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Fekete-Szegö Problem of Functions Associated with Hyperbolic Domains

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Abstract. In the field of Geometric Function Theory, one can not deny the importance of analytic and univalent functions. The characteristics of these functions including their taylor series expansion, their coefficients in these representations as well as their associated functional inequalities have always attracted the researchers. In particular, Fekete-Szegö inequality is one of such vastly studied and investigated functional inequality. Our main focus in this article is to investigate the Fekete-Szegö functional for the class of analytic functions associated with hyperbolic regions. To further enhance the worth of our work, we include similar problems for the inverse functions of these discussed analytic functions.

1. Introduction and preliminaries

The class $\mathcal{A}$ of analytic functions f defined in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ has its functions' representation as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Whereas $\mathcal{S}$ represents the class of univalent functions in $\mathcal{U}$. Related to the coefficients a_n of taylor series of univalent functions, the so-called Fekete-Szegö problem is considered to be a major result. It was introduced by Fekete and Szegö [2]. It is stated as:

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If f is a univalent function and can be represented as (1.1), then

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} 3-4\mu, & \text{if} \quad \mu \leq 0, \\ 1+2\exp\left(\frac{2\mu}{\mu-1}\right), & \text{if} \quad 0 \leq \mu \leq 1, \\ 4\mu-3, & \text{if} \quad \mu \geq 1. \end{cases}$$

The Fekete-Szegö problem is all about maximizing the absolute value of the functional $a_3-\mu a_2^2$. This result has been widely studied by many researchers and it is sharp. Koebe function proves the sharpness of this result. Koebe fails to be extremal for the case $0<\mu<1$ and it provides an example of an extremal problem over $\mathcal{S}$. In the similar context, a number of results related to the maximization of the non-linear functional given on left hand side of the above inequality can be found for different classes and for subclasses of univalent functions, as well. Some interesting studies of this functional by considering μ as complex number besides considering it as a real number can be found in literature. Different classified techniques have been used by authors to maximize Fekete-Szegö functional. These studies show interesting geometric characteristics of image domains for different types of functions. For more details, we refer the interested readers to [1, 2, 6, 7, 11–14, 22, 25] and the references therein.

We define the subordination of two functions f and g symbolically written as $f \prec g$, and is defined as

$$f(z)=g(w(z)), \quad z \in \mathcal{U}, \tag{1.2}$$

where w is a schwarz function such that $w(0)=0,\ |w(z)|<1$ for $z \in \mathcal{U}$. We now define another class of analytic functions p satisfying $p(0)=1$ and $p \prec \frac{1+z}{1-z},\ z \in \mathcal{U}$. We denote this class with P. For details, see [4].

In the study of domains, the idea of conic domains was given by Goodman [3] in 1991. This remarkable initiation opened a new horizon of research. Goodman introduced the image domain of analytic functions as parabolic regions. In the same context, he defined the class UCV of uniformly convex functions as:

$$UCV=\left\{f \in \mathcal{A}: \Re\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right)>0,\ z,\zeta \in \mathcal{U}\right\}.$$

Independently, Rønning [23, 24], and Ma and Minda [13] defined the one variable characterization of the above defined class. This was another achievement in the study of domains. The characterization is given as follows:

$$UCV=\left\{f \in \mathcal{A}: \Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \mathcal{U}\right\}.$$

The above given characterization introduced a domain, conic (parabolic) domain, given as $\Omega = \{w : \Re w > |w-1|\}$, which was an absolutely new discovery in this line. Inspired by the work going on, Kanas and Wiśniowska [10] defined $k-$uniformly convex functions as:

$$k-UCV = \left\{ f \in \mathcal{A} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > k \left| \frac{zf''(z)}{f'(z)} \right|, \; z \in \mathcal{U} \right\}.$$

With this, the most general conic domain Ω_k, given as follows, was introduced. This covers all types of conic regions, i.e., parabolic as well as hyperbolic and elliptic regions.

$$\Omega_k = \{w : \Re w > k|w-1|, \; k \geq 0\}.$$

With a variation in the values of k, this domain Ω_k, produces different image domains. This gives the right half plane as image domain for $k = 0$, and the hyperbolic regions when $0 < k < 1$. For $k = 1$, we obtain parabolic region from this generalization and regions become elliptical if $k > 1$. We refer to [9, 10] for more details of these domains. Noor and Malik [20] gave a breakthrough by removing the deficiency of fixed sizes of domains by generalizing this domain Ω_k. Since the conic regions, presented by the domain Ω_k have their fixed sizes. That is, this formulation does not allow these regions to be contracted or expanded. For this, they defined the following domain

(1.3)

$$\Omega_k(a,b) = \left\{ u + iv : (u-a)^2 > k^2 \left[(u-a+b-1)^2 + v^2 + 2b(1-b) \right] \right\}.$$

The beauty of this domain $\Omega_k(a,b)$ is that, now the conic regions can attain any size. Related to this, the following class of functions takes all values from the above domain $\Omega_k(a,b)$, $0 < k < 1$ (The generalized hyperbolic regions).

Definition 1.1. [20] A function $p(z)$ is said to be in the class $k-P(a,b)$, if and only if,

$$p(z) \prec (a+b) + (1-b)\widetilde{p}_k(z), \tag{1.4}$$

where

$$\widetilde{p}_k(z) = 1 + \frac{2}{1-k^2} \sinh^2 \left[\left(\frac{2}{\pi} \arccos k \right) \operatorname{arctanh} \sqrt{z} \right],$$

$0 < k < 1$ and a, b must be chosen accordingly, as:

$$b \in \begin{cases} \left[\frac{1}{2k^2-1}, 1 \right), & \text{when} \quad 0 < k < \frac{1}{\sqrt{2}}, \\ \\ (-\infty, 1), & \text{when} \quad \frac{1}{\sqrt{2}} \leq k < 1, \end{cases} \tag{1.5}$$

and

$$\frac{k^2(1-b)}{1-k^2}-\frac{k\sqrt{k^2(1-b)^2+(1-k^2)(1-b^2)}}{k^2-1}\leq a \tag{1.6}$$
$$<1-\frac{k^2(1-b)}{k^2-1}+\frac{k\sqrt{k^2(1-b)^2+(1-k^2)(1-b^2)}}{k^2-1}.$$

For more details about the function $\widetilde{p}_k(z)$, we refer the readers to [5, 9, 10, 15–18, 21].

We may relate it with the work done before. The restriction of the domain as $\Omega_k(0,0)=\Omega_k$, gives the conic domain that was given by Kanas and Wiśniowska [9, 10]. Using this important relation, we make an interesting observation that it connects some already known classes of analytic functions. Some of these are:

(i) $k-P(a,b)\subset P(\beta)$, the class of functions with real part greater than β, where
$$\beta=a+\frac{k^2(1-b)-k\sqrt{k^2(1-b)^2+(1-k^2)(1-b^2)}}{k^2-1}.$$

(ii) $k-P(0,0)=\mathcal{P}(\widetilde{p}_k)$, which is the class introduced by Kanas and Wiśniowska [9, 10].

We proceed in our literature review and include the definitions of the class of generalized $k-$uniformly convex functions, denoted by $k-UCV(a,b)$ and the class of corresponding $k-$starlike functions, denoted by $k-ST(a,b)$. Both of these classes will appear in next section of this article. The definitions are given as follows.

Definition 1.2. [20] A function $f\in\mathcal{A}$ is said to be in the class $k-UCV(a,b)$, where $0<k<1$ and a and b satisfy (1.6) and (1.5), if and only if,
$$\left[\Re\left\{\frac{(zf'(z))'}{f'(z)}-a\right\}\right]^2>k^2\left[\left|\frac{(zf'(z))'}{f'(z)}-a+b-1\right|^2+2b(1-b)\right],$$
or equivalently,
$$\frac{(zf'(z))'}{f'(z)}\in k-P(a,b). \tag{1.7}$$

Definition 1.3. [20] A function $f\in\mathcal{A}$ is said to be in the class $k-ST(a,b)$, where $0<k<1$ and a and b satisfy (1.6) and (1.5), if and only if,
$$\left[\Re\left\{\frac{zf'(z)}{f(z)}-a\right\}\right]^2>k^2\left[\left|\frac{zf'(z)}{f(z)}-a+b-1\right|^2+2b(1-b)\right],$$

or in other words,

$$\frac{zf'(z)}{f(z)} \in k - P(a,b). \tag{1.8}$$

It can easily be seen that

$$f(z) \in k - UCV(a,b) \quad \Leftrightarrow \quad zf'(z) \in k - ST(a,b).$$

It is clear that $k-UCV(0,0) = k-UCV$ and $k-ST(0,0) = k-ST$. The well-known classes of k-uniformly convex and corresponding k-starlike functions respectively, introduced by Kanas and Wiśniowska [9, 10].

The above mentioned contribution by the well known researchers proves that it has attracted the attention towards it. Before we identify the importance of work done in this paper, we would like to give a brief review of the study of Fekete-Szegö inequality. In 1994, Ma and Minda [13] solved the Fekete-Szegö problem for the class of uniformly convex functions, whereas Kanas [8] found the maximum bound of the functional $\left|a_3 - \mu a_2^2\right|$ for the functions of class $\mathcal{P}(\widetilde{p}_k)$. Later, Mishra and Gochhayat [19] contributed by solving the same problem for the functions of classes $k-UCV$ and $k-ST$. Inspired and motivated by the research going on in this area of research, we solve the classical Fekete-Szegö problem for the functions of classes $k-P(a,b)$, $k-UCV(a,b)$ and $k-ST(a,b)$. For the main results of this paper, we need the following useful lemmas. The proofs of these lemmas are given in the respective references.

Lemma 1.4 ([13])**.** *If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is a function with positive real part in $\mathcal{U}$, then, for any complex number μ,*

$$\left|p_2 - \mu p_1^2\right| \leq 2\max\{1, |2\mu - 1|\},$$

and the result is sharp for the functions

$$p_0(z) = \frac{1+z}{1-z} \text{ or } p_*(z) = \frac{1+z^2}{1-z^2}, \quad (z \in \mathcal{U}).$$

Lemma 1.5. *[13] If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is a function with positive real part in $\mathcal{U}$, then, for any real number v,*

$$\left|p_2 - v p_1^2\right| \leq \begin{cases} -4v + 2, & v \leq 0, \\ 2, & 0 \leq v \leq 1, \\ 4v - 2, & v \geq 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < v < 1$, then, the equality holds if and only if $p(z) = \frac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, the equality holds if and *only if,*

$$p(z) = \left(\frac{1+\eta}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\eta}{2}\right)\frac{1-z}{1+z}, \quad (0 \le \eta \le 1),$$

or one of its rotations. If $v = 1$, then, the equality holds if and only if $p(z)$ is reciprocal of one of the functions which equality holds in the case of $v = 0$. Although the above upper bound is sharp, when $0 < v < 1$, it can be improved as follows:

$$\left|p_2 - vp_1^2\right| + |p_1|^2 \le 2, \quad \left(0 < v \le \frac{1}{2}\right),$$

and

$$\left|p_2 - vp_1^2\right| + (1-v)|p_1|^2 \le 2, \quad \left(\frac{1}{2} < v \le 1\right).$$

2. Main Results

Theorem 2.1. *Let $p \in k - P(a,b)$ where $0 < k < 1$, and a, b are taken according to (1.5) and (1.6). Also consider the form*

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Then, for a complex number μ, we have

$$\left|p_2 - \mu p_1^2\right| \le \frac{2|1-b|T^2}{1-k^2}.\max\left(1, \left|\frac{2T^2\mu}{1-k^2}(1-b) - \frac{2+T^2}{3}\right|\right), \tag{2.1}$$

and for real number μ, we have

(2.2)

$$\left|p_2 - \mu p_1^2\right| \le \frac{|1-b|T^2}{1-k^2}$$
$$\times \begin{cases} \frac{4}{3} + \frac{2T^2}{3} - \frac{4(1-b)T^2}{1-k^2}\mu, & \mu \le \frac{1-k^2}{6(1-b)}\left(1 - \frac{1}{T^2}\right), \\ 2, & \frac{1-k^2}{6(1-b)}\left(1 - \frac{1}{T^2}\right) \le \mu \le \frac{1-k^2}{6(1-b)}\left(\frac{5}{T^2} + 1\right), \\ -\frac{4}{3} - \frac{2T^2}{3} + \frac{4(1-b)T^2}{1-k^2}\mu, & \mu \ge \frac{1-k^2}{6(1-b)}\left(\frac{5}{T^2}\% + 1\right). \end{cases}$$

These results are sharp and the equality in (2.1) holds for the functions

$$p_1(z) = 1 + a + \frac{2(1-b)}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos k\right)\operatorname{arctanh}\sqrt{z}\right], \tag{2.3}$$

or

$$p_2(z) = 1 + a + \frac{2(1-b)}{1-k^2} \sinh^2 \left[\left(\frac{2}{\pi} \arccos k \right) \operatorname{arctanh}(z) \right]. \tag{2.4}$$

When $\mu < \frac{1-k^2}{6(1-b)} \left(1 - \frac{1}{T^2}\right)$ *or* $\mu > \frac{1-k^2}{6(1-b)} \left(\frac{5}{T^2} + 1\right)$, *the equality in* (2.2) *holds for the function* $p_1(z)$ *or one of its rotations. If*

$$\frac{1-k^2}{6(1-b)} \left(1 - \frac{1}{T^2}\right) < \mu < \frac{1-k^2}{6(1-b)} \left(\frac{5}{T^2} + 1\right),$$

then the equality in (2.2) *holds for the function* $p_2(z)$ *or one of its rotations. If* $\mu = \frac{1-k^2}{6(1-b)} \left(1 - \frac{1}{T^2}\right)$, *the equality in* (2.2) *holds for the function*

$$p_3(z) = \left(\frac{1+\eta}{2}\right) p_1(z) + \left(\frac{1-\eta}{2}\right) p_1(-z), \quad (0 \leq \eta \leq 1), \tag{2.5}$$

or one of its rotations. If $\mu = \frac{1-k^2}{6(1-b)} \left(\frac{5}{T^2} + 1\right)$, *then, the equality in* (2.2) *holds for the function* $p(z)$ *which is reciprocal of one of the functions which equality holds in the case for* $\mu = \frac{1-k^2}{6(1-b)} \left(1 - \frac{1}{T^2}\right)$.

Proof. For $h \in P$ of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

we consider

$$h(z) = \frac{1 + w(z)}{1 - w(z)},$$

where $w(z)$ is such that $w(0) = 0$ and $|w(z)| < 1$. It follows easily that (2.6)

$$\begin{aligned} w(z) &= \frac{h(z) - 1}{h(z) + 1} \\ &= \frac{(1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots) - 1}{(1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots) + 1} \\ &= \frac{1}{2} c_1 z + \left(\frac{1}{2} c_2 - \frac{1}{4} c_1^2\right) z^2 + \left(\frac{1}{2} c_3 - \frac{1}{2} c_2 c_1 + \frac{1}{8} c_1^3\right) z^3 + \cdots . \end{aligned}$$

Now, if $\widetilde{p}_k(w(z)) = 1 + R_1(k) w(z) + R_2(k) w^2(z) + R_3(k) w^3(z) + \cdots$, then from (2.6), one may have

$$\begin{aligned} \widetilde{p}_k(w(z)) &= 1 + R_1(k) w(z) + R_2(k) w^2(z) + R_3(k) w^3(z) + \cdots \\ &= 1 + R_1(k) \left(\frac{1}{2} c_1 z + \left(\frac{1}{2} c_2 - \frac{1}{4} c_1^2\right) z^2 \right. \\ &\quad \left. + \left(\frac{1}{2} c_3 - \frac{1}{2} c_2 c_1 + \frac{1}{8} c_1^3\right) z^3 + \cdots \right) \end{aligned}$$

$$+ R_2(k)\left(\frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{2}c_3 - \frac{1}{2}c_2c_1 + \frac{1}{8}c_1^3\right)z^3 + \cdots\right)^2$$

$$+ R_3(k)\left(\frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{2}c_3 - \frac{1}{2}c_2c_1 + \frac{1}{8}c_1^3\right)z^3 + \cdots\right)^3 + \cdots,$$

where $R_1(k)$, $R_2(k)$ and $R_3(k)$ are given by

$$R_1(k) = \frac{2T^2}{1-k^2},$$

$$R_2(k) = \frac{2T^2}{3(1-k^2)}\left(2+T^2\right),$$

$$R_3(k) = \frac{2T^2}{9(1-k^2)}\left(\frac{23}{5} + 4T^2 + \frac{2}{5}T^4\right),$$

and $T = T(k) = \frac{2}{\pi}\arccos(k)$, $0 < k < 1$, (see [8]). Using these, the above series reduces to

(2.7)

$$\widetilde{p}_k(w(z)) = 1 + \frac{T^2}{1-k^2}c_1 z + \frac{T^2}{1-k^2}\left(\frac{T^2-1}{6}c_1^2 + c_2\right)z^2 + \frac{T^2}{1-k^2}$$
$$\left[\left(\frac{2}{45} - \frac{1}{18}T^2 + \frac{1}{90}T^4\right)c_1^3 + \left(-\frac{1}{3} + \frac{1}{3}T^2\right)c_1c_2 + c_3\right]z^3 + \cdots.$$

Since $p \in k - P(a,b)$, $0 < k < 1$, from relations (1.2), (1.4) and (2.7), one may have

(2.8)

$$p(z) = (a+b) + (1-b)\widetilde{p}_k(w(z)),$$
$$= 1 + a + \frac{T^2}{1-k^2}(1-b)c_1 z + \frac{T^2(1-b)}{1-k^2}$$
$$= \left(c_2 + \frac{T^2-1}{6}c_1^2\right)z^2 + \frac{T^2}{1-k^2}(1-b),$$
$$\left[\left(\frac{2}{45} - \frac{1}{18}T^2 + \frac{1}{90}T^4\right)c_1^3 + \left(-\frac{1}{3} + \frac{1}{3}T^2\right)c_1c_2 + c_3\right]z^3 + \ldots.$$

If

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

then by equating coefficients of like powers of z, we have

$$p_1 = \frac{T^2}{1-k^2}(1-b)\,c_1,$$

$$p_2 = \frac{T^2(1-b)}{1-k^2}\left(c_2 + \frac{T^2-1}{6}c_1^2\right).$$

Now for complex number μ, consider

$$p_2 - \mu p_1^2 = \frac{T^2(1-b)}{1-k^2}\left(c_2 - \left(\frac{1-T^2}{6} + \frac{T^2\mu}{1-k^2}(1-b)\right)c_1^2\right).$$

This implies that

$$\left|p_2 - \mu p_1^2\right| = \frac{T^2|1-b|}{1-k^2}\left|c_2 - \left(\frac{1-T^2}{6} + \frac{T^2\mu}{1-k^2}(1-b)\right)c_1^2\right|. \tag{2.9}$$

Now by using Lemma 1.4, we have

$$\left|p_2 - \mu p_1^2\right| \le \frac{T^2|1-b|}{1-k^2}.2\max(1, |2v-1|),$$

where

$$v = \frac{1-T^2}{6} + \frac{T^2\mu}{1-k^2}(1-b).$$

This leads us to the required inequality (2.1) and applying Lemma 1.5 to the expression (2.9) for real number μ, we get the required inequality (2.2). □

For $a = 0,\ b = 0$, the above result reduces to the following form.

Corollary 2.2. *Let $p \in k - P(0,0) = \mathcal{P}(\widetilde{p}_k)$, $0 < k < 1$, and of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Then, for a complex number μ, we have*

$$\left|p_2 - \mu p_1^2\right| \le \frac{2T^2}{1-k^2}.\max\left(1, \left|\mu\frac{2T^2}{(1-k^2)} - \frac{T^2}{3} - \frac{2}{3}\right|\right), \tag{2.10}$$

and for real number μ, we have

(2.11)

$$\left|p_2 - \mu p_1^2\right| \le \frac{T^2}{1-k^2}$$
$$\times \begin{cases} \frac{4}{3} + \frac{2}{3}T^2 - \frac{4\mu T^2}{1-k^2}, & \mu \le -\frac{1-k^2}{6T^2} + \frac{1-k^2}{6}, \\ 2, & -\frac{1-k^2}{6T^2} + \frac{1-k^2\%}{6} \le \mu \le \frac{5(1-k^2)}{6T^2} + \frac{1-k^2}{6}, \\ -\frac{4}{3} - \frac{2}{3}T^2 + \frac{4\mu T^2}{1-k^2}, & \mu \ge \frac{5(1-k^2)}{6T^2} + \frac{1-k^2}{6}. \end{cases}$$

These results are sharp.

In [7, 8], Kanas studied the class $\mathcal{P}(\widetilde{p}_k)$ which consists of functions taking all the values from the conic domain Ω_k. Kanas [8] found the bound of Fekete-Szegö functional for the class $\mathcal{P}(\widetilde{p}_k)$ whose particular case for $0<k<1$ is as follows:
Let $p(z)=1+b_1z+b_2z^2+b_3z^3+\cdots\in\mathcal{P}(\widetilde{p}_k)$, $0<k<1$. Then, for a real number μ, we have

$$\left|b_2-\mu b_1^2\right|\leq\frac{2T^2}{1-k^2}\begin{cases}1-\mu\frac{2T^2}{1-k^2}, & \mu\leq 0,\\ 1, & \mu\leq 0,\\ 1+(\mu-1)\frac{2T^2}{1-k^2}, & \mu\geq 1.\end{cases}\tag{2.12}$$

We observe that Corollary 2.2 gives more refined bounds of Fekete-Szegö functional $\left|p_2-\mu p_1^2\right|$ for the functions of class $\mathcal{P}(\widetilde{p}_k)$, $0<k<1$ as compared to that from (2.12).

Theorem 2.3. *Let $f\in k-UCV(a,b)$ where $0<k<1$, and a, b are taken according to* (1.5) *and* (1.6). *Also consider the form*

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n.$$

Then, for a real number μ, we have

(2.13)

$$\left|a_3-\mu a_2^2\right|\leq\frac{T^2\left|1-b\right|}{6(1-k^2)}$$
$$\times\begin{cases}\frac{4}{3}+\frac{2}{3}T^2+(2-3\mu)\frac{2T^2(1-b)}{1-k^2}, & \mu\leq\frac{2}{3}-\frac{1-k^2}{9(1-b)}\left(\frac{1}{T^2\%}-1\right),\\ 2, & \frac{2}{3}-\frac{1-k^2}{9(1-b)}\left(\frac{1}{T^2}-1\right)\leq\mu\leq\frac{2}{3}+\frac{1-k^2}{9(1-b)}\left(\frac{5}{T^2}+1\right),\\ -\frac{4}{3}-\frac{2}{3}T^2-(2-3\mu)\frac{2T^2(1-b)}{1-k^2}, & \mu\geq\frac{2}{3}+\frac{1-k^2}{9(1-b)}\left(\frac{5}{T^2}+1\right).\end{cases}$$

Proof. If $f(z)\in k-UCV(a,b)$, $0\leq k<1$, then it follows from (1.2), (1.4) and (1.7) that

$$\frac{\left(zf'(z)\right)'}{f'(z)}=(a+b)+(1-b)\,\widetilde{p}_k(w(z)).$$

This implies by using (2.8) that

(2.14)
$$\frac{\left(zf'(z)\right)'}{f'(z)}=1+a+\frac{T^2(1-b)}{1-k^2}c_1z$$

$$+\frac{T^2(1-b)}{1-k^2}\left(c_2+\frac{T^2-1}{6}c_1^2\right)z^2+\frac{T^2}{1-k^2}(1-b)$$
$$\left[\left(\frac{2}{45}-\frac{1}{18}T^2+\frac{1}{90}T^4\right)c_1^3+\left(-\frac{1}{3}+\frac{1}{3}T^2\right)c_1c_2+c_3\right]z^3+\cdots.$$

If

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,$$

then one may have

$$\frac{(zf'(z))'}{f'(z)}=1+2a_2z+\left(6a_3-4a_2^2\right)z^2+\left(12a_4-18a_2a_3+8a_2^3\right)z^3+\cdots. \tag{2.15}$$

From (2.14) and (2.15), comparison of like powers of z gives

$$a_2=\frac{T^2(1-b)}{2(1-k^2)}c_1, \tag{2.16}$$

and

$$a_3=\frac{T^2(1-b)}{6(1-k^2)}\left(c_2-\left(\frac{1-T^2}{6}-\frac{T^2(1-b)}{1-k^2}\right)c_1^2\right). \tag{2.17}$$

Now, for a real number μ, we consider

$$\begin{aligned}\left|a_3-\mu a_2^2\right|&=\frac{T^2|1-b|}{6(1-k^2)}\left|c_2-\left(\frac{1-T^2}{6}-\frac{T^2(1-b)}{1-k^2}+\frac{3T^2\mu}{2(1-k^2)}(1-b)\right)c_1^2\right|\\&=\frac{T^2|1-b|}{6(1-k^2)}\left|c_2-\left(\frac{1-T^2}{6}-\left(1-\frac{3}{2}\mu\right)\frac{T^2(1-b)}{1-k^2}\right)c_1^2\right|.\end{aligned}$$

By applying Lemma 1.5, we have the required result. The inequality (2.13) is sharp and equality holds for

$$\mu<\frac{2}{3}-\frac{1-k^2}{9(1-b)}\left(\frac{1}{T^2}-1\right),$$

or

$$\mu>\frac{2}{3}+\frac{1-k^2}{9(1-b)}\left(\frac{5}{T^2}+1\right),$$

when $f(z)$ is $f_1(z)$ or one of its rotations, where $f_1(z)$ is defined such that $\frac{(zf_1'(z))'}{f_1'(z)}=p_1(z)$. If

$$\frac{2}{3}-\frac{1-k^2}{9(1-b)}\left(\frac{1}{T^2}-1\right)<\mu<\frac{2}{3}+\frac{1-k^2}{9(1-b)}\left(\frac{5}{T^2}+1\right),$$

then, the equality holds for the function $f_2(z)$ or one of its rotations, where $f_2(z)$ is defined such that $\frac{\left(zf_2'(z)\right)'}{f_2'(z)} = p_2(z)$. If

$$\mu = \frac{2}{3} - \frac{1-k^2}{9(1-b)}\left(\frac{1}{T^2} - 1\right),$$

the equality holds for the function $f_3(z)$ or one of its rotations, where $f_3(z)$ is defined such that $\frac{\left(zf_3'(z)\right)'}{f_3'(z)} = p_3(z)$. If

$$\mu = \frac{2}{3} + \frac{1-k^2}{9(1-b)}\left(\frac{5}{T^2} + 1\right),$$

then, the equality holds for $f(z)$, which is such that $\frac{(zf'(z))'}{f'(z)}$ is reciprocal of one of the function such that equality holds in the case of

$$\mu = \frac{2}{3} - \frac{1-k^2}{9(1-b)}\left(\frac{1}{T^2} - 1\right).$$

□

By taking $a = 0$ and $b = 0$ in Theorem 2.3, we have the following corollary which is proved by Mishra and Gochhayat [19].

Corollary 2.4. *Let $f \in k-UCV(0,0) = k-UCV$, $0 \le k < 1$ and of the form* (1.1)*. Then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{T^2}{6(1-k^2)}$$
$$\times \begin{cases} \frac{4}{3} + \frac{2T^2}{3} + (2-3\mu)\frac{2T^2}{1-k^2}, & \mu \le \frac{2}{3} - \frac{1-k^2}{9T^2} + \frac{1-k^2}{9}, \\ 2, & \frac{2}{3} - \frac{1-k^2}{9T^2} + \frac{1-k^2}{9} \le \mu \le \frac{2}{3} + \frac{5\left(1-k^2\right)}{9T^2} + \frac{1-k^2}{9}, \\ -\frac{4}{3} - \frac{2T^2}{3} - (2-3\mu)\frac{2T^2}{1-k^2}, & \mu \ge \frac{2}{3} + \frac{5\left(1-k^2\right)}{9T^2} + \frac{1-k^2}{9}. \end{cases}$$

Theorem 2.5. *If $f(z) \in k-ST(a,b)$ where $0 < k < 1$, and a, b are taken according to* (1.5) *and* (1.6)*. Also consider the form*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Then for a real number μ, we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{T^2\left|1-b\right|}{2(1-k^2)}$$

$$\times\begin{cases} \frac{4}{3}+\frac{2}{3}T^2+(1-2\mu)\,\frac{4T^2(1-b)}{1-k^2}, & \mu\leq\frac{1}{2}+\frac{\left(T^2-1\right)\left(1-k^2\right)}{12T^2(1-b)}, \\ 2, & \frac{1}{2}+\frac{\left(T^2-1\right)\left(1-k^2\right)}{12T^2(1-b)}\leq\mu\leq\frac{1}{2}+\frac{\left(5+T^2\right)\left(1-k^2\right)}{12T^2(1-b)}, \\ -\frac{4}{3}-\frac{2}{3}T^2-(1-2\mu)\,\frac{4T^2(1-b)}{1-k^2}, & \mu\leq\frac{1}{2}+\frac{\left(T^2-1\right)\left(1-k^2\right)}{12T^2(1-b)}. \end{cases}$$

This result is sharp.

Proof. The proof follows similarly as in Theorem 2.3. □

By taking $a=0$ and $b=0$ in Theorem 2.5, we have the following corollary which is proved by Mishra and Gochhayat [19].

Corollary 2.6. *Let $f\in k-ST\,(0,0)=k-ST,\ 0<k<1$ and of the form* (1.1). *Then, for a real number μ,*

$$\left|a_3-\mu a_2^2\right|\leq\frac{T^2}{1-k^2}$$

$$\times\begin{cases} \frac{2+T^2}{3}+(1-2\mu)\,\frac{2T^2}{1-k^2}, & \mu\leq\frac{1}{2}+\frac{\left(T^2-1\right)\left(1-k^2\right)}{\%12T^2}, \\ 1, & \frac{1}{2}+\frac{\left(T^2-1\right)\left(1-k^2\right)}{12T^2}\leq\mu\leq\frac{1}{2}+\frac{\left(5+T^2\right)\left(1-k^2\right)}{12T^2}, \\ -\frac{2+T^2}{3}-(1-2\mu)\,\frac{2T^2}{1-k^2}, & \mu\geq\frac{1}{2}+\frac{\left(5+T^2\right)\left(1-k^2\right)}{12T^2}. \end{cases}$$

Now we consider the inverse function $\mathcal{F}$ which maps regions presented by (1.3) to the open unit disk $\mathcal{U}$, defined as $\mathcal{F}\,(w)=\mathcal{F}\,(f\,(z))=z,\ z\in\mathcal{U}$ and we find the following coefficient bound for inverse functions.

Theorem 2.7. *Let $w=f\,(z)\in k-UCV\,(a,b)$ where $0<k<1,$ and $a,\ b$ are taken according to* (1.5) *and* (1.6). *Also let*

$$\mathcal{F}\,(w)=f^{-1}\,(w)=w+\sum_{n=2}^{\infty}d_nw^n.$$

Then,

$$|d_n|\leq\frac{2\,|1-b|\,T^2}{n\,(n-1)\,(1-k^2)},\quad (n=2,3)\,.$$

Proof. Since $\mathcal{F}\,(w)=\mathcal{F}\,(f\,(z))=z$, so it is easy to see that

$$d_2=-a_2,\qquad d_3=2a_2^2-a_3,\qquad d_4=-a_4+5a_2a_3-5a_2^3.$$

By using (2.16) and (2.17), one can have

$$d_2=-\frac{T^2\,(1-b)}{2\,(1-k^2)}c_1, \tag{2.18}$$

and

$$d_3 = \frac{T^2(1-b)}{6(1-k^2)}\left[\left(\frac{2T^2(1-b)}{1-k^2} + \frac{1-T^2}{6}\right)c_1^2 - c_2\right]. \tag{2.19}$$

Now, from (2.18) and (2.19), one can have

$$|d_2| \leq \frac{|1-b|\,T^2}{1-k^2},$$

and

$$\begin{aligned} |d_3| \leq{}& \frac{|1-b|\,T^2}{6(1-k^2)}\left|\frac{1}{6} - \frac{T^2}{6} + \frac{2(1-b)T^2}{1-k^2}\right|\left|c_2 - c_1^2\right| \\ &+ \frac{|1-b|\,T^2}{6(1-k^2)}\left|\frac{5}{6} + \frac{T^2}{6} - \frac{2(1-b)T^2}{1-k^2}\right||c_2|. \end{aligned}$$

Application of the bounds $\left|c_2 - c_1^2\right| \leq 2$ and $|c_2| \leq 2$ (see Lemma 1.5 for $v = 1$ and $v = 0$) gives $|d_3| \leq \frac{|1-b|T^2}{3(1-k^2)}$. □

Theorem 2.8. *Let $w = f(z) \in k - UCV(a, b)$ where $0 < k < 1$, and a, b are taken according to (1.5) and (1.6). Also let*

$$\mathcal{F}(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n.$$

Then, for a real number μ, we have

$$\left|d_3 - \mu d_2^2\right| \leq \frac{|1-b|\,T^2}{3(1-k^2)}$$
$$\times \begin{cases} \frac{2}{3} + \frac{1}{3}T^2 - (4-3\mu)\frac{T^2(1-b)}{1-k^2}, & \mu \geq \frac{4}{3} + \frac{(1-k^2)(1-T^2)}{9T^2(1-b)}, \\ 1, & \frac{4}{3} - \frac{(1-k^2)(5+T^2)}{9T^2(1-b)} \leq \mu \leq \frac{4}{3} + \frac{(1-k^2)(1-T^2)}{9T^2(1-b)}, \\ -\frac{2}{3} - \frac{1}{3}T^2 + (4-3\mu)\frac{T^2(1-b)}{1-k^2}, & \mu \leq \frac{4}{3} - \frac{(1-k^2)(5+T^2)}{9T^2(1-b)}. \end{cases}$$

This result is sharp.

Proof. The proof follows directly from (2.18), (2.19) and Lemma 1.5. □

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