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Numerical Reckoning Fixed Points in $CAT(0)$ Spaces

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Abstract. In this paper, first we use an example to show the efficiency of M iteration process introduced by Ullah and Arshad [24] for approximating fixed points of Suzuki generalized nonexpansive mappings. Then by using M iteration process, we prove some strong and $\Delta-$convergence theorems for Suzuki generalized nonexpansive mappings in the setting of $CAT(0)$ Spaces. Our results are the extension, improvement and generalization of many known results in $CAT(0)$ spaces.

1. Introduction

The well-known Banach contraction theorem uses Picard iteration process for approximation of fixed points. Some of the other well-known iterative processes are Mann [14], Ishikawa [12], S [2], Noor [17], Abbas [1], SP [19], Moudafi [16], S* [13], CR [6], Normal-S [21], Picard Mann [15], Picard-S [11], Thakur New [23] and so on. These iteration processes are also used to approximate fixed point in CAT(0) spaces (see e.g. [9, 10, 25]). Recently Ullah and Arshad [24] introduced new iteration process known as M iteration process. They proved that M iteration process is faster than the well-known iteration processes like Picard-S and S iteration processes.

In [24], the authors developed an example of Suzuki generalized nonexpansive mapping which is not nonexpansive and use it to show the efficiency of M iteration process. They also proved some weak and strong convergence theorems for Suzuki generalized nonexpansive mappings in the setting of uniformaly convex Banach spaces.

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Motivated by above, in this paper first we develop a new example of Suzuki generalized nonexpansive mapping and compare M iteration process with Picard-S iteration process and S iteration process using numerical values. Graphic representation is also given. After this we prove some strong and Δ-convergence theorems in the setting of $CAT(0)$ spaces for the sequence generated by M iteration process.

2. Preliminaries

Let (X, d) be a metric space. A geodesic form x to y in X is a map c from closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$ and $d(c(t), c(t^{'})) = |t - t^{'}|$ for all $t, t^{'} \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a geodesic (or metric) segment joining x and y. The space (X, d) is said to be geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each x, $y \in X$, which we will denote by $[x, y]$, called the segment joining x to y.

A geodesic triangle $\Delta(x_{1,}x_{2,}x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the triangle $\Delta(x_{1,}x_{2,}x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_{1,}x_{2,}x_3) := \Delta(\bar{x}_{1,}\bar{x}_{2,}\bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be $CAT(0)$ space if all geodesic tiangles of appropriate size satisfy the following comparison axiom.

$CAT(0)$: Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \Delta$ and comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$, we have

$$d(x, y) \leq d_{E^2}(\bar{x}, \bar{y}).$$

If $x, y_{1,}y_2$ are points in $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$ then the $CAT(0)$ inequality implies

$$\text{(CN)} \qquad d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$

This is the (CN) inequality of Burhat and Tits [4].

A $CAT(0)$ space may be regarded as a metric version of Hilbert Space. Following is the extended version of parallelogram law:

$$(2.1) \quad d(z, \alpha x \oplus (1-\alpha)y)^2 \leq \alpha d(x, z)^2 + (1-\alpha)d(z, y)^2 - \alpha(1-\alpha)d(x, y)^2,$$

for any $\alpha \in [0, 1]$, $x, y \in X$.

If $\alpha = \frac{1}{2}$, then the inequality (2.1) becomes the CN inequality.

In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality. Complete $CAT(0)$ spaces are often called Hadmard space. For more on these spaces, see [3, 5].

We recall the following result of Dhompongsa and Panyanak [9].

Lemma 2.1. *For $x, y \in X$ and $\alpha \in [0,1]$, there exists a unique point $z \in [x,\, y]$ such that*

$$d(x,z) = \alpha d(x,y) \text{ and } d(y,z) = (1-\alpha)d(x,y). \tag{2.2}$$

Notation $(1-\alpha)x \oplus \alpha y$ is used for the unique point z satisfying (2.2). Note that a subset C of X is called convex if $(1-\alpha)x \oplus \alpha y \in C$ for all $x, y \in C$ and $\alpha \in [0,1]$.

Lemma 2.2. *For $x, y, z \in X$ and $\alpha \in [0,1]$, we have*

$$d(z, ax \oplus (1-\alpha)y) \le \alpha d(z,x) + (1-\alpha)d(z,y), \tag{2.3}$$

for all $z \in X$.

Let C be a nonempty closed convex subset of a $CAT(0)$ space X and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r(x, \{x_n\}) = \lim \sup_{n\to\infty} d(x_n, x).$$

The asymptotic radius of $\{x_n\}$ relative to C is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\},$$

and the asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

Proposition 2.3 ([7], Proposition 5)**.** *It is known that, in a $CAT(0)$ space, $A(C, \{x_n\})$ consists of exactly one point.*

We now recall the definitions of strong and Δ-convergance in a $CAT(0)$ space.

Definition 2.4. A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be Δ-convergence to $x \in X$ if x is the unique asymptotic center of $\{u_x\}$ for every subsequence $\{u_x\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-lim of $\{x_n\}$.

Recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r\{u_x\}$ for every subseuence $\{u_x\}$ of $\{x_n\}$. Since in a $CAT(0)$ space every regular sequence is Δ-converges, we see that, every bounded sequence in X has a Δ-convergence subsequence.

Definition 2.5. A $CAT(0)$ space X is said to satisfy the Opial′s property [9] if for each sequence $\{x_n\}$ in X, Δ-converges to $x \in X$, we have

$$\lim_{n\to\infty} \sup d(x_n, x) < \lim_{n\to\infty} \sup d(x_n, y),$$

for all $y \in X$ such that $y \neq x$.

A point p is called a fixed point of a mapping T if $T(p) = p$ and $F(T)$ represents the set of all fixed points of mapping T. Let C be a nonempty subset of a $CAT(0)$ space X.

A mapping $T : C \to C$ is called contraction if there exists $\alpha \in (0,1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y),$$

for all $x, y \in C$.

A mapping $T : C \to C$ is called nonexpansive if for all $x, y \in C$ we have,

$$d(Tx, Ty) \leq d(x, y),$$

and quasi-nonexpansive if for all $x \in C$ and $p \in F(T)$, we have

$$d(Tx, p) \leq d(x, p).$$

In 2008, Suzuki [18] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called condition (C). A mapping $T : C \to C$ is said to satisfy condition (C) if for all $x, y \in C$, we have

$$\frac{1}{2} d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq d(x, y). \tag{2.4}$$

Suzuki [22] showed that the mapping satisfying condition (C) is weaker than nonexpansiveness and stronger than quasi nonexpansiveness. The mapping satisfy condition (C) is called Suzuki generalized nonexpansive mapping.

Suzuki [22] obtained fixed point theorems and convergence theorems for Suzuki generalized nonexpansive mappings. In 2011, Phuengrattana [18] proved convergence theorems for Suzuki generalized nonexpansive mappings using the Ishikawa iteration process in uniformly convex Banach spaces and $CAT(0)$ spaces. Recently, fixed point theorems for Suzuki generalized nonexpansive mapping have been studied by a number of authors see e.g. [23] and references therein.

We now list some properties of Suzuki generalized nonexpansive mappings.

Proposition 2.6. *Let C be a nonempty subset of a $CAT(0)$ space X and $T : C \to C$ be any mapping. Then*

(i) *If T is nonexpansive then T is Suzuki generalized nonexpansive mapping.*

(ii) *If T is Suzuki generalized nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.*

(iii) *If T is Suzuki generalized nonexpansive mapping, then*

$$d(x, Ty) \leq 3d(Tx, x) + d(x, y)$$

for all $x, y \in C$ [22].

Lemma 2.7 ([8], Proposition 2.1)**.** *If C is a closed convex subset of a complete $CAT(0)$ space X and if $\{x_n\}$ is a bounded sequence in C then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.8 ([18])**.** *Every bounded sequence in a complete $CAT(0)$ space always has a Δ-convergent subsequence.*

Lemma 2.9 ([18], Proposition 3.7)**.** *Let C is a closed convex subset of a complete $CAT(0)$ space X and let $T : C \to X$ be a Suzuki generalized nonexpansive mapping. Then the conditions $\{x_n\}$ Δ-converges to x and $d(Tx_n, x_n) \to 0$ implies $x \in C$ and $Tx = x$.*

Lemma 2.10 ([22])**.** *Let T be a mapping on a subset C of a $CAT(0)$ space X with the Opial property. Assume that T is a Suzuki generalized nonexpansive mapping. If $\{x_n\}$ Δ-converges to z and*

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0, \quad \Rightarrow \quad Tz = z.$$

That is $I - T$ is demiclosed at zero.

Lemma 2.11 ([22])**.** *Let C be a weakly compact convex subset of a $CAT(0)$ space X. Let T be a mapping on C. Assume that T is a Suzuki generalized nonexpansive mapping. Then T has a fixed point.*

Lemma 2.12 ([17], Lemma 1.3)**.** *Suppose that X is a $CAT(0)$ space and $\{t_n\}$ is any real sequence such that*

$$0 < p \le t_n \le q < 1,$$

for all $n \ge 1$. Let $\{x_n\}$ and $\{y_n\}$ be any two sequences of X such that

$$\limsup_{n\to\infty} d(x_n, 0) \ \le r, \qquad \limsup_{n\to\infty} d(y_n, 0) \le r,$$

and

$$\limsup_{n\to\infty} d(t_n x_n \oplus (1 - t_n) y_n) = r,$$

hold for some $r \ge 0$. Then

$$\lim{}_{n\to\infty} d(x, y_n) = 0.$$

Let $n \ge 0$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$. Ullah and Arshad [24] introduced a new three-step iteration process known as M iteration process, defined as:

$$\left\{ \begin{array}{l} x_0 \in C; \\ z_n = (1 - \alpha_n)\, x_n + \alpha_n T x_n; \\ y_n = T z_n; \\ x_{n+1} = T y_n. \end{array} \right. \tag{2.5}$$

Following is the example of Suzuki generalized nonexpansive mapping which is not nonexpansive.

Example 2.13. Define a mapping $T : [0,1] \to [0,1]$ by

$$Tx = \begin{cases} 1-x & x \in \left[0, \frac{1}{12}\right); \\ \\ \frac{x+2}{3} & x \in \left[\frac{1}{12}, 1\right]. \end{cases}$$

We need to prove that T is Suzuki generalized nonexpansive mapping but not nonexpansive.

If $x = \frac{1}{13}$, $y = \frac{1}{12}$ we see that

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty| \\ &= \left|1 - \frac{1}{13} - \frac{25}{36}\right| \\ &= \frac{107}{468} \\ &> \frac{1}{156} \\ &= d(x, y). \end{aligned}$$

Hence T is not nonexpansive mapping.

To verify that T is Suzuki generalized nonexpansive mapping, consider the following cases:

Case I: Let $x \in \left[0, \frac{1}{12}\right)$, then

$$\frac{1}{2}d(x, Tx) = \frac{1-2x}{2} \in \left(\frac{5}{12}, \frac{1}{2}\right].$$

For $\frac{1}{2}d(x, Tx) \leq d(x, y)$, we must have $\frac{1-2x}{2} \leq y - x$, i.e., $\frac{1}{2} \leq y$, hence $y \in \left[\frac{1}{2}, 1\right]$. We have

$$d(Tx, Ty) = \left|\frac{y+2}{3} - (1-x)\right| = \left|\frac{y+3x-1}{3}\right| < \frac{1}{12},$$

and

$$d(x; y) = |x - y| > \left|\frac{1}{12} - \frac{1}{2}\right| = \frac{5}{12}.$$

Hence

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \quad \Rightarrow \quad d(Tx, Ty) \leq d(x, y).$$

Case II: Let $x \in \left[\frac{1}{12}, 1\right]$, then

$$\frac{1}{2}d(x, Tx) = \frac{1}{2}\left|\frac{x+2}{3} - x\right| = \frac{2-2x}{6} \in \left[0, \frac{11}{36}\right].$$

For $\frac{1}{2}d(x, Tx) \leq d(x, y)$, we must have $\frac{2-2x}{6} \leq |y - x|$, which gives two possibilities:

(a) Let $x < y$, then

$$\frac{2-2x}{6} \leq y - x \quad \Rightarrow \quad y \geq \frac{2+4x}{6}$$
$$\Rightarrow \quad y \in \left[\frac{7}{18}, 1\right] \subset \left[\frac{1}{12}, 1\right].$$

So

$$d(Tx, Ty) = \left|\frac{x+2}{3} - \frac{y+2}{3}\right| = \frac{1}{2}d(x,y) \leq d(x,y).$$

Hence

$$\frac{1}{2}d(x, Tx) \leq d(x,y) \quad \Rightarrow \quad d(Tx, Ty) \leq d(x,y).$$

(b) Let $x > y$, then

$$\frac{2-2x}{6} \leq x - y \quad \Rightarrow \quad y \leq x - \frac{2-2x}{6} = \frac{8x-2}{6}$$
$$\Rightarrow \quad y \in \left[-\frac{4}{18}, 1\right].$$

Since $y \in [0,1]$, so

$$y \leq \frac{8x-2}{6} \quad \Rightarrow \quad x \in \left[\frac{1}{4}, 1\right].$$

So the case is $x \in \left[\frac{1}{4}, 1\right]$ and $y \in [0,1]$.
Now $x \in \left[\frac{1}{4}, 1\right]$ and $y \in \left[\frac{1}{12}, 1\right]$ is already included in (a).
So let $x \in \left[\frac{1}{4}, 1\right]$ and $y \in \left[0, \frac{1}{12}\right)$, then

$$d(Tx, Ty) = \left|\frac{x+2}{3} - (1-y)\right|$$
$$= \left|\frac{x+3y-1}{3}\right|.$$

For convenience, first we consider $x \in \left[\frac{1}{4}, \frac{1}{2}\right]$ and $y \in \left[0, \frac{1}{12}\right)$, then $\mathrm{d}(Tx, Ty) \leq \frac{1}{12}$ and $\mathrm{d}(x,y) > \frac{2}{12}$. Hence $\mathrm{d}(Tx, Ty) \leq d(x,y)$.
Next consider $x \in \left[\frac{1}{2}, 1\right]$ and $y \in \left[0, \frac{1}{12}\right)$, then $\mathrm{d}(Tx, Ty) \leq \frac{1}{12}$ and $\mathrm{d}(x,y) > \frac{5}{12}$. Hence $\mathrm{d}(Tx, Ty) \leq d(x,y)$. So

$$\frac{1}{2}d(x, Tx) \leq d(x,y) \quad \Rightarrow \quad d(Tx, Ty) \leq d(x,y).$$

Hence T is Suzuki generalized nonexpansive mapping.

In Table 1, some of the values of the sequences generated by M, Picard-S and S iteration processes are given. We can easily see the efficiency of M iteration process. Graphic representation is given in Figure 1.

TABLE 1. Sequences generated by M, Picard-S and S iteration processes

	M	Picard-S	S
x_0	0.9	0.9	0.9
x_1	0.991688625725994	0.988888888888889	0.966666666666667
x_2	0.999514134612753	0.998895566981943	0.990060102837099
x_3	0.999980464268334	0.999896209964084	0.997197669030279
x_4	0.999999512143476	0.999990573260599	0.999236434108505
x_5	0.999999994342934	0.999999163272635	0.999796675250441
x_6	1	0.999999926951357	0.999946747538955
x_7	1	0.999999993701947	0.999986226158885
x_8	1	0.999999999462288	0.999996472072917
x_9	1	0.999999999954451	0.999999103464931
x_{10}	1	0.999999999996166	0.999999773635541

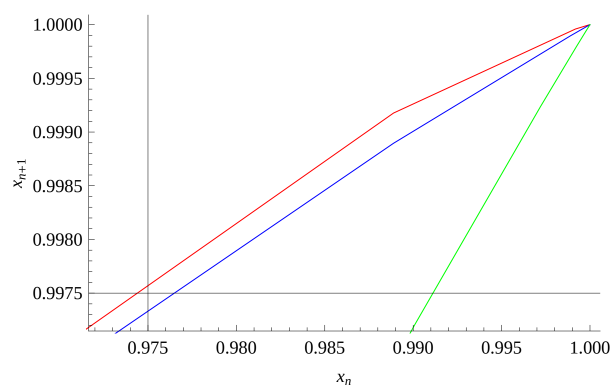

FIGURE 1. Convergence of iterative sequences generated by M (red line), Picard-S (blue line) and S (green line) iteration processes to the fixed point 1 of mapping T defined in Example 2.13.

3. Convergence Results for Suzuki Generalized Nonexpansive Mappings

In this section, we prove some strong and Δ-convergence theorems for the sequence generated by M iteration process in the setting of $CAT(0)$ spaces. M iteration process in the launguge of $CAT(0)$ spaces is given by

$$\begin{cases} x_0 \in C; \\ z_n = (1-\alpha_n)\,x_n \oplus \alpha_n T x_n; \\ y_n = T z_n; \\ x_{n+1} = T y_n. \end{cases} \tag{3.1}$$

Theorem 3.1. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X, and let $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by* (3.1)*, then $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F(T)$.*

Proof. Let $p \in F(T)$ and $z \in C$. Since T is Suzuki generalized nonexpansive mapping, so

$$\frac{1}{2}d(p, Tp) = 0 \leq d(p, z),$$

implies that $d(Tp, Tz) \leq d(p, z)$.

So by Proposition 2.6 (ii), we have

$$\begin{aligned} d(z_n, p) &= d((1-\beta_n)x_n \oplus \beta_n T x_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(Tx_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{3.2}$$

By using (3.2), we get

$$\begin{aligned} d(y_n, p) &= d(Tz_n, p) \\ &\leq d(z_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{3.3}$$

Similarly by using (3.3), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(Ty_n, p) \\ &\leq d(y_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{3.4}$$

This implies that $\{d(x_n, p)\}$ is bounded and non-increasing for all $p \in F(T)$. Hence $\lim\limits_{n\to\infty}(x_n, p)$ exists, as required. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X, and let $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (3.1) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$. Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Theorem 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} (x_n, p) = r. \tag{3.5}$$

From (3.2) and (3.5), we have

$$\limsup_{n\to\infty} d(z_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = r. \tag{3.6}$$

By Proposition 2.6 (ii) we have

$$\limsup_{n\to\infty} d(Tx_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = r. \tag{3.7}$$

On the other hand by using S Iteration Process, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(Ty_n, p) \\ &\leq d(y_n, p) \\ &= d(Tz_n, p) \\ &\leq d(z_n, p). \end{aligned}$$

Therefore

$$r \leq \liminf_{n\to\infty} d(z_n, p). \tag{3.8}$$

By (3.6) and (3.8) we get

$$\begin{aligned} r &= \lim_{n\to\infty} d(z_n, p) \\ &= \lim_{n\to\infty} d(((1-\beta_n)x_n \oplus \beta_n Tx_n), p). \end{aligned} \tag{3.9}$$

We have that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Let $p \in A(C, \{x_n\})$. By Proposition 2.6 (iii), we have

$$\begin{aligned} r(Tp, \{x_n\}) &= \limsup_{n\to\infty} d(x_n, Tp) \\ &\leq \limsup_{n\to\infty} (3d(Tx_n, x_n) + d(x_n, p)) \\ &\leq \limsup_{n\to\infty} d(x_n, p) \\ &= r(p, \{x_n\}). \end{aligned}$$

This implies that $Tp \in A(C, \{x_n\})$. So by Proposition 2.3, $A(C, \{x_n\})$ is singleton, and we have $Tp = p$. Hence $F(T) \neq \emptyset$. □

Now we are in the position to prove Δ-convergence theorem.

Theorem 3.3. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X, and let $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (3.1) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$. Then $\{x_n\}$ Δ-converges to a fixed point of T .*

Proof. Since $F(T) \neq \emptyset$, by Theorem 3.2 we have that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. We now let $w_w\{x_n\} := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $w_w\{x_n\} \subset F(T)$. Let $u \in w_w\{x_n\}$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.7 and Lemma 2.8 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n \{v_n\} = v \in C$. Since $\lim_{n\to\infty} d(v_n, Tv_n) = 0$, then $v \in F(T)$ by Lemma 2.9. We claim that $u = v$. Suppose not, since T is a Suzuki generalized nonexpansive mapping and $v \in F(T)$, $\lim_n d(x_n, v)$ exists by Theorem 3.1. Then by uniqueness of asymptotic centers,

$$\begin{aligned}
\lim_n supd(v_n, v) &< \lim_n supd(v_n, u) \\
&\leq \limsup_n d(u_n, u) \\
&< \limsup_n d(u_n, v) \\
&= \limsup_n d(x_n, v)) \\
&= \limsup_n d(v_n, v),
\end{aligned}$$

which is a contradiction, and hence $u = v \in F(T)$. To show that $\{x_n\}$ Δ-converges to a fixed point of T, it is sufices to show that $w_w\{x_n\}$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 2.7 and Lemma 2.8 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $c \in F(T)$. We can complete the proof by showing $x = v$. Suppose not, since $\{d(x_n, v)\}$ is convergent, then by the uniqueness of asymptotic centers,

$$\begin{aligned}
\lim_n supd(v_n, v) &< \limsup_n d(v_n, x) \\
&\leq \limsup_n d(x_n, x) \\
&< \limsup_n d(x_n, v) \\
&= \limsup_n d(v_n, v)
\end{aligned}$$

which is a contradiction, and hence the conclusion follows. □

Next we prove the strong convergence theorem.

Theorem 3.4. *Let C be a nonempty compact convex subset of a complete $CAT(0)$ space X, and let $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (3.1) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2.11, we have that $F(T) \neq \emptyset$ so by Theorem 3.1 we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Since C is compact, so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p for some $p \in C$. By Proposition 2.6(iii), we have

$$d(x_{n_k}, Tp) \leq 3d(Tx_{n_k}, x_{n_k}) + d(x_{n_k}, p), \text{ for all } n \geq 1.$$

Letting $k \to \infty$, we get $Tp = p$, *i.e.*, $p \in F(T)$. By Theorem 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists for every $p \in F(T)$, so x_n converges strongly to p. □

Senter and Dotson [20] introduced the notion of condition (I) as follows:

A mapping $T : C \to C$ is said to satisfy condition (I), if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf_{p\in F(T)} d(x, p)$.

Now we prove the strong convergence theorem using condition (I).

Theorem 3.5. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X, and let $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (3.1) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$ such that $F(T) \neq \emptyset$. If T satisfy condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Theorem 3.1, we have $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$ and so $\lim_{n\to\infty} d(x_n, F(T))$ exists. Assume that $\lim_{n\to\infty} d(x_n, p) = r$ for some $r \geq 0$. If $r = 0$ then the result follows. Suppose $r > 0$, from the hypothesis and condition (I), we have

$$f(d(x_n, F(T))) \leq d(Tx_n, x_n). \tag{3.10}$$

Since $F(T) \neq \emptyset$, so by Theorem 3.2, we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. So (3.10) implies that

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0. \tag{3.11}$$

Since f is nondecreasing function, so from (3.11) we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Thus, we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\} \subset F(T)$ such that

$$d(x_{n_k}, y_k) < \frac{1}{2^k}, \quad \text{for all } k \in \mathbb{N}.$$

So by using (3.4), we get

$$d(x_{n_{k+1}}, y_k) \leq d(x_{n_k}, y_k) < \frac{1}{2^k}.$$

Hence

$$\begin{aligned} d(y_{k+1}, y_k) &\leq d(y_{k+1}, x_{k+1}) + d(x_{k+1}, y_k) \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \to 0, \quad \text{as } k \to \infty. \end{aligned}$$

This shows that $\{y_k\}$ is a Cauchy sequence in $F(T)$ and so it converges to a point p. Since $F(T)$ is closed, therefore $p \in F(T)$ and then $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} d(x_n, p)$ exists, we have that $x_n \to p \in F(T)$ and the proof is complete. □

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