rangle.$$

Since $z_0 \in N_{\lambda,\mu}$ and $z_0 \notin N^0_{\lambda,\mu}$, we get

$$\langle \phi'_{\lambda,\mu}(z_0), z_0\rangle \neq 0.$$

Hence $\gamma = 0$ and this completes the proof. □

Lemma 2.3. *Set*

$$\Lambda_1 = \left(\frac{p-q}{p^*-q}\right)^{\frac{p}{p^*-q}} \left(\frac{p^*-q}{p^*-p}|\Omega|^{\frac{p^*-q}{p^*}}\right)^{-\frac{p}{p-q}} S^{\frac{N}{p}+\frac{q}{p-q}} > 0, \tag{2.7}$$

then for (λ,μ) *satisfying*

$$0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_1,$$

we have $N^0_{\lambda,\mu} = \emptyset$.

Proof. Suppose opposite, i.e., there exist $\lambda, \mu > 0$ with

$$0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_1,$$

such that $N^0_{\lambda,\mu} \neq \emptyset$. Then for $z \in N^0_{\lambda,\mu}$, by (2.2), we have

$$\| z \|^p = \frac{p^*-q}{p-q} L(z), \qquad \| z \|^p = \frac{p^*-q}{p^*-p} K_{\lambda,\mu}(z).$$

Then we have

$$\| z \| \geq \left(\frac{p-q}{p^*-q} S^{\frac{p^*}{p}}\right)^{\frac{1}{p^*-p}},$$

and similar to the proof of Lemma 2.1, (see (2.3)) by the Hölder inequality and the Sobolev embedding theorem, one can get

$$\| z \| \leq \left(\frac{p^*-q}{p^*-p} S^{-\frac{q}{p}} |\Omega|^{\frac{p^*-q}{p^*}} \right)^{\frac{1}{p-q}} \left(\lambda^{\frac{p}{q-p}} + \mu^{\frac{p}{q-p}} \right)^{\frac{1}{p}}.$$

This implies

$$\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} \geq \left(\frac{p-q}{p^*-q} \right)^{\frac{p}{p^*-p}} \left(\frac{p^*-q}{p^*-p} |\Omega|^{\frac{p^*-q}{p^*}} \right)^{-\frac{p}{p-q}} S^{\frac{N}{p}+\frac{q}{p-q}} = \Lambda_1,$$

that is a contradiction. Therefore, we can conclude that there exists a positive number Λ_1 such that for

$$0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_1,$$

we have $N^0_{\lambda,\mu} = \emptyset$. □

Let

$$\Theta_\Lambda = \left\{ (\lambda, \mu) \in \mathbb{R}^2 \backslash (0,0) : 0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda \right\},$$

and $\Lambda_0 = \left(\frac{q}{p}\right)^{\frac{p}{p-q}} \Lambda_1 < \Lambda_1$. If $(\lambda, \mu) \in \Theta_{\Lambda_1}$, by Lemma 2.3, we have $N_{\lambda,\mu} = N^+_{\lambda,\mu} \cup N^-_{\lambda,\mu}$. Define

$$\theta_{\lambda,\mu} = \inf_{z \in N_{\lambda,\mu}} J_{\lambda,\mu}(z),$$
$$\theta^+_{\lambda,\mu} = \inf_{z \in N^+_{\lambda,\mu}} J_{\lambda,\mu}(z),$$
$$\theta^-_{\lambda,\mu} = \inf_{z \in N^-_{\lambda,\mu}} J_{\lambda,\mu}(z).$$

Theorem 2.4. (i) *If* $(\lambda, \mu) \in \Theta_{\Lambda_1}$, *then* $\theta_{\lambda,\mu} \leq \theta^+_{\lambda,\mu} < 0$;

(ii) *If* $(\lambda, \mu) \in \Theta_{\Lambda_0}$, *then there exists*

$$d_0 = d_0(\lambda, \mu, p, q, N, S, |\Omega|) > 0,$$

such that $\theta^-_{\lambda,\mu} > d_0$.

Proof. (i) For $z \in N^+_{\lambda,\mu}$, by (2.2), we have

$$K_{\lambda,\mu}(z) > \frac{p^*-p}{p^*-q} \| z \|^p,$$

and so,

$$\begin{aligned} J_{\lambda,\mu}(z) &= \left(\frac{1}{p}-\frac{1}{p^*}\right)\| z \|^p - \left(\frac{1}{q}-\frac{1}{p^*}\right) K_{\lambda,\mu}(z) \\ &< \left(\frac{1}{p}-\frac{1}{p^*}\right)\| z \|^p - \left(\frac{1}{q}-\frac{1}{p^*}\right) \frac{p^*-p}{p^*-q}\| z \|^p \\ &< \frac{p^*-p}{p^*}\left(\frac{1}{p}-\frac{1}{q}\right)\| z \|^p < 0. \end{aligned}$$

Thus, from the definition of $\theta_{\lambda,\mu}$ and $\theta^+_{\lambda,\mu}$, we can deduce that $\theta_{\lambda,\mu} \leq \theta^+_{\lambda,\mu} < 0$.

(ii) For $z \in N^-_{\lambda,\mu}$, by (2.2) we have $\frac{p-q}{p^*-q}\| z \|^p < L(z)$. Moreover, using (2.4), we get $L(z) \leq S^{-\frac{p^*}{p}} \| z \|^p$. This implies that

$$\| z \| \geq \left(\frac{p-q}{p^*-q}\right)^{\frac{1}{p^*-p}} S^{\frac{N}{p^2}}.$$

By (2.3), we have

$$\begin{aligned} J_{\lambda,\mu}(z) &\geq \| z \|^q \Bigg(\frac{p^*-p}{p^*p}\| z \|^{p-q} \\ &\quad - \frac{p^*-q}{p^*q} S^{-\frac{q}{p}} |\Omega|^{\frac{p^*-q}{p^*}} \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}\,\frac{p-q}{p}} \Bigg) \\ &\geq \left(\frac{p-q}{p^*-q}\right)^{\frac{q}{p^*-p}} S^{\frac{qN}{p^2}} \Bigg(\frac{p^*-p}{p^*p} \left(\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-p}} S^{\frac{(p-q)N}{p^2}} \\ &\quad - \frac{p^*-q}{p^*q} S^{-\frac{q}{p}} |\Omega|^{\frac{p^*-q}{p^*}} (\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})^{\frac{p-q}{p}} \Bigg). \end{aligned}$$

Thus, if $(\lambda, \mu) \in \Theta_{\Lambda_0}$, then for $z \in N^-_{\lambda,\mu}$ we have

$$J_{\lambda,\mu}(z) > d_0 = d_0(\lambda, \mu, p, q, N, S, |\Omega|) > 0.$$

□

For each $z \in W$ such that $L(z) > 0$, let

$$t_{\max} = \left(\frac{(p-q)\| z \|^p}{(p^*-q)L(z)}\right)^{\frac{1}{p^*-p}} > 0. \tag{2.8}$$

Then the following lemma holds.

Lemma 2.5. *Assume that $(\lambda, \mu) \in \Theta_{\Lambda_1}$, then for every $z \in W$ with $L(z) > 0$, one has the following:*

(i) *if* $K_{\lambda,\mu}(z) \leq 0$, *then there exists* $t^- = t^-(z) > t_{\max}$ *such that* $t^- z \in N^-_{\lambda,\mu}$ *and*

$$J_{\lambda,\mu}(t^- z) = \sup_{t \geq 0} J_{\lambda,\mu}(tz); \tag{2.9}$$

(ii) *if* $K_{\lambda,\mu}(z) > 0$, *then there exist*

$$0 < t^+ = t^+(z) < t_{\max} < t^- = t^-(z),$$

such that $t^\pm z \in N^\pm_{\lambda,\mu}$ *and*

$$J_{\lambda,\mu}(t^+ z) = \inf_{0 \leq t \leq t_{\max}} J_{\lambda,\mu}(tz); \qquad J_{\lambda,\mu}(t^- z) = \sup_{t \geq 0} J_{\lambda,\mu}(tz). \tag{2.10}$$

Proof. Fix $z \in W$ with $L(z) > 0$. Let

$$m(t) = t^{p-q} \| z \|^p - t^{p^*-q} L(z),$$

for $t \geq 0$. Clearly $m(0) = 0$ and $m(t) \to -\infty$ as $t \to \infty$. Since

$$m'(t) = (p-q) t^{p-q-1} \| z \|^p - (p^* - q) t^{p^*-q-1} L(z),$$

there exists a unique $t_{\max} > 0$ such that $m(t)$ achieves its maximum at $t_{\max} > 0$, increasing for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, \infty)$. Clearly, $tz \in N^+_{\lambda,\mu}$(or $N^-_{\lambda,\mu}$) if and only if $m'(t) > 0$(or < 0). Moreover

(2.11)

$$\begin{aligned} m(t_{\max}) &= \left(\frac{(p-q) \| z \|^p}{(p^* - q) L(z)} \right)^{\frac{p-q}{p^*-p}} \| z \|^p - \left(\frac{(p-q) \| z \|^p}{(p^* - q) L(z)} \right)^{\frac{p^*-q}{p^*-p}} L(z) \\ &= \| z \|^q \left[\left(\frac{p-q}{p^*-q} \right)^{\frac{p-q}{p^*-p}} - \left(\frac{p-q}{p^*-q} \right)^{\frac{p^*-q}{p^*-p}} \right] \left(\frac{\| z \|^{p^*}}{L(z)} \right)^{\frac{p-q}{p^*-p}} \\ &\leq \| z \|^q \left(\frac{p-q}{p^*-q} \right)^{\frac{p-q}{p^*-p}} \left(\frac{p^*-q}{p^*-q} \right) \left(S^{\frac{p^*}{p}} \right)^{\frac{p-q}{p^*-p}}. \end{aligned}$$

(i) $K_{\lambda,\mu}(z) \leq 0$, then there exists a unique $t^- > t_{\max}$ such that $m(t^-) = K_{\lambda,\mu}(z)$ and $m'(t^-) < 0$. Now,

$$(p-q)(t^-)^p \| z \|^p - (p^* - q)(t^-)^{p^*} L(z) = (t^-)^{q+1} m'(t^-) < 0,$$

and

$$\langle J'_{\lambda,\mu}(t^- z), (t^- z) \rangle = (t^-)^q [m(t^-) - K_{\lambda,\mu}(z)] = 0.$$

Thus, $t^- z \in N^-_{\lambda,\mu}$. Since for $t > t_{\max}$, we have $m'(t) < 0$ and $m''(t) < 0$, subsequently,

$$J_{\lambda,\mu}(t^- z) = \sup_{t \geq 0} J_{\lambda,\mu}(tz).$$

(ii) $K_{\lambda,\mu}(z) > 0$. For $(\lambda,\mu) \in \Theta_{\Lambda_1}$, by (2.10) we have

$$\begin{aligned} m(0) &= 0 \\ &< K_{\lambda,\mu}(z) \\ &\leq S^{-\frac{q}{p}} |\Omega|^{\frac{p^*-q}{p^*}} \left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}} \parallel z \parallel^q \\ &\leq \parallel z \parallel^q \left(\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-p}} \left(\frac{p^*-q}{p^*-q}\right) \left(S^{\frac{p^*}{p}}\right)^{\frac{p-q}{p^*-p}}, \end{aligned}$$

there exists unique t^+ and t^- such that $0 < t^+ = t^+(z) < t_{\max} < t^- = t^-(z)$,

$$m(t^+) = K_{\lambda,\mu}(z) = m(t^-), \qquad m'(t^+) > 0 > m'(t^-).$$

We have $t^{\pm} z \in N^{\pm}_{\lambda,\mu}$, and

$$J_{\lambda,\mu}(t^- z) \geq J_{\lambda,\mu}(tz) \geq J_{\lambda,\mu}(t^+ z), \qquad \forall t \in [t^+, t^-],$$

$$J_{\lambda,\mu}(t^+ z) \leq J_{\lambda,\mu}(tz), \qquad \forall t \in [0, t_{\max}].$$

Thus

$$J_{\lambda,\mu}(t^+ z) = \inf_{0 \leq t \leq t_{\max}} J_{\lambda,\mu}(tz); \quad J_{\lambda,\mu}(t^- z) = \sup_{t \geq 0} J_{\lambda,\mu}(tz).$$

□

3. Proof of Theorem 1.1

At first, we give the following definitions about $(PS)_c$ -sequence and introduce the Brézis-Lieb lemma (see [9]) as a remark.

Definition 3.1. Let $c \in \mathbb{R}$, W be a Banach space and $J \in C^1(W, \mathbb{R})$.

(i) $\{z_n\}$ is a $(PS)_c$-sequence in W for J if $J(z_n) = c + o(1)$ and $J'(z_n) = o(1)$ strongly in W^{-1} as $n \to \infty$.

(ii) We say that J satisfies the $(PS)_c$ condition if any $(PS)_c$-sequence $\{z_n\}$ in W for J has a convergent subsequence.

Remark 3.2. Let $z_n \in W$ such that

(i) $\parallel z_n \parallel \leq$ constant,

(ii) $z_n \to z_0^+$ almost everywhere in Ω, then

$$\parallel \bar{z}_n \parallel^p \longrightarrow \parallel z_n \parallel^p - \parallel z_0^+ \parallel^p, \tag{3.1}$$

as $n \to \infty$ where $\bar{z}_n = z_n - z_0^+$.

Proposition 3.3. (i) *If $(\lambda,\mu) \in \Theta_{\Lambda_1}$, then there exists a $(PS)_{\theta_{\lambda,\mu}}$-sequence $\{z_n\} \subset N_{\lambda,\mu}$ in W for $J_{\lambda,\mu}$.*

(ii) *If* $(\lambda,\mu)\in\Theta_{\Lambda_0}$, *then there exists a* $(PS)_{\theta^-_{\lambda,\mu}}$*-sequence* $\{z_n\}\subset N^-_{\lambda,\mu}$ *in* W *for* $J_{\lambda,\mu}$.

Proof. The proof is almost the same as that in [15]. □

Now, we prove the existence of a local minimum for $J_{\lambda,\mu}$ on $N^+_{\lambda,\mu}$.

Theorem 3.4. *If* $(\lambda,\mu)\in\Theta_{\Lambda_1}$, *then* $J_{\lambda,\mu}$ *has a minimizer* z_0^+ *in* $N^+_{\lambda,\mu}$ *and satisfies the following:*

(i) $J_{\lambda,\mu}(z_0^+)=\theta^+_{\lambda,\mu}=\theta_{\lambda,\mu}<0$;

(ii) z_0^+ *is a positive solution of* (1.1).

Proof. By Proposition 3.3 (i), there exists a minimizing sequence $\{z_n\}$ for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$ such that

$$J_{\lambda,\mu}(z_n)=\theta_{\lambda,\mu}+o(1),\qquad J'_{\lambda,\mu}(z_n)=o(1). \tag{3.2}$$

Then, by Lemma 2.1, there exists a subsequence $\{z_n=(u_n,v_n)\}$ and $(z_0^+)=(u_0^+,v_0^+)\in W$ such that

$$\begin{cases} u_n\rightharpoonup u_0^+, v_n\rightharpoonup v_0^+ & \text{weakly in } W_0^{1,p}(\Omega),\\ u_n\to u_0^+, v_n\to v_0^+ & \text{almost everywhere in } \Omega,\\ u_n\to u_0^+, v_n\to v_0^+ & \text{strongly in } L^s(\Omega)\ (1\le s<p^*),\end{cases} \tag{3.3}$$

as $n\to\infty$. It is easy to see that

$$K_{\lambda,\mu}(z_n)=K_{\lambda,\mu}(z_0^+)+o(1),\quad \text{as } n\to\infty. \tag{3.4}$$

First, we prove that z_0^+ is a nontrivial solution of (1.1). By (3.1) and (3.2), we can deduce that z_0^+ is a weak solution of (1.1). By (2.2)

$$\begin{aligned} J_{\lambda,\mu}(z_n)&=\frac{p^*-p}{pp^*}\parallel z_n\parallel^p-\frac{p^*-q}{qp^*}K_{\lambda,\mu}(z_n)\\ &\ge-\frac{p^*-q}{qp^*}K_{\lambda,\mu}(z_n),\end{aligned}$$

if $n\to\infty$, one can get

$$K_{\lambda,\mu}(z_0^+)\ge-\frac{qp^*}{p^*-q}\theta_{\lambda,\mu}>0.$$

Thus, $z_0^+\in N_{\lambda,\mu}$ is a nontrivial solution of (1.1). Now, we claim that $z_n\to z_0^+$ strongly in W and $J_{\lambda,\mu}(z_0^+)=\theta_{\lambda,\mu}$. By applying Fatou's lemma and $z_0^+ \in N_{\lambda,\mu}$, we have

$$\begin{aligned}\theta_{\lambda,\mu} &\le J_{\lambda,\mu}(z_0^+)\\ &= \frac{1}{N}\parallel z_0^+ \parallel^p - \frac{p^*-q}{qp^*}K_{\lambda,\mu}(z_0^+)\\ &\le \liminf_{n\to\infty}\left(\frac{1}{N}\parallel z_n \parallel^p - \frac{p^*-q}{qp^*}K_{\lambda,\mu}(z_n)\right)\\ &\le \liminf_{n\to\infty} J_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu},\end{aligned}$$

This implies that $J_{\lambda,\mu}(z_0^+) = \theta_{\lambda,\mu}$ and

$$\lim_{n\to\infty}\parallel z_n \parallel^p = \parallel z_0^+ \parallel^p .$$

Let $\bar{z}_n = z_n - z_0^+$, then by Remark 3.2 (see (3.1)) we get

$$\parallel \bar{z}_n \parallel^p = \parallel z_n \parallel^p - \parallel z_0^+ \parallel^p .$$

Therefore, $z_n \to z_0^+$ strongly in W. Next, we show that $z_0^+ \in N^+_{\lambda,\mu}$. Suppose that $z_0^+ \in N^-_{\lambda,\mu}$, then by Lemma 2.5, there exists unique t_0^+ and t_0^- such that $t_0^\pm z_0^\pm \in N^\pm_{\lambda,\mu}$ and $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt}J_{\lambda,\mu}(t_0^+ z_0^+) = 0, \qquad \frac{d^2}{dt^2}J_\lambda(t_0^+ z_0^+) > 0,$$

there exists $t_0^+ < \bar{t} \le t_0^-$ such that $J_{\lambda,\mu}(t_0^+ z_0^+) < J_{\lambda,\mu}(\bar{t} z_0^+)$. Again by Lemma 2.5 we have

$$J_{\lambda,\mu}(t_0^+ z_0^+) < J_{\lambda,\mu}(\bar{t} z_0^+) \le J_{\lambda,\mu}(t_0^- z_0^+) = J_{\lambda,\mu}(z_0^+),$$

which contradicts $J_{\lambda,\mu}(z_0^+) = \theta^+_{\lambda,\mu}$. Thus $z_0^+ \in N^+_{\lambda,\mu}$. Since $J_{\lambda,\mu}(z_0^+) = J_{\lambda,\mu}(|z_0^+|)$ and $|z_0^+| \in N^+_{\lambda,\mu}$, by Lemma 2.2 we deduce that z_0^+ is a nontrivial nonnegative solution of (1.1). By the maximum principle, it follows that $u_0^+ > 0, v_0^+ > 0$ in Ω . □

Proof of Theorem 1.1. By Theorem 3.4, we get that for all $\lambda, \mu > 0$ and $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_1$ (or $(\lambda,\mu) \in \Theta_{\Lambda_1}$), (1.1) has a positive solution $z_0^+ \in N^+_{\lambda,\mu}$. □

4. Proof of Theorem 1.2

For the existence of a second positive solution of system (1.1), we will need here a stronger condition. In this section, at the first we will find the range of c where $(PS)_c$ condition holds for $J_{\lambda,\mu}$.

Lemma 4.1. *Assume that* $\{z_n\} \subset W$ *is a* $(PS)_c$*-sequence for* $J_{\lambda,\mu}$ *and* $z_n \rightharpoonup z$ *in* W *, then* z *is a critical point of* $J_{\lambda,\mu}$*, and there exists a* $C_0 = C_0(p, q, N, S, |\Omega|) > 0$ *such that*

$$J_{\lambda,\mu} \geq -C_0 \left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} \right).$$

Proof. Let $z_n = (u_n, v_n)$ and assume that $\{z_n\}$ be a $(PS)_c$-sequence for $J_{\lambda,\mu}$ with $z_n \rightharpoonup z$ in W, it is easy to deduce that $J'_{\lambda,\mu}(z) = 0$, so $\langle J'_{\lambda,\mu}(z), z\rangle = 0$ and by (2.1)

$$\int_\Omega |u|^\alpha |v|^\beta dx = \| z \|^p - K_{\lambda,\mu}(z).$$

Consequently as (2.4),

$$J_{\lambda,\mu}(z) = \frac{p^* - p}{pp^*} \| z \|^p - \frac{p^* - q}{qp^*} K_{\lambda,\mu}(z).$$

By the Hölder inequality and the Sobolev embedding theorem, we have

$$J_{\lambda,\mu}(z) \geq \frac{1}{N} \| z \|^p - \frac{p^* - q}{qp^*} S^{-\frac{q}{p}} |\Omega|^{\frac{p^*-q}{p^*}}$$
$$\times \left[\lambda \left(\int_\Omega (|\nabla u|^p + a(x)|u|^p) dx \right) + \mu \left(\int_\Omega (|\nabla v|^p + a(x)|v|^p) dx \right) \right]^{\frac{q}{p}}.$$

Finally, by the Young inequality, one can get

$$J_{\lambda,\mu}(z) \geq \frac{1}{N} \| z \|^p - \frac{1}{N} \| z \|^p - C_0 \left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} \right)$$
$$= C_0 (\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}),$$

where $C_0 = C_0(p, q, N, S, |\Omega|) > 0$. □

Lemma 4.2. *Suppose that* $\{z_n\} \subset W$ *is a* $(PS)_c$*-sequence for* $J_{\lambda,\mu}$*, then* $\{z_n\}$ *is bounded in* W*.*

Proof. Suppose opposite, that $\| z_n \| \to \infty$. Let

$$z_n^* = (u_n^*, v_n^*) = \frac{z_n}{\| z_n \|} = \left(\frac{u_n}{\| u_n \|}, \frac{v_n}{\| v_n \|} \right), \tag{4.1}$$

$z_n^* \rightharpoonup z^* = (u^*, v^*)$ in W. This implies that $u_n^* \to u^*, v_n^* \to v^*$ strongly in $L^s(\Omega)$ for all $1 \leq s < p^*$ and

$$K_{\lambda,\mu}(z_n^*) = K_{\lambda,\mu}(z^*) + o(1). \tag{4.2}$$

Now, since $\{z_n\} \subset W$ is a $(PS)_c$-sequence for $J_{\lambda,\mu}$ and $\| z_n \| \to \infty$, we have

$$\frac{\| z_n^* \|^p}{p} - \frac{\| z_n \|^{q-p}}{q} K_{\lambda,\mu}(z_n^*) - \frac{\| z_n \|^{p^*-p}}{p^*} L(z_n^*) = o(1), \tag{4.3}$$

and

$$\| z_n^* \|^p - \| z_n \|^{q-p} K_{\lambda,\mu}(z_n^*) - \| z_n \|^{p^*-p} L(z_n^*) = o(1). \tag{4.4}$$

From (4.2) and (4.4), one can get

$$\| z_n^* \|^p = \frac{p(p^*-q)}{q(p^*-p)} \| z \|^{q-p} K_{\lambda,\mu}(z_n^*) + o(1). \tag{4.5}$$

Since $1 < q < p$ and $\| z_n \| \to \infty$, (4.5) implies that $\| z_n^* \|^p \to \infty$, as $n \to \infty$, which contradict $\| z_n^* \|^p = 1$. □

Lemma 4.3. *Let*

$$C_{\lambda,\mu} = \frac{1}{N} (S_{\alpha,\beta})^{\frac{N}{p}} - C_0 \left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} \right),$$

where C_0 is the positive constant given in Lemma 4.1, then $J_{\lambda,\mu}$ satisfies the $(PS)_c$ condition with $c \in (-\infty, C_{\lambda,\mu})$.

Proof. Let $\{z_n\} \subset W$ be a $(PS)_c$-sequence for $J_{\lambda,\mu}$ with $c \in (-\infty, C_{\lambda,\mu})$. By Lemma 4.2 we have that $\{z_n\}$ is bounded in W. This implies that $z_n \rightharpoonup z$ up to a subsequence, when z is a critical point of $J_{\lambda,\mu}$. Furthermore we may assume

$$\begin{cases} u_n \rightharpoonup u, v_n \rightharpoonup v & \text{in } W_0^{1,p}(\Omega), \\ u_n \to u, v_n \to v & \text{a.e on } \Omega, \\ u_n \to u \ , \ v_n \to v & \text{in } L^s(\Omega) \, (1 \le s < p^*). \end{cases} \tag{4.6}$$

Hence we have $J'_{\lambda,\mu}(z) = 0$ and

$$K_{\lambda,\mu}(z_n) = K_{\lambda,\mu}(z) + o(1). \tag{4.7}$$

Let $\tilde{u}_n = u_n - u$, $\tilde{v}_n = v_n - v$, and $\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n)$. Then by Remark 3.2 (see (3.1)), we obtain

$$\| \tilde{z}_n \|^p = \| z_n \|^p - \| z \|^p + o(1), \tag{4.8}$$

and by an argument of [12], Lemma 2.1

$$L(\tilde{z}_n) = L(z_n) - L(z) + o(1). \tag{4.9}$$

Since $J_{\lambda,\mu}(z_n) = c + o(1)$ and (4.7) and (4.9), we deduce that

$$\frac{1}{p} \| \tilde{z}_n \|^p - \frac{1}{p^*} L(\tilde{z}_n) = c - J_{\lambda,\mu}(z) + o(1), \tag{4.10}$$

and

$$\| \tilde{z}_n \|^p - L(\tilde{z}_n) = o(1).$$

Thus, we may assume that

$$\| \tilde{z}_n \|^p \to h, \qquad L(\tilde{z}_n) \to h. \tag{4.11}$$

Assume that $h > 0$; by the definition of $S_{\alpha,\beta}$ and (4.11), one can get

$$S_{\alpha,\beta} h^{\frac{p}{p^*}} = S_{\alpha,\beta} \lim_{n\to\infty} L(\tilde{z}_n)^{\frac{p}{p^*}} \\ \leq \| \tilde{z}_n \|^p = h,$$

which implies that $h \geq (S_{\alpha,\beta})^{\frac{N}{p}}$. By (4.10) and (4.11), we have

$$c = \left(\frac{1}{p} - \frac{1}{p^*}\right) h + J_{\lambda,\mu}(z),$$

then by Lemma 4.1, we get

$$c \geq \frac{1}{N}(S_{\alpha,\beta})^{\frac{N}{p}} - C_0 \left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right) = C_{\lambda,\mu},$$

which is a contradiction. Hence $h = 0$; that is $z_n \to z$ strongly in W. □

Next, we will establish the existence of a local minimum for $J_{\lambda,\mu}$ on $N^-_{\lambda,\mu}$ to obtain a second positive solution of system (1.1). We point the following fact as a remark that will use in the next lemma.

Remark 4.4. Let $A, B > 0$, then we get

$$\sup_{t\geq 0}\left(\frac{t^p}{p}A - \frac{t^{\alpha+\beta}}{\alpha+\beta}B\right) = \frac{1}{N}A\left(\frac{A}{B}\right)^{\frac{N-p}{p}} \\ = \frac{1}{N}A\left(\frac{A}{B^{\frac{p}{p^*}}}\right)^{\frac{N}{p}}.$$

Precisely, setting

$$f(t) = \frac{t^p}{p}A - \frac{t^{\alpha+\beta}}{\alpha+\beta}B,$$

we have $f'(t) = t^{p-1}A - t^{\alpha+\beta-1}B$, the result obtains by an easy calculation.

Lemma 4.5. *There exist a nonnegative function $z \in W \setminus \{(0,0)\}$ and $\Lambda^* > 0$ such that for $(\lambda,\mu) \in \Theta_{\Lambda^*}$, we have*

$$\sup_{t\geq 0} J_{\lambda,\mu}(tz) < c_{\lambda,\mu}, \tag{4.12}$$

where $c_{\lambda,\mu}$ is the constant given in Lemma 4.3. In particular, $\theta^-_{\lambda,\mu} < c_{\lambda,\mu}$, for all $(\lambda,\mu) \in \Theta_{\Lambda^*}$.

Proof. Since $0 \in \Omega$, we know that there exists $\rho_0 > 0$ such that $B^N(0, 2\rho_0) \subset \Omega$. Now, we consider the functional $I : W \longrightarrow \mathbb{R}$ defined by

$$I(z) = \frac{1}{p} \| z \|^p - \frac{1}{\alpha+\beta} L(z),$$

for all $z \in W$, and define a cut-off function $\eta(x) \in C_0^\infty(\Omega)$ such that

$$\eta(x) = \begin{cases} 1 & |x| < \rho_0, \\ 0 & |x| > 2\rho_0, \end{cases}$$

where $0 \le \eta \le 1$ and $|\nabla \eta| \le C$. For $\varepsilon > 0$, let

$$u_\varepsilon(x) = \frac{\eta(x)}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}. \tag{4.13}$$

Step 1. We show that

$$\sup_{t\ge 0} I_{\lambda,\mu}(tz_0) \le \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{p}} + O(\varepsilon^{\frac{N-p}{p}}).$$

From [13] (Lemma 4.3), we have

$$\left(\int_\Omega |u_\varepsilon|^{p^*} dx\right)^{\frac{p}{p^*}} = \varepsilon^{-\frac{N-p}{p}} |U|^p_{L^{p^*}(\mathbb{R}^N)} + O(\varepsilon),$$

$$\int_\Omega |\nabla u_\varepsilon|^p dx = \varepsilon^{-\frac{N-p}{p}} |\nabla U|^p_{L^p(\mathbb{R}^N)} + O(1),$$

$$\frac{\displaystyle\int_\Omega |\nabla u_\varepsilon|^p dx}{\left(\displaystyle\int_\Omega |u_\varepsilon|^{p^*} dx\right)^{\frac{p}{p^*}}} = S + O\left(\varepsilon^{\frac{N-p}{p}}\right), \tag{4.14}$$

where

$$U(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{-\frac{N-p}{p}} \in W^{1,p}\left(\mathbb{R}^N\right).$$

Set $u_0 = \sqrt[p]{\alpha} u_\varepsilon$, $v_0 = \sqrt[p]{\beta} u_\varepsilon$ and $z_0 \in W$. Then from Remark 4.4, (2.4) and (4.14), we conclude that

$$\begin{aligned}
\sup_{t\ge 0} I_{\lambda,\mu}(tz_0) &\le \frac{1}{N}\left(\frac{(\alpha+\beta)\displaystyle\int_\Omega |\nabla u_\varepsilon|^p dx}{\left(\alpha^{\frac{\alpha}{p}}\beta^{\frac{\beta}{p}}\displaystyle\int_\Omega |u_\varepsilon|^{p^*} dx\right)^{\frac{p}{p^*}}}\right)^{\frac{N}{p}} \\
&\le \frac{1}{N}\left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right)^{\frac{N}{p}} \left(S + O\left(\varepsilon^{\frac{N-p}{p}}\right)\right)^{\frac{N}{p}} \\
&= \frac{1}{N}\left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right)^{\frac{N}{p}} \left(S^{\frac{N}{p}} + O\left(\varepsilon^{\frac{N-p}{p}}\right)\right) \\
&= \frac{1}{N}\left(S_{\alpha,\beta}\right)^{\frac{N}{p}} + O\left(\varepsilon^{\frac{N-p}{p}}\right).
\end{aligned}$$

Step 2. We claim that if we set $\varepsilon = \left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)^{\frac{p}{N-p}}$, then there exists $\Lambda^* > 0$, such that for $(\lambda,\mu) \in \Theta_{\Lambda^*}$ we have $\sup\limits_{t\geq 0} J_{\lambda,\mu}(tz) < C_{\lambda,\mu}$.

Let C_0 be the positive constant given in Lemma 4.1. We can choose $\delta_1 > 0$ such that for all $(\lambda,\mu) \in \Theta_{\delta_1}$, we have

$$C_{\lambda,\mu} = \frac{1}{N}\left(S_{\alpha,\beta}\right)^{\frac{N}{p}} - C_0\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right) > 0.$$

Using the definition of $J_{\lambda,\mu}$ and z_0, we get

$$J_{\lambda,\mu}(tz_0) \leq \frac{t^p}{p}\| z_0 \|^p = \frac{\alpha+\beta}{p} t^p |\nabla u_\varepsilon|^p_{L^p(\mathbb{R}^N)}, \quad \forall t \geq 0,\ \lambda,\mu > 0,$$

which implies that there exists $t_0 \in (0,1)$ satisfying

$$\sup_{0\leq t\leq t_0} J_{\lambda,\mu}(tz_0) < C_{\lambda,\mu}, \quad \forall(\lambda,\mu) \in \Theta_{\delta_1}.$$

Using the definition of $J_{\lambda,\mu}$ and z_0 and by $\alpha,\beta > 1$, we have

$$\begin{aligned}
\sup_{t\geq t_0} J_{\lambda,\mu}(tz_0) &= \sup_{t\geq t_0}\left(I(tz_0) - \frac{t^q}{q}K_{\lambda,\mu}(z_0)\right) \\
&\leq \frac{1}{N}\left(S_{\alpha,\beta}\right)^{\frac{N}{p}} + 0\left(\varepsilon^{\frac{N-p}{p}}\right) - \frac{{t_0}^q}{q}\left(\alpha^{\frac{q}{p}}\lambda + \beta^{\frac{q}{p}}\mu\right)\int_{B^N(0,\rho_0)} |u_\varepsilon|^q dx \\
&\leq \frac{1}{N}\left(S_{\alpha,\beta}\right)^{\frac{N}{p}} + 0\left(\varepsilon^{\frac{N-p}{p}}\right) - \frac{{t_0}^q}{q}\left(\lambda+\mu\right)\int_{B^N(0,\rho_0)} |u_\varepsilon|^q dx.
\end{aligned}$$

Let $0 < \varepsilon \leq {\rho_0}^{\frac{p}{p-1}}$, by (4.13) we have

$$\begin{aligned}
\int_{B^N(0,\rho_0)} |u_\varepsilon|^q dx &= \int_{B^N(0,\rho_0)} \frac{1}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{q\frac{N-p}{p}}} dx \\
&\geq \int_{B^N(0,\rho_0)} \frac{1}{\left(2\rho_0^{\frac{p}{p-1}}\right)^{q\frac{N-p}{p}}} dx \\
&= C_1 \\
&= C_1(N,p,q,\rho_0).
\end{aligned}$$

Then for all $\varepsilon = \left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)^{\frac{p}{N-p}} \in (0, {\rho_0}^{\frac{p}{p-1}})$, one can get

$$\sup_{t\geq t_0} J_{\lambda,\mu}(tz_0) \leq \frac{1}{N}\left(S_{\alpha,\beta}\right)^{\frac{N}{p}} + 0\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right) - \frac{{t_0}^q}{q}C_1(\lambda+\mu).$$

Hence, we can choose $\delta_2 > 0$ such that for all $(\lambda,\mu) \in \Theta_{\delta_2}$, we have

$$0\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right) - \frac{{t_0}^q}{q}C_1(\lambda+\mu) < C_0\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right).$$

If we set $\Lambda^* = \min\left\{\delta_1, \rho_0{}^{\frac{N-p}{p-1}}, \delta_2\right\}$ and $\varepsilon = \left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)^{\frac{p}{N-p}}$, then for $(\lambda, \mu) \in \Theta_{\Lambda^*}$, we have

$$\sup_{t \geq 0} J_{\lambda,\mu}(tz_0) < C_{\lambda,\mu}. \tag{4.15}$$

Step 3. We prove that $\theta^-_{\lambda,\mu} < C_{\lambda,\mu}$ for all $(\lambda, \mu) \in \Theta_{\Lambda^*}$.

By the definition of z_0 and u_ε, we have

$$L(z_0) > 0, \qquad K_{\lambda,\mu}(z_0) > 0.$$

Combining this with Lemma 2.5(ii), from the definition of $\theta^-_{\lambda,\mu}$, we obtain that there exists $t_0 > 0$ such that $t_0 z_0 \in N^-_{\lambda,\mu}$ and

$$\theta^-_{\lambda,\mu} \leq J_{\lambda,\mu}(tz_0) \leq \sup_{t \geq 0} J_{\lambda,\mu}(tz_0) < C_{\lambda,\mu},$$

for all $(\lambda, \mu) \in \Theta_{\Lambda^*}$. □

Theorem 4.6. *If $(\lambda, \mu) \in \Theta_{\Lambda_2}$, then $J_{\lambda,\mu}$ has a minimizer $z_0{}^-$ in $N^-_{\lambda,\mu}$ which satisfies the following*

(i) $J_{\lambda,\mu}(z_0^-) = \theta^-_{\lambda,\mu}$;

(ii) z_0^- *is a positive solution of* (1.1),

where $\Lambda_2 = \min\{\Lambda^*, \Lambda_0\}$, Λ^* is the same as in Lemma 4.5.

Proof. If $(\lambda, \mu) \in \Theta_{\Lambda_0}$, then by Proposition 3.3, there exists a $(PS)_{\theta^-_{\lambda,\mu}}$-sequence $\{z_n\} \subset N^-_{\lambda,\mu}$ in W for $J_{\lambda,\mu}$. From Lemmas 4.3 and 4.5 and Theorem 2.4 (ii), for $(\lambda, \mu) \in \Theta_{\Lambda^*}$, $J_{\lambda,\mu}$ satisfies $(PS)_{\theta^-_{\lambda,\mu}}$ condition and $\theta^-_{\lambda,\mu} \in (0, C_{\lambda,\mu})$. By Lemma 2.1 and from coercivity of $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$, we get that $\{z_n\}$ is bounded in W. Therefore, there exists a subsequence still denoted by $\{z_n\}$ and a nontrivial solution $z_0^- \in N^-_{\lambda,\mu}$ such that $z_n \rightharpoonup z_0^-$ weakly in W. Finally by the same arguments as in the proof of Theorem 3.4, for all $(\lambda, \mu) \in \Theta_{\Lambda_2}$, we have that z_0^- is a positive solution of (1.1). □

Proof of Theorem 1.2. By Theorems 3.4 and 4.6, we obtain that for all $\lambda, \mu > 0$ and $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Lambda_2 < \Lambda_1$ (or $(\lambda, \mu) \in \Theta_{\Lambda_2}$), (1.1) has two positive solutions z_0^+, z_0^- with $z_0^\pm \in N^\pm_{\lambda,\mu}$. Since $N^+_{\lambda,\mu} \cap N^-_{\lambda,\mu} = \emptyset$, this implies that z_0^+ and z_0^- are distinct. This completes the proof. □

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[1] Department of Basic Sciences, Babol Noushirvani University of Technology, 47148-71167, Babol, Iran.
E-mail address: s.khademloo@nit.ac.ir

[2] Department of Basic Sciences, Babol Noushirvani University of Technology, 47148-71167, Babol, Iran.
E-mail address: s.khanjany@gmail.com