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Common Fixed Point Theory in Modified Intuitionistic Probabilistic Metric Spaces with Common Property (E.A.)

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Abstract. In this paper, we define the concepts of modified intuitionistic probabilistic metric spaces, the property (E.A.) and the common property (E.A.) in modified intuitionistic probabilistic metric spaces.

Then, by the common property (E.A.), we prove some common fixed point theorems in modified intuitionistic Menger probabilistic metric spaces satisfying an implicit relation.

1. Introduction and Preliminaries

Probabilistic metric spaces (abbreviated, PM spaces) have been introduced and studied in 1942 by Karl Menger in [19]. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a PM space corresponds to the situation when we do not know exactly the distance between two points, and we know only probabilities of possible values of this distance. In fact the study of such spaces received an impetus with the pioneering works of Schweizer and Sklar ([23] and [24]). The study of fixed point theorems in PM spaces is also a topic of recent interest and forms an active direction of research. Sehgal et al. [25] made the first ever effort in this direction. Since then, several authors have already studied fixed point and common fixed point theorems in PM spaces ([10, 11, 16, 20, 28]). Kutukcu et. al [15] introduced the notion of intuitionistic Menger spaces with the help of t-norms and t-conorms as a generalization of Menger space due to Menger [19]. Further, they introduced the notion of Cauchy

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sequences and found a necessary and sufficient condition for an intuitionistic Menger space to be complete. Sharma et. al [26] introduced the new concepts of subcompatibility and subsequential continuity which are respectively weaker than occasionally weak compatibility and reciprocal continuity in intuitionistic Menger space to prove a common fixed point theorem. Goudarzi [8] introduced a definition of the contraction which is a the generalization of some old definitions and he proved a result in generalized fixed point theory on intuitionistic Menger spaces. Many authors have studied intuitionistic Menger spaces (see [2, 9, 12, 17, 18, 27, 30]).
In 1986, Jungck [13] introduced the notion of the compatible maps for a pair of self-maps in a metric space. In 1991, Mishra [21] introduced the notion of compatible mappings in probabilistic metric spaces. It is seen that most of the common fixed point theorems for contraction mappings invariably require a compatibility condition besides assuming continuity of at least one of the mappings. Later on, Singh et. al [28] introduced the notion of weakly compatible mappings and proved some fixed point theorems in Menger spaces. In 2002, Aamri et. al [1] defined a property (E.A) for self-maps which contained the class of noncompatible maps. Recently, Kubiaczyk et. al [14] proved some common fixed point theorems under some strict contractive conditions for weakly compatible mappings satisfying the property (E.A) due to Aamri et. al [1]. Subsequently, there are a number of results which contained the notions of property (E.A) and common property (E.A) in Menger spaces (see [3, 6, 7, 29]).

In this paper, we first define the concept of modified intuitionistic probabilistic metric spaces (abbreviated, IPM space), then we utilize the notation of the property (E.A.) and common property (E.A) to prove some common fixed point theorems in modified intuitionistic Menger PM spaces (abbreviated, IMPM space). We show that if A, B, S and T be four self-mappings of an IMPM space $(X, \mu, \mathcal{T})$ such that the pairs (A, S) and (B, T) share the common property $(E.A.)$ and the subsets $S(X)$ and $T(X)$ of X are closed, then under certain conditions A, B, S and T have a unique common fixed point in X. Also, we show that if A, B, S and T be four self-mappings of an IMPM space $(X, \mu, \mathcal{T})$ such that the pair (A, S) (or (B, T)) satisfies the property $(E.A.)$ and $A(X) \subset T(X)$ (or $B(X) \subset S(X)$), then under certain conditions A, B, S and T have a unique common fixed point in X.

Next we shall recall some well-known definitions and results in the theory of probabilistic metric spaces which are used later in this paper.

Definition 1.1. A distance distribution function is a function $F : \mathbb{R} \to [0.1]$, that is nondecreasing and left continuous on $\mathbb{R}$, moreover, $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

The set of all the distance distribution functions ($d.d.f.$) such that $F(0) = 0$, is denoted by Δ^+. In particular for every $a \in \mathbb{R}$, ϵ_a is the ($d.d.f.$) defined by

$$\epsilon_a(x) = \begin{cases} 1 & x > a, \\ 0 & x \leq a. \end{cases}$$

The space Δ^+ is partially ordered by the usual pointwise ordering of functions and the maximal element for Δ^+ in this order is ϵ_0.

Definition 1.2. A nondistance distribution function is a function $L : \mathbb{R} \to [0, 1]$, that is nonincreasing and left continuous on $\mathbb{R}$, moreover, $\inf_{t\in\mathbb{R}} L(t) = 0$ and $\sup_{t\in\mathbb{R}} L(t) = 1$.

The family of all nondistance distribution functions ($n.d.f.$) such that $L(0) = 1$, is denoted by Γ^+. In particular for every $a \in \mathbb{R}$, ζ_a is the ($n.d.f.$) defined by

$$\zeta_a(x) = \begin{cases} 0 & x > a, \\ 1 & x \leq a. \end{cases}$$

Unfortunately in many papers, the nondistance distribution function is defined as $\inf_{t\in\mathbb{R}} L(t) = 1$ and $\sup_{t\in\mathbb{R}} L(t) = 0$, which contradicts by the definitions of supremum and infimum of a function. The space Γ^+ is partially ordered by the usual pointwise ordering of functions and the minimal element for Γ^+ in this order is ζ_0.

The functions $F_{p,q}(t)$ and $L_{p,q}(t)$ denote the degree of nearness and the degree of non-nearness between p and q with respect to t, respectively.

The collection of all pairs $(s_1, s_2) \in \Delta^+ \times \Gamma^+$ such that $s_1(t) + s_2(t) \leq 1$ for all $t \in \mathbb{R}$, will be denoted by Λ. That is,

$$\Lambda = \{(s_1, s_2) \ : \ s_1 \in \Delta^+, \ s_2 \in \Gamma^+ \text{ and } s_1(t) + s_2(t) \leq 1, \ \textit{for all } t \in \mathbb{R}\}.$$

We denote its unit by $1_\Lambda = (\varepsilon_0, \zeta_0)$. We endow the product space $\Delta^+ \times \Gamma^+$ with the following partial order:

$$(x, y) \leq (u, v) \quad \Leftrightarrow \quad x(t) \leq u(t),\ y(t) \geq v(t),$$

for all $(x, y), (u, v) \in \Delta^+ \times \Gamma^+$ and for all $t \in \mathbb{R}$.

Definition 1.3. A modified intuitionistic probabilistic metric space (abbreviated, IPM space) is an ordered pair (X, μ), where X is a nonempty set and $\mu : X \times X \to \Lambda$ ($\mu(p, q)$ is denoted by $\mu_{p,q}$), satisfies the following conditions:

(i) $\mu_{p,q}(t) = 1_\Lambda(t)$ iff $p = q$,
(ii) $\mu_{p,q}(t) = \mu_{q,p}(t)$,

(iii) $\mu_{p,q}(t) = 1_\Lambda(t)$ and $\mu_{q,r}(s) = 1_\Lambda(s)$, then $\mu_{p,r}(s+t) = 1_\Lambda(s+t)$, for every $p, q, r \in X$ and $t, s \geq 0$. In this case, $\mu_{p,q}$ is called a modified intuitionistic probabilistic metric on X.

Lemma 1.4 ([5]). *Consider the set L^* and the operation $\leq_{L^*}$ defined by*

$$L^* = \{(a_1, a_2) : a_1, a_2 \in [0,1] \quad \textit{and} \quad a_1 + a_2 \leq 1\},$$

$(a_1, a_2) \leq_{L^*} (b_1, b_2)$ iff $a_1 \leq b_1$ and $b_2 \leq a_2$, for every $(a_1, a_2), (b_1, b_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.

We denote its units by $0_{L^*} = (0,0)$ and $1_{L^*} = (1,0)$. For $z_i = (x_i, y_i) \in L^*, i = 1, \ldots, n$, if $c_i \in [0,1]$ such that

$$\sum_{i=1}^{n} c_i = 1,$$

then it is easy to see that

$$\begin{aligned} c_1(x_1, y_1) + \cdots + c_n(x_n, y_n) &= \sum_{i=1}^{n} c_i(x_i, y_i) \\ &= \left(\sum_{i=1}^{n} c_i x_i, \sum_{i=1}^{n} c_i y_i \right) \in L^*. \end{aligned}$$

Definition 1.5. A continuous triangular norm (abbreviated, continuous t-norm) is a binary operation $T : [0,1] \times [0,1] \to [0,1]$ which is continuous, commutative, associative and nondecreasing with respect to each variable and has 1 as the unit element i.e., $T(1, x) = x$, for all $x \in [0,1]$. Basic examples of continuous t-norms are $T_M(a,b) = \min\{a,b\}$ and $T_P(a,b) = ab$.

Definition 1.6. A binary operation $S : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular conorm (abbreviated, continuous t-conorm) if S is continuous commutative, associative and nondecreasing with respect to each variable and has 0 as the unit element i.e., $S(0, x) = x$, for all $x \in [0,1]$.

Let T be a t-norm. If $T^* : [0,1] \times [0,1] \to [0,1]$ is defined by $T^*(a,b) = 1 - T(1-a, 1-b)$, for all $a, b \in [0,1]$, then T^* is a t-conorm (t-conorm of T). Two typical examples of continuous t-conorms are $S_M(a,b) = \max\{a,b\}$ and $S_L(a,b) = \min\{1, a+b\}$.

Using the lattice $(L^*, \leq_{L^*})$, these definitions can straightforwardly be extended.

Definition 1.7. A triangular norm (briefly, t-norm) on L^* is a mapping $\mathcal{T} : L^* \times L^* \to L^*$ satisfying the following conditions for all $a, b, c, d \in L^*$:

(i) $\mathcal{T}(a, 1_{L^*}) = a$,
(ii) $\mathcal{T}(a, b) = \mathcal{T}(b, a)$,
(iii) $\mathcal{T}(a, \mathcal{T}(b, c)) = \mathcal{T}(\mathcal{T}(a, b), c)$,
(iv) if $a \leq_{L^*} c$ and $b \leq_{L^*} d$, then $\mathcal{T}(a, b) \leq_{L^*} \mathcal{T}(c, d)$.

Definition 1.8. A continuous t-norm $\mathcal{T}$ on L^* is called continuous t-representable iff there exist a continuous t-norm T and a continuous t-conorm S on $[0, 1]$ such that, for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in L^*$,

$$\mathcal{T}(a, b) = (T(a_1, b_1), S(a_2, b_2)).$$

Now, we define a sequence $\{\mathcal{T}^n\}$ recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x_1, \ldots, x_{n+1}) = \mathcal{T}(\mathcal{T}^{n-1}(x_1, \ldots, x_n), x_{n+1}),$$

for $n \geq 2$ and $x_i \in L^*$.

Definition 1.9. A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \to L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0, 1]$ is a decreasing mapping $N : [0, 1] \to [0, 1]$ satisfying $N(0) = 1$ and $N(1) = 0$. The standard negator on $[0, 1]$ defined as (for all $x \in [0, 1]$) $N_s(x) = 1 - x$.

Definition 1.10. A modified intuitionistic Menger PM space (briefly, IMPM space) is a triplet $(X, \mu, \mathcal{T})$, where (X, μ) is an IPM space and $\mathcal{T}$ is a continuous t-representable on L^* such that for all $x, y, z \in X$ and $t, s > 0$:

$$\mu_{x,y}(t + s) \geq_{L^*} \mathcal{T}(\mu_{x,z}(t), \mu_{z,y}(s)).$$

Remark 1.11. In an IMPM space $(X, \mu, \mathcal{T})$, if $\mu_{x,y} = (F_{x,y}, L_{x,y})$ such that $F_{x,y} \in \Delta^+$ and $L_{x,y} \in \Gamma^+$, since $F_{x,y}(.)$ is nondecreasing and $L_{x,y}(.)$ is nonincreasing for all $x, y \in X$, then by the partial order on Λ, $\mu_{x,y}(.)$ is a nondecreasing function on Λ for all $x, y \in X$.

Example 1.12. Let (X, d) be a metric space. For all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$, denote $\mathcal{T}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ and for all $x, y \in X$, let $F_{x,y}$ and $L_{x,y}$ be $(d.d.f.)$ and $(n.d.f.)$ respectively defined as follows:

$$\begin{aligned} \mu_{x,y}(t) &= (F_{x,y}(t), L_{x,y}(t)) \\ &= \left(\frac{pt^n}{pt^n + qd(x, y)}, \frac{qd(x, y)}{pt^n + qd(x, y)} \right), \end{aligned}$$

for all $p, q, n, t \in (0, \infty)$. With a simple calculation, we see that $(X, \mu, \mathcal{T})$ is an IMPM space.

Example 1.13. Let $X = \mathbb{N}$ and $\mathcal{T}(a,b) = (\max\{0, a_1 + b_1 - 1\}, a_2 + b_2 - a_2 b_2)$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$ and for all $x, y \in X$, let $F_{x,y}$ and $L_{x,y}$ be $(d.d.f.)$ and $(n.d.f.)$, respectively defined as follows:

$$\begin{aligned} \mu_{x,y}(t) &= (F_{x,y}(t), L_{x,y}(t)) \\ &= \begin{cases} \left(\dfrac{x}{y}, \dfrac{y-x}{y}\right) & x \leq y, \\ \left(\dfrac{y}{x}, \dfrac{x-y}{x}\right) & y \leq x, \end{cases} \end{aligned}$$

for all $t > 0$. Then it is easy to see that $(X, \mu, \mathcal{T})$ is an IMPM space.

Remark 1.14. Every probabilistic Menger space (X, F, T) is an IMPM space of the form $(X, \mu, \mathcal{T})$ such that $\mathcal{T} = (T, S)$ and $\mu = (F, L)$ where S is the associated t-conorm, i.e., $S(a,b) = 1 - T(1-a, 1-b)$ for any $a, b \in [0,1]$ and $L_{x,y} = 1 - F_{x,y}$ for any $x, y \in X$.

Definition 1.15. Let $(X, \mu, \mathcal{T})$ be an IMPM space. For $t > 0$, define the open ball $B(x, r, t)$ with center $x \in X$ and radius $r \in (0,1)$ is defined as

$$B(x, r, t) = \{y \in X : \mu_{x,y}(t) >_{L^*} (N_s(r), r)\},$$

Where N_s is the standard negator.

Similar to the proof of [22], we obtain the following result:

Proposition 1.16. *Let $(X, \mu, \mathcal{T})$ be an IMPM space. Define $\tau_\mu = \{A \subseteq X :$ for each $x \in A$, there exist $t > 0$ and $r \in (0,1)$ such that $B(x,r,t) \subseteq A\}$. Then τ_μ is a Hausdorff topology on X.*

Proposition 1.17. *Every open ball $B(x, r, t)$ is an open set.*

Proof. Similar to the proof of Theorem 3.2 of [22], the desired result can be archived. □

Remark 1.18. From Proposition 1.17 and Proposition 1.16, every modified intuitionistic probabilistic metric μ on X generates a topology τ_μ on X which has as a bases of the family of open sets of the form $\{B(x, r, t) : x \in X, r \in (0,1), t > 0\}$.

Definition 1.19. A sequence (x_n) in an IMPM space $(X, \mu, \mathcal{T})$ is said to be convergent to x in X, if $\mu_{x_n, x}(t) \to 1_{L^*}$, whenever $n \to \infty$ for every $t > 0$.

Definition 1.20. A sequence (x_n) in an IMPM space $(X, \mu, \mathcal{T})$ is called a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that

$$\mu_{x_n, x_m}(t) >_{L^*} (N_s(\epsilon), \epsilon),$$

for each $m, n \geq n_0$, Where N_s is the standard negator.

An IMPM space is said to be complete if and only if every Cauchy sequence is convergent.

Proposition 1.21. *The limit of a convergent sequence in an IMPM space $(X, \mu, \mathcal{T})$ is unique.*

Proof. This is trivial. □

Definition 1.22. Let f and g be two self-mappings from an IMPM space $(X, \mu, \mathcal{T})$. Then the pair (f, g) is said to be weakly compatible if they commute at their coincidence points, that is, $fx = gx$ implies that $fgx = gfx$.

Definition 1.23. Let f and g be two self-mappings from an IMPM space $(X, \mu, \mathcal{T})$. Then the pair (f, g) is said to be compatible if

$$\lim_{n\to\infty} \mu_{fgx_n,gfx_n}(t) = 1_{L^*}, \quad \forall\, t > 0,$$

whenever (x_n) is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x \in X.$$

Definition 1.24. Let f and g be two self-mappings from an IMPM space $(X, \mu, \mathcal{T})$. Then the pair (f, g) is said to be non-compatible if there exists at least one sequence (x_n) in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x \in X,$$

but

$$\lim_{n\to\infty} \mu_{fgx_n,gfx_n}(t) \neq 1_{L^*},$$

or non-existent for at least one $t > 0$.

Proposition 1.25. *If self-mappings f and g of an IMPM space $(X, \mu, \mathcal{T})$ are compatible, then they are weakly compatible.*

Proof. Let $f, g : X \to X$ be compatible mappings and $x \in X$ be a coincidence point of f and g. Let $x_n = x$, then since f and g are compatible mappings, we have $\lim_{n\to\infty} fx_n = fx = gx = \lim_{n\to\infty} gx_n$ and for all $t > 0$,

$$\lim_{n\to\infty} \mu_{fgx,gfx}(t) = \lim_{n\to\infty} \mu_{fgx_n,gfx_n}(t) = 1_{L^*},$$

so $fgx = gfx$, therefore f and g are weakly compatible. □

The converse is not true as seen in the following example.

Example 1.26. Let $(X, \mu, \mathcal{T})$ be an IMPM space, where $X = [0, 2]$ and

$$\mu_{x,y}(t) = \left(\frac{t}{t + |x - y|}, \frac{|x - y|}{t + |x - y|} \right),$$

for all $t > 0$ and $x, y \in X$. Denote $\mathcal{T}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$. Define self-maps f and g on X as follows:

$$f(x) = \begin{cases} 2 & 0 \le x \le 1, \\ \dfrac{x}{2} & 1 < x \le 2, \end{cases}, \qquad g(x) = \begin{cases} 2 & x = 1, \\ \dfrac{x+3}{5} & x \ne 1. \end{cases}$$

Then we have $g1 = f1 = 2$ and $g2 = f2 = 1$. Also $gf1 = fg1 = 1$ and $gf2 = fg2 = 2$. Thus the pair (f, g) is weakly compatible. But if $\left(x_n = 2 - \frac{1}{2n}\right)$, then $fx_n = 1 - \dfrac{1}{4n}$, $gx_n = 1 - \dfrac{1}{10n}$. Thus $fx_n \to 1$, $gx_n \to 1$. Further $gfx_n = \dfrac{4}{5} - \dfrac{1}{20n}$, $fgx_n = 2$. Now

$$\begin{aligned} \lim_{n\to\infty} \mu_{fgx_n, gfx_n}(t) &= \lim_{n\to\infty} \mu_{2, \frac{4}{5} - \frac{1}{20n}}(t) \\ &= \left(\frac{t}{t + \frac{6}{5}}, \frac{\frac{6}{5}}{t + \frac{6}{5}} \right) \\ &<_{L^*} 1_{L^*}, \end{aligned}$$

for any $t > 0$. Hence the pair (f, g) is not compatible.

Definition 1.27. Let f and g be two self-mappings of an IMPM space $(X, \mu, \mathcal{T})$. Then the pair (f, g) satisfy the property $(E.A.)$ if there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} \mu_{fx_n, u}(t) = \lim_{n\to\infty} \mu_{gx_n, u}(t) = 1_{L^*},$$

for some $u \in X$ and for all $t > 0$.

Example 1.28. Let $(X, \mu, \mathcal{T})$ be the IMPM space, where $X = \mathbb{R}$, for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$, $\mathcal{T}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ and

$$\mu_{x,y}(t) = \left(\frac{t}{t + |x - y|}, \frac{|x - y|}{t + |x - y|} \right),$$

for every $x, y \in X$ and $t > 0$. Define self-maps f and g on X as follows:

$$fx = 2x + 1, \qquad gx = x + 2.$$

Consider the sequence $\left(x_n = 1 + \dfrac{1}{n}\right)$, $n = 1, 2, \ldots$. Thus we have

$$\lim_{n\to\infty} \mu_{fx_n, 3}(t) = \lim_{n\to\infty} \mu_{gx_n, 3}(t) = 1_{L^*},$$

for every $t > 0$. Then f and g satisfy the property $(E.A.)$.

In the next example, we show that there do exist pairs of mappings which do not share the property $(E.A.)$.

Example 1.29. Let $(X, \mu, \mathcal{T})$ be the IMPM space, where $\mathcal{T}(a,b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$, $X = \mathbb{R}$ and

$$\mu_{x,y}(t) = \left(\frac{t}{t + |x-y|}, \frac{|x-y|}{t + |x-y|} \right),$$

for every $x, y \in X$ and $t > 0$. Define self-maps f and g on X as $fx = x + 1$, $gx = x + 2$. If there exists a sequence (x_n) such that

$$\lim_{n\to\infty} \mu_{fx_n,u}(t) = \lim_{n\to\infty} \mu_{gx_n,u}(t) = 1_{L^*},$$

for some $u \in X$, then

$$\lim_{n\to\infty} \mu_{fx_n,u}(t) = \lim_{n\to\infty} \mu_{x_n+1,u}(t) = \lim_{n\to\infty} \mu_{x_n,u-1}(t) = 1_{L^*},$$

and

$$\lim_{n\to\infty} \mu_{gx_n,u}(t) = \lim_{n\to\infty} \mu_{x_n+2,u}(t) = \lim_{n\to\infty} \mu_{x_n,u-2}(t) = 1_{L^*}.$$

So, $x_n \to u - 1$ and $x_n \to u - 2$ which is a contradiction by the Proposition 1.21. Hence f and g do not satisfy the property $(E.A.)$.

Definition 1.30. Two pairs (f, S) and (g, T) of self-mappings of an IMPM space $(X, \mu, \mathcal{T})$ are said to satisfy the common property $(E.A.)$ if there exist two sequences (x_n) and (y_n) in X such that

$$\lim_{n\to\infty} \mu_{fx_n,u}(t) = \lim_{n\to\infty} \mu_{Sx_n,u}(t) = \lim_{n\to\infty} \mu_{gy_n,u}(t) = \lim_{n\to\infty} \mu_{Ty_n,u}(t) = 1_{L^*},$$

for some $u \in X$ and for all $t > 0$.

Definition 1.31. Two finite families of self-mappings $(f_i)_{i=1}^m$ and $(g_k)_{k=1}^n$ of a set X are said to be pairwise commuting if:

(i) $f_i f_j = f_j f_i \quad i, j \in \{1, 2, \dots, m\}$,
(ii) $g_k g_l = g_l g_k \quad k, l \in \{1, 2, \dots, n\}$,
(iii) $f_i g_k = g_k f_i \quad i \in \{1, 2, \dots, m\}$ and $k \in \{1, 2, ..., n\}$.

Let Ψ be the set of all continuous functions $F : {L^*}^6 \to L^*$, satisfying the following conditions:

(F1) if $F(u, v, u, v, v, u) \geq_{L^*} 0_{L^*}$ or $F(u, v, v, u, u, v) \geq_{L^*} 0_{L^*}$ for all $u, v >_{L^*} (0, 1)$, then $u \geq_{L^*} v$,
(F2) $F(u, u, 1_{L^*}, 1_{L^*}, u, u) \geq_{L^*} 0_{L^*}$ implies that $u \geq_{L^*} 1_{L^*}$,

for all $u = (u_1, u_2), v = (v_1, v_2) \in L^*$.

Example 1.32. Define

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = 15t_1 - 13t_2 + 5t_3 - 7t_4 + t_5 - t_6,$$

then $F \in \Psi$.

Example 1.33. Define

$$F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1-\frac{1}{2}t_2-\frac{5}{6}t_3+\frac{1}{3}t_4-t_5+t_6,$$

then $F\in\Psi$.

Example 1.34. Define

$$F(t_1,t_2,t_3,t_4,t_5,t_6)=18t_1-16t_2+8t_3-10t_4+t_5-t_6,$$

then $F\in\Psi$.

Proposition 1.35. [4, 2.5.3] *If (X,F,T) is a probabilistic Menger space and T is continuous, then probabilistic distance function F is a lower semi continuous function of points, i.e. for every fixed point $t>0$, if $x_n\to x$ and $y_n\to y$, then*

$$\liminf_{n\to\infty}F_{x_n,y_n}(t)=F_{x,y}(t).$$

2. Main Results

The following lemma is proved to interrelate the property $(E.A.)$ with common the property $(E.A.)$ in the setting of IMPM spaces.

Lemma 2.1. *Let A,B,S and T be four self-mappings of an IMPM space $(X,\mu,\mathcal{T})$ satisfying the following conditions:*

(i) *the pair (A,S) (or (B,T)) satisfies the property $(E.A.)$,*
(ii) *$A(X)\subset T(X)$, (or $B(X)\subset S(X)$),*
(iii) *$B(y_n)$ converges for every sequence (y_n) in X whenever $T(y_n)$ converges (or $A(x_n)$ converges for every sequence (x_n) in X whenever $S(x_n)$ converges),*
(iv) *for all $x,y\in X, s>0, F\in\Psi$,*

$$F\left(\mu_{Ax,By}(s),\mu_{Sx,Ty}(s),\mu_{Ty,By}(s),\mu_{Sx,Ax}(s),\right.\\ \left.\mu_{Ax,Ty}(s),\mu_{Sx,By}(s)\right)\geq_{L^*}0_{L^*}. \tag{2.1}$$

Then the pairs (A,S) and (B,T) share the common property $(E.A.)$.

Proof. Since the pair (A,S) enjoys the property $(E.A.)$, there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=z,\quad \text{for some } z\in X.$$

Let

$$\mu_{x,y}(s)=(F_{x,y}(s),L_{x,y}(s)),$$

where $F\in\Delta^+$ and $L\in\Gamma^+$. Now, by Proposition 1.35, we have

$$\liminf_{n\to\infty}F_{Ax_n,Sx_n}(s)=1.$$

Since

$$\limsup_{n\to\infty} F_{Ax_n,Sx_n}(s) \geq \liminf_{n\to\infty} F_{Ax_n,Sx_n}(s) = 1,$$

then

$$\lim_{n\to\infty} F_{Ax_n,Sx_n}(s) = 1.$$

Let $G = 1 - L$, then $G \in \Delta^+$ and in the same way we can show that

$$\lim_{n\to\infty} G_{Ax_n,Sx_n}(s) = 1.$$

Hence

$$\lim_{n\to\infty} L_{Ax_n,Sx_n}(s) = 0,$$

therefore

$$\lim_{n\to\infty} \mu_{Ax_n,Sx_n}(s) = (1,0) = 1_{L^*}.$$

Since $A(X) \subset T(X)$, so for the sequence (x_n) there exists a sequence (y_n) in X such that $Ax_n = Ty_n$. We also have

$$\lim_{n\to\infty} \mu_{Ax_n,Ty_n}(s) = (1,0) = 1_{L^*} = \lim_{n\to\infty} \mu_{Ty_n,Sx_n}(s).$$

Now by the triangle inequality for μ and arbitrary $\epsilon > 0$ we have

$$\mu_{Ax_n,By_n}(s) \geq \mathcal{T}\left(\mu_{Ax_n,Ty_n}(\epsilon), \mu_{Ty_n,By_n}(s-\epsilon)\right). \tag{2.2}$$

Now by taking the $\liminf$ as $n \to \infty$, we have

$$\begin{aligned}
\liminf_{n\to\infty} \mu_{Ax_n,By_n}(s) &\geq \mathcal{T}\left(\liminf_{n\to\infty} \mu_{Ax_n,Ty_n}(\epsilon), \liminf_{n\to\infty} \mu_{Ty_n,By_n}(s-\epsilon)\right) \\
&= \mathcal{T}\left(1_{L^*}, \liminf_{n\to\infty} \mu_{Ty_n,By_n}(s-\epsilon)\right) \\
&= \liminf_{n\to\infty} \mu_{Ty_n,By_n}(s-\epsilon),
\end{aligned}$$

since ϵ is arbitrary, so we have

$$\liminf_{n\to\infty} \mu_{Ax_n,By_n}(s) \geq \liminf_{n\to\infty} \mu_{Ty_n,By_n}(s).$$

If we change the roles of Ax_n and Ty_n, we get

$$\liminf_{n\to\infty} \mu_{Ax_n,By_n}(s) \leq \liminf_{n\to\infty} \mu_{Ty_n,By_n}(s),$$

therefore,

$$\liminf_{n\to\infty} \mu_{Ax_n,By_n}(s) = \liminf_{n\to\infty} \mu_{Ty_n,By_n}(s).$$

Similarly, we can show that

$$\liminf_{n\to\infty} \mu_{Sx_n,By_n}(s) = \liminf_{n\to\infty} \mu_{Ty_n,By_n}(s).$$

Instead of lim inf, if we take the lim sup of (2.2), then by the same method we can show that

$$\begin{aligned}\limsup_{n\to\infty} \mu_{Ax_n,By_n}(s) &= \limsup_{n\to\infty} \mu_{Sx_n,By_n}(s)\\ &= \limsup_{n\to\infty} \mu_{Ty_n,By_n}(s).\end{aligned}$$

Now, we show that

$$\lim_{n\to\infty} \mu_{By_n,z}(s) = 1_{L^*}.$$

By using inequality (2.1), we have

$$\begin{aligned}F(\mu_{Ax_n,By_n}(s), \mu_{Sx_n,Ty_n}(s), \mu_{Ty_n,By_n}(s), \mu_{Sx_n,Ax_n}(s)\\ , \mu_{Ax_n,Ty_n}(s), \mu_{Sx_n,By_n}(s)) \geq_{L^*} 0_{L^*}.\end{aligned} \tag{2.3}$$

Now, by taking the lim inf as $n \to \infty$, we get

$$\begin{aligned}F(\liminf_{n\to\infty} \mu_{Ax_n,By_n}(s), \liminf_{n\to\infty} \mu_{Sx_n,Ty_n}(s), \liminf_{n\to\infty} \mu_{Ty_n,By_n}(s),\\ \liminf_{n\to\infty} \mu_{Sx_n,Ax_n}(s), \liminf_{n\to\infty} \mu_{Ax_n,Ty_n}(s), \liminf_{n\to\infty} \mu_{Sx_n,By_n}(s)) \geq_{L^*} 0_{L^*}.\end{aligned}$$

So, we have

$$\begin{aligned}F(\liminf_{n\to\infty} \mu_{Ax_n,By_n}(s), 1_{L^*}, \liminf_{n\to\infty} \mu_{Ax_n,By_n}(s), 1_{L^*}\\ , 1_{L^*}, \liminf_{n\to\infty} \mu_{Ax_n,By_n}(s)) \geq_{L^*} 0_{L^*}.\end{aligned}$$

Since

$$\begin{aligned}\liminf_{n\to\infty} \mu_{Ax_n,By_n}(s) &= \liminf_{n\to\infty} \mu_{Ax_n,By_n}(s)\\ &= \liminf_{n\to\infty} \mu_{Ax_n,By_n}(s),\end{aligned}$$

then, by using (F1), we get

$$\liminf_{n\to\infty} \mu_{Ax_n,By_n}(s) \geq_{L^*} 1_{L^*},$$

for all $s > 0$, so that $\liminf_{n\to\infty} F_{Ax_n,By_n}(s) = 1$. Let $\mathcal{T} = (T, S)$ where T is a continuous t-norm and S is a continuous t-conorm. Now, by triangle inequality for F and arbitrary $\epsilon > 0$, we have

$$F_{Ax_n,By_n}(s) \geq T(F_{Ax_n,z}(\epsilon), F_{z,By_n}(s-\epsilon)).$$

Now, by taking the lim inf as $n \to \infty$, we get

$$\begin{aligned}\liminf_{n\to\infty} F_{Ax_n,By_n}(s) &\geq T\left(\liminf_{n\to\infty} F_{Ax_n,z}(\epsilon), \liminf_{n\to\infty} F_{z,By_n}(s-\epsilon)\right)\\ &= T\left(1, \liminf_{n\to\infty} F_{z,By_n}(s-\epsilon)\right)\\ &= \liminf_{n\to\infty} F_{z,By_n}(s-\epsilon),\end{aligned}$$

since ϵ is arbitrary, so we have

$$\liminf_{n\to\infty} F_{Ax_n,By_n}(s) \geq \liminf_{n\to\infty} F_{z,By_n}(s).$$

If we change the roles of Ax_n and z, we get

$$\liminf_{n\to\infty} F_{Ax_n,By_n}(s) \leq \liminf_{n\to\infty} F_{z,By_n}(s),$$

therefore,

$$\liminf_{n\to\infty} F_{z,By_n}(s) = \liminf_{n\to\infty} F_{Ax_n,By_n}(s) = 1.$$

Since

$$\limsup_{n\to\infty} F_{z,By_n}(s) \geq \liminf_{n\to\infty} F_{z,By_n}(s) = 1,$$

then

$$\lim_{n\to\infty} F_{z,By_n}(s) = 1.$$

Since

$$\begin{aligned}\limsup_{n\to\infty} \mu_{Ax_n,By_n}(s) &= \limsup_{n\to\infty} \mu_{Sx_n,By_n}(s)\\ &= \limsup_{n\to\infty} \mu_{Ty_n,By_n}(s),\end{aligned}$$

if we take the lim sup of (2.3) and use (F1), then we get

$$\limsup_{n\to\infty} \mu_{Ax_n,By_n}(s) \geq_{L^*} 1_{L^*},$$

for all $s > 0$, so we get

$$\limsup_{n\to\infty} L_{Ax_n,By_n}(s) \leq 0.$$

Since

$$\liminf_{n\to\infty} L_{Ax_n,By_n}(s) \leq \limsup_{n\to\infty} L_{Ax_n,By_n}(s) \leq 0,$$

and $L \in \Gamma^+$, then we have

$$\lim_{n\to\infty} L_{Ax_n,By_n}(s) = 0.$$

Now, by triangle inequality for L, in the same way we can show that

$$\lim_{n\to\infty} L_{z,By_n}(s) = \lim_{n\to\infty} L_{Ax_n,By_n}(s),$$

so

$$\lim_{n\to\infty} L_{z,By_n}(s) = 0,$$

therefore

$$\lim_{n\to\infty} \mu_{z,By_n}(s) = (1,0) = 1_{L^*},$$

then

$$\lim_{n\to\infty} By_n = z.$$

By the same method we can show that

$$\lim_{n\to\infty} Ty_n = z,$$

which shows that the pairs (A,S) and (B,T) share the common property $(E.A.)$. □

Our next result is a common fixed point theorem via the common property $(E.A.)$.

Theorem 2.2. *Let A,B,S and T be four self-mappings of an IMPM space $(X,\mu,\mathcal{T})$ satisfying the condition (2.1). Suppose that*

(i) *the pairs (A,S) and (B,T) share the common property $(E.A.)$,*
(ii) *$S(X)$ and $T(X)$ are closed subsets of X.*

Then the pair (A,S) as well as (B,T) have a coincidence point. Moreover, A,B,S and T have a unique common fixed point in X provided that both the pairs (A,S) and (B,T) are weakly compatible.

Proof. Since the pairs (A,S) and (B,T) share the common property $(E.A.)$, there exist two sequences (x_n) and (y_n) in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z, \quad \text{for some } z\in X.$$

Since $S(X)$ is a closed subset of X, therefore

$$\lim_{n\to\infty} Sx_n = z \in S(X).$$

Also, there exists a point $u\in X$ such that $Su=z$. By the proof of the above lemma, we can show that

$$\lim_{n\to\infty} \mu_{Su,Ty_n}(s) = \lim_{n\to\infty} \mu_{Su,By_n}(s) = \lim_{n\to\infty} \mu_{By_n,Ty_n}(s) = 1_{L^*},$$

and

$$\liminf_{n\to\infty} \mu_{Au,Ty_n}(s) = \liminf_{n\to\infty} \mu_{Au,By_n}(s) = \mu_{Au,Su}(s).$$

Now we show that $\mu_{Au,Su}(s)=1_{L^*}$. By using inequality (2.1), we have

$$F(\mu_{Au,By_n}(s), \mu_{Su,Ty_n}(s), \mu_{Ty_n,By_n}(s), \mu_{Su,Au}(s) \\ , \mu_{Au,Ty_n}(s), \mu_{Su,By_n}(s)) \geq_{L^*} 0_{L^*},$$

now taking the lim inf as $n\to\infty$, we get

$$F\left(\liminf_{n\to\infty} \mu_{Au,By_n}(s), 1_{L^*}, 1_{L^*}, \mu_{Au,Su}(s), \liminf_{n\to\infty} \mu_{Au,Ty_n}(s), 1_{L^*}\right) \geq_{L^*} 0_{L^*}.$$

Using $(F1)$, we get

$$\liminf_{n\to\infty} \mu_{Au,By_n}(s) \geq_{L^*} 1_{L^*}, \quad \text{for all } s>0,$$

so that

$$\liminf_{n\to\infty} \mu_{Au,By_n}(s) = 1_{L^*}.$$

Therefore $\mu_{Au,Su}(s)=1_{L^*}$, that is $Au=z=Su$. Thus, u is a coincidence point of the pair (A,S). Since $T(X)$ is a closed subset of X, therefore

$$\lim_{n\to\infty} Ty_n = z \in T(X).$$

So, there exists a point $w \in X$ such that $Tw = z$. By the same method we can show that w is a coincidence point of the pair (B, T).

Since $Au = z = Su$ and the pair (A, S) is weakly compatible, therefore $Az = ASu = SAu = Sz$. Now we need to show that z is a common fixed point of the pair (A, S), so we show that $\mu_{Az,z}(s) = 1_{L^*}$. By using inequality (2.1), we have

$$F(\mu_{Az,Bw}(s), \mu_{Sz,Tw}(s), \mu_{Tw,Bw}(s), \mu_{Sz,Az}(s), \mu_{Az,Tw}(s), \mu_{Sz,Bw}(s)) \geq_{L^*} 0_{L^*},$$

implying thereby

$$F(\mu_{Az,z}(s), \mu_{Az,z}(s), 1_{L^*}, 1_{L^*}, \mu_{Az,z}(s), \mu_{Az,z}(s)) \geq_{L^*} 0_{L^*}.$$

Using (F2), we get $\mu_{Az,z}(s) \geq_{L^*} 1_{L^*}$, for all $s > 0$, so that $\mu_{Az,z}(s) = 1_{L^*}$, that is $Az = z$ which shows that z is a common fixed point of the pair (A, S).
Also $Bw = z = Tw$ and the pair (B, T) is weakly compatible, therefore $Bz = BTw = TBw = Tz$. By the same method we can show that z is a common fixed point of the pair (B, T). Uniqueness of the common fixed point is an easy consequence of inequality (2.1) (in view of condition (F2)). □

Corollary 2.3. *The conclusions of Theorem 2.2 remain true if the condition (ii) of Theorem 2.2 is replaced by the following condition:*
(ii') $\overline{A(X)} \subset T(X)$ *and* $\overline{B(X)} \subset S(X)$.

Proof. Since the pairs (A, S) and (B, T) share the common property $(E.A.)$, there exist two sequences (x_n) and (y_n) in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z, \quad \textit{for some } z \in X.$$

Since $\overline{B(X)} \subset S(X)$ and $\lim_{n\to\infty} By_n = z$, therefore $z \in \overline{B(X)} \subset S(X)$. So, there exists a point $u \in X$ such that $Su = z$. The rest of the proof can be completed on the lines of Theorem 2.2. □

Corollary 2.4. *The conclusions of Theorem 2.2 remain true if the conditions (ii) is replaced by the following condition:*
(ii") $A(X)$ and $B(X)$ are closed subsets of X, $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

Proof. Since $A(X)$ and $B(X)$ are closed subsets of X, so $\overline{B(X)} = B(X) \subseteq S(X)$ and $\overline{A(X)} = A(X) \subseteq T(X)$ and by Corollary 2.3 the result follows. □

Theorem 2.5. *Let A, B, S and T be four self-mappings of an IMPM space $(X, \mu, \mathcal{T})$ satisfying the condition* (2.1). *Suppose that*

(i) *the pair (A, S) (or (B, T)) satisfies the property $(E.A.)$,*
(ii) *$A(X) \subset T(X)$ (or $B(X) \subset S(X)$),*

(iii) *$B(y_n)$ converges for every sequence y_n in X whenever $T(y_n)$ converges (or $A(x_n)$ converges for every sequence x_n in X whenever $S(x_n)$ converges),*

(iv) *$S(X)$ (or $T(X)$) be a closed subset of X.*

Then the pair (A, S) as well as (B, T) have a coincidence point. Moreover, A, B, S and T have a unique common fixed point in X provided that the pairs (A, S) and (B, T) are weakly compatible.

Proof. By Lemma 2.1 and by Theorem 2.2 the result follows. □

By choosing A, B, S, and T suitably, one can deduce corollaries for a pair as well as trio of mappings. As a simple we drive the following corollary for a pair of mappings.

Corollary 2.6. *Let A and S be two self-mappings of an IMPM space $(X, \mu, \mathcal{T})$ satisfying the following conditions:*

(i) *the pair (A, S) satisfies the property $(E.A.)$ and $A(x_n)$ converges for every sequence (x_n) in X whenever $S(x_n)$ converges,*

(ii) *$S(X)$ is a closed subset of X,*

(iii) *for all $x, y \in X, s > 0$ and $F \in \Psi$,*

$$F(\mu_{Ax,Ay}(s), \mu_{Sx,Sy}(s), \mu_{Sy,Ay}(s), \mu_{Sx,Ax}(s), \mu_{Ax,Sy}(s), \mu_{Ay,Sx}(s)) \geq_{L^*} 0_{L^*}.$$

Then the pair (A, S) has a coincidence point. Moreover, A and S have a unique common fixed point in X provided that the pair (A, S) is weakly compatible.

As an application of Theorem 2.2, we can have the following result for four finite families of self-mappings. While proving this result, we utilize Definition 1.30 which is a natural extension of the commutativity condition to two finite families of mappings.

Theorem 2.7. *Let $\{A_1, A_2, \ldots, A_m\}, \{B_1, B_2, \ldots, B_p\}, \{S_1, S_2, \ldots, S_n\}$ and $\{T_1, T_2, \ldots, T_q\}$ be four finite families of self-mappings of an IMPM space $(X, \mu, \mathcal{T})$ with $A = A_1A_2 \ldots A_m$, $B = B_1B_2 \ldots B_p, S = S_1S_2 \ldots S_n$ and $T = T_1T_2 \ldots T_q$ satisfying inequality (2.1). Suppose that*

(i) *the pairs (A, S) and (B, T) share the common property $(E.A.)$,*

(ii) *$S(X)$ and $T(X)$ are closed subsets of X,*

(iii) *the pairs of families (A_i, S_k) and (B_r, T_t) commute pairwise, where $i \in \{1, \ldots, m\}$, $k \in \{1, \ldots, n\}, r \in \{1, \ldots, p\}$ and $t \in \{1, \ldots, q\}$.*

Then A_i, S_k, B_r and T_t have a unique common fixed point.

Proof. Since for all $i \in \{1, \ldots, m\}, k \in \{1, \ldots, n\}, r \in \{1, \ldots, p\}$ and $t \in \{1, \ldots, q\}$, $A_iS_k = S_kA_i$ and $B_rT_t = T_tB_r$, hence $AS = SA$ and $BT = TB$, therefore by Theorem 2.2, A, B, S and T have a unique common fixed point z in X. Now we show that z is a fixed point of A_i, B_r, S_k, T_t. To prove this, we have

$$\begin{aligned}
A(A_i z) &= (A_1A_2 \dots A_m)(A_i z) \\
&= (A_1A_2 \dots A_{m-1})(A_mA_i z) \\
&= (A_1A_2 \dots A_{m-1})(A_iA_m z) \\
&= (A_1A_2 \dots A_m - 2)(A_{m-1}A_iA_m z) \\
&= (A_1A_2 \dots A_{m-2})(A_iA_{m-1}A_m z) \\
&\ \ \vdots \\
&= A_1A_i(A_2A_3 \dots A_m z) \\
&= A_iA_1A_2A_3 \dots A_m z \\
&= A_iAz \\
&= A_i z.
\end{aligned}$$

This shows that A_iz is a fixed point of A and so $A_iz = z$. By a simillar way, we can show that A_iz and S_rz are fixed points of (A, S) and B_kz and T_tz are fixed points of (B, T). Now by uniqueness of the common fixed point we have $z = A_iz = S_rz = B_kz = T_tz$, so z is a common fixed point of A_i, B_r, S_k, T_t. □

By setting $A_1 = A_2 = \cdots = A_m = A, B_1 = B_2 = \cdots = B_p = B, S_1 = S_2 = \cdots = S_n = S$ and $T_1 = T_2 = \cdots = T_q = T$ in Theorem 2.7, we deduce the following:

Corollary 2.8. *Let A, B, S and T be four self-mappings of an IMPM space $(X, \mu, \mathcal{T})$ such that the pairs (A^m, S^n) and (B^p, T^q) share the common property (E.A.) and also satisfy the condition (for all $x, y \in X, s > 0, F \in \Psi$ and for all $m, n, p, q \geq 2$),*

$$\begin{aligned}
(2.4)\quad F(&\mu_{A^m x, B^p y}(s), \mu_{S^n x, T^q y}(s), \mu_{T^q y, B^p y}(s), \mu_{S^n x, A^m x}(s), \\
&\mu_{A^m x, T^q y}(s), \mu_{S^n x, B^p y}(s)) \geq_{L^*} 0_{L^*}.
\end{aligned}$$

where m, n, p and q are positive integers. If $S^n(X)$ and $T^q(X)$ are closed subsets of X, then A, B, S and T have a unique common fixed point provided that $AS = SA$ and $BT = TB$.

Example 2.9. Let $X = [0, 1]$. For all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$, denote $\mathcal{T}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ and let

$$\mu_{x,y}(t) = \left(\frac{t}{t + |x - y|}, \frac{|x - y|}{t + |x - y|} \right),$$

for all $x, y \in X$ and $t > 0$. By a simple calculation we see that $(X, \mu, \mathcal{T})$ is an IMPM space. Define A, S, B and T by $Ax = Bx = 1$ and

$$S(x) = T(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q}, \\ \frac{1}{3}, & x \notin [0,1] \cap \mathbb{Q}. \end{cases}$$

Also define $F(t_1, t_2, t_3, t_4, t_5, t_6) = 15t_1 - 13t_2 + 5t_3 - 7t_4 + t_5 - t_6$. For all $m, n, p, q \geq 2$ and $x, y \in X$ and $t > 0$, the inequality (2.4) holds. The remaining requirements of Corollary 2.8 can be easily verified and 1 is the unique common fixed point of A, S, B and T.
However, this implicit function F does not hold for the maps A, S, B and T in respect of Theorem 2.2. Otherwise, with $x = 0$ and $y = \frac{1}{\sqrt{2}}$, we get

$$\begin{aligned} F(\mu_{Ax,By}(st), &\mu_{Sx,Ty}(t), \mu_{Ty,By}(t), \mu_{Sx,Ax}(t), \mu_{Ax,Ty}(t), \mu_{Sx,By}(t)) \\ &= 15(1,0) - 13\left(\frac{t}{t+\frac{2}{3}}, \frac{\frac{2}{3}}{t+\frac{2}{3}}\right) + 5\left(\frac{t}{t+\frac{2}{3}}, \frac{\frac{2}{3}}{t+\frac{2}{3}}\right) - 7(1,0) \\ &\quad + \left(\frac{t}{t+\frac{2}{3}}, \frac{\frac{2}{3}}{t+\frac{2}{3}}\right) - (1,0) \\ &= \left(\frac{\frac{14}{3}}{t+\frac{2}{3}}, \frac{\frac{-14}{3}}{t+\frac{2}{3}}\right) \notin L^*. \end{aligned}$$

Thus Corollary 2.8 is a partial generalization of Theorem 2.2 and can be situationally useful.

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