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Admissible Vectors of a Covariant Representation of a Dynamical System

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Abstract. In this paper, we introduce admissible vectors of covariant representations of a dynamical system which are extensions of the usual ones, and compare them with each other. Also, we give some sufficient conditions for a vector to be admissible vector of a covariant pair of a dynamical system. In addition, we show the existence of Parseval frames for some special subspaces of $L^2(G)$ related to a uniform lattice of G.

1. Introduction

The extended coefficients of a square integrable unitary representation of a locally compact group, called *wavelet* or *voice transform* [3], are important tools for initiate new Banach spaces [6]. Each of these transforms are corresponding to a vector called *admissible vector*. The notions voice transform and admissible vector are very useful in study of frames and wavelets [7]. On the other hand, dynamical system is a concept with a long history which has connections with many branches of mathematical analysis. Corresponding to each dynamical system, a crossed product C^*-algebra is defined whose representations are in one-to-one correspondence with covariant representations of the associated dynamical system [8, Proposition 2.40]. In this paper we initiate the notion of admissible vector for a covariant pair of representatins corresponding to the action of a locally compact group G on a C^*-algebra A, and compare this concept with the classical one. Also, we give some

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sufficient conditions for a vector to be admissible vector for a covariant pair of a dynamical system. In addition, we prove the existence of Parseval frames for some special subspaces of $L^2(G)$ related to a uniform lattice of G.

2. Preliminaries

In this paper G is a locally compact group with a (left) Haar measure m. For each $1 \leq p < \infty$, we denote $L^p(G) := L^p(G, m)$. For a Hilbert space $\mathcal{H}$, the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$ is denoted by $B(\mathcal{H})$. The space of all unitary operators in $B(\mathcal{H})$ is denoted by $U(\mathcal{H})$. Any continuous homomorphism $U : G \to U(\mathcal{H})$, in which $U(\mathcal{H})$ is endowed with the strong operator topology, is called a *unitary representation* of G on $\mathcal{H}$ (for more details see [1]). For each $x \in G$ we denote $U_x := U(x)$. The left regular representation $\tau : G \to U(L^2(G, m))$ is defined by $\tau_x f(y) := f(x^{-1}y)$, where $f \in L^2(G, m)$ and $x, y \in G$.

Definition 2.1. Let U be a unitary representaion of G on $\mathcal{H}$. A vector $\eta \in \mathcal{H}$ is called *admissible* if there exists a positive constant number B such that for each $\xi \in \mathcal{H}$,

$$\int_G |\langle \xi, U_x \eta \rangle|^2 \, dm(x) \leq B\|\xi\|^2.$$

Let U be a unitary representaion of G on $\mathcal{H}$ and $\eta \in \mathcal{H}$ be an admissible vector for U. The mapping $V_\eta : \mathcal{H} \to L^2(G, m)$ is defined by $V_\eta(\xi)(x) := \langle \xi, U_x \eta \rangle$. In the case that V_η is an isometry, it is called a *voice* or generalized *continuous wavelet transform*.

Remark 2.2. If $\eta \in \mathcal{H}$ is an admissible vector for representation U of G, then easily one can prove that $f * (V_\eta \xi) = V_\eta(U(f)(\xi))$ for all $\xi \in \mathcal{H}$ and $f \in L^1(G)$, where

$$U(f) := \int_G f(x) U_x \, dm(x).$$

In this paper A is a C^*-algebra and the space of all $*$-automorphisms of A is denoted by $\mathrm{Aut}(A)$.

Definition 2.3. Let G be a locally compact group and A be a C^*-algebra. Any continuous homomorphism $\alpha : G \to \mathrm{Aut}(A)$ is called an *action* of G on A. In this case, the triple (A, G, α) is called a *dynamical system*.

If α is an action of G on A, for each $x \in G$ we denote $\alpha_x := \alpha(x)$. So, for each $a \in A$ the mapping $x \mapsto \alpha_x(a)$ from G to A is continuous.

$(\mathbb{C}, G, \mathrm{Id})$ and $(A, \{e\}, \mathrm{Id})$ are trivial examples of dynamical systems. See [8] for more examples.

Definition 2.4. Let (A, G, α) be a dynamical system. Let π be a representation of A on a Hilbert space $\mathcal{H}$ and U be a unitary representation of G on the same Hilbert space $\mathcal{H}$. The pair (π, U) is called a *covariant representation* of (A, G, α) if for all $x \in G$ and $a \in A$,

$$\pi(\alpha_x(a))\,U_x = U_x \pi(a).$$

Example 2.5. Let G act on $C_0(G)$ via

$$\mathrm{L}_x(f)(y) := f(x^{-1}y),$$

where $x, y \in G$ and $f \in C_0(G)$. Also, assume that the representation $\pi : C_0(G) \to B(L^2(G))$ is defined by

$$\pi(f)g := fg,$$

where $f \in C_0(G)$ and $g \in L^2(G)$. Then, (π, τ) is a covariant pair for the dynamical system $(C_0(G), G, \mathrm{L})$, where τ is the left regular representation of G [8, Example 2.12].

3. Main Results

In this section, we introduce the main notion of this paper and give related results.

Definition 3.1. Let (π, U) be a covariant representation of a dynamical system (A, G, α) on a Hilbert space $\mathcal{H}$. A vector $\eta \in \mathcal{H}$ is called an *admissible vector* for (π, U), if there exist an element $a \in A$ and a constant positive number B such that for all $\xi \in \mathcal{H}$,

$$\int_G |\langle \pi(\alpha_x(a))\xi, U_x\eta\rangle|^2 \; dm(x) \leq B\|\xi\|^2.$$

Remark 3.2. Let (π, U) be a covariant pair on $\mathcal{H}$. If η is an admissible vector for U and there exists an element a in A with $\pi(a) = \mathrm{Id}_{\mathcal{H}}$, then η is an admissible vector for (π, U).

The above definition generalizes the notion of admissible vectors of unitary representations of a locally compact group. Precisely, a vector $\eta \in \mathcal{H}$ is an admissible vector for a unitary representation U of G (as in Definition 2.1) if and only if it is an admissible vector for a covariant representation of the dynamical system $(\mathbb{C}, G, \mathrm{Id})$ (as in Definition 3.1), where Id is the trivial action of G on $\mathbb{C}$.

Proposition 3.3. *Let (π, U) be a covariant representation of a dynamical system (A, G, α) on a Hilbert space $\mathcal{H}$. Then, $\eta \in \mathcal{H}$ is an admissible vector for (π, U) if and only if for some $a \in A$, $\pi(a)\eta$ is an admissible vector for U.*

Proof. For each $x \in G$, $a \in A$ and $\xi \in \mathcal{H}$ we have

$$\langle \pi(\alpha_x(a^*))\xi, U_x\eta\rangle = \langle \xi, \pi(\alpha_x(a))U_x\eta\rangle = \langle \xi, U_x\pi(a)\eta\rangle,$$

and so the proof is complete. □

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K} \subseteq B(\mathcal{H})$. The commutant space of $\mathcal{K}$ is defined by

$$\mathcal{K}' := \{T \in B(\mathcal{H}) : \text{ for all } S \in \mathcal{K}, TS = ST\}.$$

If (π, U) is a covariant representation of (A, G, α) on $\mathcal{H}$, a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ is called invariant under (π, U) if for each $x \in G$ and $a \in A$, $\pi(a)(\mathcal{H}_0) \subseteq \mathcal{H}_0$ and $U_x(\mathcal{H}_0) \subseteq \mathcal{H}_0$ [8, page 47].

Proposition 3.4. *Let (π, U) be a covariant pair for a dynamical system (A, G, α).*

(i) *If $T \in \pi(A)' \cap U(G)'$ and η is an admissible vector for (π, U), then $T\eta$ is also an admissible vector.*

(ii) *If $\mathcal{H}_0$ is a closed invariant subspace under (π, U) with orthogonal projection $P_{\mathcal{H}_0}$ and η is an admissible vector for (π, U), then $P_{\mathcal{H}_0}\eta$ is an admissible vector for (π, U) resricted to $\mathcal{H}_0$.*

Proof. (i) Let $x \in G$, $a \in A$ and $\xi \in \mathcal{H}$. Since $T \in \pi(A)' \cap U(G)'$, we have

$$\langle \pi(\alpha_x(a^*))\xi, U_xT\eta\rangle = \langle T^*\xi, U_x\pi(a)\eta\rangle.$$

This implies that $T\eta$ is an admissible vector for (π, U).

(ii) If $\mathcal{H}_0$ is invariant under (π, U), then $\mathcal{H}_0^\perp$ is also invariant under (π, U). There are $\eta_1 \in \mathcal{H}_0$ and $\eta_2 \in \mathcal{H}_0^\perp$ such that $\eta = \eta_1 + \eta_2$. So, for each $a \in A$ and $\xi \in \mathcal{H}_0$ we have

$$\begin{aligned}
\langle \pi(\alpha_x(a))\xi, U_x\eta\rangle &= \langle \xi, U_x\pi(a)\eta\rangle \\
&= \langle \xi, U_x\pi(a)\eta_1\rangle + \langle \xi, U_x\pi(a)\eta_2\rangle \\
&= \langle \xi, U_x\pi(a)P_{\mathcal{H}_0}\eta\rangle \\
&= \langle \pi(\alpha_x(a))\xi, U_xP_{\mathcal{H}_0}\eta\rangle.
\end{aligned}$$

This completes the proof.

□

Proposition 3.5. *Let A be a unital C^*-algebra and (π, U) be a covariant representation of a dynamical system (A, G, α) on a Hilbert space $\mathcal{H}$. If $\eta \in \mathcal{H}$ is an admissible vector for (π, U), then for each invertible element $b \in A$, $\pi(b)\eta$ is also an admissible vector for (π, U). In particular, for each $x \in A$ and $\lambda \in \mathbb{C} - \sigma(x)$, $\lambda\eta - \pi(x)\eta$ is an admissible vector for (π, U).*

Proof. By hypothesis, there are $B > 0$ and $a \in A$ such that for each $\xi \in \mathcal{H}$,

$$\int_G |\langle \pi(\alpha_x(a))\xi, U_x\eta\rangle|^2 \, dm(x) \le B\|\xi\|^2.$$

Hence, for each invertible element $b \in A$ we have

$$\begin{aligned}
&\int_G \left|\langle \pi(\alpha_x((b^*)^{-1}a))\xi, U_x\pi(b)\eta\rangle\right|^2 \, dm(x) \\
&\qquad = \int_G \left|\langle \pi(\alpha_x((b^*)^{-1}a))\xi, \pi(\alpha_x(b))U_x\eta\rangle\right|^2 \, dm(x) \\
&\qquad = \int_G \left|\langle \pi(\alpha_x(b^*))\pi(\alpha_x((b^*)^{-1}a))\xi, U_x\eta\rangle\right|^2 \, dm(x) \\
&\qquad = \int_G |\langle \pi(\alpha_x(a)\xi, U_x\eta\rangle|^2 \, dm(x) \\
&\qquad \le B\|\xi\|^2,
\end{aligned}$$

and the proof is complete. □

Notation 3.6. *If (π, U) is a covariant pair on $\mathcal{H}$ and $\eta \in \mathcal{H}$ is an admissible vector for (π, U), we denote*

$$W_\eta := \overline{span}\{U_x\pi(a)\eta : \, x \in G \text{ and } a \in A\}.$$

Example 3.7. Let $(C_0(G), G, \mathrm{L})$ be the dynamical system introduced in Example 2.5 with covariant pair (π, τ). Then, for an admissible vector $\varphi \in L^2(G)$ we have

$$\begin{aligned}
W_\varphi &= \overline{\mathrm{span}}\{\tau_x\pi(f)\varphi : x \in G \text{ and } f \in C_0(G)\} \\
&= \overline{\mathrm{span}}\{\mathrm{L}_x f\tau_x\varphi : x \in G \text{ and } f \in C_0(G)\} \\
&= \overline{\mathrm{span}}\{g\tau_x\varphi : x \in G \text{ and } g \in C_0(G)\}.
\end{aligned}$$

Lemma 3.8. *Let (π, U) be a covariant pair on $\mathcal{H}$ with an admissible vector η. Then, the closed subspace W_η of $\mathcal{H}$ is invariant under (π, U).*

Proof. Let $x, y \in G$ and $a, b \in A$. Then, by Definition 2.4 we have

$$\begin{aligned}
\pi(a)U_x\pi(b)\eta &= U_x\pi(\alpha_{x^{-1}}(a))\pi(b)\eta \\
&= U_x\pi(\alpha_{x^{-1}}(a)b)\eta \in W_\eta.
\end{aligned}$$

Also,

$$U_xU_y\pi(a)\eta = U_{xy}\pi(a)\eta \in W_\eta.$$

So, W_η is invariant under (π, U). □

By Proposition 3.4 and Lemma 3.8 we can conclude:

Corollary 3.9. *Let η be an admissible vector for a given covariant pair (π, U). Then, $P_{W_\eta}\eta$ is an admissible vector for (π, U) restricted to W_η.*

Definition 3.10. Let U be a unitary representation of a locally compact group G on $\mathcal{H}$. If there are a vector $\eta \in \mathcal{H}$ and constant numbers $A, B > 0$ such that for every $\xi \in \mathcal{H}$,

$$A\|\xi\|^2 \leq \int_G |\langle \xi, U_x\eta\rangle|^2 \, dm(x) \leq B\|\xi\|^2,$$

U is called a *frame representation.* In the case that $A = B$, U is called *tight frame.* In particular, if $A = B = 1$, U is called *Parseval frame.*

Trivially, every frame representation has an admissible vector.

The following result is an immediate conclusion of [4, Proposition 4.25] and Example 3.7.

Corollary 3.11. *If (π, τ) is the covariant representation of $(C_0(G), G, L)$ as in Example 2.5, and φ is an admissible vector for (π, τ), then the space*

$$\overline{span}\{g\tau_x\varphi : x \in G \text{ and } g \in C_0(G)\}$$

has a Parseval frame.

We recall that a discrete subgroup Γ of G is called a *uniform lattice* if the quotient space G/Γ is compact.

Theorem 3.12. *Let π be a unitary representation of G on $\mathcal{H}_\pi$ with an admissible vector η and $\mathcal{H} := L^2(G) * V_\eta(\eta)$. Let Γ be a uniform lattice for G such that for each $f \in \mathcal{H}$,*

$$\sum_{x\in\Gamma} |f(x)|^2 = \|f\|_2^2.$$

Then, $\eta \otimes V_\eta(\eta)$ is the admissible vector for the induced representation $ind_G^{G\times\Gamma}(\pi)$.

Proof. By [2, Theorem 2.56], $V_\eta\eta$ is an admissible vector for $\tau|_\Gamma$, the restricted left regular representation τ on Γ. So, by [5, Corollary 5.3], $V_\eta\eta \otimes \eta$ is an admissible vector for $\tau|_\Gamma \otimes \pi$. Finally, because of [1, Proposition 7.26] the proof is complete. □

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