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SKEW LEFT BRACES OF NILPOTENT TYPE

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Abstract. We study series of left ideals of skew left braces that are analogs of upper central series of groups. These concepts allow us to define left and right nilpotent skew left braces. Several results related to these concepts are proved and applications to infinite left braces are given. Indecomposable solutions of the Yang–Baxter equation are explored using the structure of skew left braces.

Contents

Introduction

The Yang–Baxter equation was first introduced in the field of statistical mechanics. It depends on the idea that in some scattering situations particles may preserve their momentum while changing their quantum internal states. The equation states that a matrix R satisfies

$$(R\otimes \mathrm{id})(\mathrm{id}\otimes R)(R\otimes \mathrm{id}) = (\mathrm{id}\otimes R)(R\otimes \mathrm{id})(\mathrm{id}\otimes R).$$

In one dimensional quantum systems, R is the scattering matrix and if it satisfies the Yang–Baxter equation then the system is integrable. The Yang–Baxter equation also appears in topology and algebra mainly because its connections with braid groups. It takes its name from independent work of Yang [34] and Baxter [6].

In [10] Drinfeld observed that the Yang–Baxter equation also makes sense even if R is not a linear operator $V\otimes V\to V\otimes V$ but a map $X\times X\to X\times X$ where X is a set. In this context, non-degenerate solutions are interesting. Recall that a set-theoretic solution of the Yang–Baxter equation (X,r) is said to be *non-degenerate* if $r(x,y)=(\sigma_x(y),\tau_y(x))$ for all $x,y\in X$, where σ_x and τ_y are permutations of X. Permutation solutions are examples of non-degenerate solutions: if X is a set, σ

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and τ are permutations of X such that $\sigma\tau = \tau\sigma$ and $r(x, y) = (\sigma(y), \tau(x))$, then the pair (X, r) is a non-degenerate set-theoretic solution of the Yang–Baxter equation.

The first papers where non-degenerate set-theoretic solutions were studied are [11, 15, 21, 31]. It turns out that set-theoretic solutions have connections for example with groups of I-type, Bieberbach groups, bijective 1-cocycles, Garside theory, etc. In [23] Rump found a new and unexpected connection between set-theoretic solutions and Jacobson radical rings: such rings produce involutive solutions. This observation was the key for introducing a new algebraic structure that generalizes Jacobson radical rings. To strengthen the connection with rings, Rump conceived the name *braces*. New connections appeared, for example with regular subgroups and Hopf–Galois extensions [2], left orderable groups [5], flat manifolds [25].

In [18] braces were generalized to skew left braces and this structure was used to produce and study not necessarily involutive solutions. Skew braces are useful for studying regular subgroups and Hopf–Galois extensions, bijective 1-cocycles, rings and near-rings, triply factorized groups, see for example [30].

A skew left brace is a set with two compatible group structures. One of these groups is known as the *multiplicative group*; the other as the *additive group*. The terminology used in the theory of Hopf–Galois extensions suggest that the additive group determines the *type* of the skew left brace. For example, skew left braces of abelian type are Rump's braces, i.e. those braces with abelian additive group.

A skew left brace is a triple $(A, +, \circ)$, where $(A, +)$ and $(A, \circ)$ are (not necessarily abelian) groups such that the compatibility

$$a \circ (b + c) = a \circ b - a + a \circ c$$

holds for all $a, b, c \in A$.

Radical rings are certain examples of skew left braces where the additive group $(A, +)$ is abelian. For $a, b \in A$ one defines

$$a * b = -a + a \circ b - b.$$

In the case of Jacobson radical rings, the operation $*$ is the multiplication of the ring. For arbitrary skew left braces the operation $*$ is not associative, but nevertheless the relation with radical rings suggests how to translate ideas from ring theory to the world of skew left braces.

In [11, Definition 3.1] Etingof, Schedler and Soloviev introduced multipermutation solutions. As the name suggests, such solutions are generalizations of permutation solutions. Gateva–Ivanova's strong conjecture [13, 2.28(I)] states that square-free involutive non-degenerate set-theoretic solutions are multipermutation solutions. Despite the conjecture was proved to be false [32], it was the motivation of several original investigations [8, 9, 14, 26].

Braces provide a powerful algebraic framework to work with set-theoretic solutions. In [18, Theorem 3.1] it is proved that for a skew left brace A, the map

$$r_A \colon A \times A \to A \times A, \quad r_A(a, b) = (-a + a \circ b, (-a + a \circ b)' \circ a \circ b),$$

is a non-degenerate set-theoretic solution of the Yang–Baxter equation. In [30, Theorem 4.5] it is proved that if (X, r) be a non-degenerate solution of the Yang–Baxter equation, then there exists a unique skew left brace structure over the group

$$G = G(X, r) = \langle X : x \circ y = u \circ v \text{ whenever } r(x, y) = (u, v)\}$$

such that

$$\begin{tikzcd} X\times X \arrow[r, "r"] \arrow[d, "\iota\times\iota"'] & X\times X \arrow[d, "\iota\times\iota"] \\ G\times G \arrow[r, "r_G"] & G\times G \end{tikzcd}$$

where $\iota\colon X\to G(X,r)$ is the canonical map. Moreover, the pair $(G(X,r),\iota)$ has the following universal property: if B is a skew left brace and $f\colon X\to B$ is a map such that

$$\begin{tikzcd} X\times X \arrow[r, "r"] \arrow[d, "f\times f"'] & X\times X \arrow[d, "f\times f"] \\ B\times B \arrow[r, "r_B"] & B\times B \end{tikzcd}$$

then there exists a unique skew left brace homomorphism $\phi\colon G(X,r)\to B$ such that

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & G \arrow[d, "\phi"] \\ & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G\times G \arrow[r, "r_G"] \arrow[d, "\phi\times\phi"'] & G\times G \arrow[d, "\phi\times\phi"] \\ B\times B \arrow[r, "r_B"] & B\times B \end{tikzcd}$$

commute.

To study Gateva–Ivanova's strong conjecture Rump introduced two sequences of left ideals [23]. These sequences turned out to be very important for understanding the structure of Rump's braces. It makes sense to consider similar sequences in the context of skew left braces. The right series of a skew left brace A is defined as the sequence

$$A\supseteq A^{(2)}\supseteq A^{(3)}\supseteq\cdots,$$

where $A^{(n+1)}=A^{(n)}*A$ denotes the additive subgroup of A generated by elements of the form $x*a$ for $a\in A$ and $x\in A^{(n)}$. Each $A^{(n)}$ is an ideal of A, see Proposition 2.1. A skew left brace is said to be *right nilpotent* if there is a positive integer n such that $A^{(n)}=0$. The left series of a skew left brace A is the sequence

$$A\supseteq A^2\supseteq A^3\supseteq\cdots,$$

where $A^{n+1}=A*A^n$. Each A^n is a left ideal of A, see Proposition 2.2. A skew left brace is said to be *left nilpotent* if there is a positive integer n such that $A^n=0$. Following [28] we also define the sequence

$$A\supseteq A^{[2]}\supseteq A^{[3]}\supseteq\cdots,$$

where $A^{[1]}=A$ and $A^{[n+1]}$ is the additive subgroup of A generated by all the elements from the sets $A^{[i]}*A^{[n+1-i]}$ for $1\leq i\leq n$. Each $A^{[n]}$ is a left ideal of A, see Proposition 2.28. A skew left brace A is said to be *strongly nilpotent* if there is a positive integer n such that $A^{[n]}=0$. Theorem 2.30 states that a skew left brace A is strongly nilpotent if and only if it is left and right nilpotent.

The sequence of the $A^{[n]}$ is a basic tool for understanding problems similar to Köthe conjecture in the context of skew left braces, see Section 2.

One of our main goals is to study the connection between left and right nilpotency and the structure of skew left braces. We study the connection between right nilpotent skew left braces and multipermutation solutions (see Theorem 2.20). We also explore the cases where the groups of the skew left brace are nilpotent. If the skew left brace is finite and both groups are nilpotent, then it is possible to write the skew left brace as a direct product of skew left braces of prime-power order (see

Corollary 4.3). This result is then applied to study infinite skew left braces with multiplicative group isomorphic to $\mathbb{Z}$ (see Theorems 5.5 and 5.6); this answers [30, Question A.10]. Left nilpotent skew left braces are also studied. We show that a finite skew left brace with nilpotent additive group is left nilpotent if and only if its multiplicative group is nilpotent (see Theorem 4.8).

This paper is organized as follows. In Section 1 basic definitions are recalled. These definitions include skew left braces, left ideals and ideals. In Section 2 we define left and right nilpotent skew left braces. In Theorem 2.8 and Proposition 2.26 we prove analogs of Hirsch's theorem for nilpotent groups. We use these results in Theorem 2.20 to prove that, under mild assumptions, a skew left brace is right nilpotent if and only if it has finite multipermutation level (see Theorem 2.20). In Section 3 we deal with perfect skew left braces. Using wreath product of skew left braces to show that one cannot produce an analog of Grün's lemma for skew left braces. Skew left braces of nilpotent type are studied in Section 4. Our first result is Corollary 4.3, where we prove that if both groups of a finite skew left brace are nilpotent, then the skew left brace is a direct product of sub skew left braces of prime-power size. Theorem 4.6 is the consequence of applying Hall's results from [19] to the case of skew left braces. We prove in Theorem 4.8 that a finite skew left brace of nilpotent type is left nilpotent if and only if its multiplicative group is nilpotent. In particular, skew left braces of prime-power size are left nilpotent. In Section 5 we prove that skew left braces of abelian type with infinite-cyclic multiplicative group are trivial, see Theorem 5.5. Theorem 5.8 shows that this result cannot be extended to skew left braces. Finally, in Section 6 some of our results are used for studying indecomposable set-theoretic solutions. In this section we give a positive answer to [29, Question 5.6].

1. Preliminaries

Recall that a *skew left brace* is a triple $(A,+,\circ)$, where $(A,+)$ and $(A,\circ)$ are (not necessarily abelian) groups such that

$$a\circ(b+c)=a\circ b-a+a\circ c$$

holds for all $a,b,c\in A$. The group $(A,\circ)$ will be the *multiplicative group* of the skew left brace and $(A,+)$ will be the *additive group* of the skew left brace. We write a' to denote the inverse of a with respect to the circle operation $\circ$. A skew left brace $(A,+,\circ)$ such that $a\circ b=a+b$ for all $a,b\in A$ is said to be *trivial.*

Definition 1.1. *Let $\mathcal{X}$ be a property of groups. A skew left brace A is said to be of $\mathcal{X}$-type if its additive group belongs to $\mathcal{X}$.*

Rump's braces are skew left braces of abelian type.

Convention 1.2. *Skew left braces of abelian type will be called* left braces.

If A is a skew left brace, the multiplicative group acts on the additive group by automorphisms. The map $\lambda\colon(A,\circ)\to\operatorname{Aut}(A,+)$, $a\mapsto\lambda_a$, where $\lambda_a(b)=-a+a\circ b$, is a group homomorphism, see [18, Corollary 1.10].

Remark 1.3. *Let A be a skew left brace. Then the following formulas hold:*

$$a\circ b=a+\lambda_a(b),\qquad a+b=a\circ\lambda_a^{-1}(b),\qquad \lambda_a(a')=-a.$$

Let A be a skew left brace. For $a, b \in A$ let

$$a * b = \lambda_a(b) - b = -a + a \circ b - b.$$

The following identities are easily verified:

$$a * (b + c) = a * b + b + a * c - b, \tag{1.1}$$

$$(a \circ b) * c = (a * (b * c)) + b * c + a * c, \tag{1.2}$$

These identities are similar to the usual commutator identities.

Theorem 1.4. *Let A be an additive (not necessarily abelian) group. A skew left brace structure over A is equivalent to an operation $A \times A \to A$, $(a, b) \mapsto a * b$, such that $a * (b + c) = a * b + b + a * c - b$ holds for all $a, b, c \in A$, and the operation $a \circ b = a + a * b + b$ turns A into a group.*

Proof. It is straightforward. □

A *left ideal* of a skew left brace A is a subgroup I of the additive group of A such that $\lambda_a(I) \subseteq I$ for all $a \in A$. It is not hard to prove that a left ideal is a subgroup of the multiplicative group of the skew left brace. An *ideal* of A is a left ideal I of A such that $a \circ I = I \circ a$ and $a + I = I + a$ for all $a \in A$.

Definition 1.5. *For a skew left brace A let*

$$\mathrm{Fix}(A) = \{a \in A : \lambda_x(a) = a \textit{ for all } x \in A\}.$$

Proposition 1.6. *Let A be a skew left brace. Then* $\mathrm{Fix}(A)$ *is a left ideal of A.*

Proof. A routine calculation proves that $\mathrm{Fix}(A)$ is a subgroup of the additive group of A. Clearly $\lambda_x(\mathrm{Fix}(A)) \subseteq \mathrm{Fix}(A)$ for all $x \in A$. □

The following example shows that in general $\mathrm{Fix}(A)$ is not an ideal:

Example 1.7. *Consider the semidirect product $A = \mathbb{Z}/(3) \rtimes \mathbb{Z}/(2)$ of the trivial braces $\mathbb{Z}/(3)$ and $\mathbb{Z}/(2)$ via the non-trivial action of $\mathbb{Z}/(2)$ over $\mathbb{Z}/(3)$. We have*

$$\lambda_{(x,y)}(a, b) = (x, y)(a, b) - (x, y) = (x + (-1)^y a, y + b) - (x, y) = ((-1)^y a, b).$$

Hence $\mathrm{Fix}(A) = \{(0, b) \mid b \in \mathbb{Z}/(2)\}$. *Clearly* $\mathrm{Fix}(A)$ *is not a normal subgroup of the multiplicative group of A. Thus* $\mathrm{Fix}(A)$ *is not an ideal of A.*

If X and Y are subsets of a skew left brace A, $X * Y$ is the additive subgroup of A generated by elements of the form $x * y$, $x \in X$ and $y \in Y$, i.e.

$$X * Y = \langle x * y : x \in X \, , y \in Y \rangle_+.$$

Lemma 1.8. *Let A be a skew left brace. A subgroup I of the additive group of A is a left ideal of A if and only if $A * I \subseteq I$.*

Proof. Let $a \in A$ and $x \in I$. If I is a left ideal, then $a * x = \lambda_a(x) - x \in I$. Conversely, if $A * I \subseteq I$, then $\lambda_a(x) = a * x + x \in I$. □

Lemma 1.9. *Let A be a skew left brace. A normal subgroup I of the additive group of A is an ideal of A if and only $\lambda_a(I) \subseteq I$ for all $a \in A$ and $I * A \subseteq I$.*

Proof. Let $x \in I$ and $a \in A$. Assume first that I is invariant under the action of λ and that $I * A \subseteq I$. Then

$$\begin{aligned} a \circ x \circ a' &= a + \lambda_a(x \circ a') \\ &= a + \lambda_a(x + \lambda_x(a')) = a + \lambda_a(x) + \lambda_a\lambda_x(a') + a - a \\ &= a + \lambda_a(x + \lambda_x(a') - a') - a = a + \lambda_a(x + x * a') - a. \end{aligned} \tag{1.3}$$

and hence I is an ideal.

Conversely, assume that I is an ideal. Then $I * A \subseteq I$ since

$$\begin{aligned} x * a &= -x + x \circ a - a \\ &= -x + a \circ (a' \circ x \circ a) - a = -x + a + \lambda_a(a' \circ x \circ a) - a \in I. \end{aligned}$$

This completes the proof. □

The socle of a skew left brace A is defined as

$$\operatorname{Soc}(A) = \{x \in A \colon x \circ a = x + a \text{ and } x + a = a + x, \text{ for all } a \in A\}.$$

Clearly $\operatorname{Soc}(A) = \ker(\lambda) \cap Z(A, +)$. In [18, Lemma 2.5] it is proved that $\operatorname{Soc}(A)$ is an ideal of A.

Lemma 1.10. *Let A be a skew left brace and $a \in \operatorname{Soc}(A)$. Then $b + b \circ a = b \circ a + b$ and $\lambda_b(a) = b \circ a \circ b'$ for all $b \in A$.*

Proof. Let $b \in A$. Since $b' \circ (b \circ a + b) = a - b'$ and $b' \circ (b + b \circ a) = -b' + a$, the first claim follows since $a \in Z(A, +)$. To prove the second claim one computes:

$$b \circ a \circ b' = b \circ (a \circ b') = b \circ (a + b') = b \circ a - b = -b + b \circ a = \lambda_b(a),$$

and the lemma is proved. □

2. Left and right nilpotent skew left braces

Let A be a skew left brace. Following [23] one defines $A^{(1)} = A$ and for $n \geq 1$

$$A^{(n+1)} = A^{(n)} * A = \langle x * a : x \in A^{(n)},\, a \in A \rangle_+,$$

where $\langle X \rangle_+$ denotes the subgroup of the additive group of A generated by the subset X. The series $A^{(1)} \supseteq A^{(2)} \supseteq A^{(3)} \supseteq \cdots \supseteq A^{(n)} \cdots$ is the *right series* of A.

Proposition 2.1. *Let A be a skew left brace. Each $A^{(n)}$ is an ideal of A.*

Proof. We want to prove that for each $n \in \mathbb{N}$, $A^{(n)}$ is a normal subgroup of $(A, +)$, that $\lambda_a(A^{(n)}) \subseteq A^{(n)}$ for all $a \in A$ and that $A^{(n)}$ is a normal subgroup of $(A, \circ)$. We proceed by induction on n. The case $n = 1$ is trivial. We assume that the claim is true for some $n \geq 1$. We first prove that $A^{(n+1)}$ is a normal subgroup of $(A, +)$. Let $a, b \in A$ and $x \in A^{(n)}$. Then $a + x * b - a \in A^{(n+1)}$ since

$$\begin{aligned} a + x * b - a &= a + \lambda_x(b) - b - a \\ &= a + \lambda_x(b) - (a + b) = a + \lambda_x(-a + a + b) - (a + b) \\ &= a + \lambda_x(-a) + \lambda_x(a + b) - (a + b) = -x * a + x * (a + b). \end{aligned}$$

Now we prove that $A^{(n+1)}$ is an ideal. Let $a, b \in A$ and $x \in A^{(n)}$. Then

$$\begin{aligned} \lambda_a(x * b) &= \lambda_a(\lambda_x(b) - b) = \lambda_a\lambda_x(b) - \lambda_a(b) \\ &= \lambda_{a \circ x \circ a'}(\lambda_a(b)) - \lambda_a(b) = (a \circ x \circ a') * \lambda_a(b) \in A^{(n+1)} \end{aligned} \tag{2.1}$$

since $a \circ x \circ a' \in A^{(n)}$ by the inductive hypothesis. From this it immediately follows that $\lambda_a(A^{(n+1)}) \subseteq A^{(n+1)}$. Now let $y \in A^{(n+1)}$. By using (1.3) one obtains that $a \circ y \circ a' = a + \lambda_a(y + y * a') - a \in A^{(n+1)}$. Thus the result follows by induction. □

Let A be a skew left brace. Following [23] one defines $A^1 = A$ and for $n \geq 1$

$$A^{n+1} = A * A^n = \langle a * x : a \in A,\, x \in A^n \rangle_+.$$

The series $A^1 \supseteq A^2 \supseteq A^3 \supseteq \cdots \supseteq A^n \supseteq \cdots$ is the *left series* of A.

Proposition 2.2. *Let A be a skew left brace. Each A^n is a left ideal of A.*

Proof. We proceed by induction on n. The case $n = 1$ is trivial, so we may assume that the result is true for some $n \geq 1$. Let $a, b \in A$ and $x \in A^n$. By the inductive hypothesis, $\lambda_a(x) \in A^n$ and hence

$$\lambda_a(b * x) = (a \circ b \circ a') * \lambda_a(x) \in A^{n+1},$$

where the equality follows by (2.1). This implies that $\lambda_a(A^{n+1}) \subseteq A^{n+1}$. Thus the result follows by induction. □

The second term of the left series is particularly important:

Proposition 2.3. *Let A be a skew left brace. Then A^2 is the smallest ideal of A such that A/A^2 is a trivial skew left brace.*

Proof. Since $A^2 = A^{(2)}$, A^2 is an ideal by Proposition 2.1. Let I be an ideal of A and $\pi\colon A \to A/I$ be the canonical map. Then A/I is trivial as a skew left brace if and only if $\lambda_a(b) - b \in I$ for all $a, b \in A$. Since this condition is equivalent to $A^2 \subseteq I$, the claim follows. □

Definition 2.4. *A skew left brace A is said to be* right nilpotent *if $A^{(m)} = 0$ for some $m \geq 1$.*

Lemma 2.5. *Let $f\colon A \to B$ be a surjective homomorphism of skew left braces. Then $f(A^{(k)}) = B^{(k)}$ for all k. In particular, if A is right nilpotent, then B is right nilpotent.*

Proof. We proceed by induction on k. The case $k = 1$ is trivial. Let us assume that the result is valid for some $k \geq 1$. Since $f(A^{(k)}) = B^{(k)}$,

$$f(A^{(k+1)}) = f(A^{(k)} * A) = f(A^{(k)}) * f(A) = B^{(k)} * B = B^{(k+1)}.$$

From this the second claim follows. □

Lemma 2.6. *Let A be a right nilpotent skew left brace and $B \subseteq A$ be a sub skew left brace. Then B is right nilpotent.*

Proof. By induction, $B^{(k)} \subseteq A^{(k)}$ for all k. Hence the claim follows. □

Lemma 2.7. *Let $A_1, \dots, A_k$ be right nilpotent skew left braces. Then the direct product $A_1 \times \cdots \times A_k$ is right nilpotent.*

Proof. It is enough to prove the lemma in the case where $k = 2$. This case is trivial since $(a, b) * (c, d) = (a * c, b * d)$. □

Theorem 2.8. *Let A be a right nilpotent skew left brace of nilpotent type and I be a non-zero ideal of A. Then $I \cap \operatorname{Soc}(A) \neq 0$.*

Proof. Since $(A,+)$ is nilpotent and each $I\cap A^{(k)}$ is a normal subgroup of $(A,+)$, it follows from [22, 5.2.1] that $I\cap A^{(k)}\cap Z(A,+)\neq 0$ whenever $I\cap A^{(k)}\neq 0$. Let $m=\max\{k\in\mathbb{N}: I\cap A^{(k)}\cap Z(A,+)\neq 0\}$. Since

$$(I\cap A^{(m)}\cap Z(A,+))*A\subseteq I\cap (A^{(m)}*A)=I\cap A^{(m+1)}=0,$$

it follows that $I\cap A^{(m)}\cap Z(A,+)\subseteq I\cap \operatorname{Soc}(A)$. □

Corollary 2.9. *Let A be a non-zero right nilpotent skew left brace of nilpotent type. Then* $\operatorname{Soc}(A)\neq 0$.

Proof. It follows directly from Theorem 2.8 □

Corollary 2.10. *Let A be a right nilpotent skew left brace of nilpotent type and I be a minimal ideal of A. Then $I\subseteq \operatorname{Soc}(A)$.*

Proof. Since $I\cap\operatorname{Soc}(A)$ is a non-zero ideal of A by Theorem 2.8, $I\cap\operatorname{Soc}(A)=I$ by the minimality of I. □

Definition 2.11. *Let A be a skew left brace. A* s-series *of A is a sequence*

$$A=I_0\supseteq I_1\supseteq I_2\supseteq\cdots\supseteq I_n=0$$

of ideals of A such that $I_{j-1}/I_j\subseteq\operatorname{Soc}(A/I_j)$ for each $j\in\{1,\dots,n\}$.

Remark 2.12. *Let A be a left brace. Rump in* [23] *defined the socle series* $\operatorname{Soc}_n(A)$ *of A as follows:* $\operatorname{Soc}_0(A)=0$ *and, for $n\geq 1$,*

$$\operatorname{Soc}_n(A)=\{x\in A\colon x*y\in\operatorname{Soc}_{n-1}(A)\}.$$

There are examples of nonzero left braces A such that $\operatorname{Soc}_n(A)=0$ *for all positive integers n.*

Definition 2.13. *Let A be a skew left brace. We define* $\operatorname{Soc}_0(A)=0$ *and, for $n\geq 1$,* $\operatorname{Soc}_n(A)$ *is the ideal of A containing* $\operatorname{Soc}_{n-1}(A)$ *such that*

$$Soc_n(A)/\operatorname{Soc}_{n-1}(A)=\operatorname{Soc}(A/\operatorname{Soc}_{n-1}(A)).$$

Note that this definition coincides with the definition of Rump for left braces.

Lemma 2.14. *Let A be a skew left brace and let $A=I_0\supseteq I_1\supseteq I_2\supseteq\cdots\supseteq I_n=0$ be a s-series for A. Then $A^{(i+1)}\subseteq I_i$ for all i.*

Proof. We proceed by induction on i. The case $i=0$ is trivial, so let us assume that the result holds for some $i\geq 0$. Let $\pi\colon A\to A/I_{i+1}$ be the canonical map. Since $\pi(I_i)\subseteq\operatorname{Soc}(A/I_{i+1})$, $\pi(I_i*A)=\pi(I_i)*\pi(A)=0$ and hence $I_i*A\subseteq I_{i+1}$. The inductive hypothesis then implies that $A^{(i+2)}=A^{(i+1)}*A\subseteq I_i*A\subseteq I_{i+1}$. Thus the result follows by induction. □

Lemma 2.15. *Let A be a skew left brace. Then A admits a s-series if and only if there exists a positive integer n such that $A=\operatorname{Soc}_n(A)$.*

Proof. Suppose that there exists a positive integer n such that $A=\operatorname{Soc}_n(A)$. Then

$$A=\operatorname{Soc}_n(A)\supseteq\operatorname{Soc}_{n-1}(A)\supseteq\cdots\supseteq\operatorname{Soc}_0(A)=0,$$

is a s-series.

Conversely, suppose that A admits a s-series. Let

$$A=I_0\supseteq I_1\supseteq I_2\supseteq\cdots\supseteq I_n=0$$

be a s-series of A. We shall prove that $I_{n-j} \subseteq \operatorname{Soc}_j(A)$ by induction on j. For $j = 0$, $I_n = 0 = \operatorname{Soc}_0(A)$. Suppose that $j > 0$ and $I_{n-j+1} \subseteq \operatorname{Soc}_{j-1}(A)$. Since $I_{n-j}/I_{n-j+1} \subseteq \operatorname{Soc}(A/I_{n-j+1})$, $I_{n-j} * A \subseteq I_{n-j+1} \subseteq \operatorname{Soc}_{j-1}(A)$, by the induction hypothesis. Furthermore, for all $x \in A$ and all $y \in I_{n-j}$, $x + y - x - y \in I_{n-j+1} \subseteq \operatorname{Soc}_{j-1}(A)$. Therefore $I_{n-j} \subseteq \operatorname{Soc}_j(A)$. Hence $A = I_0 = \operatorname{Soc}_n(A)$ and the result follows. □

Lemma 2.16. *A skew left brace of nilpotent type is right nilpotent if and only if it admits a s-series.*

Proof. Let A be a skew left brace of nilpotent type. If A admits a s-series, then A is right nilpotent by Lemma 2.14.

Conversely, suppose that A is right nilpotent. There exists a positive integer such that $A^{(m)} = 0$. We shall prove that A admits a s-series by induction on m. For $m = 1$, $A = A^{(1)} = 0$ is a s-series. Suppose that $m > 1$ and that the result is true for $m - 1$. Consider $\bar{A} = A/A^{(m-1)}$. Since $\bar{A}^{(m-1)} = 0$, by the induction hypothesis $\bar{A}$ admits a s-series. Thus there is a sequence

$$A = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n = A^{(m-1)}$$

of ideals of A such that $I_{j-1}/I_j \subseteq \operatorname{Soc}(A/I_j)$ for each $j \in \{1, \dots, n\}$. Since $A^{(m)} = 0$, we have that $A^{(m-1)} \subseteq \ker(\lambda)$. Since A is of nilpotent type, there exists a positive integer s such that $\gamma_s^+(A) = 0$, where $\gamma_i^+(A)$ denotes the lower central series of the additive group of A, that is $\gamma_1^+(A) = A$ and $\gamma_{i+1}^+(A) = [A, \gamma_i^+(A)]_+$, for all positive integers i. Let $I_{n+j-1} = A^{(m-1)} \cap \gamma_j^+(A)$ for $j = 1, \dots, s$. Note that I_{n+j-1} is a normal subgroup of the additive group of A invariant by λ_x, for all $x \in A$, and $I_{n+j-1} * A = 0$, for all $j = 1, \dots, s$, because $A^{(m-1)} \subseteq \ker(\lambda)$. By Lemma 1.9, I_{n+j-1} is an ideal of A, for all $j = 1, \dots, s$. Note that $I_{n+j-1}/I_{n+j} \subseteq Z(A/I_{n+j}, +)$, for all $j = 1, \dots, s-1$. Therefore, since $I_{n+j-1} \subseteq \ker(\lambda)$, we have that $I_{n+j-1}/I_{n+j} \subseteq \operatorname{Soc}(A/I_{n+j})$, for all $j = 1, \dots, s-1$. Hence

$$A = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n = A^{(m-1)} \supseteq I_{n+1} \supseteq \cdots \supseteq I_{n+s-1} = 0$$

is a s-series of A, and the result follows by induction. □

Proposition 2.17. *Let A be a skew left brace such that $A/\operatorname{Soc}(A)$ is right nilpotent. Then A is right nilpotent.*

Proof. Note that $(A/\operatorname{Soc}(A))^{(k)} = 0$ if and only if $A^{(k)} \subseteq \operatorname{Soc}(A)$ by the definition of the factor brace. Then $A^{(k+1)} = A^{(k)} * A \subseteq \operatorname{Soc}(A) * A = 0$ as required. □

Proposition 2.18. *Let I be an ideal of a skew left brace A such that $I \cap A^2 = 0$. Then I is a trivial skew left brace.*

Proof. Since $I * A \subseteq I \cap A^2 = 0$, $I \subseteq \ker \lambda$. From this the claim follows. □

A skew left brace A has *finite multipermutation level* if the sequence S_n defined as $S_1 = A$ and $S_{n+1} = S_n/\operatorname{Soc}(S_n)$ for $n \geq 1$, reaches zero.

Proposition 2.19. *Let A be a skew left brace. Then A has finite multipermutation level if and only if A admits a s-series.*

Proof. Let $S_1 = A$ and $S_{n+1} = S_n/\operatorname{Soc}(S_n)$, for $n \geq 1$. We shall prove that $S_{n+1} \simeq A/\operatorname{Soc}_n(A)$, by induction on n. For $n = 0$, it is clear since $\operatorname{Soc}_0(A) = 0$.

Suppose that $n > 0$ and the result is true for $n-1$. Hence, by the induction hypothesis,

$$\begin{aligned} S_n &= S_{n-1}/\mathrm{Soc}(S_{n-1}) \simeq (A/\mathrm{Soc}_{n-1}(A))/\mathrm{Soc}(A/\mathrm{Soc}_{n-1}(A)) \\ &= (A/\mathrm{Soc}_{n-1}(A))/(\mathrm{Soc}_n(A)/\mathrm{Soc}_{n-1}(A)) \simeq A/\mathrm{Soc}_n(A). \end{aligned}$$

Therefore $S_n = 0$ if and only if $A = \mathrm{Soc}_n(A)$. Now the result follows from Lemma 2.15. □

Theorem 2.20. *Let A be a skew left brace. Then A has finite multipermutation level, if and only if A is right nilpotent and $(A,+)$ is nilpotent.*

Proof. Suppose that A has finite multipermutation level. We proceed by induction on the multipermutation level n. The case $n = 1$ is trivial. Let A be a skew left brace of finite multipermutation level $n+1$. Since $A/\mathrm{Soc}(A)$ has multipermutation level n, the inductive hypothesis implies that $(A/\mathrm{Soc}(A))^{(m)} = 0$ for some m and $(A/\mathrm{Soc}(A),+)$ is nilpotent. This implies that $A^{(m)} \subseteq \mathrm{Soc}(A)$ and hence $A^{(m+1)} = 0$, furthermore, since $\mathrm{Soc}(A)$ is central in $(A,+)$, we have that $(A,+)$ is nilpotent. Conversely, suppose that A is right nilpotent and $(A,+)$ is nilpotent. By Lemma 2.16, A admits a s-series. Thus the result follows by Proposition 2.19. □

The following example shows that the assumption on the nilpotency of the additive group of the skew left brace is needed for Theorem 2.20.

Example 2.21. *Let A be a non-zero skew left brace such that $a \circ b = a + b$ for all $a, b \in A$. Then $A^{(2)} = 0$, thus A is right nilpotent. But if $Z(A,+) = 0$, then $\mathrm{Soc}(A) = 0$ and A does not have finite multipermutation level. For example, we can take $(A,+) = (A,\circ)$ any non-abelian simple group.*

Definition 2.22. *A skew left brace A is said to be* left nilpotent *if $A^m = 0$ for some $m \geq 1$.*

Lemma 2.23. *Let $f\colon A \to B$ be a surjective homomorphism of skew left braces. Then $f(A^k) = B^k$ for all k. In particular, if A is left nilpotent, then B is left nilpotent.*

Proof. It is similar to the proof of Lemma 2.5. □

Lemma 2.24. *Let A be a left nilpotent skew left brace and $B \subseteq A$ be a sub skew left brace. Then B is left nilpotent.*

Proof. It is similar to the proof of Lemma 2.6. □

Lemma 2.25. *Let $A_1, \dots, A_k$ be left nilpotent skew left braces. Then the direct product $A_1 \times \cdots \times A_k$ is left nilpotent.*

Proof. It is similar to the proof of Lemma 2.7. □

Proposition 2.26. *Let A be a left nilpotent skew left brace and I be a non-zero left ideal of A. Then $I \cap \mathrm{Fix}(A) \neq 0$.*

Proof. Let $m = \max\{k : I \cap A^k \neq 0\}$. Since $A * (I \cap A^m) \subseteq I \cap A^{m+1} = 0$, it follows that there exists a non-zero $x \in I \cap A^m$ such that $a * x = 0$ for all $a \in A$. Thus $0 \neq x \in \mathrm{Fix}(A) \cap I$. □

Corollary 2.27. *Let A be a non-zero skew left brace. If A is left nilpotent, then $\mathrm{Fix}(A) \neq 0$.*

Proof. Apply Proposition 2.26 with $I = A$. □

Another important series of ideals was defined in [28] for braces. Let A be a skew left brace. Let $A^{[1]} = A$ and for $n \geq 1$ let $A^{[n+1]}$ be the additive subgroup of A generated by elements from $\{A^{[i]} * A^{[n+1-i]} : 1 \leq i \leq n\}$, i.e.

$$A^{[n+1]} = \left\langle \bigcup_{i=1}^{n} A^{[i]} * A^{[n+1-i]} \right\rangle_{+}$$

for all $n \geq 2$. One easily proves by induction that $A^{[1]} \supseteq A^{[2]} \supseteq \cdots$.

Proposition 2.28. *Let A be a skew left brace. Each $A^{[n]}$ is a left ideal of A.*

Proof. Each $A^{[n]}$ is a subgroup of $(A,+)$. Since $A * A^{[n]} \subseteq A^{[n+1]} \subseteq A^{[n]}$, the claim follows from Lemma 1.8. □

We will show that there exists a left brace A such that $A^{[n]} = A^{[n+1]}$ are non-zero for some positive integer n and $A^{[n+2]} = 0$.

Example 2.29. *Let*

$$G = \langle r, s : r^8 = s^2 = 1,\ srs = r^7 \rangle \simeq \mathbb{D}_{16},$$
$$X = \langle a, b : 8a = 2b = 0,\ a + b = b + a \rangle \simeq \mathbb{Z}/(8) \times \mathbb{Z}/(2).$$

The group G acts by automorphisms on X via

$$r \cdot a = a + b, \quad r \cdot b = 4a + b, \quad s \cdot a = 3a, \quad s \cdot b = 4a + b.$$

A direct calculation shows that the map $\pi \colon G \to X$ given by

$$\begin{array}{cccc}
1 \mapsto 0, & r \mapsto a, & r^2 \mapsto 2a + b, & r^3 \mapsto 7a + b, \\
r^4 \mapsto 4a, & r^5 \mapsto 5a, & r^6 \mapsto 6a + b, & r^7 \mapsto 3a + b, \\
rs \mapsto 6a, & r^2 s \mapsto 7a, & r^3 s \mapsto b, & r^4 s \mapsto 5a + b, \\
r^5 s \mapsto 2a, & r^6 s \mapsto 3a, & r^7 s \mapsto 4a + b, & s \mapsto a + b,
\end{array}$$

is a bijective 1-cocycle. Therefore there exists a left brace A with additive group isomorphic to X and multiplicative group isomorphic to G. The addition of A is that of X and the multiplication is given by

$$x \circ y = \pi(\pi^{-1}(x)\pi^{-1}(y)), \quad x, y \in X.$$

Since

$$a * a = -a + a \circ a - a = -a + (2a + b) - a = b,$$
$$(5a + b) * a = -(5a + b) + (5a + b) \circ a - a = -(5a + b) + b - a = 2a,$$

it follows that $A^{[2]}$ contains $\langle 2a, b \rangle_+ = \{0, 2a, 4a, 6a, b, 2a + b, 4a + b, 6a + b\}$, the additive subgroup of $(A,+)$ generated by $2a$ and b. Therefore $A^{[2]} = \langle 2a, b \rangle_+$ since $A^{[2]} \neq A$ (this can be proved by hand or using Theorem 4.8). Routine calculations prove that

$$A^{[3]} = \{0, 2a + b, 4a, 6a + b\}, \qquad A^{[4]} = A^{[5]} = \{0, 4a\}, \qquad A^{[6]} = \{0\}.$$

The relation between the sequence of the $A^{[n]}$ and the left and right series is given in the following theorem.

Theorem 2.30. *Let A be a skew left brace. The following statements are equivalent:*

(1) $A^{[\alpha]}=0$ *for some* $\alpha\in\mathbb{N}$.
(2) $A^{(\beta)}=0$ *and* $A^{\gamma}=0$ *for some* $\beta,\gamma\in\mathbb{N}$.

Proof. To prove that (1) $\implies$ (2) one proves by induction that $A^n\subseteq A^{[n]}$ and $A^{(n)}\subseteq A^{[n]}$ for all positive integer n. Let us prove that (2) $\implies$ (1). We proceed by induction on β. If $\beta\in\{1,2\}$, then $0=A^{(2)}=A^2=A^{[2]}$ and the result is true. Fix $\beta\in\mathbb{N}$ and suppose that the result holds for this β, so for every γ there exists $\alpha=\alpha(\gamma)$ depending on γ such that $A^{[\alpha]}=0$. We need to show that $A^{(\beta+1)}=0$ and $A^{\gamma}=0$ imply that $A^{[n]}$ for some n. Let $n>\alpha(\gamma)$. Every element of $A^{[n]}$ is a sum of elements from $A^{[i]}*A^{[j]}$, where $i+j=n$ and $1\leq i\leq n-1$. Note that if $\alpha(\gamma)\leq i\leq n-1$, $a_i\in A^{[i]}$ and $a_{n-i}\in A^{[n-i]}$, then by the inductive hypothesis applied to the quotient $A/A^{(\beta)}$, $a_i\in A^{(\beta)}$ and thus $a_i*a_{n-i}\in A^{(\beta+1)}=0$. Hence we may assume that the elements of $A^{[n]}$ are sums of elements from $A^{[i]}*A^{[j]}$ for $1\leq i<\alpha(\gamma)$ and $j\geq n-\alpha(\gamma)$ such that $i+j=n$. Then

$$A^{[n]}\subseteq A*A^{[n-\alpha(\gamma)]}\subseteq A^2.$$

Applying the same argument for $n'=n-\alpha(\gamma)$ we obtain that $A^{[n']}\subseteq A*A^{[n'-\alpha(\gamma)]}$ provided that $n'>\alpha(\gamma)$. Therefore

$$A^{[n]}\subseteq A*(A*A^{[n-2\alpha(\gamma)]})\subseteq A^3$$

provided that $n>2\alpha(\gamma)$. Continuing in this way we obtain that $A^{[n]}\subseteq A^k$ provided that $n>(k-1)\alpha(\gamma)$. Then it follows that $A^{[(\gamma-1)\alpha(\gamma)+1]}\subseteq A^{\gamma}=0$. □

Definition 2.31. *A skew left brace is said to be* left nil *if for every* $a\in A$ *we have* $a*(a*(a*\cdots))=0$ *(for sufficiently large number of brackets in this equation).*

It is known that every finite left nil left brace is left nilpotent, see [27]. Right nil skew left braces are defined in a similar fashion.

Definition 2.32. *A skew left brace* A *is said to be* strongly nilpotent *if there is a positive integer* n *such that* $A^{[n]}=0$.

Definition 2.33. *A skew left brace* A *is said to be* strongly nil *if for every* $a\in A$ *there is a positive integer* $n=n(a)$ *such that any* $*$*-product of* n *copies of* a *is zero.*

We do not know the answer to the following questions:

Question 2.34. *Let* A *be a finite right nil skew left brace. Is* A *right nilpotent?*

Question 2.35. *Let* A *be a finite strongly nil skew left brace. Is* A *strongly nilpotent?*

3. Perfect skew left braces

A skew left brace A is said to be perfect if $A^2=A$. Let G be a perfect group, that is $G=[G,G]$. By Grün's Lemma (see [17, page 3]), $Z(G/Z(G))=\{1\}$. Let B be the skew left brace with multiplicative group G and addition defined by $a+b=ba$, for all $a,b\in G$. In B we have $a*b=-a+ab-b=b^{-1}aba^{-1}$, for all $a,b\in G$. Hence the multiplicative group of $B*B$ is $[G,G]$. Note also that $x\in\operatorname{Soc}(B)$ if and only if $1=a^{-1}xax^{-1}$, for all $a\in G$. Thus $\operatorname{Soc}(B)=Z(G)$. Since $Z(G/Z(G))=\{1\}$, we have that $\operatorname{Soc}(B/Soc(B))=\{1\}$.

Thus the following question appears to be natural.

Question 3.1. *Let* A *be a perfect skew left brace. Is* $\operatorname{Soc}(A/Soc(A))=0$*?*

We shall see that the answer is negative.

Let B_1 and B_2 be skew left braces. Recall that the wreath product $B_2 \wr B_1$ of the left braces B_2 and B_1 is a left brace which is the semidirect product of left braces $H_2 \rtimes B_1$, where $H_2 = \{f \colon B_1 \longrightarrow B_2 \mid |\{g \in B_1 \mid f(g) \neq 0\}| < \infty\}$ is a left brace with the operations $(f_1 \circ f_2)(g) = f_1(g) \circ f_2(g)$ and $(f_1 + f_2)(g) = f_1(g) + f_2(g)$, for all $f_1, f_2 \in B_2$ and $g \in B_1$, and the action of $(B_1, \circ)$ on H_2 is given by the homomorphism $\sigma \colon (B_1, \circ) \longrightarrow \mathrm{Aut}(H_2, +, \circ)$ defined by $\sigma(g)(f)(x) = f(g' \circ x)$, for all $g, x \in B_1$ and $f \in H_2$. Recall that the operations of $H_2 \rtimes B_1$ are

$$(h_2, b_1) \circ (k_2, c_1) = (h_2 \circ b_1(k_2), b_1 \circ c_1),$$
$$(h_2, b_1) + (k_2, c_1) = (h_2 + k_2, b_1 + c_1),$$

where we denote $\sigma(b_1)(k_2)$ simply by $b_1(k_2)$.

The wreath product of left braces appears in [9, Corollary 1] (see also [4, Section 4]). This construction also works for skew left braces (see [30, Corollary 2.39]).

Theorem 3.2. *Let B be a finite perfect left skew left brace. Let p be an odd prime non-divisor of the order of B. Let $T = \mathbb{Z}/(p)$ be the trivial left brace of order p. Then the subbrace $W \rtimes B$ of $T \wr B$, where $W = \{f \colon B \longrightarrow T \mid \sum_{b \in B} f(b) = 0\}$, is perfect and* $\mathrm{Soc}(W \rtimes B) = W \times \{0\}$.

Proof. Note that

$$(h_1, b_1) * (h_2, b_2) = (-h_1 + h_1 \circ b_1(h_2) - h_2, b_1 * b_2), \tag{3.1}$$

for all $h_1, h_2 \in W$ and all $b_1, b_2 \in B$. In particular,

$$\{0\} \times B = \{0\} \times (B * B) = (\{0\} \times B) * (\{0\} \times B) \subseteq (W \rtimes B) * (W \rtimes B).$$

Let $f_{b_1} : B \longrightarrow T$ be the function defined by $f_{b_1}(b_2) = \delta_{b_1, b_2}$ (the Kronecker delta), for all $b_1, b_2 \in B$. Note that $\{f_b - f_0 \mid b \in B\}$ generates the additive group of W. Since p is not a divisor of $|B|$, there exists $|B|^{-1} \in \mathbb{Z}/(p) = T$, and $f_0 - |B|^{-1} \sum_{b \in B} f_b \in W$ (note that the additive group of W is a vector space over $\mathbb{Z}/(p)$). Now we have

$$(0, b_1) * (f_0 - |B|^{-1} \sum_{b \in B} f_b, 0) = (b_1(f_0 - |B|^{-1} \sum_{b \in B} f_b) - (f_0 - |B|^{-1} \sum_{b \in B} f_b), 0)$$

and

$$\begin{aligned} b_1(f_0 - |B|^{-1} \sum_{b \in B} f_b)(b_2) &= (f_0 - |B|^{-1} \sum_{b \in B} f_b)(b_1' \circ b_2) \\ &= f_0(b_1' \circ b_2) - |B|^{-1} \sum_{b \in B} f_b(b_1' \circ b_2) \\ &= f_{b_1}(b_2) - |B|^{-1} \\ &= f_{b_1}(b_2) - |B|^{-1} \sum_{b \in B} f_b(b_2) \\ &= (f_{b_1} - |B|^{-1} \sum_{b \in B} f_b)(b_2) \end{aligned}$$

for all $b_1, b_2 \in B$. Hence

$$(0, b_1) * (f_0 - |B|^{-1} \sum_{b \in B} f_b, 0) = (f_{b_1} - f_0, 0),$$

for all $b_1 \in B$. Thus $W \times \{0\} \subseteq (W \rtimes B) * (W \rtimes B)$. Hence

$$W \rtimes B = W \times \{0\} + \{0\} \times B \subseteq (W \rtimes B) * (W \rtimes B).$$

Therefore $W \rtimes B$ is perfect.

Let $(h_1, b_1) \in \mathrm{Soc}(W \rtimes B)$. Then, by (3.1), $h_1 \circ b_1(h_2) = h_1 + h_2$ and $b_1 * b_2 = 0$, for all $h_2 \in W$ and all $b_2 \in B$. Hence

$$h_1(b) + h_2(b) = h_1(b) \circ b_1(h_2)(b) = h_1(b) \circ h_2(b_1' \circ b) = h_1(b) + h_2(b_1' \circ b),$$

and thus

$$h_2(b) = h_2(b_1' \circ b),$$

for all $h_2 \in W$ and all $b \in B$. In particular, $(f_{b_1} - f_0)(b_1) = (f_{b_1} - f_0)(0)$. Suppose that $b_1 \neq 0$. Then $1 = (f_{b_1} - f_0)(b_1) = (f_{b_1} - f_0)(0) = -1$ in $\mathbb{Z}/(p)$, a contradiction because p is odd. Hence $b_1 = 0$. Note that by (3.1),

$$(h_1, 0) * (h_2, b_2) = (-h_1 + h_1 \circ 0(h_2) - h_2, b_1 * 0) = (-h_1 + h_1 \circ h_2 - h_2, 0) = (0, 0),$$

for all $h_1, h_2 \in W$ and $b_2 \in B$. Hence $\mathrm{Soc}(W \rtimes B) = W \times \{0\}$, and the result follows. □

Note that in Theorem 3.2, if B is a left brace, then $W \rtimes B$ also is a left brace. The following result answers Question 3.1 in the negative:

Corollary 3.3. *For every positive integer n, there exists a finite perfect left brace B such that $\mathrm{Soc}(B/\mathrm{Soc}_n(B)) \neq \{0\}$.*

Proof. We shall prove the result by induction on n. For $n = 1$, let B_0 be a finite simple non-trivial left brace (see [3]). Then by Theorem 3.2, there exists a perfect finite left brace B_1 with non-zero socle. By Theorem 3.2, there exists a perfect finite left brace B_2 with non-zero socle such that $B_2/\mathrm{Soc}(B_2) \cong B_1$. Therefore $\mathrm{Soc}(B_2/\mathrm{Soc}_1(B_2)) \cong \mathrm{Soc}(B_1) \neq \{0\}$, and this proves the result for $n = 1$.

Suppose that $n > 1$ and that there exists a perfect finite left brace B_n with $\mathrm{Soc}(B_n/\mathrm{Soc}_{n-1}(B_n)) \neq \{0\}$. By Theorem 3.2, there exists a perfect finite left brace B_{n+1} such that $\mathrm{Soc}(B_{n+1}) \neq \{0\}$ and $B_{n+1}/\mathrm{Soc}(B_{n+1}) \cong B_n$. Hence

$$\begin{aligned}\mathrm{Soc}(B_{n+1}/\mathrm{Soc}_n(B_{n+1})) &\cong (\mathrm{Soc}((B_{n+1}/\mathrm{Soc}(B_{n+1}))/(\mathrm{Soc}_n(B_{n+1})/\mathrm{Soc}(B_{n+1})) \\ &\cong \mathrm{Soc}(B_n/\mathrm{Soc}_{n-1}(B_n)) \neq \{0\},\end{aligned}$$

by the inductive hypothesis. By induction the result follows. □

4. Skew braces of nilpotent type

We first prove that if both groups of a finite skew left brace A are nilpotent, then A can be decomposed as a direct product of skew left braces of prime-power size. A similar result was proved by Byott in the context of Hopf–Galois extensions, see [7, Theorem 1].

Lemma 4.1. *Let A be a skew left brace such that the additive group is a direct sum of ideals I_1, I_2, that is $A = I_1 + I_2$ and $I_1 \cap I_2 = \{0\}$. Then the map $f : A \to I_1 \times I_2$ defined by $f(a_1 + a_2) = (a_1, a_2)$, for all $a_1 \in I_1$ and $a_2 \in I_2$, is an isomorphism of skew left braces.*

Proof. Recall that the operations of the skew left brace $I_1 \times I_2$ are defined componentwise. Clearly f is an isomorphism of the additive groups of A and $I_1 \times I_2$. Let $a_1 \in I_1$ and $a_2 \in I_2$. Since I_1 and I_2 are ideals we have that

$$a_1 + a_2 - a_1 - a_2, a_1 * a_2, a_2 * a_1 \in I_1 \cap I_2 = \{0\},$$

thus $a_1 + a_2 = a_2 + a_1$ and $a_1 \circ a_2 = a_1 + a_2 = a_2 \circ a_1$. Hence

$$\begin{aligned} f((a_1 + a_2) \circ (b_1 + b_2)) &= f(a_1 \circ a_2 \circ b_1 \circ b_2) = f(a_1 \circ b_1 \circ a_2 \circ b_2) \\ &= f(a_1 \circ b_1 + a_2 \circ b_2) = (a_1 \circ b_1, a_2 \circ b_2) \\ &= (a_1, a_2) \circ (b_1, b_2) = f(a_1 + a_2) \circ f(b_1 + b_2), \end{aligned}$$

for all $a_1, b_1 \in I_1$ and $a_2, b_2 \in I_2$. Therefore, the result follows. □

Theorem 4.2. *Let n be a positive integer. Let A be a skew left brace such that the additive group is a direct sum of ideals $I_1, \dots, I_n$, that is every element $a \in A$ is uniquely written as $a = a_1 + \dots + a_n$, with $a_j \in I_j$ for all j. Then the map $f : A \to I_1 \times \dots \times I_n$ defined by $f(a_1 + \dots + a_n) = (a_1, \dots, a_n)$, for all $a_j \in I_j$, is an isomorphism of skew left braces.*

Proof. We shall prove the result by induction on n. For $n = 1$, it is clear. Suppose that $n > 1$ and that the result is true for $n - 1$. Let $A_1 = I_1 + \dots + I_{n-1}$. Then A_1 is an ideal of A and A is the direct sum of the ideals A_1 and I_n. By Lemma 4.1, the map $f_1 : A \to A_1 \times I_n$ defined by $f(a + a_n) = (a, a_n)$, for all $a \in A_1$ and $a_n \in I_n$, is an isomorphism of skew left braces. By the induction hypothesis, the map

$$f_2 : A_1 \to I_1 \times \dots \times I_{n-1}, \quad f_2(a_1 + \dots + a_{n-1}) = (a_1, \dots, a_{n-1}),$$

is an isomorphism of skew left braces. Therefore $f = (f_2 \times \mathrm{id}) \circ f_1 : A \to I_1 \times \dots \times I_n$ is an isomorphism of skew left braces and $f(a_1 + \dots + a_n) = (a_1, \dots, a_n)$, for all $a_j \in I_j$. Therefore, the result follows by induction. □

Corollary 4.3. *Let A be a finite skew left brace such that $(A, +)$ and $(A, \circ)$ are nilpotent. Let $I_1, \dots, I_n$ be the distinct Sylow subgroups of the additive group of A. Then $I_1, \dots, I_n$ are ideals of A and the map $f : A \to I_1 \times \dots \times I_n$ defined by $f(a_1 + \dots + a_n) = (a_1, \dots, a_n)$, for all $a_j \in I_j$, is an isomorphism of skew left braces.*

Proof. Since $(A, +)$ is nilpotent, for every prime divisor p of the order of A, there is a unique Sylow p-subgroup I of $(A, +)$. Hence I is a normal subgroup of $(A, +)$, and $\lambda_a(b) \in I$ for all $a \in A$ and $b \in B$. Thus I is a left ideal of A and thus it is a Sylow p-subgroup of $(A, \circ)$. Since $(A, \circ)$ is nilpotent, I is the unique Sylow p-subgroup of $(A, \circ)$ and, thus, it is normal in $(A, \circ)$. Therefore I is an ideal of A. Hence $I_1, \dots, I_n$ are ideals of A and clearly the additive group of A is the direct sum of $I_1, \dots, I_n$. The result follows by Theorem 4.2. □

Let A be a skew left brace. Let G be the multiplicative group of A and X be the additive group of A. Since G acts on X by automorphisms, one forms the semidirect product $\Gamma = X \rtimes G$ with multiplication

$$(x, g)(y, h) = (x + \lambda_g(y), g \circ h).$$

Identifying each $g \in G$ with $(0,g) \in \Gamma$ and each $x \in X$ with $(x,0) \in \Gamma$, we see that

$$\begin{aligned}[g,x] &= gxg^{-1}x^{-1} = (0,g)(x,0)(0,g')(-x,0)\\ &= (\lambda_g(x),g)(-\lambda_g^{-1}(x),g') = (\lambda_g(x)-x,0) = \lambda_g(x)-x = g*x.\end{aligned}$$

Let

$$\begin{aligned} X_0 &= X = A^1,\\ X_{n+1} &= [G,X_n] = A^{n+2} \quad \text{for } n \geq 0. \end{aligned} \tag{4.1}$$

Thus the elements of the left series of A are indeed iterated commutators of the group Γ. This observation has strong consequences. Our first application is the following useful result, which was proved by Rump for classical braces using different methods (see the corollary after Proposition 2 of [23]).

Proposition 4.4. *Let p be a prime and A be skew left brace of size p^m. Then A is left nilpotent.*

Proof. Let G be the multiplicative group of A and X be the additive group of A. Since the semidirect product $\Gamma = A \rtimes G$ is a p-group, it is nilpotent. Thus there exists k such that the k-repeated commutator $[\Gamma,\Gamma,\dots,\Gamma]$, where Γ appears k-times, is trivial. Since

$$A^k = [G,\dots,G,X] \subseteq [\Gamma,\dots,\Gamma],$$

it follows that A is left nilpotent. □

The following results follow immediately from theorems of P. Hall:

Lemma 4.5. *Let A be finite skew left brace such that $A^3 = 0$. Then the additive group of A^2 is abelian. In fact A^2 is a trivial brace.*

Proof. The first part follows by [19, Theorem 6]. Note that $(A^2)^2 \subseteq A^3 = 0$, hence $a \circ b = a + b$ for all $a,b \in A^2$, and the result follows. □

Theorem 4.6. *Let A be left nilpotent skew left brace. Then the following statements hold:*

1. *The additive group of A^2 is locally nilpotent.*
2. *The multiplicative group of $A/\ker\lambda$ is locally nilpotent.*

Proof. Since each element of the left series of A is a repeated commutator, the first claim follows from Hall's theorem [19, Theorem 4]. To prove the second claim, we use the notation above Proposition 4.4. Let $K = [G,X]G \subseteq \Gamma$ and $H = [G,X]X$. Let C be the centralizer of H in K. Then by [19, Theorem 4], K/C is locally nilpotent. Note that, since X is normal in Γ, $H = X$. Hence $G \cap C$ is the centralizer of X in G, that is

$$\begin{aligned} G \cap C &= \{g \in G \mid gxg^{-1} = x, \text{ for all } x \in X\}\\ &= \{g \in A \mid \lambda_g(x) = x, \text{ for all } x \in A\} = \ker\lambda. \end{aligned}$$

Thus $(GC)/C \cong G/(G \cap C) = G/\ker\lambda$ is locally nilpotent. □

We shall introduce some notation. Let A be a skew left brace. We denote by $\gamma^+(a,b) = a + b - a - b$ the commutator of a,b in $(A,+)$, for all $a,b \in A$. Let B,C be two subgroups of $(A,+)$. We define $\gamma^+(B,C) = \langle \gamma^+(b,c) \mid b \in B,\ c \in C\rangle_+$, the additive subgroup generated by the elements $\gamma^+(b,c)$, for $b \in B$ and $c \in C$. We also write $*(a,b) = a*b$ for all $a, b \in A$, and $*(B, C) = B*C = \langle *(b,c) \mid b \in B,\ c \in C\rangle_+$. Let M be the free monoid with basis $\{\gamma^+, *\}$. Then the elements of M are words in the alphabet $\{\gamma^+, *\}$, that is, if $m \in M$ then

$$m = \epsilon_1\epsilon_2 \cdots \epsilon_s,$$

for some non-negative integer s and $\epsilon_i \in \{\gamma^+, *\}$. In this case, we say that m has degree s and we write $\deg(m) = s$. Furthermore, if $s > 0$, we define

$$m(a_1a_2 \dots a_{s+1}) = \epsilon_1(a_1, \epsilon_2(a_2, \dots (\epsilon_s(a_s, a_{s+1})) \dots)),$$

for all $a_1, \dots, a_{s+1} \in A$, and if $A_1, \dots, A_{s+1}$ are subgroups of $(A, +)$, we define

$$m(A_1A_2 \dots A_{s+1}) = \epsilon_1(A_1, \epsilon_2(A_2, \dots (\epsilon_s(A_s, A_{s+1})) \dots)).$$

Finally we denote by $A_1(t)$ the word $A_1A_1 \dots A_1$ of length t in the letter A_1. We order M with the degree-lexicographic order, extending $* < \gamma^+$. Note that if $m_2 > m_1$ are elements of M, then

$$\begin{aligned} & m_2(a_1 \dots a_{\deg(m_2)+1}) + m_1(b_1 \dots b_{\deg(m_1)+1}) \\ = \quad & m_1(b_1 \dots b_{\deg(m_1)+1}) + m_2(a_1 \dots a_{\deg(m_2)+1}) \\ & -\gamma^+(-m_1(b_1 \dots b_{\deg(m_1)+1}), -m_2(a_1 \dots a_{\deg(m_2)+1})). \end{aligned}$$

In particular, the elements of the additive subgroup generated by

$$\{m(A(\deg(m) + 1)) \mid m \in M, \text{ with } \deg(m) \geq t\}$$

are of the form $a_1 + a_2 + \cdots + a_s$, where $a_i \in m_i(A(\deg(m_i) + 1))$, $\deg(m_1) \geq t$ and $m_1 < \cdots < m_s$. We denote this additive subgroup by

$$\sum_{\{m \in M \mid \deg(m) \geq t\}} m(A(\deg(m) + 1)).$$

Lemma 4.7. *Let A be a skew left brace. Let $G_1 = \ker \lambda$, and for $i > 1$, let $G_i = [A, G_{i-1}] = \langle a \circ b \circ a' \circ b' \mid a \in A,\ b \in G_{i-1}\rangle$. Let M be the free monoid with basis $\{\gamma^+, *\}$. Then*

$$G_n \subseteq \sum_{\{m \in M \mid \deg(m) \geq n-1\}} m(A(\deg(m) + 1)).$$

Proof. Let $a \in \ker \lambda$ and $g \in A$. Note that

$$\begin{aligned} g \circ a \circ g' \circ a' &= g \circ (a + g') + a' \\ &= g \circ a - g - a = g + \lambda_g(a) - g - a. \end{aligned} \tag{4.2}$$

We shall prove the result by induction on n. For $n = 1$,

$$G_1 = \ker \lambda \subseteq A = \sum_{\{m \in M \mid \deg(m) \geq 0\}} m(A(\deg(m) + 1)).$$

Let $n > 1$ and suppose that

$$G_{n-1} \subseteq \sum_{\{m \in M \mid \deg(m) \geq n-2\}} m(A(\deg(m) + 1)).$$

Let $g \in A$ and $a \in G_{n-1}$. Then since G_{n-1} is a subgroup of $\ker \lambda$, by (4.2) we have

$$g \circ a \circ g' \circ a' = g + \lambda_g(a) - g - a.$$

Let $a_m \in m(A(\deg(m) + 1))$ be such that

$$a = a_{m_1} + a_{m_2} + \cdots + a_{m_s},$$

with $\deg(m_1) \geq n-2$ and $m_1 < \cdots < m_s$. We have

$$\begin{aligned} g \circ a \circ g' \circ a' &= g + \lambda_g(a) - g - a \\ &= g + \lambda_g(a_{m_1} + a_{m_2} + \cdots + a_{m_s}) - g - (a_{m_1} + a_{m_2} + \cdots + a_{m_s}) \\ &= g + \lambda_g(a_{m_1}) + \cdots + \lambda_g(a_{m_s}) - g - (a_{m_1} + a_{m_2} + \cdots + a_{m_s}) \\ &= g + (g * a_{m_1} + a_{m_1}) + \cdots + (g * a_{m_s} + a_{m_s}) - g \\ & \quad -(a_{m_1} + a_{m_2} + \cdots + a_{m_s}). \end{aligned}$$

We shall prove that

$$\begin{aligned} & g + (g * a_{m_1} + a_{m_1}) + \cdots + (g * a_{m_s} + a_{m_s}) - g \\ & \quad -(a_{m_1} + a_{m_2} + \cdots + a_{m_s}) \in \sum_{\{m \in M \mid \deg(m) \geq n-1\}} m(A(\deg(m)+1)) \end{aligned}$$

by induction on s. For $s = 1$ we have

$$\begin{aligned} g + g * a_{m_1} + a_{m_1} - g - a_{m_1} &= \gamma^+(g, g * a_{m_1}) + g * a_{m_1} + g + a_{m_1} - g - a_{m_1} \\ &= \gamma^+(g, g * a_{m_1}) + g * a_{m_1} + \gamma^+(g, a_{m_1}). \end{aligned}$$

Since $\gamma^+(g, g * a_{m_1}) \in \gamma^+ * m_1(A(\deg(m_1)+3))$, $g * a_{m_1} \in *m_1(A(\deg(m_1)+2))$ and $\gamma^+(g, a_{m_1}) \in \gamma^+ m_1(A(\deg(m_1)+2))$, we have that

$$g + g * a_{m_1} + a_{m_1} - g - a_{m_1} \in \sum_{\{m \in M \mid \deg(m) \geq n-1\}} m(A(\deg(m)+1)).$$

Suppose that $s > 1$ and $g + (g * a_{m_1} + a_{m_1}) + \cdots + (g * a_{m_{s-1}} + a_{m_{s-1}}) - g - (a_{m_1} + a_{m_2} + \cdots + a_{m_{s-1}}) \in \sum_{\{m \in M \mid \deg(m) \geq n-1\}} m(A(\deg(m)+1))$.

We have that

$$\begin{aligned} & g + (g * a_{m_1} + a_{m_1}) + \cdots + (g * a_{m_s} + a_{m_s}) - g \\ & \qquad -(a_{m_1} + a_{m_2} + \cdots + a_{m_s}) \\ & \quad = g + (g * a_{m_1} + a_{m_1}) + \cdots + (g * a_{m_{s-1}} + a_{m_{s-1}}) \\ & \qquad + g * a_{m_s} + a_{m_s} - g - a_{m_s} + g - g - (a_{m_1} + a_{m_2} + \cdots + a_{m_{s-1}}) \\ & \quad = g + (g * a_{m_1} + a_{m_1}) + \cdots + (g * a_{m_{s-1}} + a_{m_{s-1}}) \\ & \qquad + g * a_{m_s} - \gamma^+(-g, a_{m_s}) - g - (a_{m_1} + a_{m_2} + \cdots + a_{m_{s-1}}) \\ & \quad = g + (g * a_{m_1} + a_{m_1}) + \cdots + (g * a_{m_{s-1}} + a_{m_{s-1}}) - g \\ & \qquad -(a_{m_1} + a_{m_2} + \cdots + a_{m_{s-1}}) \\ & \qquad + g * a_{m_s} - \gamma^+(-g, a_{m_s}) \\ & \qquad -\gamma^+(a_{m_1} + a_{m_2} + \cdots + a_{m_{s-1}} + g, \gamma^+(-g, a_{m_s}) - g * a_{m_s}) \\ & \qquad \in \sum_{\{m \in M \mid \deg(m) \geq n-1\}} m(A(\deg(m)+1)). \end{aligned}$$

Hence

$$g \circ a \circ g' \circ a' \in \sum_{\{m \in M \mid \deg(m) \geq n-1\}} m(A(\deg(m)+1)).$$

Note that $\sum_{\{m \in M \mid \deg(m) \geq n-1\}} m(A(\deg(m)+1))$ is a left ideal of A. Therefore

$$G_n \subseteq \sum_{\{m \in M \mid \deg(m) \geq n-1\}} m(A(\deg(m)+1)),$$

and the result follows by induction. □

The following result generalizes [28, Theorem 1].

Theorem 4.8. *Let A be a finite skew left brace with nilpotent additive group. Then A is left nilpotent if and only if the multiplicative group of A is nilpotent.*

Proof. Let us first assume that $(A,\circ)$ and $(A,+)$ are nilpotent. By Corollary 4.3, the skew left brace A is the direct product of skew left braces with prime-power orders. By Proposition 4.4 all such skew left braces are left nilpotent, hence A is left nilpotent by Lemma 2.25.

Suppose now that $(A,+)$ is nilpotent and A is left nilpotent. There exist positive integers n_1,n_2 such that $A^{n_1}=0$ and $\gamma^+_{n_2}(A)=0$, where $\gamma^+_{j+1}(A)=(\gamma^+)^j(A(j+1))$, using the notation above Lemma 4.7. By Theorem 4.6, we know that the multiplicative group of $A/\ker\lambda$ is nilpotent. Let $\gamma_1(A)=A$ and for $i>1$ let

$$\gamma_i(A)=[A,\gamma_{i-1}(A)]=\langle a\circ b\circ a'\circ b'\mid a\in A,\ b\in\gamma_{i-1}(A)\rangle.$$

Thus there exists a positive integer k such that $\gamma_k(A)\subseteq\ker\lambda$. Using the notation in the proof of Lemma 4.7, we have that $\gamma_{k+j}(A)\subseteq G_{j+1}$ for every nonnegative integer j. Hence, by Lemma 4.7

$$\gamma_{k+n_1n_2}(A)\subseteq G_{n_1n_2+1}\subseteq\sum_{\{m\in M\mid\deg(m)\geq n_1n_2\}}m(A(\deg(m)+1)).$$

Let $m\in M$ be an element with $\deg(m)\geq n_1n_2$. Note that if γ^+ appears t times in m, then $m(A(\deg(m)+1))\subseteq(\gamma^+)^t(A(t+1))$. In particular, if $t\geq n_2$, then $m(A(\deg(m)+1))=0$. Suppose that γ^+ appears at most n_2-1 times in m. In this case, there exist $m_1,m_2\in M$ such that $m=m_1(*)^{n_1}m_2$. In this case,

$$\begin{aligned}m(A(\deg(m)+1))&=m_1(*)^{n_1}(A(\deg(m_1)+n_1)m_2(A(\deg(m_2)+1)))\\&\subseteq m_1(*)^{n_1}(A(\deg(m_1)+n_1+1))\\&=m_1(A(\deg(m_1)A^{n_1}))=0.\end{aligned}$$

Hence $\gamma_{k+n_1n_2}(A)=0$. Therefore the multiplicative group of A is nilpotent, and the result follows. □

The assumption on the nilpotency of the additive group in Theorem 4.8 is needed (see Example 2.21).

Corollary 4.9. *Let A be a finite skew left brace of size p^n for some prime number p and some positive integer n. Then either A is the trivial brace of order p or it is not simple.*

Proof. By Theorem 4.8, A is left nilpotent. In particular, if $A\neq 0$, then $A^2\neq A$. Since A^2 is an ideal either A is not simple or $A^2=0$. Assume that $A^2=0$. In this case, $a\circ b=a+b$ for all $a,b\in A$. Therefore $[A,A]$ is a proper an ideal of A. Hence, either A is not simple or $[A,A]=0$. Assume that $A^2=[A,A]=0$. In this case A is a trivial brace and the result follows. □

Lemma 4.10. *Let A be a finite skew left brace with nilpotent additive group. Let p and q distinct prime numbers and let P and Q be Sylow subgroups of $(A,+)$ of sizes p^n and q^m, respectively. Then P, Q and $P+Q$ are left ideals of A.*

Proof. Let us first prove that P is a left ideal. Since $(A,+)$ is nilpotent, P is a normal subgroup of $(A,+)$. Let $a\in A$ and $x\in P$. Then $\lambda_a(x)\in P$ since λ_a is a group homomorphism. Similarly one proves that Q is a left ideal. From this it follows that $P+Q$ is a left ideal. □

The following is based on [27, Theorem 5(1)]. However, the proof is completely different.

Theorem 4.11. *Let A be a finite skew left brace with nilpotent additive group. Let p and q distinct prime numbers and let A_p and A_q be Sylow subgroups of $(A,+)$ of sizes p^n and q^m, respectively. If p does not divide q^t-1 for all $t\in\{1,\dots,m\}$, then $A_p*A_q=0$. In particular, $\lambda_x(y)=y$ for all $x\in A_p$ and $y\in A_q$.*

Proof. By Lemma 4.10 A_p, A_q and A_p+A_q are left ideals of A. In particular, A_p+A_q is a skew subbrace of A and A_p and A_q are Sylow subgroups of $(A_p+A_q,\circ)$. By Sylow's theorem, the number n_p of Sylow p-subgroups of the multiplicative group of A_p+A_q is

$$n_p=[A_p+A_q:N]\equiv 1 \bmod p,$$

where $N=\{g\in A_p+A_q : g\circ A_p\circ g'=A_p\}$ is the normalizer of A_p in the multiplicative group of A_p+A_q. Since $[A_p+A_q:N]=q^s$ for some $s\in\{0,\dots,m\}$ and p does not divide q^t-1 for all $t\in\{1,\dots,n\}$, it follows that $s=0$ and hence A_p is a normal subgroup of the multiplicative group of A_p+A_q. Thus A_p is an ideal of the skew left brace A_p+A_q. Since A_p is an ideal of A_p+A_q and A_q is a left ideal, we have that $A_p*A_q\subseteq A_p\cap A_q=0$, and the result follows. □

Corollary 4.12. *Let A be a skew left brace of size $p_1^{\alpha_1}\cdots p_k^{\alpha_k}$, where $p_1<p_2<\cdots<p_k$ are prime numbers and $\alpha_1,\dots,\alpha_k$ are positive integers. Assume that the additive group of A is nilpotent. Let A_j be the Sylow p_j- subgroups of the additive group of A. Assume that, for some $j\leq k$, p_j does not divide $p_i^{t_i}-1$ for all $t_i\in\{1,\dots\alpha_i\}$ for all $i\neq j$. Then $\operatorname{Soc}(A_j)\subseteq\operatorname{Soc}(A)$.*

Proof. Write $A=A_1+\cdots+A_k$. Let $a\in\operatorname{Soc}(A_j)$ and $b\in A$. Hence there exist elements $b_k\in A_k$ such that $b=b_1+\cdots+b_k$. By Theorem 4.11, $\lambda_a(b_i)=b_i$, for all $i\neq j$. Then $\lambda_a(b)=\lambda_a(b_1)+\cdots+\lambda_a(b_k)=b_1+\cdots+b_k=b$ and hence $a\in Soc(A)$. Thus the result follows. □

5. Braces with cyclic multiplicative group

As a consequence of a result of P. Hall one proves that the multiplicative group of a finite left brace is solvable. The following example shows that this fact does not hold for infinite left braces:

Example 5.1. *Let K be a field of characteristic 0. The Jacobson radical of $M_2(K[[x]])$ is $J=M_2(xK[[x]])$. Thus $(J,+,\circ)$ is a two-sided brace, where*

$$A\circ B=AB+A+B$$

for all $A,B\in J$. The map $f\colon J\longrightarrow\operatorname{GL}(M_2(K[[x]]))$ defined by $f(A)=I_2+A$, for all $A\in J$, where I_2 is the identity 2×2 matrix, is a monomorphism of groups:

$$f(A\circ B)=I_2+AB+A+B=(I_2+A)(I_2+B)=f(A)f(B).$$

By [33, Lemma 2.8]*, the subgroup of $(J,\circ)$ generated by*

$$\begin{pmatrix}0&0\\x&0\end{pmatrix},\quad\begin{pmatrix}0&x\\0&0\end{pmatrix}$$

is free of rank two. Therefore $(J,\circ)$ is not solvable.

The following result is due to Rump (see the proof of [24, Proposition 6]).

Theorem 5.2 (Rump)**.** *Let A be a left brace of type $\mathbb{Z}$. Then either A is the trivial brace isomorphic to $\mathbb{Z}$ or its multiplication is defined by*

$$(ma)\circ(na) = ((-1)^m n + m)a, \tag{5.1}$$

for all $m, n \in \mathbb{Z}$, where $a \in A$ is a fixed generator of its additive group.

Now we shall study the left braces with multiplicative group isomorphic to $\mathbb{Z}$. Note that these are two-sided braces. Thus this is equivalent to the study of the Jacobson radical rings such that its circle group is isomorphic to $\mathbb{Z}$.

Lemma 5.3. *Let R be a Jacobson radical ring with its circle group isomorphic to $\mathbb{Z}$. Then the additive group of R is finitely generated.*

Proof. Let $x \in R$ be a generator of the circle group $(R, \circ)$ of R. Let $y \in R$ the inverse of x in $(R, \circ)$. Then

$$R = \left\{\sum_{i=1}^{n}\binom{n}{i}x^i \mid n \geq 1\right\} \cup \left\{\sum_{i=1}^{n}\binom{n}{i}y^i \mid n \geq 1\right\} \cup \{0\}.$$

Since $x + y \in R$ one of the following conditions holds:

(1) $x + y = 0$. In this case, $0 = xy = -x^2$, thus $R = \{zx \mid z \in \mathbb{Z}\}$ and the additive group of R is isomorphic to $\mathbb{Z}$.

(2) There exists a positive integer n such that

$$x + y = \sum_{i=1}^{n}\binom{n}{i}x^i.$$

Since $y \neq 0$, we have that $n > 1$. Then

$$0 = xy + x + y = x\left(-x + \sum_{i=1}^{n}\binom{n}{i}x^i\right) + \sum_{i=1}^{n}\binom{n}{i}x^i.$$

Hence $R = x\mathbb{Z} + x^2\mathbb{Z} + \cdots + x^n\mathbb{Z}$.

(3) There exists a positive integer n such that

$$x + y = \sum_{i=1}^{n}\binom{n}{i}y^i.$$

Since $x \neq 0$, we have that $n > 1$. Then

$$0 = yx + x + y = y\left(-y + \sum_{i=1}^{n}\binom{n}{i}y^i\right) + \sum_{i=1}^{n}\binom{n}{i}y^i.$$

Hence $R = y\mathbb{Z} + y^2\mathbb{Z} + \cdots + y^n\mathbb{Z}$.

Therefore the result follows. □

Theorem 5.4. *Let R be a Jacobson radical ring with its circle group isomorphic to $\mathbb{Z}$. Then $ab = 0$ for all $a, b \in R$.*

Proof. Let p be a prime number. Since the additive group of R is infinite and finitely generated, pR is a proper ideal of R and R/pR has order p^m for some positive integer m. Since the only simple left brace of order a power of p is the trivial brace of order p, there exists a maximal ideal I_p of R such that $pR \subseteq I_p$ and R/I_p has order p. Let x be a generator of the circle group of R. Then it is clear that the circle group of I_p is generated by $x^{\circ p}$, (where $x^{\circ n} = x \circ \cdots \circ x$ (n times)).

Let $a, b \in R$. We have that $(a+I_p)(b+I_p) = I_p$ because R/I_p is a ring with zero multiplication. Hence $ab \in I_p$, for all prime numbers p. Now

$$\bigcap_{p \text{ prime}} I_p = \bigcap_{p \text{ prime}} \{x^{\circ zp} \mid z \in \mathbb{Z}\} = \{0\}.$$

Therefore $ab = 0$, and the result follows. □

As a consequence we have the following result.

Theorem 5.5. *Let A be a left brace with multiplicative group isomorphic to $\mathbb{Z}$. Then A is a trivial brace, in particular the additive group of A is isomorphic to $\mathbb{Z}$.*

A natural question arises: Is it possible to extend Theorem 5.5 to skew left braces? One can prove Theorem 5.5 for skew left braces of finite multipermutation level. However, the following result shows that nothing new is covered in this case.

Theorem 5.6. *Let A be a skew left brace with multiplicative group isomorphic to $\mathbb{Z}$. Then A has finite multipermutation level if and only if A is of abelian type.*

Proof. We proved in Theorem 5.5 that skew left braces of abelian type with multiplicative group isomorphic to $\mathbb{Z}$ have finite multipermutation level. Let us assume that A has finite multipermutation level. Let $c \in \operatorname{Soc}(A) \setminus \{0\}$. Then $c^{\circ k} = kc$ for all $k \in \mathbb{N}$. Since $(A, \circ)$ is torsion-free, $c^{\circ k} \neq c^{\circ l}$ if $k \neq l$. Observe that $\lambda_a(c^{\circ k}) = c^{\circ k}$ for all $a \in A$ and $k \in \mathbb{N}$, because $(A, \circ)$ is commutative. For $k > 0$ let I_k be the ideal of A generated by $c^{\circ k}$. Then $I_k = \{klc : l \in \mathbb{Z}\}$ and $\bigcap_{k>0} I_k = \{0\}$. Let $k > 0$. Since $c \neq 0$, A/I_k is a finite skew left brace of finite multipermutation level. By Theorem 2.20, A/I_k is of nilpotent type. Thus Corollary 4.3 implies that A/I_k is a direct product of skew left braces of prime-power size. Using results of T. Kohl [20] quoted in [30, Example A.7], such skew left braces are either of abelian type or of size 2^{α} for some α. Let us assume that A is not of abelian type and let $a, b \in A$ be such that $a + b - a - b \neq 0$. For each $k > 0$ there exists $m(k) \in \mathbb{N}$ such that

$$2^{m(k)}(a + b - a - a) \in I_k.$$

Since $I_k = \{klc : l \in \mathbb{Z}\}$, for each $a, b \in A$ and each $k > 0$ there are $m(k) \in \mathbb{N}$ and $q(k) \in \mathbb{Z}$ such that

$$2^{m(k)}(a + b - a - b) = q(k)kc.$$

Let k be an odd prime number coprime with $3q(3)$. Then

$$(2^{m(3)}q(k)k - 2^{m(k)}q(3)3)(a + b - a - b) = 0.$$

Since k is an odd prime number coprime with $3q(3)$, it follows that there exists $n \neq 0$ such that $n(a + b - a - b) = 0$. Then $nq(3)3c = 0$, a contradiction. Therefore A is a skew left brace of abelian type. □

Remark 5.7. *In* [16]*, Greenfeld showed that adjoint groups of Jacobson radical and not nilpotent algebras cannot be finite products of cyclic groups. His results are general but hold only for algebras over fields; therefore our results do not follow from* [16]*.*

The following result shows that Theorem 5.5 cannot be extended to skew left braces. Recall that the *infinite dihedral group* is the group

$$\mathbb{D}_\infty = \langle r, s : srs = r^{-1},\ s^2 = 1 \rangle \simeq \mathbb{Z} \rtimes \mathbb{Z}/(2).$$

Theorem 5.8. *There exists a skew left brace with multiplicative group isomorphic to $\mathbb{Z}$ and additive group isomorphic to the infinite dihedral group $\mathbb{D}_\infty$.*

Proof. Let $G = \langle g\rangle \simeq \mathbb{Z}$. A direct calculation shows that the operations

$$g^k + g^l = g^{k+(-1)^k l}, \quad g^k \circ g^l = g^{k+l}, \quad k, l \in \mathbb{Z},$$

turn G into a skew left brace with additive group isomorphic to the infinite dihedral group $\mathbb{D}_\infty$ and multiplicative group isomorphic to $\mathbb{Z}$. □

6. Indecomposable solutions

Let A be a skew left brace and $a \in A$. We say that the skew left brace A is generated by a if A is the smallest sub skew left brace of A containing a. Let (X, r) be a non-degenerate set-theoretic solution of the Yang–Baxter equation, where $r(x, y) = (\sigma_x(y), \tau_y(x))$. Recall from [12] that (X, r) is said to be decomposable if there exist disjoint non-empty subsets X_1 and X_2 of X such that $X = X_1 \cup X_2$ and $r(X_i \times X_j) = X_j \times X_i$ for all $i, j \leq 2$. If it is not possible to find such subsets X_1 and X_2 of X, the solution (X, r) is said to be indecomposable. By the orbit of an element $z \in X$ we will mean the smallest subset Y of X such that $z \in Y$ and $\sigma_x(y), \sigma_x^{-1}(y), \tau_x(y), \tau_x^{-1}(y) \in Y$, for all $y \in Y$ and $x \in X$. That is, if H is the subgroup of the symmetric group $\mathbb{S}(X)$ over X generated by σ_x, τ_x, for all $x \in X$, then the orbit of $z \in X$ is $Y = \{h(z) \colon h \in H\}$. Note that a non-degenerate set-theoretic solution (X, r) of the Yang–Baxter equation can be decomposed into orbits X_i, for $i \in I$, such that $r(X_i \times X_j) = X_j \times X_i$ for all $i, j \in I$. Each restriction $(X_i, r|_{X_i \times X_i})$ is again a non-degenerate set-theoretic solution of the Yang–Baxter equation. However, such restricted solutions need not to be indecomposable.

Example 6.1. *Let $(A, +, \cdot)$ be a commutative nilpotent ring with generators x, y and relations $x + x = y + y = 0$, $x^2 = y^2 = 0$. Let $(A, +, \circ)$ be the associated brace and (A, r_A) be the associated involutive non-degenerate set-theoretic solution. Then $Y = \{x, x + yx\}$ is an orbit. Observe that the solution $(Y, r|_{Y\times Y})$ is decomposable and $Y = \{x\} \cup \{x + xy\}$ is the decomposition of Y into its orbits.*

Recall that if A is a skew left brace, then its associated solution (A, r_A) is defined by $r_A(a, b) = (\sigma_a(b), \tau_b(a))$, for all $a, b \in A$, where $\sigma_a(b) = \lambda_a(b)$ and $\tau_b(a) = \lambda_a(b)' \circ a \circ b$ (see [18]). In [18, Theorem 3.1] it is proved that (A, r_A) is a non-degenerate set-theoretic solution of the Yang–Baxter equation.

Remark 6.2. *Let (X, r) be an involutive non-degenerate set-theoretic solution of the Yang–Baxter equation. We write $r(x, y) = (\sigma_x(y), \tau_y(x))$. Since r is involutive we have that $\tau_y(x) = \sigma^{-1}_{\sigma_x(y)}(x)$ and $\sigma_a(b) = \tau^{-1}_{\tau_b(a)}(b)$. Note that the orbit of $x \in X$ is*

$$O_x = \{\sigma_{y_1}\tau_{y_2}\dots\sigma_{y_{2m-1}}\tau_{y_{2m}}(x) \colon y_1, \dots y_{2m} \in X \cup \{0\}\},$$

where $0 \notin X$ and $\sigma_0 = \tau_0 = \mathrm{id}_X$. This is because

$$\sigma_y^{-1}(z) = \sigma^{-1}_{\sigma_z(\sigma_z^{-1}(y))}(z) = \tau_{\sigma_z^{-1}(y)}(z) \in O_x,$$

and

$$\tau_y^{-1}(z) = \tau^{-1}_{\tau_z(\tau_z^{-1}(y))}(z) = \sigma_{\tau_z^{-1}(y)}(z) \in O_x,$$

for all $z \in O_x$ and all $y \in X$.

Therefore our definition of orbit of an element of a non-degenerate set-theoretic solution of the Yang–Baxter equation coincides with the definition of orbit in [29, Section 2.1] *in the involutive case. It is easy to see that these definitions of orbits also coincide for finite non-degenerate set-theoretic solutions of the Yang–Baxter equation. However our definition of orbit does not coincide with the definition of orbit in* [29, Section 2.1] *for arbitrary infinite non-degenerate set-theoretic solution of the Yang–Baxter equation, as the following example shows.*

Example 6.3. *Consider the map $r\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}$ defined by $r(a,b)=(b+1,a+1)$, for all $a,b\in\mathbb{Z}$. It is easy to check that $(\mathbb{Z},r)$ is a non-degenerate set-theoretic solution of the Yang–Baxter equation. With our definition of orbit, the orbit of every element $a\in\mathbb{Z}$ is $\mathbb{Z}$. With the definition of orbit in* [29, Section 2.1]*, the orbit of $a\in\mathbb{Z}$ is*

$$X_a=\{a+n\colon n \text{ is a non-negative integer}\}.$$

Note that the restriction $r'=r|_{X_a\times X_a}\colon X_a\times X_a\to X_a\times X_a$ of r to $X_a\times X_a$ is non-bijective map. In fact (X_a,r') is a non-bijective set-theoretic solution of the Yang–Baxter equation. Furthermore, if we write $r'(b,c)=(\sigma_b(c),\tau_c(b))$, for all $b,c\in X_a$, then $\sigma_b\colon X_a\to X_a$ is not bijective.

Proposition 6.4. *Let A be a skew left brace generated (as a skew left brace) by an element x. Let $X=\{\lambda_a(x)\colon a\in A\}$. Then $A=\langle X\rangle_+=\langle X\rangle_\circ$.*

Proof. Note that $x=\lambda_0(x)\in X$. Since $\lambda:(A,\circ)\to\operatorname{Aut}(A,+)$ is a homomorphism of groups, it is clear that $\lambda_a(\langle X\rangle_+)\subseteq\langle X\rangle_+$ for all $a\in A$. Let $y,z\in\langle X\rangle_+$. We have that

$$y\circ z'=y+\lambda_y(z')=y+\lambda_y(-\lambda_{z'}(z))\in\langle X\rangle_+.$$

Hence $\langle X\rangle_+$ is a sub skew left brace of A containing x and thus $A=\langle X\rangle_+$.

Let $t\in\langle X\rangle_\circ$. Then $t=\lambda_{a_1}(x)^{\varepsilon_1}\circ\cdots\circ\lambda_{a_n}(x)^{\varepsilon_n}$, for some $a_j\in A$ and $\varepsilon_j\in\{',1\}$ (where $a^1=a$). We shall prove that $\lambda_a(t)\in\langle X\rangle_\circ$ by induction on n. For $n=1$, we may assume that $t=\lambda_{a_1}(x)'$. In this case

$$\begin{aligned}\lambda_a(t)&=\lambda_a(-\lambda_t(\lambda_{a_1}(x)))=-\lambda_{a\circ t\circ a_1}(x)\\&=\lambda_{(-\lambda_{a\circ t\circ a_1}(x))'}(\lambda_{a\circ t\circ a_1}(x))'\in\langle X\rangle_\circ\end{aligned}$$

Suppose that $n>1$ and that $\lambda_b(\lambda_{b_1}(x)^{\nu_1}\circ\cdots\circ\lambda_{b_{n-1}}(x)^{\nu_{n-1}})\in\langle X\rangle_\circ$, for all $b,b_1,\dots,b_{n-1}\in A$ and $\nu_j\in\{',1\}$. Thus

$$\begin{aligned}\lambda_a(t)&=\lambda_a(\lambda_{a_1}(x)^{\varepsilon_1}+\lambda_{\lambda_{a_1}(x)^{\varepsilon_1}}(\lambda_{a_2}(x)^{\varepsilon_2}\circ\cdots\circ\lambda_{a_n}(x)^{\varepsilon_n}))\\&=\lambda_a(\lambda_{a_1}(x)^{\varepsilon_1})+\lambda_{a\circ\lambda_{a_1}(x)^{\varepsilon_1}}(\lambda_{a_2}(x)^{\varepsilon_2}\circ\cdots\circ\lambda_{a_n}(x)^{\varepsilon_n})\\&=\lambda_a(\lambda_{a_1}(x)^{\varepsilon_1})\circ\lambda_{(\lambda_a(\lambda_{a_1}(x)^{\varepsilon_1}))'\circ a\circ\lambda_{a_1}(x)^{\varepsilon_1}}(\lambda_{a_2}(x)^{\varepsilon_2}\circ\cdots\circ\lambda_{a_n}(x)^{\varepsilon_n})\in\langle X\rangle_\circ,\end{aligned}$$

by the inductive hypothesis. Therefore $\lambda_a(\langle X\rangle_\circ)\subseteq\langle X\rangle_\circ$, for all $a\in A$.

Let $t,z\in\langle X\rangle_\circ$. We have that $-t+z=\lambda_t(t'\circ z)\in\langle X\rangle_\circ$. Hence $\langle X\rangle_\circ$ is a sub skew left brace of A containing x and thus $A=\langle X\rangle_\circ$ and the result follows. □

As a consequence we obtain a generalization of [29, Theorem 5.4] to skew left braces. In particular, by Remark 6.2, we answer in positive [29, Question 5.6].

Proposition 6.5. *Let B be a skew left brace and let $x\in B$. Let $A=B(x)$ be the smallest sub skew left brace of B containing x. Let (A,r_A) be the solution associated to the skew left brace A and let X be the orbit of x in (A,r_A). Then $(X,r_A|_{X\times X})$ is indecomposable.*

Proof. Recall that $r_A(a,b)=(\sigma_a(b),\tau_b(a))$, where $\sigma_a(b)=\lambda_a(b)$ and $\tau_b(a)=\lambda_a(b)'\circ a\circ b$, for all $a,b\in A$. By [18, Corollary 1.10], the map $\lambda\colon(A,\circ)\to\mathrm{Aut}(A,+)$ defined by $\lambda(a)=\lambda_a$, for all $a\in A$, is a homomorphism of groups. By [1, Lemma 2.4], the map $\tau\colon(A,\circ)\to\mathbb{S}(A)$ defined by $\tau(a)=\tau_a$ is an antihomomorphism of groups.

Let $X_1=\{\lambda_a(x)\colon a\in A\}$. Note that $X_1\subseteq X$. By Proposition 6.4, $A=\langle X_1\rangle_\circ$. Let $z,t\in X$. Hence there exist $a_1,\dots,a_{2n},b_1,\dots,b_{2m}\in X_1\cup X_1'\cup\{1\}$, such that

$$z=\lambda_{a_1}\tau_{a_2}\cdots\lambda_{a_{2n-1}}\tau_{a_{2n}}(x)\quad\text{and}\quad t=\lambda_{b_1}\tau_{b_2}\cdots\lambda_{b_{2m-1}}\tau_{b_{2m}}(x),$$

where $X_1'=\{y'\colon y\in X_1\}$. Thus

$$t=\lambda_{b_1}\tau_{b_2}\cdots\lambda_{b_{2m-1}}\tau_{b_{2m}}\tau_{a'_{2n}}\lambda_{a'_{2n-1}}\cdots\tau_{a'_2}\lambda_{a'_1}(z).$$

Therefore the result follows. □

Proposition 6.6. *Let (X,r) be a non-degenerate set-theoretic solution of the Yang–Baxter equation, and suppose that (X,r) is decomposable with $X=Y\cup Z$, then Y and Z are unions of orbits. In particular, every non-degenerate solution with a unique orbit is indecomposable.*

Proof. If (X,r) is a decomposable non-degenerate solution, then X is the union of two disjoint non-empty sets Y and Z such that $r(Y\times Y)=Y\times Y$, $r(Z\times Z)=Z\times Z$, $r(Y\times Z)=Z\times Y$ and $r(Z\times Y)=Y\times Z$. Therefore $r(X\times Y)=Y\times X$ and $r(Y\times X)=X\times Y$. We write $r(x,y)=(\sigma_x(y),\tau_y(x))$. Let $y\in Y$ and $x\in X$. Hence have $\sigma_x(y)\in Y$ and $\tau_x(y)\in Y$. Similarly $\sigma_x(z),\tau_x(z)\in Z$ for all $z\in Z$ and $x\in X$. Since $\sigma_x(\sigma_x^{-1}(y))=y\in Y$ for all $y\in Y$ and $x\in X$, we have that $\sigma_x^{-1}(y)\in X\setminus Z=Y$. Similarly, $\tau_x^{-1}(y)\in Y$ for all $y\in Y$ and all $x\in X$. Hence Y and Z are unions of orbits of elements. In particular, every non-degenerate solution with a unique orbit is indecomposable. □

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