Sahand Communications in Mathematical Analysis (SCMA) Vol. 14 No. 1 (2019), 199-210

http://scma.maragheh.ac.ir

DOI: 10.22130/scma.2018.72370.290

A Subclass of Analytic Functions Associated with Hypergeometric Functions

Santosh B. Joshi[1], Haridas H. Pawar[2*], and Teodor Bulboacă[3]

Abstract. In the present paper, we have established sufficient conditions for Gaussian hypergeometric functions to be in certain subclass of analytic univalent functions in the unit disc $\mathcal{U}$. Furthermore, we investigate several mapping properties of Hohlov linear operator for this subclass and also examined an integral operator acting on hypergeometric functions.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the unit disc $\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$, and are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Also, denote by $\mathcal{S}$ the subclass of $\mathcal{A}$, which consist of functions that are univalent in $\mathcal{U}$.

A function $f \in \mathcal{A}$ is said to be *starlike of order* α $(0 \le \alpha < 1)$, if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathcal{U},$$

and is said to be *convex of order* α $(0 \le \alpha < 1)$, if and only if

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathcal{U},$$

2010 *Mathematics Subject Classification.* 30C45, 30A20.

Key words and phrases. Univalent function, Starlike and convex functions, Gaussian hypergeometric function, Carlson-Shaffer operator, Coefficient estimates.

Received: 23 September 2017, Accepted: 29 April 2018.

* Corresponding author.

and we denote these classes by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively. Note that $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$, where $\mathcal{S}^*$ and $\mathcal{K}$ are the classes of *starlike and convex functions*, respectively (for details, see [12]).

The *Hadamard (or convolution) product* of the two power series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad z \in \mathcal{U},$$

is defined by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in \mathcal{U}.$$

The following class $\mathcal{D}(\alpha, \beta, \gamma)$ was defined and studied by Kulkarni [10].

Definition 1.1 ([10])**.** A function $f \in \mathcal{S}$ is said to be in the class $\mathcal{D}(\alpha, \beta, \gamma)$, if and only if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{2\gamma \left(f'(z) - \alpha\right) - \left(f'(z) - 1\right)} \right| < \beta, \ z \in \mathcal{U},$$

where $0 < \beta \leq 1$, $0 \leq \alpha < \dfrac{1}{2\gamma}$ and $\dfrac{1}{2} \leq \gamma \leq 1$.

To establish our results, we need the following lemma, which has a straightforward proof (for the detailed proof, see [10]):

Lemma 1.2 ([10])**.** *A sufficient condition for a function f defined by* (1.1) *to belongs to the class $\mathcal{D}(\alpha, \beta, \gamma)$ is that*

$$\sum_{n=2}^{\infty} \left[1 + \beta(1 - 2\gamma)\right] n a_n \leq 2\beta\gamma(1 - \alpha).$$

The *Gaussian hypergeometric function* ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n \, (b)_n}{(c)_n \, (1)_n} \, z^n, \quad z \in \mathcal{U},$$

where $a, b, c \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$ and

$$(a)_n := \begin{cases} 1, & \text{if} \quad n = 0, \\ a(a+1)(a+2)\cdots(a+n-1), & \text{if} \quad n = 1, 2, 3, \dots. \end{cases}$$

The sum of the above power series represents an analytic function in $\mathcal{U}$ and has an analytic continuation throughout the finite complex plane except at most for the cut $[1, \infty)$. We note that ${}_2F_1(a, b; c; z)$ converges for $\mathrm{Re}(c-a-b)>0$ and is connected to the *Gamma function* by the formula

$$ {}_2F_1(a,b;c;z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

Using the Gaussian hypergeometric function, let consider the functions

$$ \mathrm{G}(a,b;c;z):=z\cdot{}_2F_1(a,b;c;z),\, z\in\mathcal{U}, $$

$$ \mathrm{H}_\mu(a,b;c;z):=(1-\mu)\mathrm{G}(a,b;c;z)+\mu z\mathrm{G}'(a,b;c;z),\, z\in\mathcal{U}\quad(\mu\geq 0), $$

$$ \begin{aligned} \mathrm{J}_{\mu,\delta}(a,b;c;z):=&(1-\mu+\delta)\mathrm{G}(a,b;c;z)+(\mu-\delta)z\mathrm{G}'(a,b;c;z)\\ &+\mu\delta z^2\mathrm{G}''(a,b;c;z),\quad z\in\mathcal{U},(\mu,\delta\geq 0;\mu\geq\delta). \end{aligned} $$

These hypergeometric functions were studied extensively by various authors. We note that the mapping properties of functions $\mathrm{H}_\mu(a,b;c;z)$ and $\mathrm{J}_{\mu,\delta}(a,b;c;z)$ were investigated by Shukla and Shukla [14], Tang and Deng [18] and recently by Aouf et al. [1].

In [6], Hohlov defined the linear operator $\mathrm{I}_{a,b,c}:\mathcal{A}\to\mathcal{A}$ by the convolution product

$$ \mathrm{I}_{a,b,c}(f)(z):=[z\cdot{}_2F_1(a,b;c;z)]*f(z),\quad z\in\mathcal{U}. \tag{1.2} $$

Also, Kanas and Srivastava [8], and Srivastava and Owa [17] showed that the operator $\mathrm{I}_{a,b,c}$ is the natural extensions of the Alexander, Libera, Bernardi and Carlson-Shaffer operators.

These functions and its various generalizations have large number of applications in problems of physical science such as calculation of fields and astrophysics to quantum mechanics, in proving combinatorial identities, genarating function etc. Hypergeometric functions are also having extensive applications in statistics and probability, differential equations, number theory, geometry, topology and geometric function theory (for details see [17]).

Silverman [15] studied starlikeness and convexity properties for hypergeometric functions, and has also examined a linear operator acting on hypergeometric functions. In fact, the more generalized operators such as Dziok-Srivastava and Srivastava-Wright operators, while similar type were rather extensively studied by Dziok and Srivastava [5, 4] and Srivastava [16]. The reader can also refer to the works of Merkes and Scott [11], Carlson and Shaffer [2], Kim and Shon [9], Ruscheweyh and Singh [13], and Joshi [7] for the function ${}_2F_1(a,b;c;z)$ in connection with various subclasses of univalent functions.

In the present paper, we determine sufficient conditions for the functions $\mathrm{G}(a,b;c;z)$, $\mathrm{H}_\mu(a,b;c;z)$ and $\mathrm{J}_{\mu,\delta}(a,b;c;z)$ to be in the class $\mathcal{D}(\alpha,\beta,\gamma)$. Furthermore, we obtained the mapping properties of $\mathrm{I}_{a,b,c}$

and we considered an integral operator related to the hypergeometric function.

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that $\mu > 0$, $\delta > 0$, $\mu \geq \delta$, $0 < \beta \leq 1$, $0 \leq \alpha < \dfrac{1}{2\gamma}$ and $\dfrac{1}{2} \leq \gamma \leq 1$.

Theorem 2.1. *Let $a, b \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and let $c \in \mathbb{R}_+$, such that $c > |a| + |b| + 1$. Then, a sufficient condition for the function $\mathrm{G}(a, b; c; z)$ to be in the class $\mathcal{D}(\alpha, \beta, \gamma)$, i.e. $\mathrm{G}(a, b; c; z) \in \mathcal{D}(\alpha, \beta, \gamma)$, is that*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+\frac{|a||b|}{c-|a|-|b|-1}\right] \leq \frac{2\beta\gamma(1-\alpha)}{1+\beta(1-2\gamma)}+1. \tag{2.1}$$

Proof. Since

$$\mathrm{G}(a, b; c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n, \quad z \in \mathcal{U},$$

according to Lemma 1.2, we need to show that

$$\sum_{n=2}^{\infty} n\left[1+\beta(1-2\gamma)\right]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \leq 2\beta\gamma(1-\alpha). \tag{2.2}$$

Since $|(a)_n| \leq (|a|)_n$, then the left-hand side of (2.2) is less than or equal to

$$\sum_{n=2}^{\infty} n\left[1+\beta(1-2\gamma)\right]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} =: T_1,$$

hence

$$\begin{aligned}
T_1 \leq& \left[1+\beta(1-2\gamma)\right]\sum_{n=2}^{\infty} n\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\\
=& \left[1+\beta(1-2\gamma)\right]\left[\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}}+\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right]\\
=& \left[1+\beta(1-2\gamma)\right]\left[\frac{|a||b|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n}+\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}\right]\\
=& \left[1+\beta(1-2\gamma)\right]\left[\frac{|a||b|}{c}\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}\right.\\
&\left.+\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right]\\
=& \left[1+\beta(1-2\gamma)\right]\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[\frac{|a||b|}{(c-|a|-|b|-1)}+1\right]
\end{aligned}$$

$$-\left[1+\beta(1-2\gamma)\right].$$

This last expression is upper bounded by $2\beta\gamma(1-\alpha)$ if the assumption (2.1) holds, and this completes the proof of Theorem 2.1. □

Theorem 2.2. *Let $a, b \in \mathbb{C}^*$ and let $c \in \mathbb{R}_+$, such that $c > |a| + |b| + 2$. Then, a sufficient condition for $\mathrm{H}_\mu(a, b; c; z)$ to be in the class $\mathcal{D}(\alpha, \beta, \gamma)$, i.e. $\mathrm{H}_\mu(a, b; c; z) \in \mathcal{D}(\alpha, \beta, \gamma)$, is that*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+(1+2\mu)\frac{|a||b|}{c-|a|-|b|-1}+\mu\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}\right] \leq \frac{2\beta\gamma(1-\alpha)}{1+\beta(1-2\gamma)}+1. \tag{2.3}$$

Proof. Since the power series representation of $\mathrm{H}_\mu(a, b; c; z)$ is

$$\mathrm{H}_\mu(a, b; c; z) = z + \sum_{n=2}^{\infty}\left[1+\mu(n-1)\right]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n, \quad z \in \mathcal{U},$$

according to Lemma 1.2, we need to show that

$$\sum_{n=2}^{\infty} n\left[1+\beta(1-2\gamma)\right]\left|\left[1+\mu(n-1)\right]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \leq 2\beta\gamma(1-\alpha). \tag{2.4}$$

Since $|(a)_n| \leq (|a|)_n$, then the left-hand side of (2.4) is less than or equal to

$$\sum_{n=2}^{\infty} n\left[1+\beta(1-2\gamma)\right]\left[1+\mu(n-1)\right]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} =: T_2.$$

Hence,

$$\begin{aligned}
T_2 &\leq \left[1+\beta(1-2\gamma)\right]\sum_{n=2}^{\infty} n\left[1+\mu(n-1)\right]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= \left[1+\beta(1-2\gamma)\right]\left[\sum_{n=2}^{\infty}\left[1+\mu(n-1)\right]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}}\right. \\
&\quad \left.+\sum_{n=2}^{\infty}\left[1+\mu(n-1)\right]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right] \\
&= \left[1+\beta(1-2\gamma)\right]\left[\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right. \\
&\quad \left.+(1+2\mu)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}}+\mu\sum_{n=3}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-3}}\right]
\end{aligned}$$

$$= [1+\beta(1-2\gamma)] \left[\sum_{n=1}^{\infty} \frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} \right.$$
$$+ (1+2\mu)\frac{|a||b|}{c} \sum_{n=1}^{\infty} \frac{(|a|+1)_{n-1}(|b|+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}}$$
$$\left. +\mu \frac{|a|(|a|+1)|b|(|b|+1)}{c(c+1)} \sum_{n=2}^{\infty} \frac{(|a|+2)_{n-2}(|b|+2)_{n-2}}{(c+2)_{n-2}(1)_{n-2}} \right],$$

$$T_2 \leq [1+\beta(1-2\gamma)] \quad \left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1 \right.$$
$$+(1+2\mu)\frac{|a||b|}{c} \frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}$$
$$\left. +\mu \frac{(|a|)_2(|b|)_2}{(c)_2} \frac{\Gamma(c+2)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)} \right]$$
$$= [1+\beta(1-2\gamma)] \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left[1 + (1+2\mu)\frac{|a||b|}{(c-|a|-|b|-1)} \right.$$
$$\left. +\mu \frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} \right] - [1+\beta(1-2\gamma)].$$

If the inequality (2.3) holds, then the last expression is upper bounded by $2\beta\gamma(1-\alpha)$, and the proof of theorem is complete. □

Theorem 2.3. *Let $a,b \in \mathbb{C}^*$ and let $c \in \mathbb{R}_+$, such that $c > |a| + |b| + 3$. Then, a sufficient condition for $\mathrm{J}_{\mu,\delta}(a,b;c;z)$ to be in the class $\mathcal{D}(\alpha,\beta,\gamma)$, i.e. $\mathrm{J}_{\mu,\delta}(a,b;c;z) \in \mathcal{D}(\alpha,\beta,\gamma)$, is that*

(2.5)
$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left[1 + (1+2\mu-2\delta+4\mu\delta)\frac{|a||b|}{c-|a|-|b|-1} \right.$$
$$\left. +(\mu-\delta+5\mu\delta)\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} + \mu\delta \frac{(|a|)_3(|b|)_3}{(c-|a|-|b|-3)_3} \right]$$
$$\leq \frac{2\beta\gamma(1-\alpha)}{1+\beta(1-2\gamma)} + 1.$$

Proof. The power series representation of $\mathrm{J}_{\mu,\delta}(a,b;c;z)$ is

$$\mathrm{J}_{\mu,\delta}(a,b;c;z) = z + \sum_{n=2}^{\infty} [1+(n-1)(\mu-\delta+n\mu\delta)] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n,$$

for $z \in \mathcal{U}$ and according to Lemma 1.2, we need to show that

$$\sum_{n=2}^{\infty} n\,[1+\beta(1-2\gamma)] \left| [1+(n-1)(\mu-\delta+n\mu\delta)] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| \leq 2\beta\gamma(1-\alpha).$$

Since $|(a)_n| \leq (|a|)_n$, then the left-hand side of (2.6) is less than or equal to

$$\sum_{n=2}^{\infty} n\left[1+\beta(1-2\gamma)\right]\left[1+(n-1)(\mu-\delta+n\mu\delta)\right]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} =: T_3.$$

Hence,

$$\begin{aligned}
T_3 &\leq \left[1+\beta(1-2\gamma)\right]\sum_{n=2}^{\infty} n\left[1+(n-1)(\mu-\delta+n\mu\delta)\right]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= \left[1+\beta(1-2\gamma)\right]\left[\sum_{n=2}^{\infty} n\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right. \\
&\quad + \sum_{n=2}^{\infty} n(n-1)(\mu-\delta)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&\quad \left. + \sum_{n=2}^{\infty} n^2(n-1)(\mu\delta)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right], \\
T_3 &\leq \left[1+\beta(1-2\gamma)\right]\left[\sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right. \\
&\quad + \left[1+2\mu-2\delta+4\mu\delta\right]\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}} \\
&\quad \left. +(\mu-\delta+5\mu\delta)\sum_{n=3}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-3}} + \mu\delta\sum_{n=4}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-4}}\right] \\
&= \left[1+\beta(1-2\gamma)\right]\left[\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}\right. \\
&\quad + \left[1+2\mu-2\delta+4\mu\delta\right]\frac{|a||b|}{c}\sum_{n=1}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&\quad +(\mu-\delta+5\mu\delta)\frac{(|a|)_2(|b|)_2}{(c)_2}\sum_{n=2}^{\infty}\frac{(|a|)_{n-2}(|b|)_{n-2}}{(c)_{n-2}(1)_{n-2}} \\
&\quad \left. +\frac{(|a|)_3(|b|)_3}{(c)_3}\sum_{n=3}^{\infty}\frac{(|a|)_{n-3}(|b|)_{n-3}}{(c)_{n-3}(1)_{n-3}}\right] \\
&= \left[1+\beta(1-2\gamma)\right]\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1\right. \\
&\quad + \left[1+2\mu-2\delta+4\mu\delta\right]\frac{|a||b|}{c}\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}
\end{aligned}$$

$$+(\mu-\delta+5\mu\delta)\frac{(|a|)_2(|b|)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}$$
$$\left.+(\mu\delta)\frac{(|a|)_3(|b|)_3}{(c)_3}\frac{\Gamma(c+3)\Gamma(c-|a|-|b|-3)}{\Gamma(c-|a|)\Gamma(c-|b|)}\right],$$
$$T_3\le[1+\beta(1-2\gamma)]\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\Big[1$$
$$+[1+2\mu-2\delta+4\mu\delta]\frac{|a||b|}{(c-|a|-|b|-1)}$$
$$+(\mu-\delta+5\mu\delta)\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}$$
$$\left.+(\mu\delta)\frac{(|a|)_3(|b|)_3}{(c-|a|-|b|-3)_3}\right]-[1+\beta(1-2\gamma)].$$

This last expression is upper bounded by $2\beta\gamma(1-\alpha)$ whenever (2.5) holds, which prove our result. □

Theorem 2.4. *Let $a,b\in\mathbb{C}^*$ and let $c\in\mathbb{R}_+$, such that $c>|a|+|b|+2$. If the following inequality holds*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+\frac{3|a||b|}{c-|a|-|b|-1}+\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}\right]\le\frac{2\beta\gamma(1-\alpha)}{1+\beta(1-2\gamma)}+1, \tag{2.6}$$

then the operator $\mathrm{I}_{a,b,c}$ maps the class $\mathcal{S}^*$ to the class $\mathcal{D}(\alpha,\beta,\gamma)$, that is

$$\mathrm{I}_{a,b,c}\left(\mathcal{S}^*\right)\subset\mathcal{D}(\alpha,\beta,\gamma).$$

Proof. If $f\in\mathcal{A}$ has the form (1.1), from the definition formula (1.2) we deduce

$$\mathrm{I}_{a,b,c}(f)(z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_nz^n, z\in\mathcal{U},$$

and according to Lemma 1.2, we need to show that

$$\sum_{n=2}^{\infty}n\left[1+\beta(1-2\gamma)\right]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right||a_n|\le2\beta\gamma(1-\alpha). \tag{2.7}$$

Since $|(a)_n|\le(|a|)_n$ and using the fact that $|a_n|\le n$ if $f\in\mathcal{S}^*$ (for details, see [3]), it follows that the left-hand side of (2.7) is less than or equal to

$$\sum_{n=2}^{\infty}n^2\left[1+\beta(1-2\gamma)\right]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}=:T_4.$$

Hence,

$$
\begin{aligned}
T_4 &\le [1+\beta(1-2\gamma)] \sum_{n=2}^{\infty} n^2 \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= [1+\beta(1-2\gamma)] \left[\sum_{n=3}^{\infty} n \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}} + \sum_{n=2}^{\infty} n \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right] \\
&= [1+\beta(1-2\gamma)] \left[\sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} + 3 \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}} \right. \\
&\quad \left. + \sum_{n=3}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-3}} \right], \\
T_4 &\le [1+\beta(1-2\gamma)] \left[\sum_{n=1}^{\infty} \frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} \right. \\
&\quad + 3 \frac{|a||b|}{c} \sum_{n=1}^{\infty} \frac{(|a|+1)_{n-1}(|b|+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}} \\
&\quad \left. + \frac{(|a|)_2(|b|)_2}{(c)_n} \sum_{n=2}^{\infty} \frac{(|a|+2)_{n-2}(|b|+2)_{n-2}}{(c+2)_{n-2}(1)_{n-2}} \right] \\
&= [1+\beta(1-2\gamma)] \left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1 \right. \\
&\quad + 3 \frac{|a||b|}{c} \frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)} \\
&\quad \left. + \frac{(|a|)_2(|b|)_2}{(c)_2} \frac{\Gamma(c+2)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)} \right],
\end{aligned}
$$

and

$$
\begin{aligned}
T_4 &\le [1+\beta(1-2\gamma)] \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left[1 + 3 \frac{|a||b|}{(c-|a|-|b|-1)} \right. \\
&\quad \left. + \frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} \right] - [1+\beta(1-2\gamma)] .
\end{aligned}
$$

If the inequality (2.6) holds, this last expression is upper bounded by $2\beta\gamma(1-\alpha)$, which completes our proof. □

In the next theorem we obtained a result for the particular integral operator acting on ${}_2F_1(a,b;c;z)$, by considering the function

$$
\mathcal{G}(a,b;c;z) := \int_0^z {}_2F_1(a,b;c;t)dt.
$$

Theorem 2.5. *Let $a, b \in \mathbb{C}^*$ and let $c \in \mathbb{R}_+$, such that $c > |a| + |b|$. Then, a sufficient condition for $\mathcal{G}(a, b; c; z)$ to be in the class $\mathcal{D}(\alpha, \beta, \gamma)$, i.e. $\mathcal{G}(a, b; c; z) \in \mathcal{D}(\alpha, \beta, \gamma)$, is that*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \leq \frac{2\beta\gamma(1-\alpha)}{1+\beta(1-2\gamma)} + 1. \tag{2.8}$$

Proof. The power series expansion of $\mathcal{G}(a, b; c; z)$ is given by

$$\mathcal{G}(a, b; c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} z^n, \quad z \in \mathcal{U},$$

and according to Lemma 1.2, we need to show that

$$\sum_{n=2}^{\infty} n\left[1+\beta(1-2\gamma)\right] \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}\right| \leq 2\beta\gamma(1-\alpha). \tag{2.9}$$

Since $|(a)_n| \leq (|a|)_n$, then the left-hand side of (2.9) is less than or equal to

$$\sum_{n=2}^{\infty} n\left[1+\beta(1-2\gamma)\right] \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_n} =: T_5,$$

hence

$$\begin{aligned}
T_5 &\leq \left[1+\beta(1-2\gamma)\right] \sum_{n=2}^{\infty} n \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_n} \\
&= \left[1+\beta(1-2\gamma)\right] \sum_{n=2}^{\infty} \frac{n}{n} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= \left[1+\beta(1-2\gamma)\right] \sum_{n=1}^{\infty} \frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} \\
&= \left[1+\beta(1-2\gamma)\right] \left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1\right] \\
&= \left[1+\beta(1-2\gamma)\right] \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} - \left[1+\beta(1-2\gamma)\right].
\end{aligned}$$

Using the fact that this last expression is upper bounded by $2\beta\gamma(1-\alpha)$ if the assumption (2.8) holds, which complete the proof. □

3. Conclusion

We have obtained the sufficient conditions for the functions $\mathrm{G}(a, b; c; z)$, $\mathrm{H}_\mu(a, b; c; z)$ and $\mathrm{J}_{\mu,\delta}(a, b; c; z)$ to be in the class $\mathcal{D}(\alpha, \beta, \gamma)$. More importantly, we obtained the mapping properties of $\mathrm{I}_{a,b,c}$ and a result for the particular integral operator acting on ${}_2F_1(a, b; c; z)$. The results derived in this paper are universal in character and expected to find certain applications in the theory of special functions.

Acknowledgment. The authors wish to express their sincere thanks to the referees of this paper for several useful comments and suggestions.

References

1. M.K. Aouf, A.O. Mostafa, and H.M. Zayed, *Some constraints of hypergeometric functions belong to certain subclasses of analytic functions*, J. Egyptian Math. Soc., 24 (2016), pp. 1-6.
2. B.C. Carlson and D.B. Shaffer, *Starlike and prestarlike hypergeometric functions*, J. Math. Anal. Appl., 15 (1984), pp. 737-745.
3. P.L. Duren, *Univalent functions*, Springer-Verlag, New York, 1983.
4. J. Dziok and H.M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transforms Spec. Funct., 14 (2003), pp. 7-18.
5. J. Dziok and H.M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, App. Math. Comput., 103 (1999), pp. 1-13.
6. Y.E. Hohlov, *Operators and operations in the class of univalent functions*, Izv. Vyss. Ucebn. Zaved. Mat. (in Russian), 10 (1978) 83-89.
7. S.S. Joshi, *A certain subclass of analytic functions associated with fractional derivative operator*, Tamsui Oxf. J. Inf. Math. Sci., 24 (2008), pp. 201-214.
8. S. Kanas and H.M. Srivastava, *Linear operators associated with k-uniformly convex functions*, Integral Transforms Spec. Funct., 9 (2000), pp. 121-132.
9. J.A. Kim and K.H. Shon, *Certain properties for convolutions involving hypergeometric functions*, Int. J. Math. Math. Sci., 17 (2003), pp. 1083-1091.
10. S.R. Kulkarni, *Some problems connected with univalent functions*, Ph.D. Thesis. Shivaji University, Kolhapur, 1981.
11. E. Merkes and B.T. Scott, *Starlike hypergeometric functions*, Proc. Amer. Math. Soc., 12 (1961), pp. 885-888.
12. M.S. Robertson, *On the theory of univalent functions*, Ann. Math., 37 (1936), pp. 374-408.
13. S. Ruscheweyh and V. Singh, *On the order of starlikeness of hypergeometric functions*, J. Math. Anal. Appl., 113 (1986), pp. 1-11.
14. N. Shukla and P. Shukla, *Mapping properties of analytic function defined by hypergeometric functions*, Soochow J. Math., 25 (1999), pp. 19-36.
15. H. Silverman, *Starlike and convexity properties for hypergeometric functions*, J. Math. Anal. Appl., 172 (1993), pp. 574-581.
16. H.M. Srivastava, *Some fox-wright generalized hypergeometric functions and associated families of convolution operators*, Appl. Anal. Discrete Math., 1 (2007), pp. 56-71.
17. H.M. Srivastava and S. Owa (eds.), *Current topics in analytic function theory*, World Sci. Publ., 1992.
18. H. Tang and G.T. Deng, *Subordination and superordination preserving properties for a family of integral operators involving the noor integral operator*, J. Egyptian Math. Soc., 22 (2014), pp. 352-361.

[1]Department of Mathematics, Walchand College of Engineering, Sangli 416415, India.
E-mail address: joshisb@hotmail.com

[2]Department of Mathematics, Sveri's College of Engineering, Pandharpur 413304, India.
E-mail address: haridas_pawar007@yahoo.co.in

[3]Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania.
E-mail address: bulboaca@math.ubbcluj.ro