Sahand Communications in Mathematical Analysis (SCMA) Vol. 12 No. 1 (2018), 113-120

http://scma.maragheh.ac.ir

DOI: 10.22130/scma.2018.78500.364

# A Certain Class of Character Module Homomorphisms on Normed Algebras

Ali Reza Khoddami

Abstract. For two normed algebras $A$ and $B$ with the character space $\triangle(B) \neq \emptyset$ and a left $B-$module $X$, a certain class of bounded linear maps from $A$ into $X$ is introduced. We set $CMH_B(A,X)$ as the set of all non-zero $B-$character module homomorphisms from $A$ into $X$. In the case where $\triangle(B) = \{\varphi\}$ then $CMH_B(A,X)\bigcup\{0\}$ is a closed subspace of $L(A,X)$ of all bounded linear operators from $A$ into $X$. We define an equivalence relation on $CMH_B(A,X)$ and use it to show that $CMH_B(A,X)\bigcup\{0\}$ is a union of closed subspaces of $L(A,X)$. Also some basic results and some hereditary properties are presented. Finally some relations between $\varphi-$amenable Banach algebras and character module homomorphisms are examined.

## 1. Introduction and Preliminaries

Let $A$ be a normed algebra. Then a character on $A$ is a bounded linear functional $\varphi : A \longrightarrow \mathbb{C}$ such that $\varphi(ac) = \varphi(a)\varphi(c)$ for all $a, c \in A$. The set of all non-zero characters on the normed algebra $A$ is denoted by $\triangle(A)$. Also $\triangle(A)\bigcup\{0\}$ is called the character space of $A$.

Let $A$ be a Banach algebra. The second dual of $A$, which is denoted by $A^{**}$, is a Banach algebra with respect to the first and the second Arens products $\Box$ and $\Diamond$ respectively, which are defined as follows. For $a, b \in A$, $f \in A^*$ and $m, n \in A^{**}$,

$$\langle m\Box n, f\rangle = \langle m, n\cdot f\rangle, \quad \langle n\cdot f, a\rangle = \langle n, f\cdot a\rangle, \quad \langle f\cdot a, b\rangle = \langle f, ab\rangle,$$

and

$$\langle m\Diamond n, f\rangle = \langle n, f\cdot m\rangle, \quad \langle f\cdot m, a\rangle = \langle m, a\cdot f\rangle, \quad \langle a\cdot f, b\rangle = \langle f, ba\rangle.$$

2010 *Mathematics Subject Classification.* Primary: 46H05, 46H10; Secondary: 46B28, 46A32.

*Key words and phrases.* Character space, Character module homomorphism, Arens products, $\varphi-$amenability, $\varphi-$contractibility.

Received: 31 December 2017, Accepted: 11 April 2018.

A new version of amenability which is related to characters was introduced and investigated by E. Kaniuth and A.T.-M. Lau and J. Pym in [2]. Also M.S. Monfared independently studied this concept in [8].

Let $A$ be a Banach algebra and let $\varphi \in \triangle(A)$. Then $A$ is said to be $\varphi-$ amenable if there exists an $m \in A^{**}$ such that $\langle m, \varphi \rangle = 1$ and $\langle m, f \cdot a \rangle = \varphi(a)\langle m, f \rangle$ for all $a \in A$ and $f \in A^*$. Any such $m$ is called a $\varphi-$mean.

A Banach algebra $A$ is said to be $\varphi-$ contractible if there exists an $u \in A$ such that $\varphi(u) = 1$ and $au = \varphi(a)u$ for all $a \in A$. The concept of $\varphi-$contractibility of Banach algebras was introduced by Z. Hu, M.S. Monfared and T. Traynor [1].

Suppose that $V$ is a non-zero normed vector space and $f \in V^*$ (the dual space of $V$) is a non-zero element such that $\|f\| \leq 1$. Define $a \cdot c = f(a)c$ for all $a, c \in V$. Then,the pair $(V, \cdot)$ which we denote it by $V_f$ is an associative normed algebra. One can easily verify that $\triangle(V_f) = \{f\}$. Some basic properties of $V_f$ such as Arens regularity, amenability, weak amenability, $n-$weak amenability are investigated in [7]. Also strongly zero-product, strongly Jordan zero-product, strongly Lie zero-product preserving maps on $V_f$ are investigated in [5, 4, 3, 6].

In this article we introduce a certain class of operators from a normed algebra into a left module. Some basic results and also some hereditary properties concerning them are investigated in Sections 2 and 3 respectively. Finally some relations between $\varphi-$amenable Banach algebras and character module homomorphisms are examined in Section 4.

## 2. Main Results

In this section let $A$ and $B$ be two normed algebras and let $\triangle(B) \neq \emptyset$. Also let $X$ be a left $B-$module. So $X^*$ and $X^{**}$ with the following operations are right and left $B-$modules respectively.

$$\langle g \cdot b, x \rangle = \langle g, b \cdot x \rangle,$$
$$\langle b \cdot \Phi, g \rangle = \langle \Phi, g \cdot b \rangle,$$

for all $b \in B, x \in X, g \in X^*, \Phi \in X^{**}$.

Also one can easily verify that $X^{**}$ with the following operations is a left $B^{**}-$module.

$$\langle n \cdot \Phi, g \rangle = \langle n, \Phi \cdot g \rangle,$$
$$\langle \Phi \cdot g, b \rangle = \langle \Phi, g \cdot b \rangle,$$
$$\langle g \cdot b, x \rangle = \langle g, b \cdot x \rangle,$$

for all $b \in B, n \in B^{**}, x \in X, g \in X^*, \Phi \in X^{**}$.

**Definition 2.1.** Let $A$ and $B$ be two normed algebras and let $\triangle(B) \neq \emptyset$. Also let $X$ be a left $B-$module. We say that a bounded linear map

$T : A \longrightarrow X$ is a $B-$character module homomorphism if there exists a $\varphi \in \triangle(B)$ such that $T^*(g \cdot b) = \varphi(b)T^*(g)$ for all $b \in B$ and $g \in X^*$.

**Remark 2.2.** Note that in the case $X = B$, since $B$ is a left $B-$module, so a bounded linear map $T : A \longrightarrow B$ is a $B-$character module homomorphism if and only if there exists a $\varphi \in \triangle(B)$ such that $T^*(g \cdot b) = \varphi(b)T^*(g)$ for all $b \in B$ and $g \in B^*$.

**Example 2.3.** Let $V$ be a non-zero normed vector space and $0 \neq f \in V^*$ such that $\|f\| \leq 1$. Also let $A$ be an arbitrary normed algebra. Then every bounded linear map $T : A \longrightarrow V_f$ is a $V_f-$character module homomorphism. Indeed, for $f \in \triangle(V_f) = \{f\}$ we have,

$$T^*(g \cdot b) = T^*(f(b)g) = f(b)T^*(g), \quad (g \in V_f^*, b \in V_f).$$

**Proposition 2.4.** *Let $A$ and $B$ be two normed algebras and let $X$ be a left $B-$module. Also let $T : A \longrightarrow X$ be a $B-$character module homomorphism. Then,*

(i) *$T(A)$ is a left $B-$module such that for some $\varphi \in \triangle(B)$, $b \cdot T(a) = \varphi(b)T(a)$ for all $b \in B$ and $a \in A$.*
(ii) *If $T$ is surjective then for some $\varphi \in \triangle(B)$, $b \cdot x = \varphi(b)x$ for all $b \in B$ and $x \in X$. Moreover $X^* \cdot \ker(\varphi) = 0$.*

*Proof.* (i) Let $T : A \longrightarrow X$ be a $B-$character module homomorphism. Then there exists a $\varphi \in \triangle(B)$ such that

$$T^*(g \cdot b) = \varphi(b)T^*(g), \tag{2.1}$$

for all $b \in B$ and $g \in X^*$. So $\langle T^*(g \cdot b), a\rangle = \langle \varphi(b)T^*(g), a\rangle$ for all $a \in A, b \in B$ and $g \in X^*$. It follows that $\langle g, b \cdot T(a)\rangle = \langle g, \varphi(b)T(a)\rangle$ for all $a \in A, b \in B$ and $g \in X^*$. Hence $b \cdot T(a) = \varphi(b)T(a), a \in A, b \in B$.

(ii) The equality $b \cdot x = \varphi(b)x$, $b \in B$, $x \in X$ is an immediate consequence of part (i), because $T$ is surjective. Also the surjectivity of $T$ implies the injectivity of $T^*$. So if $b \in \ker(\varphi)$ then by (2.1) $T^*(g \cdot b) = 0$. So $g \cdot b = 0$, $b \in \ker(\varphi)$, $g \in X^*$. It follows that $X^* \cdot \ker(\varphi) = 0$.

□

**Remark 2.5.** Let $B$ be a normed algebra with $\varphi \in \triangle(B)$. Also let $X$ be a left $B-$module such that $b \cdot x = \varphi(b)x, b \in B, x \in X$. Then for each bounded linear map $T : A \longrightarrow X$ we have,

$$T^*(g \cdot b) = T^*(\varphi(b)g) = \varphi(b)T^*(g), \quad (b \in B, g \in X^*).$$

**Corollary 2.6.** *Let $A$ and $B$ be two normed algebras. Also let $T : A \longrightarrow B$ be a surjective $B-$character module homomorphism. Then there exists $\varphi \in \triangle(B)$ such that $B = B_\varphi$. Moreover $B^*B = B^*$.*

*Proof.* The equality $B = B_\varphi$ is an immediate consequence of Proposition 2.4 part (ii). Let $g \in B^*$ and $e \in B$ be an element such that $\varphi(e) = 1$. It follows that $g \cdot e = g$. So $B^* \subseteq B^*B$. Hence $B^*B = B^*$. □

## 3. Hereditary Properties

In this section we present some hereditary properties concerning character module homomorphisms.

**Theorem 3.1.** *Let $A$ and $B$ be two normed algebras and let $X$ be a left $B-$module. Also let $T : A \longrightarrow X$ be a non-zero bounded linear map. Then $T$ is a $B-$character module homomorphism if and only if $T^{**} : A^{**} \longrightarrow X^{**}$ is a $B^{**}-$(also $B-$)character module homomorphism.*

*Proof.* Let $T : A \longrightarrow X$ be a $B-$character module homomorphism. Then there exists a $\varphi \in \triangle(B)$ such that $T^*(g \cdot b) = \varphi(b)T^*(g)$ for all $b \in B, g \in X^*$. We shall show that $T^{**} : A^{**} \longrightarrow X^{**}$ is a $B^{**}-$character module homomorphism. A similar argument can be applied to show that $T^{**}$ is a $B-$character module homomorphism.
Let $\Lambda \in X^{***}$ and $n \in B^{**}$. Also let $\{g_\alpha\}_\alpha \subseteq X^*$ and $\{b_\beta\}_\beta \subseteq B$ be two nets such that $\Lambda = w^* - \lim_\alpha g_\alpha$ and $n = w^* - \lim_\beta b_\beta$. It follows that

$$\Lambda \cdot n = w^* - \lim_\alpha w^* - \lim_\beta g_\alpha \cdot b_\beta.$$

Indeed,

$$\begin{aligned}
\lim_\alpha \lim_\beta \langle g_\alpha \cdot b_\beta, \Phi\rangle &= \lim_\alpha \lim_\beta \langle \Phi \cdot g_\alpha, b_\beta\rangle \\
&= \lim_\alpha \langle n, \Phi \cdot g_\alpha\rangle = \lim_\alpha \langle n \cdot \Phi, g_\alpha\rangle \\
&= \langle n \cdot \Phi, \Lambda\rangle = \langle \Lambda, n \cdot \Phi\rangle \\
&= \langle \Lambda \cdot n, \Phi\rangle, \quad (\Phi \in X^{**}).
\end{aligned}$$

As $T^{***}$ is $w^* - w^*-$continuous so,

$$\begin{aligned}
T^{***}(\Lambda \cdot n) &= w^* - \lim_\alpha w^* - \lim_\beta T^{***}(g_\alpha \cdot b_\beta) \\
&= w^* - \lim_\alpha w^* - \lim_\beta T^*(g_\alpha \cdot b_\beta) \\
&= w^* - \lim_\alpha w^* - \lim_\beta \varphi(b_\beta) T^*(g_\alpha) \\
&= n(\varphi) T^{***}(\Lambda),
\end{aligned}$$

for all $n \in B^{**}$ and $\Lambda \in X^{***}$.

Define $\tilde{\varphi} : B^{**} \longrightarrow \mathbb{C}$ by $\tilde{\varphi}(n) = n(\varphi)$. One can easily verify that $\tilde{\varphi} \in \triangle(B^{**})$.
Also

$$T^{***}(\Lambda \cdot n) = \tilde{\varphi}(n) T^{***}(\Lambda), \quad (n \in B^{**}, \Lambda \in X^{***}).$$

For the converse, if $T^{**}: A^{**} \longrightarrow X^{**}$ is a $B^{**}-$character module homomorphism then there exists $\psi \in \triangle(B^{**})$ such that $T^{***}(\Lambda \cdot n) = \psi(n)T^{***}(\Lambda)$ for all $n \in B^{**}$ and $\Lambda \in X^{***}$. Set $\varphi = \psi\big|_B$. Clearly $\varphi$ is a multiplicative linear functional. Also $\varphi \neq 0$. Indeed, the assumption $\varphi = 0$ implies,

$$\begin{aligned}
\psi(n)T^{***}(\Lambda) &= T^{***}(\Lambda \cdot n)\\
&= w^* - \lim_{\alpha} w^* - \lim_{\beta} T^{***}(g_\alpha \cdot b_\beta)\\
&= w^* - \lim_{\alpha} w^* - \lim_{\beta} \psi(b_\beta)T^{***}(g_\alpha)\\
&= w^* - \lim_{\alpha} w^* - \lim_{\beta} \varphi(b_\beta)T^{***}(g_\alpha) = 0,
\end{aligned}$$

where, $\{g_\alpha\}_\alpha \subseteq X^*$ and $\{b_\beta\}_\beta \subseteq B$ are some nets such that $\Lambda = w^* - \lim_\alpha g_\alpha$ and $n = w^* - \lim_\beta b_\beta$.

It follows that $T^{***} = 0$. So $T = 0$, that is a contradiction. Therefore $\varphi \in \triangle(B)$.

Hence

$$\begin{aligned}
T^*(g \cdot b) &= T^{***}(g \cdot b) = \psi(b)T^{***}(g)\\
&= \varphi(b)T^*(g),
\end{aligned}$$

for all $b \in B, g \in X^*$. So $T$ is a $B-$character module homomorphism. $\square$

**Corollary 3.2.** *Let $A$ and $B$ be two normed algebras and let $X$ be a left $B-$module. Also let $T : A \longrightarrow X$ be a non-zero bounded linear map. Then $T$ is a $B-$character module homomorphism if and only if $T^{(2n)} : A^{(2n)} \longrightarrow X^{(2n)}$ is a $B^{(2n)}-$ character module homomorphism.*

**Proposition 3.3.** *Let $A, B, C$ be normed algebras and let $X$ be a left $B-$module. Also let $T : A \longrightarrow X$ be a $B-$character module homomorphism. Then for each bounded linear map $S : C \longrightarrow A$ the map $T \circ S : C \longrightarrow X$ is a $B-$character module homomorphism.*

*Proof.* As $T : A \longrightarrow X$ is a $B-$character module homomorphism so there exists $\varphi \in \triangle(B)$ such that $T^*(g \cdot b) = \varphi(b)T^*(g)$ for all $b \in B$ and $g \in X^*$. Hence

$$\begin{aligned}
(T \circ S)^*(g \cdot b) &= S^* \circ T^*(g \cdot b) = S^*(T^*(g \cdot b))\\
&= S^*(\varphi(b)T^*(g)) = \varphi(b)S^*(T^*(g))\\
&= \varphi(b)S^* \circ T^*(g)\\
&= \varphi(b)(T \circ S)^*(g),
\end{aligned}$$

for all $b \in B$ and $g \in X^*$. It follows that $T \circ S$ is a $B-$character module homomorphism. $\square$

Let $A$ and $B$ be two normed algebras and let $X$ be a left $B-$module. Set $CMH_B(A,X)$ as the set of all non-zero $B-$character module homomorphisms from $A$ into $X$.

**Proposition 3.4.** *Let $A$ and $B$ be two normed algebras and let $\triangle(B) = \{\varphi\}$. Then $CMH_B(A,X)\bigcup\{0\}$ is a closed subspace of $L(A,X)$ of all bounded linear operators from $A$ into $X$.*

*Proof.* Clearly $CMH_B(A,X)\bigcup\{0\}$ is a subspace of $L(A,X)$. We shall show that $CMH_B(A,X)\bigcup\{0\}$ is a closed subspace. For this end let

$$T \in L(A,X)\bigcap\overline{CMH_B(A,X)\bigcup\{0\}}$$

and $T \neq 0$. So there exists a sequence $\{T_n\}_n \subseteq CMH_B(A,X)$ such that $\|{T_n}^* - T^*\| = \|T_n - T\| \longrightarrow 0$.
Hence ${T_n}^*(g\cdot b) \longrightarrow T^*(g\cdot b)$. As ${T_n}^*(g\cdot b) = \varphi(b){T_n}^*(g)$ we can conclude that $\varphi(b)T^*(g) = T^*(g\cdot b), b \in B, g \in X^*$. So $T \in CMH_B(A,X)\bigcup\{0\}$. □

**Corollary 3.5.** *Let $A$ be a normed algebra and let $\triangle(A) = \{\varphi\}$. Then $CMH_A(A,A)\bigcup\{0\}$ is a closed right ideal of $L(A,A)$.*

*Proof.* The proof is an immediate consequence of Proposition 3.3 and Proposition 3.4. □

Let $A$ and $B$ be two normed algebras and let $X$ be a left $B-$module such that $CMH_B(A,X) \neq \emptyset$. If $T \in CMH_B(A,X)$ then there exists a unique element $\varphi_T \in \triangle(B)$ such that $T^*(g\cdot b) = \varphi_T(b)T^*(g)$ for all $b \in B$ and $g \in X^*$. For $T,S \in CMH_B(A,X)$ define $T \sim S$ if and only if $\varphi_T = \varphi_S$. Now we can conclude the following result.

**Proposition 3.6.** *Let $A$ and $B$ be two normed algebras and let $X$ be a left $B-$module such that $CMH_B(A,X) \neq \emptyset$. Then $\sim$ is an equivalence relation on $CMH_B(A,X)$.*

*Proof.* the proof is straightforward. □

Let $T \in CMH_B(A,X)$ and let $[T]_\sim$ be the equivalence class of $T$.

Note that in the case when $\triangle(B) = \{\varphi\}$, the set $CMH_B(A,X)\bigcup\{0\}$ is a closed subspace of $L(A,X)$. But it is not the case in general.

The following result reveals that the set $CMH_B(A,X)\bigcup\{0\}$ is the union of closed subspaces of $L(A,X)$.

**Proposition 3.7.** *Let $A$ and $B$ be two normed algebras and let $X$ be a left $B-$module such that $CMH_B(A,X) \neq \emptyset$. Then $[T]_\sim\bigcup\{0\}$ is a closed subspace of $L(A,X)$ for all $T \in CMH_B(A,X)$. Moreover*

$$CMH_B(A,X)\bigcup\{0\} = \bigcup_T \left([T]_\sim\bigcup\{0\}\right).$$

*Proof.* Clearly $[T]_\sim \bigcup\{0\}$ is a subspace of $L(A,X)$. An argument similar to the proof of Proposition 3.4 can be applied to show that $[T]_\sim \bigcup\{0\}$ is a closed subspace. As $CMH_B(A,X) = \bigcup_T [T]_\sim$ so $CMH_B(A,X) \bigcup\{0\} = \bigcup_T ([T]_\sim \bigcup\{0\})$. $\square$

## 4. The Relation Between $\varphi-$amenable Banach Algebras and Character Module Homomorphisms.

In this section we give some relations between $\varphi-$amenable, $\varphi-$contractible Banach algebras and character module homomorphisms.

**Theorem 4.1.** *Let $A$ be a reflexive $\varphi-$amenable Banach algebra. Then $CMH_A(\mathbb{C}, A) \neq \emptyset$.*

*Proof.* As $A$ is reflexive and $\varphi-$amenable Banach algebra so there exists $m \in A$ such that $\varphi(m) = 1$ and $f(am) = \varphi(a)f(m)$ for all $a \in A$ and $f \in A^*$. Define $T : \mathbb{C} \longrightarrow A$ by $T(\lambda) = \lambda m, \lambda \in \mathbb{C}$. It follows that $T^*(f) = f(m)$ for all $f \in A^*$. So

$$\begin{aligned} T^*(f \cdot a) &= f \cdot a(m) \\ &= f(am) \\ &= \varphi(a)f(m) \\ &= \varphi(a)T^*(f), \end{aligned}$$

for all $a \in A$ and $f \in A^*$. Hence $T \in CMH_A(\mathbb{C}, A)$ and $CMH_A(\mathbb{C}, A) \neq \emptyset$. $\square$

**Theorem 4.2.** *Let $A$ be a Banach algebra and let $\varphi \in \triangle(A)$. Also let $T : \mathbb{C} \longrightarrow A$ be a linear map such that $\varphi(T(1)) \neq 0$ and $T^*(f \cdot a) = \varphi(a)T^*(f)$ for all $a \in A$ and $f \in A^*$. Then $A$ is $\varphi-$amenable.*

*Proof.* Let $u = T(1)$. So $T(\lambda) = T(\lambda 1) = \lambda T(1) = \lambda u$ for all $\lambda \in \mathbb{C}$. It follows that $T^*(f) = f(u)$ for all $f \in A^*$. So the assumption $T^*(f \cdot a) = \varphi(a)T^*(f)$ implies that, $f \cdot a(u) = \varphi(a)f(u)$. Hence $f(au) = \varphi(a)f(u)$. Set $m = \frac{u}{\varphi(u)}$. So $\varphi(m) = 1$ and $f(am) = \varphi(a)f(m), a \in A, f \in A^*$. This shows that $m$ is a $\varphi-$mean. Hence $A$ is $\varphi-$amenable. $\square$

**Proposition 4.3.** *Let $A$ be a $\varphi-$contractible Banach algebra. Then $CMH_A(\mathbb{C}, A) \neq \emptyset$.*

*Proof.* Let $A$ be a $\varphi-$contractible Banach algebra. So there exists $u \in A$ such that $\varphi(u) = 1$ and $au = \varphi(a)u$ for all $a \in A$. Define $T : \mathbb{C} \longrightarrow A$ by $T(\lambda) = \lambda u$. One can easily verify that $T \in CMH_A(\mathbb{C}, A)$. So $CMH_A(\mathbb{C}, A) \neq \emptyset$. $\square$

## References

1. Z. Hu, M.S. Monfared and T. Traynor, *On character amenable Banach algebras*, Studia Math., 193 (2009), pp. 53-78.
2. E. Kaniuth, A.T.-M. Lau and J. Pym, *On character amenability of Banach algebras*, J. Math. Anal. Appl., 344 (2008), pp. 942-955.
3. A.R. Khoddami, *On maps preserving strongly zero-products*, Chamchuri J. Math., 7 (2015), pp. 16-23.
4. A.R. Khoddami, *On strongly Jordan zero-product preserving maps*, Sahand Commun. Math. Anal., 3 (2016), pp. 53-61.
5. A.R. Khoddami, *Strongly zero-product preserving maps on normed algebras induced by a bounded linear functional*, Khayyam J. Math., 1 (2015), pp. 107-114.
6. A.R. Khoddami, *The second dual of strongly zero-product preserving maps*, Bull. Iran. Math. Soc., 43 (2017), pp. 1781-1790.
7. A.R. Khoddami and H.R. Ebrahimi Vishki, *The higher duals of a Banach algebra induced by a bounded linear functional*, Bull. Math. Anal. Appl., 3 (2011), pp. 118-122.
8. M.S. Monfared, *Character amenability of Banach algebras*, Math. Proc. Camb. Phil. Soc., 144 (2008), pp. 697-706.

---

Faculty of Mathematical Sciences, Shahrood University of Technology, P. O. Box 3619995161-316, Shahrood, Iran.

*E-mail address*: khoddami.alireza@shahroodut.ac.ir