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Imbalance market real options and the valuation of storage in future energy systems

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Version March 28, 2019 submitted to Risks

Abstract: As decarbonisation progresses and conventional thermal generation gradually gives way to other technologies including intermittent renewables, there is an increasing requirement for system balancing from new and also fast-acting sources such as battery storage. In the deregulated context this raises questions of market design and operational optimisation. In this paper we assess the *real option value* of an arrangement under which an autonomous energy-limited storage unit sells incremental balancing reserve. The arrangement is akin to a perpetual American swing put option with random refraction times, where a single incremental balancing reserve action is sold at each exercise. The power used is bought in an energy imbalance market (EIM), whose price we take as a general regular one-dimensional diffusion. The storage operator's strategy and its real option value are derived in this framework by solving the twin timing problems of when to buy power and when to sell reserve. Our results are illustrated with an operational and economic analysis using data from the German Amprion EIM.

Keywords: Multiple optimal stopping; general diffusion; real option analysis; energy imbalance market

1. Introduction

In today's electric grids, power system security is managed in real time by the system operator, who coordinates electricity supply and demand in a manner that avoids fluctuations in frequency or disruption of supply (see, for example, [New Zealand Electricity Authority \(2016\)](#)). In addition the SO carries out planning work to ensure that supply can meet demand, including the procurement of non-energy or *ancillary* services such as *operating reserve*, the capacity to make near real-time adjustments to supply and demand. These services are provided principally by network solutions such as the control of large-scale generation, although from a technical perspective they can also be provided by smaller, distributed resources such as demand response or energy storage ([National Grid plc \(2016\)](#); [Xu et al. \(2016\)](#)). Such resources have strongly differing operating characteristics: when compared to thermal generation, for example, energy storage is energy limited but can respond much more quickly. Storage also has important time linkages, since each discharge necessitates a corresponding recharge at a later time.

The coming decades are expected to bring a period of "energy transition" in which markets for ancillary services will evolve, among other highly significant changes to generation, consumption and network operation. The UK government, for example, has an ambition that "*new solutions such as storage or demand-side response can compete directly with more traditional network solutions*" ([UK Office of Gas and Electricity Markets \(2017, p. 29\)](#)). In harmony the UK System Operator National Grid has recently declared its intention to "*create a marketplace for balancing that encourages new and existing*

34 providers, and all new technology types" (*National Grid plc 2017*). In anticipation of changes such as these,
35 we will examine the participation of autonomous energy storage in a future marketplace for balancing.

36 Operating reserve is typically procured via a two-price mechanism, with a reservation payment
37 plus an additional utilisation payment each time the reserve is called for (*Ghaffari and Venkatesh*
38 *2013; Just and Weber 2008*). Since the incentivisation and efficient use of operating reserve for system
39 balancing is of increasing importance with growing penetration of variable renewable generation
40 (*King et al. 2011*), several system operators have recently introduced real-time *energy imbalance markets*
41 (EIMs) in which operating reserve is pooled, including in Germany (*Ocker and Ehrhart 2017*) and
42 California (*CAISO 2016; Lenhart et al. 2016*). Such markets typically involve the submission of bids
43 and offers from several providers for reserves running across multiple time periods, which are then
44 accepted, independently in each period, in price order until the real-time balancing requirement is
45 met. As one provider can potentially be called upon over multiple consecutive periods, this reserve
46 procurement mechanism is not well suited to energy-limited reserves such as energy storage. However,
47 storage-oriented solutions are being pioneered in a number of countries including a recent tender by
48 National Grid in the UK (*National Grid plc 2016*) and various trials by state system operators in the
49 US (*Xu et al. 2016*).

50 This paper considers operating reserve contracts for energy limited storage devices such as
51 batteries. In contrast to previous work on the pricing and hedging of energy options where settlement
52 is financial (see for example *Benth et al. (2008)* and references therein), we take account of the *physical*
53 *settlement* required in system balancing, considering also the limited energy and time linkages of
54 storage. The potential physical feedback effects of such contracts are investigated by studying the
55 operational policy of the storage or *battery operator*. To address the limited nature of storage, the
56 considered reserve contract is for a fixed quantity of energy. In this way, each contract written can be
57 physically covered with the appropriate amount of stored energy. We consider a simple arrangement
58 where the system operator sets the contract parameters, namely the premia (the reservation and
59 utilisation payments) plus an EIM price level x^* at which the energy is delivered. That is, rather than
60 being the outcome of a price formation process, these parameters are set administratively. Our analysis
61 thus focuses exclusively on the timing of the battery operator's actions. This dynamic modelling
62 contrasts with previous economic studies of operating reserve in the literature, which have largely
63 been static (*Just and Weber 2008*).

64 To quantify the economic opportunity for the storage operator we use *real options* analysis. Real
65 Options analysis is the application of option pricing techniques to the valuation of non-financial or
66 "real" investments with flexibility (*Borison 2005; Dixit and Pindyck 1994*). Here the energy storage unit
67 is the real asset, and is coupled with the timing flexibility of the battery operator, who observes the
68 EIM price in real time. The arrangement may be viewed as providing the battery operator with a real
69 perpetual American put option on the reserve contract described above. This option is either of swing
70 type (called the *lifetime problem* in this paper) or of single exercise type (*single problem*). The feature that
71 sets it apart from the existing literature on swing options is the random refraction time (c.f. *Carmona*
72 *and Touzi (2008)*).

73 A key question in Real Options analyses is the specification of the driving randomness (*Borison*
74 *2005*). In this paper we model the EIM price to resemble the historical statistical dynamics of imbalance
75 prices. In common with electricity spot prices and commodity prices more generally but unlike the
76 prices of financial assets, imbalance prices typically exhibit significant mean reversion (*Ghaffari and*
77 *Venkatesh 2013; Pflug and Broussev 2009*).

78 To avoid trivial cases we impose the following, mild, *sustainability conditions* on the arrangement:

- 79 **S1.** *The battery operator has a positive expected profit from the arrangement.*
80 **S2.** *The reserve contract cannot lead to a certain financial loss for the system operator.*

81 Condition **S1** is also known as the individual rationality or participation condition (*Fudenberg*
82 *et al. 1991*). While the battery operator is assumed to be a profit maximiser, the system operator may
83 engage in the arrangement for wider reasons than profit maximisation. To acknowledge the potential

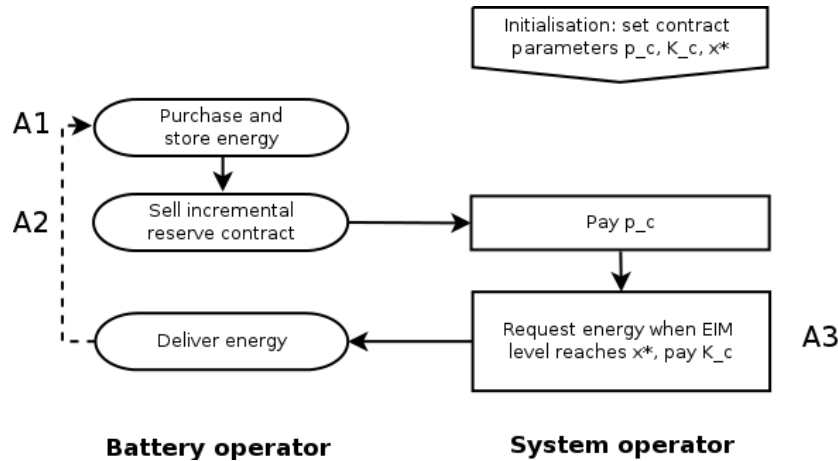


Figure 1. The sequence of actions A1–A3.

84 additional benefits provided by batteries, for example in providing response quickly and without
 85 direct emissions, condition **S2** is less strict than individual rationality.

86 By considering reserve contracts for incremental capacity (defined as an increase in generation
 87 or equivalently a decrease in load), we are able to provide complete solutions whose numerical
 88 evaluation is straightforward. Contracts for a decrease in generation, or an increase in load, lead to a
 89 fundamentally different set of optimisation problems which have been partially solved by Szabó and
 90 Martyr (2017).

91 This study extends earlier work (Moriarty and Palczewski 2017) with two important differences.
 92 Firstly, the dynamics of the imbalance price is described there by an exponential Brownian motion.
 93 In the present paper, by employing a different methodological approach we obtain explicit results
 94 for mean-reverting processes (and also other general diffusions) which better describe the statistical
 95 properties of imbalance prices (Ghaffari and Venkatesh 2013; Pflug and Broussev 2009). Secondly, the
 96 present paper takes into account deterioration of the store. Without this feature it was found that the
 97 value of storage is either very small (corresponding roughly to writing a single reserve contract) or
 98 infinite.

99 Through a benchmark case study we obtain the following economic recommendations. Firstly,
 100 investments in battery storage to provide reserve will be profitable on average for a wide range of
 101 the contract parameters. Secondly the EIM price level x^* at which energy is delivered is an important
 102 consideration. This is because as x^* increases, the EIM price reaches x^* significantly less frequently
 103 and the reserve contract starts to provide cover for rare events, resulting in infrequent power delivery
 104 and low utilisation of the battery, which may make the business case unattractive. These observations
 105 suggest that the contractual arrangement studied in this paper is more suitable for the frequent
 106 balancing of less severe imbalance.

107 1.1. Objectives

108 Given the model parameters x^* , $p_c \geq 0$ and $K_c \geq 0$, we wish to analyse the actions A1-A3 below
 109 (a graphical description of this sequence of actions is provided in Figure 1):

- 110 A1 The battery operator selects a time to purchase a unit of energy on the EIM and stores it.
 111 A2 With this physical cover in place, the battery operator then chooses a later time to sell the
 112 incremental reserve contract to the system operator in exchange for the initial premium p_c .
 113 A3 The system operator requests delivery of power when the EIM price X first lies above the level
 114 x^* and immediately receives the contracted unit of energy in return for the utilisation payment
 115 K_c .

116 Thus, the system operator obtains incremental reserve from the arrangement in preference to using
117 the EIM, when the EIM price is higher than the level x^* specified by the system operator. When
118 the sequence A1–A3 is carried out once, we refer to this as the *single problem*; when it is repeated
119 indefinitely back-to-back we refer to it as the *lifetime problem*.

120 In the lifetime problem, because storage is energy limited, action A3 must be completed before
121 the sequence A1–A3 can begin again. Thus if the arrangement is considered as a real swing put option,
122 the time between A2 and A3 is a random *refraction period* during which no exercise is possible. Note
123 that after action A3, the battery operator will perform action A1 again when the EIM price has fallen
124 sufficiently. Mathematically, therefore, we have the following objectives:

125 M1 For the single and lifetime problems, find the highest EIM price \check{x} at which the battery operator may buy
126 energy when acting optimally.

127 M2 For the single and lifetime problems, find the expected value of the total discounted cash flows (value
128 function) for the battery operator corresponding to each initial EIM price $x \geq \check{x}$.

129 We also aim to provide a straightforward numerical procedure to explicitly calculate \check{x} and the value
130 function (for $x \geq \check{x}$) in the lifetime problem.

131 1.2. Approach and related work

132 We take the EIM price to be a continuous time stochastic process $(X_t)_{t \geq 0}$. Since markets operate
133 in discrete time this is an approximation, made for analytical tractability. Nevertheless it is consistent
134 with the physical fact that the system operator's system balancing challenge is both real-time and
135 continuous.

136 Mathematically the problem is one of choosing two *optimal stopping times* corresponding to the
137 two actions A1 and A2, based on the evolution of the stochastic process X . (The reader is referred to
138 [Peskir and Shiryaev \(2006, Chapter 1\)](#) for a thorough presentation of optimal stopping problems.) We
139 centre our solution techniques around ideas of [Beibel and Lerche \(2000\)](#), who characterise optimal
140 stopping times using the Laplace transforms of first hitting times for the process X (see for example
141 [Borodin and Salminen \(2012, Section 1.10\)](#)). Methods and results from the single problem are then
142 combined with a fixed point argument for the lifetime analysis.

143 Our methodological results feed into a growing body of research on timing problems in trading.
144 In a financial context, [Zervos et al. \(2013\)](#) optimise the performance of “buy low, sell high” strategies,
145 using the same Laplace transforms to provide a candidate value function, which is later verified as
146 a solution to certain quasi-variational inequalities. An analogous strategy in an electricity market
147 using hydroelectric storage is studied in [Carmona and Ludkovski \(2010\)](#) where the authors use
148 Regression Monte Carlo methods to approximately solve the dynamic programming equations for
149 a related optimal switching problem. Our results differ from the above papers in two aspects. Our
150 analysis is purely probabilistic, leading to arguments that do not refer to the theory of PDEs and
151 quasi-variational inequalities. Secondly, our characterisation of the value function and the optimal
152 policy is explicit up to a single, one-dimensional non-linear optimisation which, as we demonstrate in
153 an empirical experiment, can be performed in milliseconds using standard scientific software. Related
154 to our lifetime analysis, [Carmona and Dayanik \(2008\)](#) apply probabilistic techniques to study the
155 optimal multiple-stopping problem for a general linear regular diffusion process and reward function.
156 However the latter work deals with a finite number of option exercises in contrast to our lifetime
157 analysis which addresses an infinite sequence of exercises via a fixed point argument. Our work thus
158 yields results with a significantly simpler and more convenient structure.

159 The contracts we consider have features in common with the *reliability options* used in Colombia,
160 Ireland and the ISO New England market and currently being introduced in Italy ([Mastropietro et al.
161 2018](#)). Reliability options pay an initial premium to a generator, usually require physical cover, and
162 have a reference market price and a *strike price* which plays a similar role to x^* . Typically the strike
163 price is set at the variable cost of the technology used to satisfy demand peaks, and the generator is

164 contracted to pay back the difference between the market price and the strike price in periods when
 165 energy is delivered and the market price is higher. However instead of being designed for system
 166 balancing, the purpose of reliability options is to ensure sufficient investment in generation capacity.

167 The remainder of the paper is organised as follows. The mathematical formulation and main
 168 tools are developed in Section 2. In the results of Section 3 we show that, for a range of price
 169 processes X incorporating mean reversion, solutions for *all* initial values x can be obtained. Further
 170 an empirical illustration using data from the German Amprion system operator is provided and
 171 qualitative implications are drawn, while Section 5 concludes. Auxiliary results are collected in the
 172 appendices.

173 2. Methodology

174 2.1. Formulation and preliminary results

In this section we characterise the real option value in the single and lifetime problems using the theory of regular one-dimensional diffusions. Denoting by $(W_t)_{t \geq 0}$ a standard Brownian motion, let $X = (X_t)_{t \geq 0}$ be a (weak) solution of the stochastic differential equation:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad (1)$$

with boundaries $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$. The solution of this equation with the initial condition $X_0 = x$ defines a probability measure \mathbb{P}^x and the related expectation operator \mathbb{E}^x . We assume that the boundaries are natural or entrance-not-exit, i.e. the process cannot reach them in finite time, and that X is a regular diffusion process, meaning that the state space $I := (a, b)$ cannot be decomposed into smaller sets from which X cannot exit. The existence and uniqueness of such an X is guaranteed if the functions μ and σ are Borel measurable in I with $\sigma^2 > 0$, and

$$\forall y \in I, \exists \varepsilon > 0 \text{ such that } \int_{y-\varepsilon}^{y+\varepsilon} \frac{1 + |\mu(\xi)|}{\sigma^2(\xi)} d\xi < +\infty, \quad (2)$$

(see Karatzas and Shreve (1991, Theorem 5.5.15); condition (2) holds if, for example, μ is locally bounded and σ is locally bounded away from zero). Necessary and sufficient conditions for the boundaries a and b to be non-exit points, i.e., natural or entrance-not-exit, are formulated in Theorem 5.5.29 of the latter book. In particular, it is sufficient that the scale function

$$p(x) := \int_c^x \exp\left(-2 \int_c^z \frac{\mu(u)}{\sigma^2(u)} du\right) dz, \quad x \in I, \quad (3)$$

175 converges to $-\infty$ when x approaches a and to $+\infty$ when x approaches b . (Here $c \in I$ is arbitrary and
 176 the condition stated above does not depend on its choice.) These conditions are mild, in the sense that
 177 they are satisfied by all common diffusion models for commodity prices, including those in Section 3.

Denote by τ_x the first time that the process X reaches $x \in I$:

$$\tau_x = \inf\{t \geq 0 : X_t = x\}. \quad (4)$$

For $r > 0$, define

$$\psi_r(x) = \begin{cases} \mathbb{E}^x\{e^{-r\tau_c}\}, & x \leq c, \\ 1/\mathbb{E}^c\{e^{-r\tau_x}\}, & x > c, \end{cases} \quad \phi_r(x) = \begin{cases} 1/\mathbb{E}^c\{e^{-r\tau_x}\}, & x \leq c, \\ \mathbb{E}^x\{e^{-r\tau_c}\}, & x > c, \end{cases} \quad (5)$$

for any fixed $c \in I$ (different choices of c merely result in a scaling of the above functions). It can be verified directly that function $\phi_r(x)$ is strictly decreasing in x while $\psi_r(x)$ is strictly increasing, and for $x, y \in I$ we have

$$\mathbb{E}^x \{e^{-r\tau_y}\} = \begin{cases} \psi_r(x)/\psi_r(y), & x < y, \\ \phi_r(x)/\phi_r(y), & x \geq y. \end{cases} \quad (6)$$

178 Since the boundaries a, b are natural or entrance-not-exit, we have $\psi_r(a+) \geq 0$, $\phi_r(b-) \geq 0$ and
179 $\psi_r(b-) = \phi_r(a+) = \infty$ (Borodin and Salminen 2012, Section II.1).

180 2.1.1. Optimal stopping problems and solution technique

The class of optimal stopping problems which we use in this paper is

$$v(x) = \sup_{\tau} \mathbb{E}^x \{e^{-r\tau} \vartheta(X_{\tau}) \mathbf{1}_{\tau < \infty}\}, \quad (7)$$

where the supremum is taken over the set of all (possibly infinite) stopping times. Here ϑ is the *payoff* function and v is the *value* function. If a stopping time τ^* exists which achieves the equality (7) we call this an *optimal* stopping time. Also, if v and ϑ are continuous then the set

$$\Gamma := \{x \in I : v(x) = \vartheta(x)\} \quad (8)$$

181 is a closed subset of I . Under general conditions (Peskir and Shiryaev 2006, Chapter 1), which are
182 satisfied by all stopping problems studied in this paper, $\tau^* = \inf\{t \geq 0 : X_t \in \Gamma\}$ is the smallest
183 optimal stopping time and the set Γ is then called the *stopping set*.

184 Appendix A contains three lemmas providing a classification of solutions to the stopping problem
185 (7) which will be used below.

186 2.1.2. Single problem

Let $(X_t)_{t \geq 0}$ denote the EIM price. We will develop a mathematical representation of actions A1–A3 (see Section 1.1) when only one reserve contract is traded. Considering A3, the time of power delivery is the first time that the EIM price exceeds a predetermined level x^* :

$$\hat{\tau}_e = \inf\{t \geq 0 : X_t \geq x^*\}.$$

Given the present level x of the EIM price, the expected net present value of the utilisation payment exchanged at time $\hat{\tau}_e$ can be expressed as follows thanks to (6):

$$h_c(x) = E^x \{e^{-r\hat{\tau}_e} K_c\} = \begin{cases} K_c, & x \geq x^*, \\ K_c \frac{\psi(x)}{\psi(x^*)}, & x < x^*. \end{cases} \quad (9)$$

Therefore, the optimal timing of action A2 corresponds to solving the following optimal stopping problem:

$$\sup_{\tau} \mathbb{E}^x \{e^{-r\tau} (p_c + h_c(X_{\tau})) \mathbf{1}_{\tau < \infty}\}.$$

Since the utilisation payment K_c obtained when the EIM price exceeds x^* is positive and constant, as is the initial premium p_c , it is best to obtain these cashflows as soon as possible. The solution of the above stopping problem is therefore trivial: the contract should be sold immediately after completing action A1, i.e. immediately after providing physical cover for the reserve contract. Optimally timing the simultaneous actions A1 and A2, the **purchase of energy and sale of the incremental reserve**

contract, is therefore the core optimisation task. It corresponds to solving the following optimal stopping problem:

$$V_c(x) = \sup_{\tau} E^x \{ e^{-r\tau} (-X_{\tau} + p_c + h_c(X_{\tau})) \mathbf{1}_{\tau < \infty} \} = \sup_{\tau} E^x \{ e^{-r\tau} h(X_{\tau}) \mathbf{1}_{\tau < \infty} \}, \quad (10)$$

where the payoff

$$h(x) = -x + p_c + h_c(x) \quad (11)$$

187 is non-smooth since h_c is non-smooth. The function $V_c(x)$ is the real option value in the single problem.

188 2.1.3. Lifetime problem formulation and notation

189 In addition to having a design life of multiple decades, thermal power stations have the primary
190 purpose of generating energy rather than providing ancillary services. In contrast electricity storage
191 technologies such as batteries have a design life of years and may be dedicated to providing ancillary
192 services. In this paper we take into account the potentially limited lifespan of electricity storage by
193 modelling a multiplicative degradation of their storage capacity: each charge-discharge cycle reduces
194 the capacity by a factor $A \in (0, 1)$.

195 We now turn to the lifetime problem. To this end, suppose that a nonnegative *continuation value*
196 $\zeta(x, \alpha)$ is also received at the same time as action A3. It is a function of the capacity of the store
197 $\alpha \in (0, 1)$ and the EIM price x , and represents the future proceeds from the arrangement.

The expected net present value of action A3 is now

$$h^{\zeta}(x, \alpha) := E^x \{ e^{-r\hat{\tau}_e} (\alpha K_c + \zeta(X_{\hat{\tau}_e}, A\alpha)) \} = \begin{cases} (\alpha K_c + \zeta(x^*, A\alpha)) \frac{\psi(x)}{\psi(x^*)}, & x < x^*, \\ \alpha K_c + \zeta(x, A\alpha), & x \geq x^*, \end{cases} \quad (12)$$

where $A \in (0, 1)$ is the multiplicative decrease of storage capacity per cycle. Here the optimal timing of action A2 may be non trivial due to the continuation value $\zeta(x, \alpha)$. We will show however that for the functions ζ of interest in this paper, it is optimal to sell the reserve contract immediately after action A1, identically as in the single problem. The timing of action A1 requires the solution of the optimal stopping problem

$$\mathcal{T}\zeta(x, \alpha) := \sup_{\tau} E^x \{ e^{-r\tau} (-\alpha X_{\tau} + \alpha p_c + h^{\zeta}(X_{\tau}, \alpha)) \mathbf{1}_{\tau < \infty} \}. \quad (13)$$

The *optimal stopping operator* \mathcal{T} makes the dependence on ζ explicit: it maps ζ onto the real option value of a selling a single reserve contract followed by continuation according to ζ . We define the **lifetime value function** \hat{V} as the limit

$$\hat{V}(x) = \lim_{n \rightarrow \infty} (\mathcal{T}^n \mathbf{0})(x, 1), \quad (14)$$

198 (if the limit exists), where \mathcal{T}^n denotes the n -fold iteration of the operator \mathcal{T} and $\mathbf{0}$ is the function
199 identically equal to 0. Thus $\mathcal{T}^n \mathbf{0}$ is the real option value of selling at most n reserve contracts under
200 the arrangement. (Note that a priori it may not be optimal to sell all n contracts in this case, since it is
201 possible to offer fewer contracts and refrain from trading afterwards by choosing $\tau = \infty$.)

202 Calculation of the lifetime value function requires the analysis of a two-argument function. We
203 will show now that this computation may be reduced to a function of the single argument x . Define
204 $\zeta_0(x, \alpha) = 0$ and $\zeta_{n+1}(x, \alpha) = \mathcal{T}\zeta_n(x, \alpha)$. We interpret $\zeta_n(x, \alpha)$ as the maximum expected wealth
205 accumulated over at most n cycles of the actions A1–A3 when the initial capacity of the store is α .

Lemma 1. We have $\zeta_n(x, \alpha) = \alpha \hat{\zeta}_n(x)$, where $\hat{\zeta}_n(x) = \zeta_n(x, 1)$. Moreover, $\hat{\zeta}_n(x) = \hat{\mathcal{T}}^n \mathbf{0}(x)$, where

$$\hat{\mathcal{T}} \hat{\zeta}(x) = \sup_{\tau} \mathbb{E}^x \{ e^{-r\tau} (-X_{\tau} + p_c + \hat{h}^{\hat{\zeta}}(X_{\tau})) \mathbf{1}_{\tau < \infty} \}, \quad (15)$$

and

$$\hat{h}^{\hat{\zeta}}(x) = \begin{cases} (K_c + A\hat{\zeta}(x^*)) \frac{\psi(x)}{\psi(x^*)}, & x < x^*, \\ K_c + A\hat{\zeta}(x), & x \geq x^*. \end{cases} \quad (16)$$

Proof. The proof is by induction. Clearly, the statement is true for $n = 0$. Assume it is true for $n \geq 0$. Then

$$\zeta_{n+1}(x, \alpha) = \mathcal{T} \zeta_n(x, \alpha) = \alpha \sup_{\tau} \mathbb{E}^x \{ e^{-r\tau} (-X_{\tau} + p_c + \frac{1}{\alpha} h^{\zeta_n}(X_{\tau}, \alpha)) \mathbf{1}_{\tau < \infty} \},$$

and

$$\frac{1}{\alpha} h^{\zeta_n}(x, \alpha) = E^x \{ e^{-r\hat{\tau}_e} (K_c + \frac{1}{\alpha} \zeta_n(X_{\hat{\tau}_e}, A\alpha)) \} = E^x \{ e^{-r\hat{\tau}_e} (K_c + A\hat{\zeta}_n(X_{\hat{\tau}_e})) \}.$$

206 Hence, $\zeta_{n+1}(x, \alpha) = \alpha \hat{\mathcal{T}} \hat{\zeta}_n(x) = \alpha \zeta_{n+1}(x, 1)$. Consequently, $\hat{\zeta}_n = \hat{\mathcal{T}}^n \mathbf{0}$. \square

Assume that $\zeta_n(x, \alpha)$ converges to $\zeta(x, \alpha)$ as $n \rightarrow \infty$. Then, clearly, $\hat{\zeta}_n$ converges to $\hat{\zeta}(x) = \zeta(x, 1)$. It is also clear that ζ is a fixed point of \mathcal{T} if and only if $\hat{\zeta}$ is a fixed point of $\hat{\mathcal{T}}$. Therefore, we have simplified the problem to that of finding a limit of $\hat{\mathcal{T}}^n \mathbf{0}(x)$. The stopping problem $\hat{\mathcal{T}} \hat{\zeta}$ will be called the *normalised* stopping problem and its payoff denoted by

$$\hat{h}(x, \hat{\zeta}) = \begin{cases} -x + p_c + \frac{\psi_r(x)}{\psi_r(x^*)} (K_c + A\hat{\zeta}(x^*)), & x < x^*, \\ -x + p_c + K_c + A\hat{\zeta}(x), & x \geq x^*. \end{cases} \quad (17)$$

207 In particular, $\hat{\mathcal{T}} \mathbf{0}$ coincides with the single problem's value function V_c .

Notation. In the remainder of this paper a caret (hat) will be used over symbols relating to the normalised lifetime problem:

$$\hat{V}(x) = \lim_{n \rightarrow \infty} \hat{\mathcal{T}}^n \mathbf{0}(x).$$

208 2.1.4. Sustainability conditions revisited

209 The sustainability conditions **S1** and **S2** introduced in Section 1 are our standing economic
210 assumptions. The next lemma, proved in the appendices, expresses them quantitatively. This makes
211 way for their use in the mathematical considerations below.

212 **Lemma 2.** When taken together, the sustainability conditions **S1** and **S2** are equivalent to the following
213 quantitative conditions:

214 **S1***: $\sup_{x \in (a, b)} h(x) > 0$, and

215 **S2***: $p_c + K_c < x^*$.

216 Notice that **S1*** is always satisfied when $a \leq 0$.

217 2.2. Three exhaustive regimes in the single problem

218 In this section we consider the single problem. Recall that the sustainability assumptions, or
219 equivalently assumptions **S1*** and **S2***, are in force. For completeness the notation and general optimal
220 stopping theory used below is presented in Appendix A.

Since the boundary a is not-exit we have $\phi_r(a+) = \infty$. When h is given by (11), the limit L of (39) is then

$$L_c := \limsup_{x \rightarrow a} \frac{-x}{\phi_r(x)}. \quad (18)$$

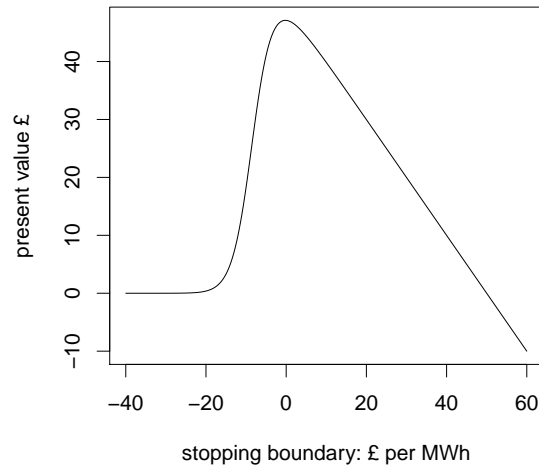


Figure 2. Sensitivity of the expected value in the single problem with respect to the stopping boundary. The EIM price is modelled as an Ornstein-Uhlenbeck process $dX_t = 3.42(47.66 - X_t)dt + 30.65dW_t$ (time measured in days, fitted to Elexon Balancing Mechanism price half-hourly data from 07/2011 to 03/2014). The interest rate $r = 0.03$, power delivery level $x^* = 60$, the initial premium $p_c = 10$, and the utilisation payment $K_c = 40$. The initial price is X_0 is set equal to x^* .

221 It can also be verified that the analogous constant R defined in (40) in the appendices satisfies $R < \infty$
 222 since, by **S2***, h is negative on $[x^*, \infty)$. The following theorem completes our aim **M2**.

223 **Theorem 1.** (Single problem) Assume that conditions **S1*** and **S2*** hold. With the definition (18) there are
 224 three exclusive cases:

(A) $L_c \leq \frac{h(x)}{\phi_r(x)}$ for some $x \implies$ there is $\hat{x} < x^*$ that maximises $\frac{h(x)}{\phi_r(x)}$, and then, for $x \geq \hat{x}$, $\tau_{\hat{x}}$ is **optimal**,
 and

$$V_c(x) = \phi_r(x) \frac{h(\hat{x})}{\phi_r(\hat{x})}, \quad x \geq \hat{x}. \quad (19)$$

225 (B) $\infty > L_c > \frac{h(x)}{\phi_r(x)}$ for all $x \implies V_c(x) = L_c \phi_r(x)$ and there is **no optimal** stopping time.

226 (C) $L_c = \infty \implies V_c(x) = \infty$ and there is **no optimal** stopping time.

227 Moreover, in cases A and B the value function V_c is continuous.

228 **Proof.** By condition **S1***, $h(y)$ is positive for some $y \in I$ and the value function $V_c(x) > 0$. For case A
 229 note first that the function h is negative on $[x^*, b)$ by **S2***, see (9) and (11). Therefore, the supremum of
 230 $\frac{h}{\phi_r}$ is positive and must be attained at some (not necessarily unique) $\hat{x} \in (a, x^*)$. The optimality of $\tau_{\hat{x}}$
 231 for $x \geq \hat{x}$ then follows from Lemma 6. Case B follows from Lemma 7 and the fact that $L_c > 0$. Lemma
 232 7 proves case C. The continuity of V_c follows from Lemma 8. \square

233 The optimal strategy in case A is of threshold type. When an arbitrary threshold strategy $\tau_{\tilde{x}}$ is
 234 used, the resulting expected value for $x \geq \tilde{x}$ is given by $\phi_r(x)h(\tilde{x})/\phi_r(\tilde{x})$. Figure 2 (whose problem
 235 data fall into case A) shows the potentially high sensitivity of the expected value of discounted cash
 236 flows for the single problem with respect to the level of the threshold \tilde{x} . It is therefore important in
 237 general to identify the optimal threshold accurately.

238 We now show that for commonly used diffusion price models, it is case A in the above theorem
 239 which is of principal interest. This is due to the mild sufficient conditions established in the following

240 lemma which are satisfied, for example, by the examples in Section 3. Although condition 2(b) in
 241 Lemma 3 is rather implicit, it may be interpreted as requiring that the process X does not ‘escape
 242 relatively quickly to $-\infty$ ’ (see Appendix D for a further discussion and examples) and it is satisfied,
 243 for example, by the Ornstein-Uhlenbeck process.

244 **Lemma 3.** *If condition S1* holds then:*

- 245 1. *The equality $L_c = 0$ implies case A of Theorem 1.*
 246 2. *Any of the following conditions is sufficient for $L_c = 0$:*

- 247 (a) $a > -\infty$,
 248 (b) $a = -\infty$ and $\lim_{x \rightarrow -\infty} \frac{x}{\phi_r(x)} = 0$.

249 **Proof.** Condition S1* ensures that h takes positive values. Hence the ratio $\frac{h(x)}{\phi_r(x)} > 0 = L_c$ for some x .
 250 For assertion 2(a), recall from Section 2 that $\phi_r(a+) = \infty$ since the boundary a is not-exit. Then we
 251 have $L_c = \limsup_{x \rightarrow a} (-x)/\phi_r(x) = 0$ as $a > -\infty$. In 2(b), the equality $L_c = 0$ is immediate from the
 252 definition of L_c . \square

253 Turning now to aim M1, we have

254 **Corollary 1.** *In the setting of Theorem 1 for the single problem, either*

(a) *the quantity*

$$\check{x} := \max \left\{ x \in I : \frac{h(x)}{\phi_r(x)} = \sup_{y \in I} \frac{h(y)}{\phi_r(y)} \right\} \quad (20)$$

- 255 *is well-defined, i.e., the set is non-empty. Then \check{x} is the highest price at which the battery operator may buy*
 256 *energy when acting optimally, and we have $\check{x} < x^*$ (this is case A); or*
 257 (b) *there is no price at which it is optimal for the battery operator to purchase energy. In this case the single*
 258 *problem’s value function may either be infinite (case C) or finite (case B).*

259 **Proof.** a) Since the maximiser \hat{x} in case A of Theorem 1 is not necessarily unique, the set in (20) may
 260 contain more than one point. Since h and ϕ_r are continuous and all maximisers lie to the left of x^* ,
 261 this set is closed and bounded from above, so \check{x} is well-defined and a maximiser in case A. For any
 262 stopping time τ with $\mathbb{P}^x\{X_\tau > \check{x}\} > 0$, it is immediate from assertion 3 of Lemma 6 that τ is not
 263 optimal for the problem $V_c(x)$, $x \geq \check{x}$. Part b) follows directly from cases B and C of Theorem 1. \square

264 Corollary 1 confirms that it is optimal for the battery operator to buy energy only when the EIM
 265 price is strictly lower than the price x^* which would trigger immediate power delivery to the system
 266 operator. Thus the battery operator (when acting optimally) does not directly conflict with the system
 267 operator’s balancing actions.

268 2.3. Two exhaustive regimes in the lifetime problem

269 Turning to the lifetime problem, we begin by letting $\hat{\zeta}(x)$ in definition (16) be a general
 270 nonnegative continuation value depending only on the EIM price x , and studying the normalised
 271 stopping problem (15) in this case (the payoff \hat{h} is therefore defined as in (17)).

272 We now wish to study the value of n cycles A1–A3, and hence the lifetime value, by iterating
 273 the operator \hat{T} . To justify this approach it is necessary to check the timing of action A2 in the lifetime
 274 problem. With the actions A1–A3 defined as in Section 1.1, recall that the timing of action A2 is trivial
 275 in the single problem: after A1 it is optimal to perform A2 immediately. Lemma 10, which may be
 276 found in the appendices, confirms that the same property holds in the lifetime problem.

277 We may now provide the following answer to objective M1 for the lifetime problem.

278 **Corollary 2.** *Assume that conditions S1* and S2* hold. In the lifetime problem with $\hat{\zeta} = \hat{V}$, either:*

(a) the quantity

$$\check{x} := \max \left\{ x \in I : \frac{\hat{h}(x, \hat{\zeta})}{\phi_r(x)} = \sup_{y \in I} \frac{\hat{h}(y, \hat{\zeta})}{\phi_r(y)} \right\} \quad (21)$$

279 is well-defined, i.e., the set is non-empty. Then \check{x} is the highest price at which the battery operator can buy
 280 energy when acting optimally in the lifetime problem, and we have $\check{x} < x^*$ (cases 1 and 2a in Lemma 9); or
 281 (b) there is no price at which it is optimal for the battery operator to purchase energy. In this case the lifetime
 282 value function may either be infinite (case 3) or finite (case 2b in Lemma 9).

283 **Proof.** The proof proceeds exactly as that of Corollary 1 with the exception of showing that $\check{x} < x^*$
 284 (this is because Lemma 9 in the appendices, which characterises the possible solution types in the
 285 lifetime problem, does not guarantee the strict inequality $\check{x} < x^*$). Assume then $\check{x} = x^*$. At the EIM
 286 price $X_t = \check{x} = x^*$ the power delivery to the system operator is immediately followed by the purchase
 287 of energy by the battery operator and this cycle can be repeated instantaneously, arbitrarily many
 288 times. However since each such cycle is loss making for the battery operator by condition **S2***, this
 289 strategy would lead to unbounded losses almost surely in the lifetime problem started at EIM price x^*
 290 leading to $\widehat{V}(x^*) = -\infty$. This would contradict the fact that $\widehat{V} > 0$, so we conclude that $\check{x} < x^*$. \square

291 Pursuing aim **M2**, we will show now that there are two regimes in the lifetime problem: either the
 292 lifetime value function is strictly greater than the single problem's value function (and the cycle A1–A3
 293 is repeated infinitely many times), or the lifetime value equals the single problem's value. Although
 294 the latter case appears counterintuitive, it is explained by the fact that the lifetime problem's value
 295 is then attained only in the limit when the purchase of energy (action A1) is made at a decreasing
 296 sequence of prices converging to a , the left boundary of the process (X_t) . In this limit, the benefit of
 297 future payoffs becomes negligible, equating the lifetime value to the single problem's value.¹

298 **Theorem 2.** *There are two exclusive regimes:*

- 299 (α) $\widehat{V}(x) > V_c(x)$ for all $x \geq x^*$,
 300 (β) $\widehat{V}(x) = V_c(x)$ for all $x \geq x^*$ (or both are infinite for all x).

301 Moreover, in regime (α) an optimal stopping time exists when the continuation value is $\hat{\zeta} = \hat{\zeta}_n = \hat{T}^n \mathbf{0}$ for
 302 $n > 0$ (that is, for a finite number of reserve contracts), and when $\hat{\zeta} = \widehat{V}$ (for the lifetime value function).

303 **Proof.** We take the continuation value $\hat{\zeta} = V_c$ in Lemma 9 from the appendices and consider separately
 304 its cases 1, 2a, 2b and 3. Firstly in case 3 we have $V_c = \infty$, implying that also $\widehat{V} = \infty$ and we have
 305 regime (β).

306 Case 2 of Lemma 9 corresponds to case B of Theorem 1, when there is no optimal stopping time
 307 in the single problem and $V_c(x) = L_c \phi_r(x)$ for all $x \in I$. Considering first case 2b and defining $\hat{\zeta}_n$ as in
 308 Lemma 11, it follows that $\hat{\zeta}_2(x) = L_c \phi_r(x) = V_c(x)$ for $x \in I$ and consequently $\widehat{V} = V_c$, which again
 309 corresponds to regime (β).

310 In case 2a of Lemma 9, suppose first that the maximiser $\hat{x} \leq x^*$ is such that $\frac{\hat{h}(\hat{x}, \hat{\zeta}_1)}{\phi_r(\hat{x})} = L_c$. Then for
 311 $x \geq x^* \geq \hat{x}$ we have $\hat{\zeta}_2(x) = \phi_r(x) \frac{\hat{h}(\hat{x}, \hat{\zeta}_1)}{\phi_r(\hat{x})} = L_c \phi_r(x)$, which also yields regime (β). On the other hand,
 312 when $\frac{\hat{h}(\hat{x}, \hat{\zeta}_1)}{\phi_r(\hat{x})} > L_c$ we have for $x \geq x^* \geq \hat{x}$ that $\hat{\zeta}_2(x) = \phi_r(x) \frac{\hat{h}(\hat{x}, \hat{\zeta}_1)}{\phi_r(\hat{x})} > L_c \phi_r(x) = \hat{\zeta}_1(x)$, and so regime
 313 (α) applies by the monotonicity of the operator \hat{T} . From the definition of \hat{h} in (17), and holding the
 314 point $\hat{x} \leq x^*$ constant, this monotonicity implies that $\frac{\hat{h}(\hat{x}, \hat{\zeta}_n)}{\phi_r(\hat{x})} > L_c$ for all $n > 1$ and that $\frac{\hat{h}(\hat{x}, \widehat{V})}{\phi_r(\hat{x})} > L_c$. We
 315 conclude that case 2a of Lemma 9 applies (rather than case 2b) for a finite number of reserve contracts
 316 and also in the lifetime problem.

¹ If the lifetime value is infinite then so is the single problem's value and they are equal in this sense. When the lifetime value is zero then it is optimal not to enter the contract, and so the single problem's value is also zero.

Considering now the maximiser \hat{x} defined in case 1 of Lemma 9, we have for $x \geq x^* \geq \hat{x}$ that

$$\hat{\zeta}_2(x) = \phi_r(x) \frac{\hat{h}(\hat{x}, \hat{\zeta}_1)}{\phi_r(\hat{x})} \geq \frac{\hat{h}(\hat{x}_0, \hat{\zeta}_1)}{\phi_r(\hat{x}_0)} > \frac{\hat{h}(\hat{x}_0, \mathbf{0})}{\phi_r(\hat{x}_0)} = \hat{\zeta}_1(x) = V_c(x),$$

317 and regime (α) again follows by monotonicity. Also, trivially, case 1 of Lemma 9 applies for $\hat{\zeta} = \hat{\zeta}_n$
 318 and $\hat{\zeta} = \hat{V}$. \square

319 The following corollary follows immediately from the preceding proof.

320 **Corollary 3.** *Regime (β) holds if and only if $\hat{T}^2 \mathbf{0}(x) = \hat{T} \mathbf{0}(x)$ for all $x \geq x^*$.*

321 To address the implicit nature of our answers to **M1** and **M2** for the lifetime problem, in the next
 322 section we provide results for the construction and verification of the lifetime value function and
 323 corresponding stopping time. For this purpose we close this section by summarising results obtained
 324 above (making use of additional results from Appendix C).

325 **Theorem 3.** *In the setting of Theorem 2 assume that regime (α) holds. Then the lifetime value function \hat{V} is
 326 continuous, is a fixed point of the operator \hat{T} and $\hat{T}^n \mathbf{0}$ converges to \hat{V} exponentially fast in the supremum norm.
 327 Moreover, there is $\check{x} < x^*$ such that $\tau_{\check{x}}$ is an optimal stopping time for $\hat{T} \hat{V}(x)$ when $x \geq \check{x}$ and, furthermore, \check{x}
 328 is the highest price at which the battery operator can buy energy when acting optimally.*

329 2.4. Construction of the lifetime value function

330 In this section we discuss a numerical procedure for solution of the lifetime problem. It is based
 331 on the problem's structure as summarised in Theorem 3. Lemma 4 provides a means of constructing
 332 the lifetime value function, together with the value \check{x} of Theorem 3, using a one-dimensional search.
 333 We assume that regime (α) of Theorem 2 holds.

334 In the circumstance when the above procedure is not followed, complementary findings in
 335 Appendix E enable one to verify if a candidate buy price \hat{x} is optimal for the lifetime problem.

Lemma 4. *The lifetime value function evaluated at x^* satisfies*

$$\hat{V}(x^*) = \max_{z \in (a, x^*)} y(z), \quad (22)$$

where

$$y(z) := \frac{-z + p_c + \frac{\psi_r(z)}{\psi_r(x^*)} K_c}{\frac{\phi_r(z)}{\phi_r(x^*)} - \frac{\psi_r(z)}{\psi_r(x^*)} A}. \quad (23)$$

Proof. Fix $z \in (a, x^*)$. In the normalised lifetime problem of Section 2.1.3, suppose that the strategy τ_z is used for each energy purchase. Writing y for the total value of this strategy under \mathbb{P}^{x^*} , by construction we have the recursion

$$y = \frac{\phi_r(x^*)}{\phi_r(z)} \left(-z + p_c + \frac{\psi_r(z)}{\psi_r(x^*)} (K_c + Ay) \right).$$

336 Rearranging, we obtain (23). By Theorem 3, there exists an optimal strategy $\tau_{\check{x}}$ of the above form under
 337 \mathbb{P}^{x^*} and (22) follows. \square

338 Hence under \mathbb{P}^{x^*} an optimal stopping level \hat{x} can be found by maximising $y(z)$ over $z \in (a, x^*)$.
 339 The value \check{x} of Theorem 3 is given by $\check{x} = \max\{x : y(x) = \max_{z \in (a, x^*)} y(z)\}$.

340 3. Results

341 The general theory presented above provides optimal stopping times for initial EIM prices $x \geq \check{x}$,
 342 where \check{x} is the highest price at which the battery operator can buy energy optimally. In this section, for
 343 specific models of the EIM price we derive optimal stopping times for *all possible* initial EIM prices
 344 $x \in I$ when the sustainability conditions **S1*** and **S2*** hold. In the examples of this section the stopping
 345 sets Γ for the single and lifetime problems take the form $(a, \check{x}]$ although, in general, stopping sets may
 346 have much more complex structure. Interestingly, the stopping sets for the single and lifetime problem
 347 are either both half-lines or both compact intervals.

348 Note that condition **S2*** is ensured by the explicit choice of parameters. Verification of condition
 349 **S1*** is straightforward by checking, for example, if the left boundary a of the interval I satisfies
 350 $a < p_c + \lim_{x \rightarrow a} \frac{\psi_r(x)}{\psi_r(x^*)} K_c$, i.e., that $\limsup_{x \rightarrow a} h(x) > 0$. In particular, **S1*** always holds if $a = -\infty$.

Our approach is to combine the above general results with the geometric method drawn from
 Section 5 of [Dayanik and Karatzas \(2003\)](#). Although Proposition 5.12 of the latter paper gives results
 for natural boundaries, we note that the same arguments apply to entrance-not-exit boundaries. In
 particular we construct the least concave majorant W of the obstacle $H : [0, \infty) \rightarrow \mathbb{R}$, where

$$H(y) := \begin{cases} \frac{\hat{h}(F^{-1}(y), \hat{\zeta})}{\phi_r(F^{-1}(y))}, & y > 0, \\ \limsup_{x \rightarrow a} \frac{\hat{h}(x, \hat{\zeta})}{\phi_r(x)} = L_c, & y = 0, \end{cases} \quad (24)$$

351 (the latter equality was given in (41) in the appendices). Here the function $F(x) = \psi_r(x)/\phi_r(x)$ is
 352 strictly increasing with $F(a+) = 0$. Writing $\hat{\Gamma}$ for the set on which W and H coincide, under appropriate
 353 conditions the smallest optimal stopping time is given by the first hitting time of the set $\Gamma := F^{-1}(\hat{\Gamma})$
 354 ([Dayanik and Karatzas 2003](#), Propositions 5.13–5.14).

The Ornstein-Uhlenbeck (OU) process is a continuous-time stochastic process with dynamics

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t, \quad (25)$$

355 where $\theta, \sigma > 0$ and $\mu \in \mathbb{R}$. It has two natural boundaries, $a = -\infty$ and $b = \infty$. This process extends
 356 the scaled Brownian motion model by introducing a mean reverting drift term $\theta(\mu - X_t)dt$. The mean
 357 reversion is commonly observed in commodity price time series and may have several causes ([Lutz
 358 2009](#)). In the present context, the mean reversion can also be interpreted as the impact on prices of
 359 the system operator's corrective balancing actions. Appendix F collects some useful facts about the
 360 Ornstein-Uhlenbeck process. In particular, when constructing W it is convenient to note that $H'' \circ F$
 361 has the same sign as $(\mathcal{L} - r)h$, where \mathcal{L} is the infinitesimal generator of X defined as in Appendix F.

362 3.1. OU price process

363 Assume now that the EIM price follows the OU process (25) so that $L_c = 0$ (see Equation (55)
 364 in Appendix F) and, by Lemma 3, case A of Theorem 1 applies. We are able to deal with the single
 365 and lifetime problems simultaneously by setting $\hat{\zeta}$ equal to 0 for the single problem and equal to (the
 366 positive function) \hat{V} in the lifetime problem. The results of Sections 2.2–2.3 yield that in both problems,
 367 the right endpoint of the set $\hat{\Gamma}$ equals $F(\check{x})$ for some $\infty < \check{x} < x^*$. Further, since ψ_r is a solution to
 368 $(\mathcal{L} - r)v = 0$ and since $\check{x} < x^*$, for $x \leq x^*$ we have

$$(\mathcal{L} - r)\hat{h}(x, \hat{\zeta}) = (\mathcal{L} - r)\left(-x + p_c + \frac{\psi_r(x)}{\psi_r(x^*)}(K_c + A\hat{\zeta}(x^*))\right) \quad (26)$$

$$= (\mathcal{L} - r)(-x + p_c) \quad (27)$$

$$= (r + \theta)x - rp_c - \theta\mu. \quad (28)$$

369 Therefore, the function $(\mathcal{L} - r)\hat{h}(\cdot, \hat{\zeta})$ is negative on $(-\infty, B_0)$ and positive on (B_0, ∞) , where $B_0 =$
 370 $\frac{rp_c + \theta\mu}{r + \theta}$. This implies that H is strictly concave on $(0, F(B_0))$ and strictly convex on $(F(B_0), \infty)$. Since

371 the concave majorant W of H cannot coincide with H in any point of convexity, so necessarily $\check{x} < B_0$
 372 and H is concave on $(0, F(\check{x}))$. Hence we conclude that W is equal to H on the latter interval and so
 373 $\Gamma = (-\infty, \check{x}]$.

374 3.2. General mean-reverting processes

The above reasoning can be extended to mean-reverting processes with general volatility

$$dX_t = \theta(\mu - X_t)dt + \sigma(X_t)dW_t$$

375 for a measurable function σ such that the above equation admits a unique solution, c.f. Section 2,
 376 and $L_c = 0$ (c.f. (24)). Recall that we assume that (X_t) has two non-exit boundaries a, b (natural or
 377 entrance-not-exit boundaries) satisfying $a < x^* < b$. Since $\mathcal{L} = \theta(\mu - x)\frac{d}{dx} + \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}$, equations
 378 (26)–(28) still apply. In particular, we see that the diffusion coefficient $\sigma(\cdot)$ does not affect the sign of (28)
 379 and thus does not influence the concavity properties of H on $(0, F(x^*))$. Proceeding as above, we argue
 380 that case A of Theorem 1 applies and the single and lifetime problems can be solved simultaneously.
 381 Particularly, the largest buy price is given by $a < \check{x} < x^*$ (different for the single and lifetime problems).
 382 Note that the form of the stopping set is purely determined by μ, θ , the left boundary a and the initial
 383 premium p_c . Obviously, the mean price level μ satisfies $\mu > a$ because a is an unreachable boundary.

384 **Lemma 5.** *If $p_c > a$, then the stopping sets for the single and lifetime problems are of the form $\Gamma = (a, \check{x}]$.*

385 **Proof.** The same arguments as in the OU case are directly applicable to the present setting and, under
 386 the assumptions of the lemma, we have $B_0 = \frac{rp_c + \theta\mu}{r + \theta} > a$. Hence for each problem the stopping set has
 387 the form $\Gamma = (a, \check{x}]$ for some $\check{x} < B_0$. \square

In the particular case of the CIR model (Cox et al. 1985)

$$dX_t = \theta(\mu - X_t)dt + \sigma\sqrt{X_t}dW_t, \quad (29)$$

388 we have $a = 0, b = \infty$. Then:

389 **Corollary 4.** *If X is the CIR process (29) with $2\theta\mu \geq \sigma^2$ and $\mu > 0$ then the boundary $a = 0$ is
 390 entrance-not-exit. Further, if $p_c > 0$ then the stopping sets for the single and lifetime problems are of the
 391 form $\Gamma = (0, \check{x}]$.*

392 **Proof.** It follows from (Cox et al. 1985, p. 391) that the condition $2\theta\mu \geq \sigma^2$ is necessary and sufficient
 393 for the boundary 0 to be entrance-not-exit. By Lemma 3, we have $L_c = 0$. An application of Lemma 5
 394 concludes. \square

Remark 1. *More generally, suppose that the imbalance price process follows*

$$dX_t = \theta(\mu - X_t)dt + \sigma X_t^\gamma dW_t,$$

395 *for some $\gamma > 0.5$. Then the left boundary $a = 0$ is entrance-not-exit for any choice of parameters $\theta, \mu, \sigma > 0$
 396 since the scale function p given in (3) converges to negative infinity at 0. Therefore, the arguments in the above
 397 corollary apply and the stopping sets for the single and lifetime problems are also of the form $\Gamma = (0, \check{x}]$.*

398 3.3. Shifted exponential price processes

399 In order to first recover and then generalise previously obtained results (Moriarty and Palczewski
400 2017), take the following shifted exponential model for the price process:

$$f(z) := D + de^{bz}, \quad (30)$$

$$X_t = f(Z_t), \quad (31)$$

401 where Z is a regular one-dimensional diffusion with non-exit (natural or entrance-not-exit) boundaries
402 a^Z and b^Z (we will use the superscripts X and Z where necessary to emphasise the dependence on the
403 stochastic process). The idea is that Z models the physical system imbalance process while f represents
404 a *price stack* of bids and offers which is used to form the EIM price. In this case the left boundary for X
405 is $a = f(a^Z) \geq D$ and, by Lemma 3, $L_c = 0$ and case A of Theorem 1 applies. Rather than working
406 with the implicitly defined process X , however, we may work directly with the process Z by setting:

$$z^* := f^{-1}(x^*), \quad (32)$$

$$h_f(z) := -f(z) + p_c + \begin{cases} \frac{\psi_r^Z(z)}{\psi_r^Z(z^*)} K_c, & z < z^*, \\ K_c, & z \geq z^*, \end{cases} \quad (33)$$

$$\hat{h}_f(z, \hat{\zeta}) := \begin{cases} -f(z) + p_c + \frac{\psi_r^Z(z)}{\psi_r^Z(z^*)} (K_c + A\hat{\zeta}(z^*)), & z < z^*, \\ -f(z) + p_c + K_c + A\hat{\zeta}(z), & z \geq z^*, \end{cases} \quad (34)$$

407 and modifying the definitions for \mathcal{T} , $\hat{\mathcal{T}}$, V_c and \hat{V} accordingly. We then have

Theorem 4. Taking definitions (30) and (32)–(34), assume that conditions **S1*** and **S2*** hold. Then

$$L_c := \limsup_{z \rightarrow a^Z} \frac{-f(z)}{\phi_r^Z(z)} = 0.$$

408 Also:

i) (Single problem) There exists $\hat{z} < z^*$ that maximises $\frac{h_f(z)}{\phi_r^Z(z)}$, the stopping time $\tau_{\hat{z}}$ is optimal for $z \geq \hat{z}$, and

$$V_c(z) = \phi_r^Z(z) \frac{h_f(\hat{z})}{\phi_r^Z(\hat{z})}, \quad z \geq \hat{z}.$$

ii) (Lifetime problem) The lifetime value function \hat{V} is continuous and a fixed point of $\hat{\mathcal{T}}$. There exists $\tilde{z} \in (\hat{z}, z^*)$ which maximises $\frac{\hat{h}(z, \hat{V})}{\phi_r^Z(z)}$ and $\tau_{\tilde{z}}$ is an optimal stopping time for $z \geq \tilde{z}$ with

$$\hat{V}(z) = \hat{\mathcal{T}}\hat{V}(z) = \phi_r^Z(z) \frac{\hat{h}(\tilde{z}, \hat{V})}{\phi_r^Z(\tilde{z})}, \quad z \geq \tilde{z}.$$

409 **Proof.** The proof follows from the one-to-one correspondence between the process X and the process
410 Z , and direct transfer from Theorems 1 and 3. \square

411 In some cases, explicit necessary and/or sufficient conditions for **S1*** may be given in terms of the
412 problem parameters. Assume that $a^Z = -\infty$ as in the examples studied below. If $p_c > D$ and $K_c \geq 0$,
413 this is sufficient for the condition **S1*** to be satisfied as then $h_f(z) \geq -f(z) + p_c > 0$ for sufficiently
414 small z . When $p_c = D$ and $K_c > 0$, it is sufficient to verify that $e^{bz} = o(\psi_r^Z(z))$ as $z \rightarrow -\infty$ since
415 then $h_f(z) = -de^{bz} + \psi_r^Z(z) K_c / \psi_r^Z(z^*)$ for $z < z^*$. On the other hand, our assumption that **S1*** holds
416 necessarily excludes parameter combinations with $p_c - D = K_c = 0$, since the reserve contract writer
417 then cannot make any profit because $h_f(z) \leq 0$ for all z .

418 In Section 3.3.1 we take Z to be the standard Brownian motion and recover results from the single
 419 problem of [Moriarty and Palczewski \(2017\)](#) (the lifetime problem is formulated differently in the latter
 420 reference, where degradation of the store is not modelled). In Section 3.3.2 we generalise to the case
 421 when Z is an OU process.

422 3.3.1. Brownian motion imbalance process

When the imbalance process $Z = W$, the Brownian motion, we have

$$(\mathcal{L} - r)\hat{h}_f(z, \hat{\zeta}) = (\mathcal{L} - r)(-f(z) + p_c) = de^{bz} \left\{ r - \frac{1}{2}b^2 \right\} + r(D - p_c).$$

423 We have several cases depending on the sign of $(D - p_c)$ and $(r - \frac{1}{2}b^2)$.

424 1. Assume first that $r > \frac{1}{2}b^2$.

- 425 (i) We may exclude the subcase $p_c \leq D$, since then $H(y) = \frac{\hat{h}(z, \hat{\zeta})}{\phi_f^Z(z)}|_{z=(F^Z)^{-1}(y)}$ is strictly convex
 426 on $(0, F^Z(z^*))$ for any $\hat{\zeta}$ and Γ cannot intersect this interval, contradicting Theorem 4 and,
 427 consequently, violating **S1*** or **S2***.
 (ii) If $p_c > D$, H is concave on $(0, F^Z(B))$ and convex on $(F^Z(B), \infty)$, where

$$B = \frac{1}{b} \log \left(\frac{r(p_c - D)}{d(r - \frac{1}{2}b^2)} \right).$$

428 By Theorem 4 and the positivity of H on $(0, F^Z(\hat{z}))$ we have $\Gamma = (-\infty, \hat{z}]$ and $\Gamma = (-\infty, \tilde{z}]$
 429 for the single and lifetime problems respectively, with $\tilde{z} < \hat{z} < B$.

430 2. Suppose that $r < \frac{1}{2}b^2$.

- 431 (i) When $p_c \geq D$, the function H is concave on $(0, \infty)$. Hence the stopping sets Γ for single and
 432 lifetime problems have the same form as in case 1(ii) above.
 433 (ii) If $p_c < D$, the function H is convex on $(0, F^Z(B))$ and concave on $(F^Z(B), \infty)$. The set Γ
 434 must then be an interval, respectively $[\hat{z}_0, \hat{z}]$ and $[\tilde{z}_0, \tilde{z}]$. For explicit expressions for the left
 435 and right endpoints for the single problem, as well as sufficient conditions for **S1***, the
 436 reader is referred to [Moriarty and Palczewski \(2017\)](#).

437 3. In the boundary case $r = \frac{1}{2}b^2$, the convexity of H is determined by the sign of the difference
 438 $D - p_c$. As above the possibility $D > p_c$ is excluded since then H is strictly convex. Otherwise H
 439 is concave and the stopping sets Γ have the same form as in case 1(ii) above.

440 3.3.2. OU imbalance process

When Z is the Ornstein-Uhlenbeck process, by adjusting d and b in the price stack function f (see
 (30)) we can restrict our analysis to the OU process with zero mean and unit volatility, that is:

$$dZ_t = -\theta Z_t dt + dW_t.$$

441 Then for $z < z^*$

$$(\mathcal{L} - r)\hat{h}_f(z, \hat{\zeta}) = (\mathcal{L} - r)(-f(z) + p_c) \tag{35}$$

$$= de^{bz} \left\{ b \left(\theta z - \frac{1}{2}b \right) + r \right\} + r(D - p_c) =: \eta(z). \tag{36}$$

Differentiating η we obtain

$$\eta'(z) = db\theta e^{bz} \left(bz + 1 + \frac{r - \frac{1}{2}b^2}{\theta} \right)$$

442 which has a unique root at $z^\diamond = \frac{1}{b} \left(\frac{\frac{1}{2}b^2 - r}{\theta} - 1 \right)$. The function η decreases from $r(D - p_c)$ at $-\infty$ until
 443 $\eta(z^\diamond) = -de^{bz^\diamond} \theta + r(D - p_c)$ at z^\diamond and then increases to positive infinity.

444 1. If $p_c \geq D$ then the function η is negative on $(-\infty, u)$, where u is the unique root of η . Hence H is
 445 concave on $(0, F^Z(u))$ and convex on $(F^Z(u), \infty)$. The stopping sets Γ for the single and lifetime
 446 problems must then be of the form $(-\infty, \hat{z}]$ and $(-\infty, \tilde{z}]$, respectively, c.f. case 1(ii) in Section
 447 3.3.1.

448 2. The case $p_c < D$ is more complex.

449 (i) Let $z^\diamond \geq z^*$. We exclude the possibility $\eta(z^*) \geq 0$, since then the function H is convex on
 450 $(0, F^Z(z^*))$ and the set Γ has empty intersection with this interval, contradicting Theorem
 451 4 and, consequently, violating **S1*** or **S2***. When $\eta(z^*) < 0$, H is convex on $(0, F^Z(u))$
 452 and concave on $(F^Z(u), F^Z(z^*))$, where u is the unique root of η on $(0, z^*)$. Therefore
 453 the stopping sets Γ for the single and lifetime problems are of the form $[\hat{z}_0, \hat{z}]$ and $[\tilde{z}_0, \tilde{z}]$,
 454 respectively, with $\min(\hat{z}_0, \tilde{z}_0) > u$, c.f. case 2(ii) in Section 3.3.1.

455 (ii) Consider now $z^\diamond < z^*$. As above we exclude the case $\eta(z^\diamond) \geq 0$, since then H is convex
 456 on $(0, F^Z(z^\diamond))$. The remaining case $\eta(z^\diamond) < 0$ implies that the stopping sets Γ have the
 457 same form as in case 2(i) above, as H is convex and then concave if $\eta(z^*) \leq 0$, and
 458 convex-concave-convex if $\eta(z^*) > 0$.

459 4. Benchmark case study and economic implications

460 In this section we use a case study to draw qualitative implications from the above results. An
 461 OU model is assumed, which captures both the mean reversion and random variability present in EIM
 462 prices, and is fitted to relevant data. The interest rate is taken to be 3% per annum, and the degradation
 463 factor for the store to be $A = 0.9999$.

464 Our data is the ‘balancing group price’ from the German Amprion system operator, which is
 465 available for every 15 minute period (AMPRION 2016). Summary statistics for the period from 1 June
 466 2012 to 31 May 2016 are presented in Table 1. To address the issue of its extreme range, which impacts
 467 the fitting of both volatility and mean reversion in the OU model, the data was truncated at the values
 468 -150 and 150. The parameters obtained by maximum likelihood fitting were then $\theta = 68.69$ (the rate of
 469 mean reversion), $\sigma = 483.33$ (the volatility), $\mu = 30.99$ (the mean-reversion level). The effect of the
 470 truncation step was to approximately halve the fitted volatility.

Table 1. Summary statistics for the 15 minute balancing group price per MWh in the German Amprion
 area, 1 June 2012 to 31 May 2016.

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
-6002.00	0.27	33.05	31.14	66.97	6344.00

471 The left panel of Figure 3, and Figure 4, show the lifetime value $\hat{V}(x^*)$, while the right panel of
 472 Figure 3 plots the stopping boundary \check{x} , which is the maximum price at which the battery operator can
 473 buy energy optimally. These values of \check{x} are significantly below the long-term mean price D , indeed
 474 the former value is negative while the latter is positive. Thus in this example the battery operator
 475 purchases energy when it is in excess supply, further contributing to balancing. To place the negative
 476 values on the stopping boundary in Figure 3 in the statistical context, recall from Table 1 that the first
 477 quartile of the price distribution is approximately zero. Indeed negative energy prices usually occur
 478 several times per day in the German EIM. In the present dataset of 1461 days there are only 11 days
 479 without negative prices and the longest observed time between negative prices is 41.5 hours.

480 We make the following empirical observations. Firstly, defining the total premium as the sum
 481 $p_c + K_c$, altering its distribution between the initial premium p_c (which is received at $x = \check{x}$) and the

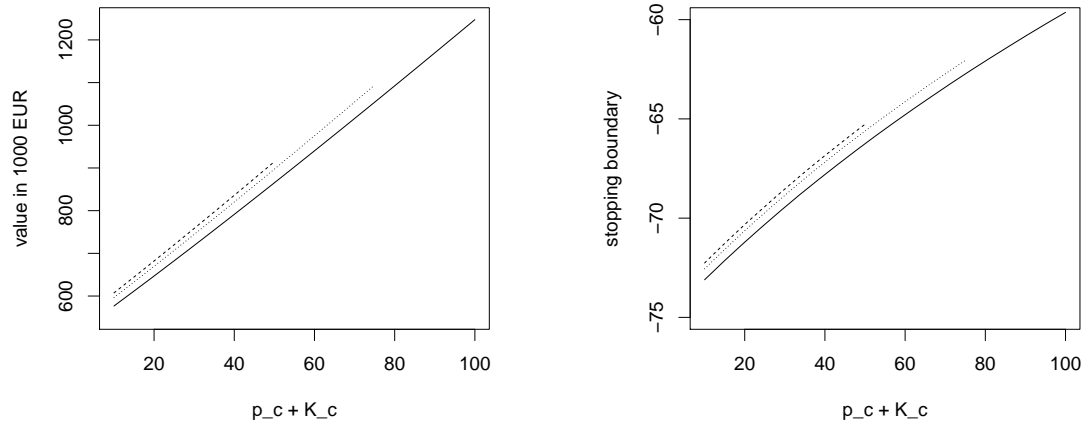


Figure 3. Results obtained with the Ornstein-Uhlenbeck model fitted in Section 4, as functions of the total premium, with interest rate 3% per annum. Solid lines: $x^* = 100$, dotted: $x^* = 75$, dashed: $x^* = 50$. Left: lifetime value $\hat{V}(x^*)$. Right: the stopping boundary \check{x} , the maximum price for which the battery operator can buy energy optimally.

482 utilisation payment K_c (which is received at $x = x^*$) results in insignificant changes to the graphs, with
 483 relative differences on the vertical axes of the order 10^{-3} (data not shown). It is for this reason that
 484 the figures are indexed by the total premium $p_c + K_c$ rather than by individual premia. Secondly, it is
 485 seen from the right hand panel of Figure 3 that the (negative valued) stopping boundary increases
 486 with the total premium, making exercise more frequent. Thus as the total premium increases, both
 487 the frequency and size of the cashflows increase, yielding a superlinear relationship in the left hand
 488 panel of Figure 3. This superlinearity is not very pronounced since the stopping boundary is relatively
 489 insensitive to the total premium in the range presented in the graphs (see the right hand panel), so that
 490 the lifetime value is driven principally by the size of the cashflows. Thirdly, the grey horizontal line of
 491 Figure 4 is placed at a level indicative of recent costs for lithium-ion batteries per megawatt hour. Thus
 492 the investment case for battery storage providing reserve is significantly positive for a wide range of
 493 the contract parameters. Finally the contours in Figure 4 have an S-shape, the marginal influence of x^*
 494 being smaller in the range $x^* < 110$ and larger for greater values of x^* (with the marginal influence
 495 eventually decreasing again in the limit of large x^*).

496 These phenomena are explained by the presence of mean reversion in the OU price model. The
 497 timings of the cashflows to the battery operator are entirely determined by the successive *passage times*
 498 of the price process between the levels x^* and \check{x} . These passage times are relatively short on average
 499 for the fitted OU model. This means that the premia are received at almost the same time under each
 500 reserve contract, and it is the total premium which drives the real option value. Further the passage
 501 times between x^* and \check{x} may be decomposed into passage times between x^* and D , and between D
 502 and \check{x} . Since the OU process is statistically symmetric about D , let us compare the distances $|\check{x} - D|$
 503 and $|x^* - D|$. From Figure 3 we have $\check{x} \approx -70$ so that $|\check{x} - D| \approx 100$. Therefore for $x^* < 110$ we have
 504 $|x^* - D| \ll |\check{x} - D|$ and the passage time between D and \check{x} , which varies little, dominates that between
 505 x^* and D . Correspondingly we observe in Figure 4 that the value function changes relatively little as
 506 x^* varies below 110. Conversely, as x^* increases beyond 110 it is the distance between x^* and D which
 507 dominates, and the value function begins to decrease relatively rapidly.

508 These results provide insights into the suitability of the considered arrangement for correcting
 509 differing levels of imbalance. As the distance between x^* and the mean level D grows, the energy
 510 price reaches x^* significantly less frequently and the reserve contract starts to provide insurance

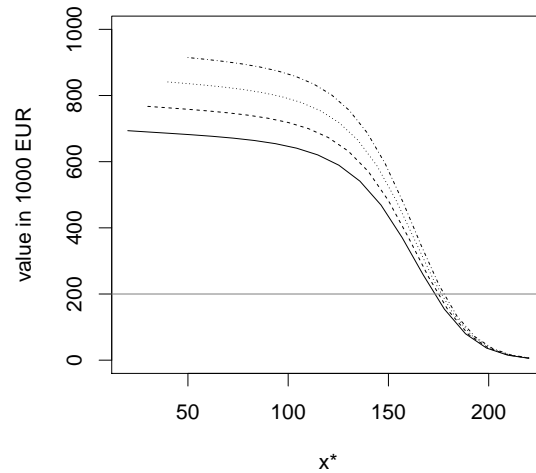


Figure 4. Lifetime value $\widehat{V}(x^*)$ as a function of x^* with the Ornstein-Uhlenbeck model fitted in Section 4, with interest rate 3% per annum. Dashed line: $p_c + K_c = 20$, solid: $p_c + K_c = 30$, dotted: $p_c + K_c = 40$, mixed: $p_c + K_c = 50$. The horizontal grey line indicates the current price of lithium-ion battery storage per MWh (IRENA 2017, Figure 33).

511 against rare events, resulting in infrequent power delivery and low utilisation of the battery. These
 512 observations suggest that the contractual arrangement studied in this paper is more suitable for the
 513 frequent balancing of less severe imbalance. In contrast, the more rapid reduction in the lifetime value
 514 for large values of x^* suggests that such arrangements based on real-time markets are not suitable for
 515 balancing relatively rare events such as large system disturbances due to unplanned outages of large
 516 generators. The system operator may prefer to use alternative arrangements, based for example on
 517 fixed availability payments, to provide security against such events.

518 5. Discussion

519 In this paper we investigate the procurement of operating reserve from energy-limited storage
 520 using a sequence of physically covered incremental reserve contracts. This leads to the pricing of a
 521 real perpetual American swing put option with a random refraction time. We model the underlying
 522 energy imbalance market price as a general linear regular diffusion, which, in particular, is capable of
 523 modelling the mean reversion present in imbalance prices. Both the optimal operational policy and
 524 the real option value of the store are characterised explicitly. Although the solutions are generally
 525 not available in an analytical form we have provided a straightforward procedure for their numerical
 526 evaluation together with empirical examples from the German energy imbalance market.

527 The results of the lifetime analysis in particular have both managerial implications for the battery
 528 operator and policy implications for the system operator. From the operational viewpoint, under the
 529 setup described in Section 1.1 we have established that the battery operator should purchase energy as
 530 soon as the EIM price falls to the level \check{x} , which may be calculated as described in Section 2.4. Further
 531 the battery operator should then sell the reserve contract immediately. Our real options valuation may
 532 be taken into account when deciding whether to invest in an energy store, and whether to sell such
 533 reserve contracts in preference to trading in other markets (for example, performing price arbitrage in
 534 the spot energy market).

535 Turning to the perspective of the system operator, we have demonstrated that the proposed
 536 arrangement can be mutually beneficial to the system operator and battery operator. More precisely,
 537 the system operator can be protected against guaranteed financial losses from the incremental capacity

538 contract purchase while the battery operator has a quantifiable profit. The analysis also provides
 539 information on feedback due to battery charging by determining the highest price \hat{x} at which the battery
 540 operator buys energy, hence identifying conditions under which the battery operator's operational
 541 strategy is aligned with system stability.

542 We address incremental reserve contracts, which are particularly valuable to the system operator
 543 when the margin of electricity generation capacity over peak demand is low. Decremental reserve may
 544 also be studied in the above framework, although the second stopping time (action A2) is non-trivial
 545 which leads to a nested stopping problem beyond the scope of the present paper. Further we assume
 546 that the energy storage unit is dedicated to providing incremental reserve contracts, so that the
 547 opportunity costs of not operating in other markets or providing other services are not modelled. The
 548 extension to a finite expiry time, the lifetime analysis with decremental reserve contracts, and also the
 549 opportunity cost of not operating in other markets would be interesting areas for further work.

550 The methodological advances of this paper reach beyond energy markets. In particular they are
 551 relevant to real options analyses of storable commodities where the timing problem over the lifetime
 552 of the store is of primary interest. The lifetime analysis via optimal stopping techniques, developed
 553 in Section 2.3, provides an example of how timing problems can be addressed for rather general
 554 dynamics of the underlying stochastic process. In this context we provide an alternative method to
 555 quasi-variational inequalities, which are often dynamics-specific and technically more involved.

556 **Funding:** This research was funded by the UK Engineering and Physical Sciences Research Council grant number
 557 EP/K00557X/2 and MNiSzW grant UMO-2012/07/B/ST1/03298.

558 **Conflicts of Interest:** The authors declare no conflict of interest.

559 Appendix A Lemmas and proofs from Section 2

The following three lemmas classify solutions to the stopping problem (7). Note that if $\sup_x \vartheta(x) \leq 0$ then no choice of the stopping time τ gives a value function greater than 0. The optimal stopping time in this case is given by $\tau = \infty$. In what follows we therefore assume

$$\sup_{x \in (a,b)} \vartheta(x) > 0. \quad (37)$$

560 These results can be derived from [Beibel and Lerche \(2000\)](#); however, for the convenience of the
 561 reader we provide simple proofs.

562 **Lemma 6.** *Assume that there exists $\hat{x} \in I$ which maximises $\vartheta(x)/\phi_r(x)$ over I . Then the value function $v(x)$
 563 is finite for all x , and for $x \geq \hat{x}$:*

- 564 1. the stopping time $\tau_{\hat{x}}$ is optimal,
- 565 2. $v(x) = \frac{\vartheta(\hat{x})}{\phi_r(\hat{x})} \phi_r(x)$,
- 566 3. any stopping time τ with $\mathbb{P}^x \{ \vartheta(X_\tau)/\phi_r(X_\tau) < \vartheta(\hat{x})/\phi_r(\hat{x}) \} > 0$ is strictly suboptimal for the problem
 567 $v(x)$.

Proof. Since ϕ_r is r -excessive ([Borodin and Salminen 2012](#), Section II.5), for any finite stopping time τ

$$\mathbb{E}^x \{ e^{-r\tau} \phi_r(X_\tau) \} \leq \phi_r(x).$$

Let now τ be a stopping time taking possibly infinite values. Let b_n be an increasing sequence converging to b with $b_1 > x$, the initial point of the process X . Then τ_{b_n} is an increasing sequence of stopping times converging to infinity and

$$\begin{aligned}\phi_r(x) &\geq \liminf_{n \rightarrow \infty} \mathbb{E}^x \{e^{-r(\tau \wedge \tau_{b_n})} \phi_r(X_{\tau \wedge \tau_{b_n}})\} \\ &\geq \mathbb{E}^x \{ \liminf_{n \rightarrow \infty} e^{-r(\tau \wedge \tau_{b_n})} \phi_r(X_{\tau \wedge \tau_{b_n}}) \} = \mathbb{E}^x \{e^{-r\tau} \phi_r(X_\tau) \mathbf{1}_{\tau < \infty}\},\end{aligned}$$

568 where $\phi_r(b-) = 0$ was used in the last equality.

For any stopping time τ

$$\begin{aligned}\mathbb{E}^x \{e^{-r\tau} \vartheta(X_\tau) \mathbf{1}_{\tau < \infty}\} &= \mathbb{E}^x \left\{ e^{-r\tau} \phi_r(X_\tau) \frac{\vartheta(X_\tau)}{\phi_r(X_\tau)} \mathbf{1}_{\tau < \infty} \right\} \\ &\leq \frac{\vartheta(\hat{x})}{\phi_r(\hat{x})} \mathbb{E}^x \{e^{-r\tau} \phi_r(X_\tau) \mathbf{1}_{\tau < \infty}\} \leq \frac{\vartheta(\hat{x})}{\phi_r(\hat{x})} \phi_r(x),\end{aligned}\tag{38}$$

569 where the final inequality follows from the first part of the proof and (37) (so $\frac{\vartheta(\hat{x})}{\phi_r(\hat{x})} > 0$). Hence, $v(x)$ is
570 finite for all $x \in I$. To prove claim 1, note from (6) that for $x \geq \hat{x}$ the upper bound is attained by $\tau_{\hat{x}}$,
571 which is therefore an optimal stopping time in the problem $v(x)$. The assumption on τ in claim 3 leads
572 to strict inequality in (38), making τ strictly suboptimal in the problem $v(x)$. \square

It is convenient to introduce the notation

$$L := \limsup_{x \rightarrow a} \frac{\vartheta(x)^+}{\phi_r(x)}.\tag{39}$$

573 Lemma 7 corresponds to cases when there is no optimal stopping time but the optimal value can be
574 reached in the limit by a sequence of stopping times.

575 **Lemma 7.**

- 576 1. If $L = \infty$ then the value function is infinite and there is no optimal stopping time.
577
578 2. If $L < \infty$ and $L > \vartheta(x)/\phi_r(x)$ for all $x \in I$, then there is no optimal stopping time and the value function
579 equals $v(x) = L\phi_r(x)$.

Proof. Assertion 1. Fix any $x \in I$. Then for any $\hat{x} < x$ we have

$$\mathbb{E}^x \{e^{-r\tau_{\hat{x}}} \vartheta(X_{\tau_{\hat{x}}})\} = \vartheta(\hat{x}) \frac{\phi_r(x)}{\phi_r(\hat{x})},$$

580 which converges to infinity for \hat{x} tending to a over an appropriate subsequence. Since the process is
581 recurrent, the point x can be reached from any other point in the state space with positive probability
582 in a finite time. This proves that the value function is infinite for all $x \in I$.

Assertion 2. Recall that due to the supremum of $\frac{\vartheta}{\phi_r}$ being strictly positive we have $L > 0$. From the proof of Lemma 6, for an arbitrary stopping time τ we have

$$\mathbb{E}^x \{e^{-r\tau} \vartheta(X_\tau) \mathbf{1}_{\tau < \infty}\} = \mathbb{E}^x \left\{ e^{-r\tau} \phi_r(X_\tau) \frac{\vartheta(X_\tau)}{\phi_r(X_\tau)} \mathbf{1}_{\tau < \infty} \right\} < L \mathbb{E}^x \{e^{-r\tau} \phi_r(X_\tau) \mathbf{1}_{\tau < \infty}\} \leq L\phi_r(x).$$

However, one can construct a sequence of stopping times that achieves this value in the limit. Take x_n such that $\lim_{n \rightarrow \infty} \vartheta(x_n)/\phi_r(x_n) = L$ and define $\tau_n = \tau_{x_n}$. Then

$$\lim_{n \rightarrow \infty} \mathbb{E}^x \{e^{-r\tau_n} \vartheta(X_{\tau_n})\} = \lim_{n \rightarrow \infty} \vartheta(x_n) \frac{\phi_r(x)}{\phi_r(x_n)} = \phi_r(x)L,$$

583 so $v(x) = \phi_r(x)L$. This together with the strict inequality above proves that an optimal stopping time
 584 does not exist. \square

The results developed in this section also have a ‘mirror’ counterpart involving

$$R := \limsup_{x \rightarrow b} \frac{\vartheta(x)^+}{\psi_r(x)} \quad (40)$$

585 rather than L . In particular, the value function is infinite if $R = \infty$, and

586 **Corollary 5.** *If $\hat{x} \in I$ maximises $\vartheta(x)/\psi_r(x)$ then for any $x \leq \hat{x}$ an optimal stopping time in the problem $v(x)$
 587 is given by $\tau_{\hat{x}}$.*

588 This also motivates the assumptions of the following lemma which collects results from [Dayanik
 589 and Karatzas \(2003, Section 5.2\)](#). Again, although those results are obtained under the assumption that
 590 both boundaries are natural, their proofs require only that they are non-exit.

591 **Lemma 8.** *Assume that $L, R < \infty$ and ϑ is locally bounded. Then the value function v is finite and continuous
 592 on (a, b) .*

593 All the stopping problems considered in this paper have a finite right-hand limit $R < \infty$. Therefore,
 594 whenever $L < \infty$, their value functions will be continuous.

595 *Proof of Lemma 2.* If **S1*** does not hold then the payoff from cycle A1–A3 is not profitable (on
 596 average) for any value of the EIM price x , so **S1** does not hold. Conversely if **S1*** holds then there
 597 exists x such that $\hat{T}0(x) \geq h(x) > 0$. For any other x' consider the following strategy: wait until the
 598 process X hits x and proceed optimally thereafter. This results in a strictly positive expected value:
 599 $\hat{T}0(x') > 0$ and by the arbitrariness of x' we have $\hat{T}0 > 0$.

600 Suppose that **S2*** holds. Then the system operator makes a profit on the reserve contract (relative
 601 to simply purchasing a unit of energy at the power delivery time $\hat{\tau}_e$, at the price $X(\hat{\tau}_e) \geq x^*$) in
 602 undiscounted cash terms. Considering discounting, the system operator similarly makes a profit
 603 provided the EIM price reaches the level x^* (or above) sufficiently quickly. Since this happens with
 604 positive probability for a regular diffusion, a certain financial loss for the system operator is excluded.
 605 When **S2*** does not hold, suppose first that $p_c + K_c > x^*$: then the system operator makes a loss in
 606 undiscounted cash terms, and if the reserve contract is sold when $x \geq x^*$ then this loss is certain. In
 607 the boundary case $p_c + K_c = x^*$ the battery operator can only make a profit by purchasing energy and
 608 selling the reserve contract when $X_t < x^*$, in which case the system operator makes a certain loss.
 609 This follows since instead of buying the reserve contract, the system operator could invest $p_c > 0$
 610 temporarily in a riskless bond, withdrawing it with interest when the EIM price rises to $x^* = p_c + K_c$.
 611 The loss in this case is equal in value to the interest payment. \square

612 Appendix B Lemmas for the lifetime problem

613 It follows from the optimal stopping theory reviewed in Section 2.1.1 and Appendix A that the
 614 following definition of an *admissible* continuation function, is natural in our setup. In particular, the
 615 final condition corresponds to the assumption that the energy purchase occurs at a price below x^* .

616 **Definition 1.** (*Admissible continuation value*) *A continuation value function $\hat{\zeta}$ is admissible if it is continuous
 617 on $(a, x^*]$ and non-negative on I , with $\frac{\hat{\zeta}(x)}{\phi_r(x)}$ non-increasing on $[x^*, b)$.*

618 The following result now characterises the possible solution types in the lifetime problem.

Lemma 9. Assume that conditions **S1*** and **S2*** hold. If $\hat{\zeta}$ is an admissible continuation value function then

$$\limsup_{x \rightarrow a} \frac{\hat{h}(x, \hat{\zeta})}{\phi_r(x)} = \limsup_{x \rightarrow a} \frac{-x}{\phi_r(x)} = L_c, \quad (41)$$

and with cases A, B, C defined just as in Theorem 1:

1. In case A, there exists $\hat{x} \leq x^*$ which maximises $\frac{\hat{h}(x, \hat{\zeta})}{\phi_r(x)}$ and $\tau_{\hat{x}}$ is an optimal stopping time for $x \geq \hat{x}$ with value function

$$v(x) = \hat{\mathcal{T}}\hat{\zeta}(x) = \phi_r(x) \frac{\hat{h}(\hat{x}, \hat{\zeta})}{\phi_r(\hat{x})}, \quad x \geq \hat{x}.$$

Denoting by \hat{x}_0 the corresponding \hat{x} in case A of Theorem 1, we have $\hat{x}_0 \leq \hat{x}$.

2. In case B, either

- a) there exists $x_L \in (a, b)$ with $\frac{\hat{h}(x_L, \hat{\zeta})}{\phi_r(x_L)} \geq L_c$: then there exists $\hat{x} \in (a, x^*]$ which maximises $\frac{\hat{h}(x, \hat{\zeta})}{\phi_r(x)}$, and

- $\tau_{\hat{x}}$ is an optimal stopping time for $x \geq \hat{x}$ with value function $v(x) = \phi_r(x) \frac{\hat{h}(\hat{x}, \hat{\zeta})}{\phi_r(\hat{x})}$ for $x \geq \hat{x}$; or

- b) there does not exist $x_L \in (a, b)$ with $\frac{\hat{h}(x_L, \hat{\zeta})}{\phi_r(x_L)} \geq L_c$: then the value function is $v(x) = L_c \phi_r(x)$ and there is no optimal stopping time.

3. In case C, the value function is infinite and there is no optimal stopping time.

Moreover, the value function v is continuous in cases A and B.

Proof. Note that

$$h(x) = \hat{h}(x, \mathbf{0}) \leq \hat{h}(x, \hat{\zeta}) = \begin{cases} h(x) + \frac{\psi_r(x)}{\psi_r(x^*)} A\hat{\zeta}(x^*), & x < x^*, \\ h(x) + A\hat{\zeta}(x), & x \geq x^*. \end{cases} \quad (42)$$

This proves (41), since $\lim_{x \rightarrow a} \psi_r(x)/\phi_r(x) = 0$. We verify from (42) and the assumptions of the lemma that $R < \infty$ in (40). Hence, whenever $L_c < \infty$ the value function v is finite and continuous by Lemma 8. As noted previously (in the proof of Theorem 1), h is negative and decreasing on $[x^*, b)$, hence the ratio $h(x)/\phi_r(x)$ is strictly decreasing on that interval. It then follows from (42) and the admissibility of $\hat{\zeta}$ that the function $x \mapsto \frac{\hat{h}(x, \hat{\zeta})}{\phi_r(x)}$ is strictly decreasing on $[x^*, b)$. Therefore the supremum of $x \mapsto \frac{\hat{h}(x, \hat{\zeta})}{\phi_r(x)}$, which is positive by (42) and **S1***, is attained on $(a, x^*]$ or asymptotically when $x \rightarrow a$. In cases 1 and 2a, the optimality of $\tau_{\hat{x}}$ for $x \geq \hat{x}$ then follows from Lemma 6. To see that $\hat{x}_0 \leq \hat{x}$ in case 1, take $x < \hat{x}_0$. Then from (42) we have

$$\frac{\hat{h}(x, \hat{\zeta})}{\phi_r(x)} = \frac{h(x)}{\phi_r(x)} + \frac{\psi_r(x)}{\phi_r(x) \psi_r(x^*)} A\hat{\zeta}(x^*) < \frac{h(\hat{x}_0)}{\phi_r(\hat{x}_0)} + \frac{\psi_r(\hat{x}_0)}{\phi_r(\hat{x}_0) \psi_r(x^*)} A\hat{\zeta}(x^*) = \frac{\hat{h}(\hat{x}_0, \hat{\zeta})}{\phi_r(\hat{x}_0)},$$

since $x \mapsto \frac{\psi_r(x)}{\phi_r(x)}$ is strictly increasing. Case 2b follows from Lemma 7 and the fact that $L_c > 0$, while Lemma 7 proves case 3. \square

Before proceeding we note the following technicalities.

Remark 2. The value function v in cases 1 and 2a of Lemma 9 satisfies the condition that $v(x)/\phi_r(x)$ is non-increasing on $[x^*, b)$. Indeed,

$$\frac{v(x)}{\phi_r(x)} = \frac{\hat{h}(\hat{x}, \hat{\zeta})}{\phi_r(\hat{x})} = \text{const.}$$

for $x \geq \hat{x}$.

Remark 3. For case 3 of Lemma 9, the assumption that $\frac{\hat{\zeta}(x)}{\phi_r(x)}$ is non-increasing on $[x^*, b)$ can be dropped.

634 **Lemma 10.** *The timing of action A2 remains trivial when the cycle A1–A3 is iterated a finite number of times.*

635 **Proof.** Let us suppose that action A1 has just been carried out in preparation for selling the first in a
 636 chain of n reserve contracts, and that the EIM price currently has the value x . Define τ_{A2} to be the time
 637 at which the battery operator carries out action A2. The remaining cashflows are (i) the first contract
 638 premium p_c (from action A2), (ii) the first utilisation payment K_c (from A3), and (iii) all cashflows
 639 arising from the remaining cycles A1–A3 (there are $n - 1$ cycles which remain available to the battery
 640 operator). The cashflows (i) and (ii) are both positive and fixed, making it best to obtain them as soon
 641 as possible. The cashflows (iii) include positive and negative amounts, so their timing is not as simple.
 642 However it is sufficient to notice that

- their expected net present value is given by an optimal stopping problem, namely, the timing of the *next* action A1:

$$\sup_{\tau \geq \sigma^*} \mathbb{E}^x \{ e^{-r\tau} h_{(iii)}(X_\tau) \mathbf{1}_{\tau < \infty} \}, \quad (43)$$

643 where $\sigma^* := \inf\{t \geq \tau_{A2} : X_t \geq x^*\}$, for some suitable payoff function $h_{(iii)}$,

- the choice $\tau_{A2} = 0$ minimises the exercise time σ^* and thus maximises the value of component (iii), since the supremum in (43) is then taken over the largest possible set of stopping times.

646 It is therefore best to set $\tau_{A2} = 0$, since this choice maximises the value of components (i), (ii) and
 647 (iii). \square

648 The next result establishes the existence of, and characterises, the lifetime value function \widehat{V} .

649 **Lemma 11.** *In cases A and B of Theorem 1,*

1. For each $n \geq 1$ the function $\hat{\zeta}_n := \hat{\mathcal{T}}^n \mathbf{0}$ is an admissible continuation value function and is decreasing on $[x^*, b)$.
2. The functions $\hat{\mathcal{T}}^n \mathbf{0}$ are strictly positive and uniformly bounded in n .
3. The limit $\hat{\zeta} = \lim_{n \rightarrow \infty} \hat{\mathcal{T}}^n \mathbf{0}$ exists and is a strictly positive bounded function. Moreover, the lifetime value function \widehat{V} coincides with $\hat{\zeta}$.
4. The lifetime value function \widehat{V} is a fixed point of $\hat{\mathcal{T}}$.

656 **Proof.** Part 1 is proved by induction. The claim is clearly true for $n = 1$. Assume it holds for n . Then
 657 Lemma 9 applies and $\hat{\zeta}_{n+1}(x)/\phi_r(x) = \hat{h}(\hat{x}, \hat{\zeta}_n)/\phi_r(\hat{x})$ for $x \geq \hat{x}$ when the optimal stopping time
 658 exists and $\hat{\zeta}_{n+1}(x)/\phi_r(x) = L_c$ otherwise. Therefore, $\hat{\zeta}_{n+1}(x) = c\phi_r(x)$ for $x \geq x^*$ and some constant
 659 $c \geq 0$. Since ϕ_r is decreasing, we conclude that $\hat{\zeta}_{n+1}$ decreases on $[x^*, b)$.

The monotonicity of $\hat{\mathcal{T}}$ guarantees that if $\hat{\mathcal{T}} \mathbf{0} > 0$ then $\hat{\mathcal{T}}^n \mathbf{0} > 0$ for every n . For the upper bound, notice that

$$\begin{aligned} \hat{\mathcal{T}} \hat{\zeta}_n(x) &= \sup_{\tau} \mathbb{E}^x \left\{ e^{-r\tau} \left(p_c - X_\tau + \mathbb{E}^{X_\tau} \left\{ e^{-r\hat{\tau}_e} (K_c + A \hat{\zeta}_n(X_{\hat{\tau}_e})) \right\} \right) \mathbf{1}_{\tau < \infty} \right\} \\ &\leq \sup_{\tau} \mathbb{E}^x \left\{ e^{-r\tau} \left(p_c - X_\tau + K_c \mathbb{E}^{X_\tau} \left\{ e^{-r\hat{\tau}_e} \right\} \right) \mathbf{1}_{\tau < \infty} \right\} + A \hat{\zeta}_n(x^*) = V_c(x) + A \hat{\zeta}_n(x^*), \end{aligned}$$

where $V_c = \hat{\mathcal{T}} \mathbf{0}$ is the value function for the single problem and the inequality follows from the fact that $\hat{\zeta}_n$ is decreasing on $[x^*, b)$. From the above we have $\hat{\zeta}_n(x) = \hat{\mathcal{T}}^n \mathbf{0}(x) \leq V_c(x) + \frac{1-A^n}{1-A} V_c(x^*)$.

Recalling that $A \in (0, 1)$ yields that the $\hat{\zeta}_n(x)$ are bounded by $V_c(x) + \frac{1}{1-A}V_c(x^*)$, so there exists a finite monotone limit $\hat{\zeta} := \lim_{n \rightarrow \infty} \hat{\zeta}_n$, and

$$\begin{aligned} \hat{\zeta}(x) &= \lim_{n \rightarrow \infty} \hat{T} \hat{\zeta}_n(x) = \sup_n \sup_{\tau} \mathbb{E}^x \left\{ e^{-r\tau} \left(p_c - X_{\tau} + \mathbb{E}^{X_{\tau}} \left\{ e^{-r\hat{t}_e} (K_c + A\hat{\zeta}_n(X_{\hat{t}_e})) \right\} \right) \mathbf{1}_{\tau < \infty} \right\} \\ &= \sup_{\tau} \lim_{n \rightarrow \infty} \mathbb{E}^x \left\{ e^{-r\tau} \left(p_c - X_{\tau} + \mathbb{E}^{X_{\tau}} \left\{ e^{-r\hat{t}_e} (K_c + A\hat{\zeta}_n(X_{\hat{t}_e})) \right\} \right) \mathbf{1}_{\tau < \infty} \right\} \\ &= \sup_{\tau} \mathbb{E}^x \left\{ e^{-r\tau} \left(p_c - X_{\tau} + \mathbb{E}^{X_{\tau}} \left\{ e^{-r\hat{t}_e} (K_c + A\hat{\zeta}(X_{\hat{t}_e})) \right\} \right) \mathbf{1}_{\tau < \infty} \right\} \\ &= \hat{T} \hat{\zeta}(x), \end{aligned}$$

660 by monotone convergence. The equality of \hat{V} and $\hat{\zeta}$ is clear from (14). \square

661 Appendix C Uniqueness of fixed points

662 Corollary 6 below establishes the uniqueness of the fixed point of \hat{T} . Lemma 13 shows that $\hat{T}^n \mathbf{0}$
663 converges exponentially fast to this unique fixed point as $n \rightarrow \infty$.

Lemma 12. *Let ζ, ζ' be two continuous non-negative functions with ζ satisfying the assumptions of Lemma 9 together with the bound $\zeta \geq \zeta'$. In the problem $\hat{T}\zeta$, assume the existence of an optimal stopping time τ^* under which stopping occurs only at values bounded above by $x' < x^*$. Then*

$$\|\hat{T}\zeta - \hat{T}\zeta'\|_{\#} \leq \rho \|\zeta - \zeta'\|_{\#},$$

where $\rho = A \frac{\psi_r(x')}{\psi_r(x^*)} < 1$ and $\|f\|_{\#} = |f(x^*)|$ is a seminorm on the space of continuous functions. Moreover,

$$0 \leq \hat{T}\zeta(x) - \hat{T}\zeta'(x) < \|\zeta - \zeta'\|_{\#}. \quad (44)$$

664 Note that in general, an optimal stopping time for $\hat{T}\zeta(x)$ depends on the initial state x . However,
665 under general conditions (cf. Section 2.1.1), $\tau^* = \inf\{t \geq 0 : X_t \in \Gamma\}$, where Γ is the stopping set.
666 Then the condition in the above lemma writes as $\Gamma \subset (a, x']$ for some $x' < x^*$.

Proof of Lemma 12. By the monotonicity of \hat{T} , for any x we have

$$\begin{aligned} 0 \leq \hat{T}\zeta(x) - \hat{T}\zeta'(x) &\leq \mathbb{E}^x \left\{ e^{-r\tau^*} \left(-X_{\tau^*} + p_c + (K_c + A\zeta(x^*)) \frac{\psi_r(X_{\tau^*})}{\psi_r(x^*)} \right) \right\} \\ &\quad - \mathbb{E}^x \left\{ e^{-r\tau^*} \left(-X_{\tau^*} + p_c + (K_c + A\zeta'(x^*)) \frac{\psi_r(X_{\tau^*})}{\psi_r(x^*)} \right) \right\} \\ &= \mathbb{E}^x \left\{ e^{-r\tau^*} A \left((\zeta(x^*) - \zeta'(x^*)) \frac{\psi_r(X_{\tau^*})}{\psi_r(x^*)} \right) \right\} \\ &= \|\zeta - \zeta'\|_{\#} A \mathbb{E}^x \left\{ e^{-r\tau^*} \frac{\psi_r(X_{\tau^*})}{\psi_r(x^*)} \right\}. \end{aligned}$$

This proves (44). Also we have

$$A \mathbb{E}^{x^*} \left\{ e^{-r\tau^*} \frac{\psi_r(X_{\tau^*})}{\psi_r(x^*)} \right\} \leq A \frac{\phi_r(x^*)}{\phi_r(x')} \frac{\psi_r(x')}{\psi_r(x^*)} \leq \rho.$$

667 \square

668 **Lemma 13.** *Assume that there exists a fixed point $\hat{\zeta}^*$ of \hat{T} in the space of continuous non-negative functions.*
669 *In the problem $\hat{T}\hat{\zeta}^*$, assume the existence of an optimal stopping time under which stopping occurs only at*
670 *values bounded above by $x' < x^*$ (c.f. the comment after the previous lemma). Then there is a constant $\rho < 1$*
671 *such that $\|\hat{\zeta}^* - \hat{T}^n \mathbf{0}\|_{\#} \leq \rho^n \|\hat{\zeta}^*\|_{\#}$ and $\|\hat{\zeta}^* - \hat{T}^n \mathbf{0}\|_{\infty} \leq \rho^{n-1} \|\hat{\zeta}^*\|_{\#}$, where $\|\cdot\|_{\infty}$ is the supremum norm.*

Proof. Clearly, $\|\hat{\zeta}^* - \mathbf{0}\|_{\#} < \infty$. By virtue of Lemma 12 we have $\|\hat{T}^n \mathbf{0} - \hat{\zeta}^*\|_{\#} \leq \rho^n \|\mathbf{0} - \hat{\zeta}^*\|_{\#}$ for $\rho = \frac{\psi_r(x')}{\psi_r(x^*)} < 1$. Hence, $\hat{T}^n \mathbf{0}$ converges exponentially fast to $\hat{\zeta}^*$ in the seminorm $\|\cdot\|_{\#}$. Using (44) we have

$$\|\hat{\zeta}^* - \hat{T}^n \mathbf{0}\|_{\infty} = \|\hat{T} \hat{\zeta}^* - \hat{T} \circ \hat{T}^{n-1} \mathbf{0}\|_{\infty} \leq \rho^{n-1} \|\hat{\zeta}^*\|_{\#}.$$

672 \square

673 **Corollary 6.** Let $\hat{\zeta}^*$ be a fixed point of \hat{T} and suppose that the problem $\hat{T} \hat{\zeta}^*$ admits an optimal stopping time
674 $\hat{\tau}^*$ satisfying $X_{\hat{\tau}^*} \leq x' < x^*$, for some constant x' . Such a fixed point $\hat{\zeta}^*$ is unique.

675 **Proof.** By Lemma 13 if $\hat{\zeta}^*$ is a fixed point satisfying the assumptions of the corollary, it is approximated
676 by $\hat{T}^n \mathbf{0}$ in the supremum norm, hence, it must be unique. \square

677 Appendix D Note on Lemma 3

678 The inequality $\lim_{x \rightarrow -\infty} \frac{-x}{\phi_r(x)} > 0$ when $a = -\infty$ asserts that the process X escapes to $-\infty$ quickly.
679 Indeed, choosing $z \in I$, we have $\mathbb{E}^z \{e^{-r\tau_x}\} = \frac{\phi_r(z)}{\phi_r(x)}$ for $x \leq z$, hence $\mathbb{E}^z \{e^{-r\tau_x}\} \geq \frac{c}{-x}$ for some constant
680 $c > 0$ and x sufficiently close to $-\infty$. To illustrate the speed of escape, assume for simplicity that X is a
681 deterministic process. Then the last inequality would imply $\tau_x \leq \frac{1}{r} (\log(-x) - \log(c))$, i.e., X escapes
682 to $-\infty$ exponentially quickly.

683 An example of a model that violates the assumptions of Lemma 3 is the negative geometric
684 Brownian motion: $X_t = -\exp((\mu - \sigma^2/2)t + \sigma W_t)$ for $\mu, \sigma > 0$. With the generator $\mathcal{A} = \frac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} +$
685 $\mu x \frac{d}{dx}$, we have $\phi_r(x) = (-x)^{\gamma_2}$ and $\psi_r(x) = (-x)^{\gamma_1}$, where $\gamma_1 < 0 < \gamma_2$ are solutions to the
686 quadratic equation $\frac{\sigma^2}{2}\gamma^2 + (\mu - \frac{\sigma^2}{2})\gamma - r = 0$, i.e., $\gamma = B \pm \sqrt{B^2 + 2\frac{r}{\sigma^2}}$ with $B = \frac{1}{2} - \frac{\mu}{\sigma^2}$. Hence,
687 $\lim_{x \rightarrow -\infty} \frac{-x}{\phi_r(x)} = \lim_{x \rightarrow -\infty} (-x)^{1-\gamma_2} > 0$ if and only if $\gamma_2 \leq 1$. It is easy to check that $\gamma_2 = 1$ for $\mu = r$
688 and γ_2 is decreasing as a function of μ . Therefore, the condition $\gamma_2 \leq 1$ is equivalent to $\mu \geq r$.

689 In summary, the negative geometric Brownian motion violates the assumptions of Lemma 3 if
690 $\mu \geq r$. If $\mu = r$ then case B of Theorem 1 applies with $L_c = 1$, while if $\mu > r$ then $L_c = \infty$ and so case C
691 applies. Both cases may be interpreted heuristically as the negative geometric Brownian motion X
692 escaping 'relatively quickly' to $-\infty$, that is, relative to the value r of the continuously compounded
693 interest rate. In the latter case this happens sufficiently quickly that the single problem's value function
694 V_c is infinite.

695 Appendix E Verification theorem for the lifetime value function

696 We now provide a verification lemma which may be used to verify if a given value \hat{x} is an optimal
697 buy price in the lifetime problem. The result is motivated by the following argument using Theorem 3.

698 We claim that for all $x \in I$, $\hat{T} \hat{V}(x)$ depends on the value function \hat{V} only through its value at
699 $x = x^*$. The argument is as follows: when the battery operator acts optimally, the energy purchase
700 occurs when the price is not greater than x^* : under \mathbb{P}^x for $x \geq x^*$, this follows directly from Theorem
701 3; under \mathbb{P}^x for $x < x^*$, the energy is either purchased before the price reaches x^* or one applies a
702 standard dynamic programming argument for optimal stopping problems (see, for example, [Peskir](#)
703 [and Shiryaev \(2006\)](#)) at x^* to reduce this to the previous case. In our setup the continuation value is
704 not received until the EIM price rises again to x^* (it is received immediately if the energy purchase
705 occurs at x^*).

706 Suppose therefore that we can construct functions $V_i : I \rightarrow \mathbb{R}$, $i = 1, 2$, with the following
707 properties:

- 708 i) $\hat{T} V_1 = V_2$,
- 709 ii) $V_1(x^*) = V_2(x^*)$,
- 710 iii) for $i = 1, 2$, the highest price at which the battery operator buys energy in the problem $\hat{T} V_i$ is not
711 greater than x^* .

712 Then we have $V_2 = \hat{\mathcal{T}}V_1 = \hat{\mathcal{T}}V_2$, so that V_2 is a fixed point of $\hat{\mathcal{T}}$.

713 We postulate the following form for V_i : given $y > 0$ take

$$V_1(x) = \hat{\zeta}_0^y(x) := \mathbf{1}_{x \leq x^*} y, \quad (45)$$

$$V_2(x) = \hat{\xi}^y(x) := \hat{\mathcal{T}}\hat{\zeta}_0^y(x). \quad (46)$$

714 For convenience define $\mathfrak{h}(x, y)$ to be the payoff in the lifetime problem when the the continuation value
715 is $\hat{\zeta}_0^y$. Thus we have

$$\mathfrak{h}(x, y) = \hat{h}(x, \hat{\zeta}_0^y), \quad (47)$$

$$\hat{\xi}^y(x) = \hat{\mathcal{T}}\hat{\zeta}_0^y(x) = \sup_{\tau} \mathbb{E}^x \{ e^{-r\tau} \mathfrak{h}(X_{\tau}, y) \mathbf{1}_{\tau < \infty} \}. \quad (48)$$

716 **Lemma 14.** *Suppose that $\hat{x} \in (a, x^*)$ satisfies the system*

$$\frac{\mathfrak{h}(\hat{x}, y)}{\phi_r(\hat{x})} = \sup_{x \in (a, x^*)} \frac{\mathfrak{h}(x, y)}{\phi_r(x)}, \quad (49)$$

$$y = \frac{\phi_r(x^*)}{\phi_r(\hat{x})} \mathfrak{h}(\hat{x}, y), \quad (50)$$

$$y > 0. \quad (51)$$

Then the function $\hat{\xi}^y$ of (48) is a fixed point of $\hat{\mathcal{T}}$, is continuous and strictly positive, and

$$\hat{\xi}^y(x) = \frac{\phi_r(x)}{\phi_r(x^*)} y, \quad \text{for } x \geq \hat{x}. \quad (52)$$

717 **Proof.** Consider first the problem (48) with $x \geq \hat{x}$. By construction $\hat{\zeta}_0^y$ is an admissible continuation
718 value in Lemma 9, and cases 1 or 2a must then hold due to the standing assumption for this section
719 that regime (a) of Theorem 2 is in force. By (49) the stopping time $\tau_{\hat{x}}$ is optimal, and the problem's
720 value function $\hat{\xi}^y$ has the following three properties. Firstly, $\hat{\xi}^y$ is continuous on I by Lemma 8.
721 Secondly, using (50) we see that $\hat{\xi}^y$ satisfies (52). This implies thirdly that $\hat{\xi}^y / \phi_r$ is constant on $[x^*, b)$
722 and establishes that $\hat{\xi}^y(x^*) = y$, giving property ii) above. Since $y > 0$ by (51), the strict positivity of
723 $\hat{\xi}^y$ everywhere follows as in part 1 of the proof of Lemma 2. Our standing assumption **S2*** implies that
724 the payoff $\mathfrak{h}(x, y)$ of (47) is negative for $x > x^*$, which establishes property iii) for problem (48).

725 The three properties of $\hat{\xi}^y$ established above make it an admissible continuation value in Lemma
726 9, so we now consider the problem $\hat{\mathcal{T}}\hat{\xi}^y$ for $x \geq \hat{x}$. Under \mathbb{P}^x for $x \geq x^*$, claim 2 of Lemma 6 prevents
727 the battery operator from buying energy at prices greater than x^* when acting optimally; under \mathbb{P}^x for
728 $x < x^*$, the dynamic programming principle mentioned above completes the argument. \square

729 The following corollary completes the verification argument, and also establishes the uniqueness
730 of the value y in Lemma 14.

731 **Corollary 7.** *Under the conditions of Lemma 14:*

732 i) *the function $\hat{\xi}^y$ coincides with the lifetime value function: $\hat{V} = \hat{\xi}^y$,*

733 ii) *there is at most one value y for which the system equations (49) and (50) has a solution $\hat{x} \in (a, x^*)$.*

734 **Proof.** i) We will appeal to Lemma 13 by refining property iii) above for the problem $\hat{\mathcal{T}}V_2 = \hat{\mathcal{T}}\hat{\xi}^y$
735 (as was done in the proof of Corollary 2). Suppose that the battery operator buys energy at the
736 price x^* . Then since the function $\hat{\xi}^y$ is a fixed point of $\hat{\mathcal{T}}$ under our assumptions, we may consider
737 $\hat{\mathcal{T}}\hat{\xi}^y(x^*) = -x^* + p_c + K_c + \hat{\xi}^y(x^*)$ and then **S2*** leads to $\hat{\mathcal{T}}\hat{\xi}^y(x^*) < \hat{\xi}^y(x^*)$ which is a contradiction.
738 Thus from Lemma 13, $\hat{\mathcal{T}}^n \mathbf{0}$ converges to $\hat{\xi}^y$ as $n \rightarrow \infty$. As the limit of $\hat{\mathcal{T}}^n \mathbf{0}$ is the lifetime value function
739 we obtain $\hat{V} = \hat{\xi}^y$.

740 ii) Assume the existence of two such values $y_1 \neq y_2$. Then (52) gives $\widehat{V}(x^*) = \widehat{\xi}^{y_1}(x^*) = y_1 \neq$
 741 $y_2 = \widehat{\xi}^{y_2}(x^*) = \widehat{V}(x^*)$, a contradiction. \square

742 We recall here that, on the other hand, the value \hat{x} in Lemma 14 may not be uniquely determined
 743 (cf. part (a) of Corollary 2). In this case the largest \hat{x} satisfying the assumptions of Lemma 14 is the
 744 highest price \check{x} at which the battery operator can buy energy optimally.

745 Appendix F Facts about the OU process

Let us temporarily fix $\mu = 0$ and $\theta = \sigma = 1$. Consider the ordinary differential equation (ODE)

$$w''(z) + \left(\nu + \frac{1}{2} - \frac{1}{4}z^2 \right) w(z) = 0.$$

There are two fundamental solutions $D_\nu(z)$ and $D_\nu(-z)$, where D_ν is a parabolic cylinder function. Assume that $\nu < 0$. This function has a multitude of representations, but the following will be sufficient for our purposes (Érdelyi et al. 1953, p. 119):

$$D_\nu(z) = \frac{e^{-z^2/4}}{\Gamma(-\nu)} \int_0^\infty e^{-zt - \frac{1}{2}t^2} t^{-\nu-1} dt.$$

Then D_ν is strictly positive. Fix $r > 0$. Define

$$\psi_r(x) = e^{\frac{(x-\mu)^2\theta}{2\sigma^2}} D_{-r/\theta} \left(-\frac{(x-\mu)\sqrt{2\theta}}{\sigma} \right), \quad \phi_r(x) = e^{\frac{(x-\mu)^2\theta}{2\sigma^2}} D_{-r/\theta} \left(\frac{(x-\mu)\sqrt{2\theta}}{\sigma} \right).$$

By direct calculation one verifies that these functions solve

$$\mathcal{L}v = rv, \tag{53}$$

where

$$\mathcal{L}v(x) = \frac{1}{2}\sigma^2 v''(x) + \theta(\mu - x)v'(x) \tag{54}$$

is the infinitesimal generator of the OU process (25). Setting $\nu = -r/\theta$ we can write

$$\psi_r(x) = \frac{1}{\Gamma(-\nu)} \int_0^\infty e^{(x-\mu)t\frac{\sqrt{2\theta}}{\sigma} - \frac{1}{2}t^2} t^{-\nu-1} dt, \quad \phi_r(x) = \frac{1}{\Gamma(-\nu)} \int_0^\infty e^{-(x-\mu)t\frac{\sqrt{2\theta}}{\sigma} - \frac{1}{2}t^2} t^{-\nu-1} dt.$$

746 Hence ψ_r is increasing and ϕ_r is decreasing in x . Also, by monotone convergence $\psi_r(-\infty) = \phi_r(\infty) = 0$
 747 and $\psi_r(\infty) = \phi_r(-\infty) = \infty$. The functions ψ_r and ϕ_r are then fundamental solutions of the equation
 748 (53). Further they are strictly convex, which can be checked by passing differentiation under the
 749 integral sign (justified by the dominated convergence theorem). Defining $F(x) = \psi_r(x)/\phi_r(x)$, then F
 750 is continuous and strictly increasing with $F(-\infty) = 0$ and $F(\infty) = \infty$.

Using the integral representation of ϕ_r and l'Hôpital's rule we have

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{-x}{\phi_r(x)} &= \lim_{x \rightarrow -\infty} \frac{-1}{\frac{1}{\Gamma(-\nu)} \int_0^\infty e^{-(x-\mu)t\frac{\sqrt{2\theta}}{\sigma} - \frac{1}{2}t^2} \left(-t\frac{\sqrt{2\theta}}{\sigma} \right) t^{-\nu-1} dt} \\ &= \frac{\sigma}{\sqrt{2\theta}} \lim_{x \rightarrow -\infty} \frac{1}{\frac{1}{\Gamma(-\nu)} \int_0^\infty e^{-(x-\mu)t\frac{\sqrt{2\theta}}{\sigma} - \frac{1}{2}t^2} t^{-\nu} dt} \\ &= \frac{\sigma}{\sqrt{2\theta}} \lim_{x \rightarrow -\infty} \frac{1}{\frac{\Gamma(-\nu+1)}{\Gamma(-\nu)} \frac{1}{\Gamma(-\nu+1)} \int_0^\infty e^{-(x-\mu)t\frac{\sqrt{2\theta}}{\sigma} - \frac{1}{2}t^2} t^{-\nu} dt} = 0, \end{aligned} \tag{55}$$

751 as the denominator is a scaled version of ϕ_r corresponding to a new \tilde{r} such that $-\tilde{r}/\theta = \nu - 1 < \nu < 0$,
 752 and so it converges to infinity when $x \rightarrow -\infty$.

753

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