

NEW LOWER BOUND FOR THE HILBERT NUMBER IN PIECEWISE QUADRATIC DIFFERENTIAL SYSTEMS

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ABSTRACT. We study the number of limit cycles bifurcating from a piecewise quadratic system. All the differential systems considered are piecewise in two zones separated by a straight line. We prove the existence of 16 crossing limit cycles in this class of systems. If we denote by $H_p(n)$ the extension of the Hilbert number to degree n piecewise polynomial differential systems, then $H_p(2) \geq 16$. As far as we are concerned, this is the best lower bound for the quadratic class. Moreover, all the limit cycles appear in one nest bifurcating from the period annulus of some isochronous quadratic centers.

1. INTRODUCTION

Consider the class of polynomial differential systems of degree n . The maximum number of isolated periodic orbits, the so-called *limit cycles*, that a polynomial differential system of degree n can have is called *Hilbert number*, $H(n)$. It is well known that linear systems have no limit cycles, then $H(1) = 0$. For $n = 2$, the problem of estimating $H(2)$ has been studied intensively during the last century. Lower bounds for $H(2)$ can be given by providing concrete examples of polynomial differential systems of degree 2. Up to now, the best result was given by Shi in [31], where he proved the existence of a quadratic system with 4 limit cycles in configuration $(3, 1)$, that is $H(2) \geq 4$. We call by $M(n)$ the maximum number of limit cycles bifurcating from a singular point as a degenerate Hopf bifurcation. Clearly, $M(n)$ is a lower bound for $H(n)$. Bautin showed in [2] that $M(2) = 3$; in [32, 33], Żołądek proved that $M(3) \geq 11$; a simpler proof was provided by Christopher in [9]. For $n = 3$, Li, Liu, and Yang proved in [21] that $H(3) \geq 13$.

In the last few years there has been an increasing interest in piecewise smooth systems. This interest has been mainly motivated by their wider range of application in modeling real phenomena (see, for instance, [1, 11]). In this paper we shall deal with the following class of piecewise vector fields

$$Z(x, y) = \begin{cases} Z^+(x, y), & h(x, y) > 0, \\ Z^-(x, y), & h(x, y) < 0, \end{cases} \quad (1)$$

where $Z^\pm = (X^\pm, Y^\pm)$ are smooth vector fields and $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a \mathcal{C}^1 function for which 0 is a regular value. In the above vector field, the discontinuity curve and the regions where Z^\pm are denoted by $\Sigma = h^{-1}(0)$ and $\Sigma^\pm = \{(x, y) \in \mathbb{R}^2 : \pm h(x, y) > 0\}$, respectively. The local trajectories of Z on Σ was stated by Filippov in [12] (see Figure 1). The points on Σ where both vectors fields simultaneously point outward or inward from Σ define the *escaping* (Σ^e) and *sliding region* (Σ^s), respectively. The

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interior of its complement on Σ defines the *crossing region* (Σ^c), and the boundary of these regions is constituted by tangential points of Z^\pm with Σ . Let $Z^{\pm h}$ denote

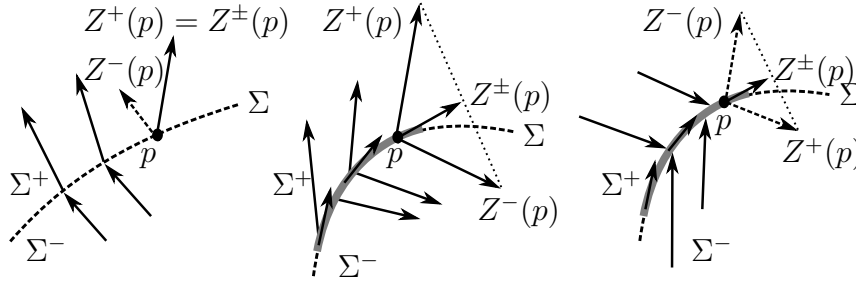


FIGURE 1. Definition of the vector field on Σ following Filippov's convention in the sewing, escaping, and sliding regions.

the derivative of the function h in the direction of the vector Z^\pm that is, $Z^\pm h(p) = \langle \nabla h(p), Z^\pm(p) \rangle$. Notice that $p \in \Sigma^c$ provided that $Z^+ h(p) \cdot Z^- h(p) > 0$, $p \in \Sigma^e \cup \Sigma^s$ provided that $Z^+ h(p) \cdot Z^- h(p) < 0$, and $p \in \Sigma$ is a tangential point of Z^\pm provided that $Z^+ h(p) Z^- h(p) = 0$. We say that $p \in \Sigma$ is a *singularity* of Z , if p is either a tangential point or a singularity of Z^+ or Z^- . We call $p \in \Sigma$ an *invisible fold* of Z^+ (resp. Z^-) if p is a tangential point of Z^+ (resp. Z^-) and $(Z^+)^2 h(p) < 0$ (resp. $(Z^-)^2 h(p) > 0$).

Analogously to the smooth case, we denote by $H_p^c(n)$ the maximum number of crossing limit cycles that piecewise polynomial differential systems of degree n admit when the curve of discontinuity is a straight line. We also denote by $M_p^c(n)$ the maximum number of crossing limit cycles bifurcating from a singular point or sliding set. Up to now, for piecewise linear systems in two zones separated by a straight line, there are no examples with more than 3 limit cycles. An example with 3 limit cycles was firstly detected numerically in [17] by Huan and Yang. Later, it was analytically proved by Llibre and Ponce in [25]. The existence of 3 limit cycles was also obtained from perturbations of a center. For instance, Buzzi et al. in [3] obtained 3 limit cycles after a seventh order piecewise linear perturbation of a linear center, and Llibre et al. in [24] obtained the same result through a first order perturbation of a piecewise linear center. We may also quote Freire et al. [14]. Consequently, for piecewise linear systems in two zones separated by a straight line we have $H_p^c(1) \geq 3$.

The averaging theory of order five for studying piecewise perturbations of the linear center was used by Llibre and Tang in [26] who provided that $H_p^c(2) \geq 8$ and $H_p^c(3) \geq M_p^c(3) \geq 13$. These are the best results so far for piecewise quadratic and cubic systems in two zones separated by a straight line. Previously, using the averaging theory of first order for studying piecewise perturbations of some quadratic isochronous systems, Llibre and Mereu in [22] obtained only 5 limit cycles. It is worthwhile to say that for quadratic polynomial systems Chicone and Jacobs in [7] proved that at most 2 limit cycles can bifurcate from any period annulus.

In this paper we shall use the averaging theory of first and second order to provide better lower bounds for the maximum number of limit cycles that piecewise quadratic systems can have. More specifically, we shall give examples satisfying $M_p^c(2) \geq 16$. Consequently, $H_p^c(3) \geq H_p^c(2) \geq 16$. Table 1 summarizes the results about the Hilbert numbers for lower degree vector fields.

Theorem 1.1. *There exists a piecewise planar quadratic differential system in two zones separated by a straight line with 16 crossing limit cycles.*

deg	PVF	PPVF
$n = 1$	$H(1) = 0$	$H_p^c(1) \geq 3$
$n = 2$	$H(2) \geq 4$	$H_p^c(2) \geq 16(8)$
$n = 3$	$H(3) \geq 13$	$H_p^c(3) \geq 16(13)$

TABLE 1. Summary of Hilbert numbers for polynomial and piecewise polynomial systems of degree n . Listed in parenthesis the known Hilbert numbers so far.

In order to prove our main result we shall proceed with a first and second order perturbation analysis of quadratic isochronous centers. In [27] the quadratic isochronous centers are classified in four families, namely S_1 , S_2 , S_3 , and S_4 . In [28] their isochronicity properties were proved as well as their linearizations. In this paper we consider the first three classes of centers, which are birational equivalent to the linear one. This property does not hold for S_4 . It is proved in [5] that any center of families S_1 , S_2 , and S_3 can be transformed, after a birational change of variables, in one of the following centers:

$$S_1 : \begin{cases} \dot{x} = -y + x^2 - y^2, \\ \dot{y} = x + 2xy. \end{cases} \quad S_2 : \begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = x + xy. \end{cases} \quad S_3 : \begin{cases} \dot{x} = -y - \frac{4}{3}x^2, \\ \dot{y} = x - \frac{16}{3}xy. \end{cases} \quad (2)$$

Here, the dot denotes the derivative with respect to time. The phase portraits of these systems are depicted in Figure 2.

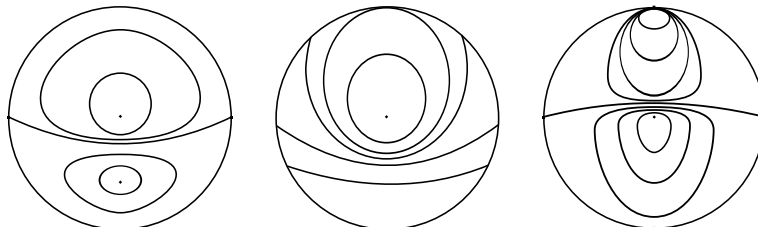


FIGURE 2. Phase portrait of systems S_1 , S_2 , and S_3 from left to right.

The first order averaging method was used in [22] to get 4 and 5 limit cycles by perturbing, respectively, the centers S_1 and S_2 inside the class of piecewise quadratic systems with two zones separated by the straight line $y = 0$. Here, due to restrictions of the employed technique, we take $\{x = 0\}$ as the curve of discontinuity for the centers S_1 and S_3 . The first and second order analysis for S_1 are performed in Propositions 3.1 and 4.1, where we get 5 and 11 limit cycles, respectively. Analogously, for S_3 Propositions 3.2 and 4.3 provide 5 and 10 limit cycles, respectively. We shall see that for the center S_2 the employed technique works whenever the curve of discontinuity is a straight line passing through the origin. This allows to reach the best result, namely $H_p^c(2) \geq 16$. In fact, proceeding with a first order analysis Proposition 3.3 provides 5, 6, and 8 limit cycles when the curve of discontinuity is $\{x = 0\}$, $\{y = 0\}$, and $\{y + \sqrt{3}x = 0\}$, respectively. Due to the difficulties in the massive computations, the second order analysis has been performed only for the case of highest ciclicity at the first order analysis, namely when the curve of discontinuity is given

by $\{y + \sqrt{3}x = 0\}$. In this case, Proposition 4.7 provides 16 limit cycles bifurcating from the origin, that is $M_p^c(2) \geq 16$. This proves our main result, Theorem 1.1.

This work is structured as follows. In Section 2, we present some basic notions and preliminary tools. In Sections 3 and 4, the first and second order analysis are performed, respectively. Finally, last section contains the integrals used in this paper.

2. PRELIMINARIES

This section is devoted to present some basic notions and preliminary tools needed to prove our main result. Firstly, we introduce some results on averaging theory of first and second orders. In fact, the limit cycles will appear from the simple zeros of some integrals (see, for instance, [18, 23]). Secondly, we recall the concepts of Extended Complete Chebyshev system (ECT-system) and Chebyshev system with accuracy (see, for instance, [29]). Then, we introduce the concept of pseudo-Hopf bifurcation, which is the birth of a limit cycle when the sliding set changes stability (see, for instance, [12, 13]). Finally, we state the Poincaré–Miranda theory, which is an extension of the intermediate value theorem, see [20].

2.1. Averaging Theory. Assume that the origin is a center equilibrium point for system (1). Consider the following perturbed piecewise polynomial vector field

$$Z_\varepsilon^\pm = \begin{cases} Z^+(x, y) + \varepsilon (P_1^+(x, y), Q_1^+(x, y)) + \varepsilon^2 (P_2^+(x, y), Q_2^+(x, y)), & \text{if } h(x, y) > 0, \\ Z^-(x, y) + \varepsilon (P_1^-(x, y), Q_1^-(x, y)) + \varepsilon^2 (P_2^-(x, y), Q_2^-(x, y)), & \text{if } h(x, y) < 0, \end{cases} \quad (3)$$

where ε is sufficiently small, P_k^\pm, Q_k^\pm are polynomials of degree n in (x, y) , for $k = 1, 2$, and $h(x, y) = y - \tan(\alpha)x$. After changing to polar coordinates, $(x, y) = (r \cos \theta, r \sin \theta)$, system (3) writes

$$\tilde{Z}_\varepsilon(\theta, r) = \begin{cases} \tilde{Z}_\varepsilon^+(\theta, r), & \text{if } \alpha < \theta < \alpha + \pi, \\ \tilde{Z}_\varepsilon^-(\theta, r), & \text{if } \alpha - \pi < \theta < \alpha. \end{cases} \quad (4)$$

Taking θ as the new independent variable, the differential system associated to the vector field (4) becomes the piecewise differential equation

$$r'(\theta) = \frac{dr}{d\theta} = \varepsilon F_1(\theta, r) + \varepsilon^2 F_2(\theta, r) + \mathcal{O}(\varepsilon^3), \quad (5)$$

with

$$F_i(\theta, r) = \begin{cases} F_i^+(\theta, r) & \text{if } \alpha < \theta < \alpha + \pi, \\ F_i^-(\theta, r) & \text{if } \alpha - \pi < \theta < \alpha, \end{cases}$$

where $F_i^\pm : [\alpha - \pi, \alpha + \pi] \times (0, \rho^*) \rightarrow \mathbb{R}$ are analytical functions 2π -periodic in the variable θ for $i = 1, 2$.

We define $\mathcal{F}_1, \mathcal{F}_2 : (0, \rho^*) \rightarrow \mathbb{R}$ as

$$\begin{aligned} \mathcal{F}_1(r) &= \int_\alpha^{\alpha+\pi} (F_1^+(\theta, r) + F_1^-(\theta - \pi, r)) d\theta, \\ \mathcal{F}_2(r) &= \int_\alpha^{\alpha+\pi} (F_2^+(\theta, r) + F_2^-(\theta - \pi, r)) d\theta \\ &\quad + \int_\alpha^{\alpha+\pi} \left(\frac{\partial}{\partial r} F_1^+(\theta, r) r_1^+(\theta, r) + \frac{\partial}{\partial r} F_1^-(\theta - \pi, r) r_1^-(\theta - \pi, r) \right) d\theta. \end{aligned} \quad (6)$$

Here, the functions $r_1^\pm : (-\pi, \pi) \times \mathbb{R}^+ \rightarrow \mathbb{R}$ are defined as

$$r_1^\pm(\theta, r) = \int_\alpha^{\alpha+\theta} F_1^\pm(\phi, r) d\phi. \quad (7)$$

Theorem 2.1 ([23]). *Consider the piecewise differential equation (5).*

- (i) *Suppose that for $\rho \in (0, \rho^*)$ with $\mathcal{F}_1(\rho) = 0$ and $\mathcal{F}'_1(\rho) \neq 0$. Then, for $|\varepsilon| > 0$ sufficiently small, there exists a 2π -periodic solution $r(\theta, \varepsilon)$ of (5) such that $r(0, \varepsilon) \rightarrow \rho$ when $\varepsilon \rightarrow 0$.*
- (ii) *Assume that $\mathcal{F}_1 = 0$. Suppose that for $\rho \in (0, \rho^*)$ with $\mathcal{F}_2(\rho) = 0$ and $\mathcal{F}'_2(\rho) \neq 0$. Then, for $|\varepsilon| > 0$ sufficiently small, there exists a 2π -periodic solution $r(\theta, \varepsilon)$ of (5) such that $r(0, \varepsilon) \rightarrow \rho$ when $\varepsilon \rightarrow 0$.*

2.2. ECT-Systems. Let $\mathcal{F} = [u_0, \dots, u_n]$ be an ordered set of functions of class \mathcal{C}^∞ on the closed interval $[a, b]$. We denote by $\mathcal{Z}(\mathcal{F})$ the maximum number of zeros counting multiplicity that any nontrivial function $v \in \text{Span}(\mathcal{F})$ can have. Here, $\text{Span}(\mathcal{F})$ is the set of functions generated by linear combinations of elements of \mathcal{F} , that is $v(s) = a_0 u_0(s) + a_1 u_1(s) + \dots + a_n u_n(s)$ where a_i , for $i = 0, 1, \dots, n$, are real numbers.

The theory of Chebyshev systems is a classical tool to study the quantity $\mathcal{Z}(\mathcal{F})$. In fact, when $\mathcal{Z}(\mathcal{F}) \leq n$, \mathcal{F} is called an *Extended Chebyshev system* or ET-system on $[a, b]$, see [19]. We say that \mathcal{F} is an *Extended Complete Chebyshev system* or an ECT-system on a closed interval $[a, b]$ if and only if for any k , $0 \leq k \leq n$, $[u_0, u_1, \dots, u_k]$ is an ET-system. In order to prove that \mathcal{F} is a ECT-system on $[a, b]$ it is sufficient and necessary to show that $W(u_0, u_1, \dots, u_k)(t) \neq 0$ on $[a, b]$ for $0 \leq k \leq n$, see also [19]. Here, $W(u_0, u_1, \dots, u_n)(t)$ denotes the Wronskian of \mathcal{F} with respect to t . That is,

$$W_n(t) = W(u_0, \dots, u_n)(t) = \det \begin{pmatrix} u_0(t) & \cdots & u_n(t) \\ u'_0(t) & \cdots & u'_n(t) \\ \vdots & \ddots & \vdots \\ u_0^{(n)}(t) & \cdots & u_n^{(n)}(t) \end{pmatrix}.$$

Furthermore, the sufficient condition to be an ECT-system also provides that each configuration of $m \leq n$ zeros, taking into account their multiplicity, is realizable.

The next theorem, proved in [29], extends the results for ECT-systems when some of the Wronskian vanish.

Theorem 2.2 ([29]). *Let $\mathcal{F} = [u_0, u_1, \dots, u_n]$ be an ordered set of analytic functions on $[a, b]$. Assume that all the ν_i zeros of the Wronskian W_i are simple for $i = 0, \dots, n$. Then, the number of isolated zeros for every element of $\text{Span}(\mathcal{F})$ does not exceed*

$$n + \nu_n + \nu_{n-1} + 2(\nu_{n-2} + \dots + \nu_0) + \nu_{n-1} + \dots + \nu_3$$

where $\nu_i = \min(2\nu_i, \nu_{i-3} + \dots + \nu_0)$, for $i = 3, \dots, n - 1$.

2.3. Pseudo-Hopf Bifurcation. In the well-known Hopf bifurcation (see, for instance, [15]) a limit cycle arises from an equilibrium point when it changes its stability. In piecewise differential systems, the pseudo-Hopf bifurcation describes the same phenomenon but when the sliding segment changes its stability. Analogously to the classical Hopf bifurcation, the proof is a direct consequence of the generalized Poincaré–Bendixson Theorem for piecewise differential systems (see, for instance, [4]).

Proposition 2.3. *Let $Z^\pm = (X^\pm(x, y), Y^\pm(x, y))$ be a \mathcal{C}^1 piecewise differential system in two zones separated by the straight line $y = 0$. Additionally, the origin is a stable*

monodromic equilibrium point and $a = (\partial Y^+ / \partial x)|_{(0,0)} > 0$. Given a real number b , we consider the perturbed system $Z_b^\pm = (X_b^\pm(x, y), Y_b^\pm(x, y))$ defined by $X_b^\pm(x, y) = X^\pm(x, y)$, and $Y_b^- = Y^-(x, y)$ and $Y_b^+ = Y^+ + b$. Then, for b small enough, the system Z_b^\pm exhibits a pseudo-Hopf bifurcation at $b = 0$ when $ab > 0$. See Figure 3.

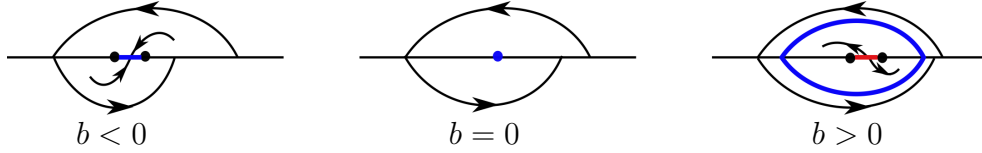


FIGURE 3. Pseudo-Hopf bifurcation.

2.4. Poincaré–Miranda Theorem. The next result is a generalization of the intermediate value theorem. It was conjectured by Poincaré in 1883 and proved by Miranda in 1940 (see, for instance, [20] and the references therein).

Theorem 2.4 ([20]). *Let a be a positive real number and $B = [-a, a]^n$ the n -dimensional cube. Let $f = (f_1, \dots, f_n) : B \rightarrow \mathbb{R}^n$ be a continuous function such that $f_i(B_i^-) < 0$ and $f_i(B_i^+) > 0$, for each $i \leq n$, where $B_i^\pm = \{(x_1, \dots, x_n) \in B : x_i = \pm a\}$. Then, there exists a point $c \in B$ such that $f(c) = 0$.*

3. FIRST ORDER PERTURBATION

In this section the first order averaging method is used to study the limit cycles of the perturbed piecewise vector field (3) when the unperturbed vector field Z_0 is a quadratic isochronous center in one of the families S_1 , S_2 , or S_3 . Regarding (3) we shall denote $Z_\varepsilon = Z_{i,\varepsilon}$ and $Z_\varepsilon^\pm = Z_{i,\varepsilon}^\pm$ in order to indicate that $Z_0 \in S_i$, for $i = 1, 2, 3$. Here, it is only considered quadratic polynomial perturbations, that is

$$P_k^\pm(x, y) = \sum_{j=0}^2 \sum_{i=0}^j p_{k,i,j-i}^\pm x^i y^{j-i} \quad \text{and} \quad Q_k^\pm(x, y) = \sum_{j=0}^2 \sum_{i=0}^j q_{k,i,j-i}^\pm x^i y^{j-i}.$$

The first order analyses for families S_1 , S_3 , and S_2 are performed in Propositions 3.1, 3.2, and 3.3, respectively. For the families S_1 and S_2 , we shall also use the ECT-system properties to study the bifurcation of limit cycles in the global interval of definition. Accordingly, Propositions 3.1 and 3.2 are concerned about upper bounds (up to a first order analysis) for the maximum number of limit cycles bifurcating from the period annulus (the so-called medium amplitude limit cycles). In the third result, Proposition 3.3, a local analysis is performed around the center point. In this case, we also see how the number of limit cycles changes when we consider different lines of discontinuity.

Before stating the main results of this section we briefly discuss the choosing of the lines of discontinuity. The birational linearizations of families S_1 and S_3 (see, for instance, [5]) transform the straight line $\{x = 0\}$ into another straight line passing through the origin. Moreover, $\{x = 0\}$ is the unique straight line for which this happens. This is the main reason for choosing $\Sigma = \{x = 0\}$ as the curve of discontinuity. The birational linearization of the family S_2 transforms straight lines passing through the origin into straight lines passing through the origin, so that we are allowed to choose any straight line passing through the origin as the curve of discontinuity. Nevertheless, in this last case, since the computations are more intricate we only study

the limit cycles bifurcating from the origin. We anticipate that all the conclusions of this section will be improved by results of the next section.

Proposition 3.1. *For $|\varepsilon| > 0$ sufficiently small the averaging method of first order predicts at most 5 crossing limit cycles for the piecewise quadratic vector field $Z_{1,\varepsilon}$ when the curve of discontinuity is the straight line $\{x = 0\}$. Moreover, this number is reached.*

Proof. In order to apply Theorem 2.1, we have to write the vector field (4) as a differential equation (5). So, we first proceed with the change of variables (see, for instance, [5])

$$x = -\frac{v}{v^2 + (u-1)^2} \quad \text{and} \quad y = -\frac{u^2 + v^2 - u}{v^2 + (u-1)^2}, \quad (8)$$

which has the following rational inverse

$$u = \frac{x^2 + y^2 + y}{x^2 + y^2 + 2y + 1} \quad \text{and} \quad v = -\frac{x}{x^2 + y^2 + 2y + 1}. \quad (9)$$

With this change of variables the differential equation S_1 becomes the linear center $(u', v') = (-v, u)$ and the line of discontinuity becomes $v = 0$.

Then, we change to polar coordinates $u = r \cos \theta$ and $v = r \sin \theta$. Taking θ as the new independent variable, (4) becomes

$$r'(\theta) = \frac{\dot{r}}{\dot{\theta}} = \varepsilon \frac{\mathcal{A}(r \cos \theta, r \sin \theta)}{\mathcal{C}(\theta, r)} + \mathcal{O}(\varepsilon^2), \quad (10)$$

where $\mathcal{C}(\theta, r) = (2r \cos \theta - r^2 - 1)^2$ and \mathcal{A} is the piecewise function

$$\mathcal{A}(r \cos \theta, r \sin \theta) = \begin{cases} \mathcal{A}^+(r \cos \theta, r \sin \theta) & \text{if } 0 < \theta \leq \pi, \\ \mathcal{A}^-(r \cos \theta, r \sin \theta) & \text{if } \pi < \theta \leq 2\pi, \end{cases} \quad (11)$$

being \mathcal{A}^\pm polynomials of degree 3.

From here we want to use the integral formulas of Section 5 to compute the averaged function \mathcal{F}_1 , as stated in (6), for $\alpha = 0$. The denominators of $F_1^+(\theta, r)$ and $F_1^-(\theta - \pi, r)$ write $(2r \cos \theta - r^2 - 1)^2$ and $(2r \cos \theta + r^2 + 1)^2$, respectively. In order to use the integrals we must apply a transformation on r in order to get the denominators written in a standard form.

Firstly, take $r = (-1 + \sqrt{1 - R^2})/R$. The denominator of $F_1^+(\theta, r)$ is transformed into $2R^2(R \cos \theta + 1)^2(R^2 + 2\sqrt{1 - R^2} - 2)$. Hence, the first part of the first averaged function

$$\int_0^\pi F_1^+(\theta, (-1 + \sqrt{1 - R^2})/R) d\theta$$

can be computed using the integrals (25) for $\alpha = 0$, $\ell = 2$, and $k \in \{0, 1, 2, 3\}$. We shall suppress it here. The original variable r is recovered by taking $R = -2r/(r^2 + 1)$.

Secondly, take $r = (1 - \sqrt{1 - R^2})/R$. The denominator of $F_1^-(\theta - \pi, r)$ is transformed into $2R^2(R \cos \theta + 1)^2(R^2 + 2\sqrt{1 - R^2} - 2)$. Hence, the second part of the first averaging function

$$\int_0^\pi F_1^-(\theta - \pi, (1 - \sqrt{1 - R^2})/R) d\theta$$

can be computed also using the integrals (25) for $\alpha = 0$, $\ell = 2$, and $k \in \{0, 1, 2, 3\}$. The original variable r is recovered by taking $R = 2r/(r^2 + 1)$.

Adding up the above integrals, we get the averaged function $\mathcal{F}_1(r)$. Proceeding with the change of parameters

$$\begin{aligned}
p_{1,0,0}^- &= p_{1,0,0}^+ - 2k_4 + \frac{k_0}{2}, \\
p_{1,0,1}^- &= p_{1,0,2}^- + p_{1,0,1}^+ - p_{1,0,2}^+ - 2k_4 + 2k_5 + \frac{k_2 + k_0}{2}, \\
p_{1,1,0}^- &= -p_{1,1,0}^+ + 2(q_{1,0,0}^- + q_{1,0,0}^+) - \frac{q_{1,0,2}^- + q_{1,2,0}^- + q_{1,0,2}^+ + q_{1,2,0}^+}{2} + \frac{k_1 - k_3}{\pi}, \\
p_{1,2,0}^- &= p_{1,2,0}^+ + \frac{q_{1,1,1}^- - q_{1,1,1}^+}{2} - 2k_4 - 2k_5, \\
q_{1,0,1}^- &= 2q_{1,0,0}^- + 2q_{1,0,0}^+ - q_{1,0,1}^+ + \frac{q_{1,0,2}^- + q_{1,2,0}^- + q_{1,0,2}^+ + q_{1,2,0}^+}{2} + \frac{k_1 + 2k_3}{\pi}, \\
q_{1,1,0}^+ &= q_{1,1,0}^- + \frac{q_{1,1,1}^+ - q_{1,1,1}^- + k_2 + k_0}{2},
\end{aligned} \tag{12}$$

we get

$$\mathcal{F}_1(r) = \sum_{n=0}^5 k_n f_n(r), \tag{13}$$

where

$$\begin{aligned}
f_0(r) &= 1, & f_1(r) &= r, & f_2(r) &= r^2, & f_3(r) &= r^3, \\
f_4(r) &= \frac{1-r^2}{r} L(r), & f_5(r) &= r(1-r^2) L(r),
\end{aligned}$$

and

$$L(r) = \log \left(\frac{1-r}{1+r} \right). \tag{14}$$

Clearly, from (12), the parameters k_n can be chosen arbitrarily.

The maximum number of simple zeros that (13) can have follows by studying the Wronskians of the ordered set $[f_0, f_1, \dots, f_5]$. Straightforward computations show that

$$\begin{aligned}
W_0(r) &= 1, & W_1(r) &= 1, & W_2(r) &= 2, & W_3(r) &= 12, \\
W_4(r) &= \frac{288}{r^5} \overline{W}_4(r), & W_5(r) &= \frac{9216(r^2+5)}{(1-r^2)^4 r^6} \overline{W}_5(r),
\end{aligned}$$

where

$$\overline{W}_4 = L(r) - \frac{2}{3} \frac{r(7r^2 - 8r^2 + 3)}{(r^2 - 1)^3}, \quad \overline{W}_5 = L(r) - \frac{2}{3} \frac{r(3r^2 - 22r^2 + 15)}{(r^2 - 1)^2(r^2 + 5)}.$$

Clearly, W_0, W_1, W_2 , and W_3 do not vanish in $(0, 1)$. Now, computing the derivative

$$\overline{W}_4'(r) = \frac{4r^4(5r^2 + 1)}{(r^2 - 1)^4} > 0,$$

as $\overline{W}_4(0) = 0$, also $W_4(r)$ does not vanish in $(0, 1)$. The same argument applies for $W_5(r)$, but using

$$\overline{W}_5'(r) = \frac{64r^6}{(r^2 - 1)^3(r^2 + 5)^2}.$$

So, the proof follows by noticing that the ordered set of functions $[f_0, \dots, f_5]$ is an ECT-system. \square

Proposition 3.2. *For $|\varepsilon| > 0$ sufficiently small the averaging method of first order predicts at most 5 crossing limit cycles for the piecewise quadratic vector field $Z_{3,\varepsilon}$ when the curve of discontinuity is the straight line $\{x = 0\}$. Moreover, this number is reached .*

Proof. We shall follow the same procedure of the proof of Proposition 3.1. The linearization stated in [5] is given by

$$x = \frac{3u}{8v+1} \quad y = \frac{3(4u^2 + 8v^2 + v)}{(8v+1)^2},$$

which has the following rational inverse

$$u = \frac{3x}{32x^2 - 24y + 9} \quad \text{and} \quad v = \frac{-4x^2 + 3y}{32x^2 - 24y + 9}.$$

Then, applying the change of variables $u = r \sin \theta$ and $v = -r \cos \theta$, and taking θ as the new independent variable, equation (5) becomes

$$r'(\theta) = \frac{\dot{r}}{\dot{\theta}} = \varepsilon \frac{\mathcal{A}(\theta, r)}{\mathcal{C}(\theta, r)} + \mathcal{O}(\varepsilon^2),$$

where $\mathcal{C}(\theta, r) = 8r \cos \theta - 1$ and \mathcal{A} is the piecewise function

$$\mathcal{A}(\theta, r) = \begin{cases} \mathcal{A}^+(r \sin \theta, -r \cos \theta) & \text{if } 0 < \theta \leq \pi, \\ \mathcal{A}^-(r \sin \theta, -r \cos \theta) & \text{if } \pi < \theta \leq 2\pi, \end{cases}$$

being \mathcal{A}^\pm polynomials of degree 6.

Now we compute the averaged function (6) for $\alpha = 0$. As in the proof of Proposition 3.1, the denominators of $F_1^+(\theta, r)$ and $F_1^-(\theta - \pi, r)$ are not written in a standard form in order to use directly the integrals of Section 5.

Firstly, take $r = -R/8$. The denominator of $F_1^+(\theta, r)$ in (6) becomes $(R \cos \theta + 1)^4$. Hence, the integral

$$\int_0^\pi F_1^+(\theta, -R/8) d\theta$$

can be computed using (25) for $\alpha = 0$, $\ell = 4$, and $k \in \{0, 1, \dots, 6\}$. The original variable r is recovered taking $R = -8r$.

Secondly, take $r = R/8$. The denominator of $F_1^-(\theta - \pi, r)$ in (6) also becomes $(R \cos \theta + 1)^4$. Hence, the integral

$$\int_0^\pi F_1^-(\theta - \pi, R/8) d\theta,$$

can be also computed using (25) for $\alpha = 0$, $\ell = 4$, and $k \in \{0, 1, \dots, 6\}$. The original variable r is recovered taking $R = 8r$.

Adding up the above integrals we obtain the first averaged function $\mathcal{F}_1(r)$, which depends on r , $\sqrt{1 - r^2}$, and $L(r)$ defined in (14). Proceeding with the change

$$r = 2\rho/(1 + \rho^2), \tag{15}$$

the averaged function writes

$$\tilde{\mathcal{F}}_1(\rho) = \sum_{n=0}^5 k_n f_n(\rho), \tag{16}$$

where

$$\begin{aligned} f_0(\rho) &= \rho, & f_1(\rho) &= \rho^2, & f_2(\rho) &= \rho^3, \\ f_3(\rho) &= \rho^4 + 1, & f_4(\rho) &= \rho^5, & f_5(\rho) &= L(\rho)/\rho. \end{aligned}$$

We remark that $L(r) = 2L(\rho)$.

The maximum number of simple zeros that (16) can have follows by studying the Wronskians of the ordered set $[f_0, f_1, \dots, f_5]$. Straightforward computations show that.

$$\begin{aligned} W_0(\rho) &= \rho, & W_1(\rho) &= \rho^2, & W_2(\rho) &= 2\rho^3, & W_3(\rho) &= 12(\rho^4 - 1), \\ W_4(\rho) &= 288\rho(\rho^4 - 5), & W_5(\rho) &= \frac{207360(1 - \rho^4)}{\rho^5} \overline{W}_5(\rho), \end{aligned}$$

where

$$\overline{W}_5(\rho) = L(\rho) - \frac{\rho(75\rho^{12} - 175\rho^{10} + 61\rho^8 + 95\rho^6 - 230\rho^4 + 140\rho^2 - 30)}{15(\rho^2 - 1)^6(\rho^2 + 1)}.$$

Clearly W_0, W_1, W_2, W_3 , and W_4 do not vanish in $(0, 1)$. The last Wronskian does not vanish either because $\overline{W}_5(0) = 0$ and the derivative

$$\overline{W}'_5(\rho) = \frac{\rho^4(\rho^4 - 5)(105\rho^8 + 105\rho^6 + 175\rho^4 - 5\rho^2 + 4)}{15(\rho^2 - 1)^7(\rho^2 + 1)^2}$$

is positive for every $\rho \in (0, 1)$. So, the proof follows by noticing that the ordered set of functions $[f_0, \dots, f_5]$ is an ECT-system. \square

The global analysis performed in the previous results cannot be performed in a straightforward way for the family S_2 . Hence, for this family we provide only a local analysis around the origin.

Proposition 3.3. *For $|\varepsilon| > 0$ sufficiently small and under the condition $P^\pm(0, 0) = Q^\pm(0, 0) = 0$, the averaging method of first order predicts at most 4, 5, or 7 limit cycles bifurcating from the origin for the quadratic vector field $Z_{2,\varepsilon}$ when the curve of discontinuity is the straight line $\{x = 0\}$, $\{y = 0\}$, or $\{y + \sqrt{3}x = 0\}$, respectively. Moreover, these numbers are reached.*

Proof. The linearization stated in [5] for family S_2 is given by

$$x = -\frac{u}{v-1} \quad \text{and} \quad y = -\frac{v}{v-1},$$

which has the following rational inverse

$$u = \frac{x}{y+1} \quad \text{and} \quad v = \frac{y}{y+1}.$$

As we have commented before, straight lines passing through the origin are transformed into straight lines passing through the origin.

Firstly, assume that $\Sigma = \{x = 0\}$. Applying the change of variable $(u, v) = (r \sin \theta, -r \cos \theta)$ and taking θ as the new independent variable we obtain the equivalent functions (10) and (11). Here, \mathcal{A}^\pm are cubic polynomials and the denominator becomes $\mathcal{C}(\theta, r) = 1 + r \cos \theta$. For this case, the first averaged function \mathcal{F}_1 is given by (6) for $\alpha = 0$. Since $\mathcal{C}(\theta, r) = 1 + r \cos \theta$ is the denominator of $F_1^+(\theta, r)$ in (6), the integrals (25) can be used directly. Nevertheless, the denominator of $F_1^-(\theta - \pi, r)$ in (6) is given by $\mathcal{C}(\theta - \pi, r) = 1 - r \cos \theta$, so it is necessary to proceed with the change $r = -R$ in order to use the integrals (25). Applying the integrals (25) for $\ell = 1$

and $k \in \{0, 1, 2, 3\}$, and going back to the original variable r we have computed the first averaged function $\mathcal{F}_1(r)$. Finally, with the change (15) and after some algebraic manipulations, we get

$$\tilde{\mathcal{F}}_1(\rho) = \sum_{n=0}^4 k_n f_n(\rho),$$

with

$$\begin{aligned} f_0(\rho) &= \frac{\rho}{(\rho^2 + 1)^2}, & f_1(\rho) &= \frac{\rho^2}{(\rho^2 + 1)^2}, \\ f_2(\rho) &= \frac{3\rho^4 + 3\rho^3 + \rho^2 + 3}{3(\rho^2 + 1)^2} + \frac{\rho^4 - \rho^2 + 1}{2(\rho^2 + 1)\rho} L(\rho), \\ f_3(\rho) &= -\frac{3\rho^2}{4(\rho^2 + 1)^2} - \frac{3\rho}{8(\rho^2 + 1)} L(\rho), & f_4(\rho) &= \frac{\rho^5}{(\rho^2 + 1)^2}, \end{aligned}$$

and L is defined in (14). Moreover, the parameters k_n can be chosen arbitrarily. The first part of the statement follows because, in a neighborhood of the origin, $f_i(\rho) = \rho^{i+1} + O(\rho^{i+2})$.

Now, assume that $\Sigma = \{y = 0\}$. The procedure for this case is similar to the previous case. We only detail the differences. The functions \mathcal{F}_1 and $\tilde{\mathcal{F}}_1$ are obtained from (6) and (25), but now for $\alpha = -\pi/2$. Thus, after some algebraic manipulations we get

$$\tilde{\mathcal{F}}_1(\rho) = \sum_{n=0}^5 k_n f_n(\rho),$$

with

$$\begin{aligned} f_0(\rho) &= \frac{\rho}{(\rho^2 + 1)^2}, & f_1(\rho) &= \frac{\rho^2}{(\rho^2 + 1)^2}, & f_2(\rho) &= \frac{\rho^3}{(\rho^2 + 1)^2}, \\ f_3(\rho) &= \frac{3\rho^2}{4(\rho^2 + 1)^2} - \frac{3\rho(\rho^2 - 1)^2}{8(\rho^2 + 1)^3} \phi\left(\rho, \frac{\pi}{2}\right), & f_4(\rho) &= \frac{\rho^5}{(\rho^2 + 1)^2}, \\ f_5(\rho) &= \frac{525\rho^4 - 490\rho^2 + 525}{768(\rho^2 + 1)^2} - \frac{(175\rho^4 + 70\rho^2 + 175)(\rho^2 - 1)^2}{512\rho(\rho^2 + 1)^3} \phi\left(\rho, \frac{\pi}{2}\right). \end{aligned} \tag{17}$$

Here, the function ϕ is defined as

$$\phi(r, \theta) = \frac{1}{\sqrt{1-r^2}} \left(\theta - 2 \arctan \left(\sqrt{\frac{1-r}{1+r}} \tan \left(\frac{\theta}{2} \right) \right) \right),$$

and the parameters k_n , $n = 0, 1, \dots, 5$, are arbitrary real numbers. The functions (17) also write $f_i(\rho) = \rho^{i+1} + O(\rho^{i+2})$. Consequently, the second part of the proof follows.

Finally, assume that $\Sigma = \{y + \sqrt{3}x = 0\}$. Again, the procedure for this case is similar to the previous cases and we shall only detail the differences. The functions \mathcal{F}_1 and $\tilde{\mathcal{F}}_1$ are obtained from (6) and (25), but now for $\alpha = -\pi/3$. Thus, after some algebraic manipulations we get

$$\tilde{\mathcal{F}}_1(\rho) = \sum_{n=0}^7 k_n f_n,$$

with

$$\begin{aligned}
f_0(\rho) &= \frac{\rho}{\rho^2 + 1}, & f_1(\rho) &= \frac{\rho^2}{(\rho^2 + 1)^2}, & f_2(\rho) &= \frac{\rho^3}{(\rho^2 + 1)^2}, & f_4(\rho) &= \frac{\rho^5}{(\rho^2 + 1)^2}, \\
f_3(\rho) &= \frac{5(54733\rho^4 + 94452\rho^2 + 54733)}{6912(\rho^2 + 1)^2} + \frac{15(1366\rho^4 + 1847\rho^2 + 1366)}{1024(\rho^2 + 1)\rho} \tilde{L}(\rho) \\
&\quad + \frac{25\sqrt{3}(236\rho^4 - 247\rho^2 + 236)(\rho^2 - 1)^2}{82944\rho(\rho^2 + 1)^3} \tilde{\phi}(\rho), \\
f_5(\rho) &= -\frac{35(21835\rho^4 + 40596\rho^2 + 21835)}{6912(\rho^2 + 1)^2} - \frac{105(550\rho^4 + 797\rho^2 + 550)}{1024(\rho^2 + 1)\rho} \tilde{L}(\rho) \\
&\quad - \frac{175\sqrt{3}(176\rho^4 - 181\rho^2 + 176)(\rho^2 - 1)^2}{82944\rho(\rho^2 + 1)^3} \tilde{\phi}(\rho), \\
f_6(\rho) &= \frac{245(227\rho^4 + 444\rho^2 + 227)}{768(\rho^2 + 1)^2} + \frac{315(122\rho^4 + 181\rho^2 + 122)}{1024(\rho^2 + 1)\rho} \tilde{L}(\rho) \\
&\quad + \frac{35\sqrt{3}(116\rho^4 - 115\rho^2 + 116)(\rho^2 - 1)^2}{9216\rho(\rho^2 + 1)^3} \tilde{\phi}(\rho), \\
f_7(\rho) &= -\frac{385(77\rho^4 + 156\rho^2 + 77)}{2304(\rho^2 + 1)^2} - \frac{3465(2\rho^4 + 3\rho^2 + 2)}{1024(\rho^2 + 1)\rho} \tilde{L}(\rho) \\
&\quad - \frac{385\sqrt{3}(8\rho^4 - 7\rho^2 + 8)(\rho^2 - 1)^2}{27648\rho(\rho^2 + 1)^3} \tilde{\phi}(\rho).
\end{aligned} \tag{18}$$

Here,

$$\tilde{L}(\rho) = \log \left(\frac{\rho^2 - \rho + 1}{\rho^2 + \rho + 1} \right), \quad \tilde{\phi}(\rho) = \phi \left(\frac{2\rho}{\rho^2 + 1}, \frac{2\pi}{3} \right) - \phi \left(-\frac{2\rho}{\rho^2 + 1}, \frac{2\pi}{3} \right).$$

Analogously to the previous cases, k_n , $n = 0, 1, \dots, 7$, are arbitrary real numbers. Here, $f_i(\rho) = \rho^{i+1} + O(\rho^{i+2})$ for $i = 0, 1, \dots, 5$, $f_6(\rho) = \rho^8 + O(\rho^9)$, and $f_7(\rho) = \rho^{10} + O(\rho^{11})$. Therefore, the ordered set of functions $[f_0, f_1, \dots, f_7]$ is an ECT-system in a neighborhood of the origin. This completes the proof of the last case. \square

Following the ideas of [10], the previous local result can be numerically improved to a global one. Taking linear combinations of the functions (18) one may try to get an ordered set of 8 functions which is an ECT-system with accuracy (see, for instance, [29]). For instance, it can be checked numerically that the ordered set $[f_0, f_1, f_2, f_3, f_4, f_6 + f_7, f_7, f_5]$ has all Wronskians non-vanishing except W_5, W_6 which vanish once. From Theorem 2.2, we conclude that \mathcal{F} has at most 9 simple zeros. We shall see that a second order analysis allow us to overcome this number of limit cycles.

The next result is a technical lemma describing the existence of a pseudo-Hopf bifurcation for $Z_{2,\varepsilon}$.

Lemma 3.4. *Consider the piecewise vector fields $Z_{i,\varepsilon}$, $i = 1, 2, 3$, under the assumption $P^\pm(0, 0) = Q^\pm(0, 0) = 0$. For all curve of discontinuity given by $\{h(x, y) = Ax + By = 0\}$, there exists a constant perturbation such that a small limit cycle bifurcates from the origin in a pseudo-Hopf bifurcation.*

Proof. The unperturbed vector fields have a monodromic equilibrium point. This property remains under the assumption $P^\pm(0, 0) = Q^\pm(0, 0) = 0$. Then, the proof follows directly from Proposition 2.3. \square

The conclusions on hyperbolic limit cycles of $Z_{2,\varepsilon}$ provided by Proposition 3.3 have assumed that $P^\pm(0,0) = Q^\pm(0,0) = 0$. So, from Lemma 3.4, the parameters $P^\pm(0,0)$ and $Q^\pm(0,0)$ can be used to get a pseudo-Hopf bifurcation for $Z_{2,\varepsilon}^\pm$, which adds an extra limit cycle to each case of Proposition 3.3. This is the content of the next result. It is worthwhile to say that this is the best result so far obtained after a first order analysis for piecewise quadratic system in two zones separated by a straight line.

Corollary 3.5. *For $|\varepsilon| > 0$ sufficiently small, the maximum number of limit cycles that the system $Z_{2,\varepsilon}$ can have in any neighborhood of origin is at least 5, 6, and 8 when the curve of discontinuity is $\{x = 0\}$, $\{y = 0\}$, and $\{y + \sqrt{3}x = 0\}$, respectively.*

4. SECOND ORDER PERTURBATION

In this section, in order to extend the previous results, we perform a second order analysis on piecewise quadratic perturbations of quadratic isochronous centers from the families S_1 , S_2 , and S_3 (see (2)). More specifically, we shall apply the averaging method of second order to study the limit cycles of $Z_{i,\varepsilon}$, $i = 1, 2, 3$. Due to the difficulties in the massive second order computations, we only perform a local study. Despite this, we shall get the best lower bounds so far for the maximum number of limit cycles of $Z_{i,\varepsilon}$, $i = 1, 2, 3$, which are 11, 16, and 10, respectively. This proves our main result, Theorem 1.1.

In Propositions 4.1 and 4.3, we provide conditions such that the second averaged functions associated to $Z_{1,\varepsilon}$ and $Z_{3,\varepsilon}$ are linear with respect to the parameters and have the highest possible rank. Under these conditions the origin is a zero of maximal finite multiplicity for \mathcal{F}_2 . Moreover, we shall see that \mathcal{F}_2 satisfies the versal unfolding property at the origin guaranteeing then the existence of the highest possible number of simple zeros near the origin and, consequently, limit cycles for $Z_{1,\varepsilon}$ and $Z_{3,\varepsilon}$. The second order analysis for centers of the family S_2 is much more difficult and the procedure used for the families S_1 and S_3 cannot be followed straightly for S_2 . In this case, some computer assisted analyses will be needed in order to use the Poincaré–Miranda theorem, that is Theorem 2.4, to obtain analytically the existence of 16 limit cycles of $Z_{2,\varepsilon}$ bifurcating from the origin. This is the content of Proposition 4.7.

Proposition 4.1. *For $|\varepsilon| > 0$ sufficiently small, the maximum number of limit cycles that $Z_{1,\varepsilon}$ can have in any neighborhood of the origin is at least 11 when the curve of discontinuity is $\{x = 0\}$.*

Proof. Assume that $P^\pm(0,0) = Q^\pm(0,0) = 0$. Under such condition, as in Proposition 2.3 or Lemma 3.4, an extra limit cycle can always be obtained from a pseudo-Hopf bifurcation. So, the rest of the proof consists in applying the second order averaging method, Theorem 2.1(ii), to get at least 10 limit cycles bifurcating from the origin. We notice that such special condition on the perturbation will guarantee that the averaged functions \mathcal{F}_1 and \mathcal{F}_2 are well defined at the origin.

The proof is structured in two parts. Firstly, we provide the expression of the function $\mathcal{F}_2(r)$. Secondly, we study the Taylor series of \mathcal{F}_2 around $r = 0$ in order to obtain the highest number of independent monomials.

The first part will follow the same steps as in the proof of Proposition 3.1. In fact, the function \mathcal{F}_1 is given by (13). Then, imposing conditions such that $\mathcal{F}_1 \equiv 0$, that is $k_0 = \dots = k_5 = 0$ in (12), we compute the second averaged function \mathcal{F}_2 from (6) for $\alpha = 0$. Proceeding with the changes of variables (8) and (9) the denominators of

the functions F_1^\pm and F_2^\pm write $(1 + R \cos \psi)^2$. Hence, the integrals

$$\begin{aligned} r_1^+(\theta, R) &= \int_0^\theta F_1^+(\theta, (-1 + \sqrt{1 - R^2})/R) d\theta, \\ r_1^-(\theta - \pi, R) &= \int_0^\theta F_1^-(\theta - \pi, (1 - \sqrt{1 - R^2})/R) d\theta, \end{aligned}$$

can be computed using the expressions $\{\mathcal{S}, \mathcal{C}\}_{k,\ell}^{\alpha=0}$ (see (25)) for $\ell = 2$, and $k \in \{0, 1, 2, 3\}$. We notice that $\mathcal{F}_2 = \mathcal{F}_2^{(1)} + \mathcal{F}_2^{(2)}$ in (6) has two summands. The first one, which depends linearly on the parameters of second order terms ($p_{2,i,j}^\pm, q_{2,i,j}^\pm$ in F_2^\pm), has the same form as \mathcal{F}_1 . Indeed, changing the first index 1 to 2 of all the parameters $p_{1,i,j}^\pm, q_{1,i,j}^\pm$ in (12) we see that $\mathcal{F}_2^{(1)}$ becomes \mathcal{F}_1 . Consequently, $\mathcal{F}_2^{(1)}$ writes as (13) for some new parameters k_0, k_1, \dots, k_5 . The second summand, which depends quadratically on the remaining parameters of first order terms ($p_{1,i,j}^\pm, q_{1,i,j}^\pm$ in F_1), can also be obtained using the integrals from Section 5. Indeed, in order to get $\mathcal{F}_2^{(2)}$ the integrals

$$\begin{aligned} \mathcal{G}^+(R) &= \int_0^\pi \left(\frac{\partial}{\partial R} F_1^+(\theta, (-1 + \sqrt{1 - R^2})/R) r_1^+(\theta, R) \right) d\theta, \\ \mathcal{G}^-(R) &= \int_0^\pi \left(\frac{\partial}{\partial R} F_1^-(\theta - \pi, (1 - \sqrt{1 - R^2})/R) r_1^-(\theta - \pi, R) \right) d\theta, \end{aligned}$$

can be computed using the expressions $\{s, c\}_{k,\ell}^\lambda, \{s, c\}_{k,\ell}^\phi, \{s, c\}_{k,\ell}^\theta, \{s, c\}_{k,\ell}^{\alpha=0}$, for $\ell = 3$ with $k \in \{0, 1, 2, 3, 4\}$, and $\{s, c\}_{k,\ell}^{\alpha=0}$ for $\ell = 4$ with $k \in \{0, 1, 2, 3, 4, 5, 6\}$. Finally, taking $R = -2r/(1 + r^2)$ and $R = 2r/(1 + r^2)$ in $\mathcal{G}^+(R)$ and $\mathcal{G}^-(R)$, respectively, we get back the original variable r . Hence, the second averaged function writes

$$\mathcal{F}_2(r) = \mathcal{F}_2^{(1)}(r) + \mathcal{F}_2^{(2)}(r) = \mathcal{F}_2^{(1)}(r) + \mathcal{G}^+(-2r/(1 + r^2)) + \mathcal{G}^-(2r/(1 + r^2)). \quad (19)$$

Now, from Lemma 5.1, expression (13), and applying the change of parameters

$$\begin{aligned} c_0 &= p_{1,1,1}^- + p_{1,1,1}^+ + 2q_{1,2,0}^- + 2q_{1,2,0}^+, \\ c_1 &= 2p_{1,0,1}^+ - 2p_{1,0,2}^+ - 2p_{1,2,0}^+ + 2q_{1,1,0}^- - q_{1,1,1}^- + q_{1,1,1}^+, \\ c_2 &= p_{1,1,0}^+ + q_{1,0,1}^+, \\ c_3 &= -2p_{1,1,0}^+ + 2q_{1,0,1}^+ - 2q_{1,0,2}^+ - 2q_{1,2,0}^+, \end{aligned}$$

the second averaged function (19) writes

$$\begin{aligned} \mathcal{F}_2(r) &= \frac{H_0(r)}{r^2} + \frac{(1 - r^2)H_1(r)}{r^3} \log \left(\frac{1 - r}{1 + r} \right) \\ &\quad + H_2(r) \log \left(\frac{(1 - r^2)^2(1 + r^2)}{(1 + 6r^2 + r^4)^3} \right) + H_3(r) \Phi_0^0 \left(\frac{2r}{r^2 + 1} \right), \end{aligned} \quad (20)$$

where Φ_0^0 is defined in (27), $H_0(r)$ and $H_1(r)$ are polynomials of degrees 7 and 6, respectively, satisfying

$$H_0(0) = 2H_1(0), \quad H_0'(0) = 2H_1'(0), \quad H_0''(0) = -\frac{8}{3}H_1(0) + 2H_1''(0). \quad (21)$$

Moreover, H_0 and H_1 depend quadratically on the parameters $c_i, p_{1,i,j}^\pm, q_{1,i,j}^\pm$ and linearly on the new parameters k_i . The other two functions write

$$H_2(r) = -\frac{(r^2 - 1)^2 \pi}{16r} c_0 c_1,$$

$$H_3(r) = \frac{r^2 - 1}{8r(r^2 + 1)} c_0 (c_3 r^4 - 2c_3 r^2 - 2c_2).$$

The above conditions imply that $\mathcal{F}_2(r) = O(r)$. This concludes the first part of this proof.

Now, we compute the Taylor series of \mathcal{F}_2 given in (20) around $r = 0$. So,

$$\mathcal{F}_2(r) = \sum_{i=0}^n f_i r^{i+1} + O(r^{n+2}).$$

Here, the coefficients f_i are quadratic functions in the variables $c_i, p_{1,i,j}^\pm, q_{1,i,j}^\pm$ and linear in the variables $k_i, p_{1,0,1}^+, q_{1,0,2}^+, q_{1,2,0}^-, \dots$. Assuming that $c_0 = 1, 2c_2 + c_3 - 2p_{1,1,1}^+ - 4q_{1,0,2}^- - 4q_{1,0,2}^+ = 1, 116c_2 + 58c_3 - 116p_{1,1,1}^+ - 230q_{1,2,0}^- - 230q_{1,2,0}^+ - 59 = 1, -2c_2 - c_3 + 2p_{1,1,1}^+ + 5 = 1$, the system of equations

$$\{f_0 = c_4, f_1 = c_5, f_2 = c_6, f_3 = c_7, f_4 = c_9, f_5 = c_8, f_7 = c_{10}, f_9 = c_{11}\}$$

has a unique solution. Accordingly, all the perturbation parameters depend only on $\{c_1, \dots, c_{11}\}$. So, the second order averaged function writes

$$\mathcal{F}_2(r) = \sum_{i=0}^{13} g_i r^{i+1} + O(r^{15}),$$

with $g_0 = c_4, g_1 = c_5, g_2 = c_6, g_3 = c_7, g_4 = c_9, g_5 = c_8, g_7 = c_{10}, g_9 = c_{11}$

$$g_6 = 3517699860675\pi c_1 + c_4 - c_6 + c_9,$$

$$g_8 = -12593243758095\pi c_1,$$

$$g_{10} = 61075412843445\pi c_1,$$

$$g_{11} = -786432c_2 + \frac{63045632}{3}c_3 + \frac{3632}{1287}c_5 - \frac{3632}{1287}c_7 + \frac{1211}{429}c_8 - \frac{109}{39}c_{10} + \frac{7}{3}c_{11},$$

$$g_{12} = -304692133550805\pi c_1,$$

$$g_{13} = -2484794504c_2 + 16745167364c_3 + \frac{7088319}{36608}c_5 - \frac{7088319}{36608}c_7$$

$$+ \frac{56833457}{292864}c_8 - \frac{1230915}{6656}c_{10} + \frac{209569}{2048}c_{11}.$$

Notice that $(g_0, g_1, \dots, g_{13})$ is a linear function on the variable (c_1, \dots, c_{11}) . Since its rank with respect to (c_1, \dots, c_{11}) is 11, there exists a change of variables $(c_1, \dots, c_{11}) \mapsto (d_1, \dots, d_{11})$ such that

$$\mathcal{F}_2(r) = r \left(\sum_{i=1}^{11} d_i U_{a_i}(r) + O(r^{14}) \right),$$

where $U_{a_i}(r) = r^{a_i} + O(r^{14})$, $a_i = i - 1, i = 1, 2, \dots, 8, a_9 = 9, a_{10} = 11$, and $a_{11} = 13$. Since \mathcal{F}_2 is analytic at $r = 0$, the Weierstrass Preparation Theorem (see, for instance,

[8]) implies that there exists an analytic function F such that $F(0) \neq 0$ and

$$\mathcal{F}_2(r) = rF(r) \sum_{i=1}^{11} d_i r^{a_i},$$

This proof follows by noticing that the parameters d_i , $i = 1, 2, \dots, 11$, can be chosen (small) in order that the function $\mathcal{F}_2(r)$ has 10 simple zeros near the origin. \square

Remark 4.2. *There are two main difficulties in studying the maximum number of simple zeros of (20). First, Φ_0^0 is an integral function that cannot be expressed with simple functions. Second, the parameter coefficients of the polynomials H_0 and H_1 have a quadratic dependence on the parameters of $Z_{1,\varepsilon}$, consequently the ECT-systems theory cannot be directly applied.*

Similar difficulties as pointed out by Remark 4.2 will also appear in the next two propositions.

Proposition 4.3. *For $|\varepsilon| > 0$ sufficiently small, the maximum number of limit cycles that $Z_{3,\varepsilon}$ can have in any neighborhood of the origin is at least 10 when the curve of discontinuity is $\{x = 0\}$.*

Proof. The proof follows the same steps as the proof of Proposition 4.1. We recall that the first order analysis has been performed in Proposition 3.2. Again, an extra limit cycle can be obtained from a pseudo-Hopf bifurcation, so we may assume that $P^\pm(0,0) = Q^\pm(0,0) = 0$. Then, the proof will consist in applying the second order averaging method to get at least 9 limit cycles bifurcating from the origin.

Firstly, using (7) for $\alpha = 0$, the functions $r_1^+(\theta, R)$ and $r_1^-(\theta - \pi, R)$ write

$$\begin{aligned} r_1^+(\theta, R) &= \int_0^\theta F_1^+(\theta, -R/8) d\theta, \\ r_1^-(\theta - \pi, R) &= \int_0^\theta F_1^-(\theta - \pi, R/8) d\theta. \end{aligned}$$

The above integrals can be computed using the expressions from Section 5, $\{\mathcal{S}, \mathcal{C}\}_{k,\ell}^{\alpha=0}$ for $\ell = 4$, and $k = 0, 1, \dots, 6$. Then, the second summand of the second averaged function, $\mathcal{F}_2^{(2)}$, follows from the integrals

$$\begin{aligned} \mathcal{G}^+(R) &= \int_0^\pi \left(\frac{\partial}{\partial R} F_1^+(\theta, -R/8) r_1^+(\theta, R) \right) d\theta, \\ \mathcal{G}^-(R) &= \int_0^\pi \left(\frac{\partial}{\partial R} F_1^-(\theta - \pi, R/8) r_1^-(\theta - \pi, R) \right) d\theta, \end{aligned}$$

which can be computed using the expressions from Section 5, $\{s, c\}_{k,\ell}^\lambda$, $\{s, c\}_{k,\ell}^\phi$, $\{s, c\}_{k,\ell}^\theta$, $\{s, c\}_{k,\ell}^{\alpha=0}$ for $\ell = 5$ with $k = 0, 1, \dots, 7$, and $\{s, c\}_{k,\ell}^{\alpha=0}$ with $\ell = 8$ and $k = 0, 1, \dots, 12$. So, going back to the original variable r we get

$$\mathcal{F}_2(r) = \mathcal{F}_2^{(1)}(r) + \mathcal{F}_2^{(2)}(r) = \mathcal{F}_2^{(1)}(r) + \mathcal{G}^+(-8r) + \mathcal{G}^-(8r).$$

Again, from Lemma 5.1 and expression (16) we get

$$\mathcal{F}_2(r) = \frac{H_0(r)}{r^2(1-r^4)^2} + \frac{H_1(r)}{r^3(1+r^2)} \log \left(\frac{1-r}{1+r} \right).$$

Here, H_0 and H_1 are polynomials of degree 13 and 8, respectively, and satisfy the relations (21). As previously, with these conditions, we have $\mathcal{F}_2(r) = O(r)$.

Now, computing the Taylor series of \mathcal{F}_2 around $r = 0$ we get

$$\mathcal{F}_2(r) = \sum_{i=1}^n f_i r^i + O(r^{n+1}). \quad (22)$$

In order to simplify the expression of $\mathcal{F}_2(r)$, we introduce the new parameters,

$$\begin{aligned} c_1 &= 16 q_{1,1,0}^- + 3 q_{1,1,1}^- - 16 q_{1,1,0}^+ - 3 q_{1,1,1}^+, \\ c_2 &= -16 q_{1,1,0}^- + 4 p_{1,0,2}^+ - 3 q_{1,1,1}^- + 16 q_{1,1,0}^+ + 3 q_{1,1,1}^+, \\ c_3 &= -48 q_{1,1,0}^- + 12 p_{1,0,2}^+ + 8 p_{1,2,0}^+ - 9 q_{1,1,1}^- + 48 q_{1,1,0}^+ + 7 q_{1,1,1}^+, \\ c_4 &= 184 p_{1,1,1}^+ + 21 q_{1,0,2}^- + 71 q_{1,0,2}^+, \\ c_5 &= q_{1,0,2}^- + q_{1,0,2}^+, \\ c_6 &= q_{1,2,0}^- + q_{1,2,0}^+, \\ c_7 &= 200 p_{1,1,0}^+ + 579 p_{1,1,1}^+ - 825 q_{1,2,0}^- - 1300 q_{1,0,1}^+, \\ c_8 &= 16 p_{1,0,1}^- - 16 p_{1,0,1}^+ - 3 q_{1,1,1}^- + 3 q_{1,1,1}^+, \\ c_9 &= 16 p_{1,0,1}^+ - 3 q_{1,1,1}^+, \\ c_{10} &= q_{1,1,0}^+ + p_{1,0,1}^+, \\ c_{11} &= 800 p_{1,1,0}^+ - 1413 p_{1,1,1}^+ - 800 q_{1,0,1}^+, \\ c_{12} &= -\frac{789}{6125} p_{1,1,1}^+ - \frac{1}{24500} c_4 + \frac{71}{24500} c_5 + \frac{184}{18375} c_6 + \frac{4}{67375} c_7 - \frac{2}{28875} c_{11}. \end{aligned}$$

We see that the coefficients f_i in (22) depend linearly on the second order parameters, $p_{2,i,j}^\pm$ and $q_{2,i,j}^\pm$, and quadratically on the new parameters c_i . Under the assumption $P^\pm(0,0) = Q^\pm(0,0) = 0$, the first averaged function, studied in Proposition 3.2, provides only the first 5 linearly independent coefficients. Thus, there exists a transformation on the parameters space such that $f_i = d_i$ for $i = 1, \dots, 5$, and, under the condition $c_1 = 1, c_5 = c_6 = c_8 = c_{10} = 0$, the system

$$\{f_6 = d_6, f_7 = d_7, f_8 = d_8, f_9 = d_9, f_{10} = d_{10}\}$$

has a unique solution with respect to $\{c_2, c_3, c_4, c_7, c_9\}$. Therefore, we get

$$\mathcal{F}_2(r) = \sum_{i=1}^{10} d_i r^i + O(r^{11}).$$

In this case, the multiplicity of the origin cannot be increased because the coefficients d_{11} and d_{12} depend linearly on the parameters $\{d_1, \dots, d_{10}\}$. In fact,

$$\begin{aligned} d_{11} &= d_5 + d_7 - d_9 \\ d_{12} &= -\frac{76}{715} d_2 - \frac{37}{65} d_4 + \frac{502}{715} d_6 + \frac{116}{65} d_8 - \frac{2}{5} d_{10}. \end{aligned}$$

Finally, this proof follows by noticing that the parameters $d_i, i = 1, 2, \dots, 9$, can be chosen (small) in order that the function $\mathcal{F}_2(r)$ has 9 simple zeros near the origin. \square

Next technical results, whose proofs are straightforward, provide lower and upper bounds for the values that a polynomial of n variables take in a n dimensional polyhedron. Moreover, they will be useful for proving the last proposition of this section concerning the second order analysis of the system $Z_{2,\varepsilon}$.

Lemma 4.4. Consider $h > 0$, $p > 0$, q real numbers such that $p \in [\underline{p}, \overline{p}]$, with $\underline{p}\overline{p} > 0$, and $q \in [\underline{q}, \overline{q}]$, with $\underline{q}\overline{q} > 0$.

(i) Then, $\sigma^\ell(q, p) \leq qp \leq \sigma^r(q, p)$,

$$\text{where } \sigma^\ell(q, p) = \begin{cases} q\underline{p}, & \text{if } q > 0, \\ q\overline{p}, & \text{if } q < 0, \end{cases} \text{ and } \sigma^r(q, p) = \begin{cases} q\overline{p}, & \text{if } q > 0, \\ q\underline{p}, & \text{if } q < 0. \end{cases}$$

(ii) If $u_j \in [-h, h]$, for $j = 1, \dots, n$, and denoting $u^i = u_1^{i_1} u_2^{i_2} \cdots u_n^{i_n}$ for $i = (i_1, \dots, i_n) \neq 0$, we have $\chi^\ell(q, u^i) \leq qu^i \leq \chi^r(q, u^i)$, where

$$\chi^\ell(q, u^i) = \begin{cases} 0, & \text{if } q > 0 \text{ and } i_k \text{ even for all } k = 1, \dots, n, \\ -\underline{q} h^{i_1 + \dots + i_n}, & \text{if } q > 0 \text{ and } i_k \text{ odd for some } k = 1, \dots, n, \\ \underline{q} h^{i_1 + \dots + i_n}, & \text{if } q < 0, \end{cases}$$

and

$$\chi^r(q, u^i) = \begin{cases} \overline{q} h^{i_1 + \dots + i_n}, & \text{if } q > 0, \\ 0, & \text{if } q < 0 \text{ and } i_k \text{ even for all } k = 1, \dots, n, \\ -\underline{q} h^{i_1 + \dots + i_n}, & \text{if } q < 0 \text{ and } i_k \text{ odd for some } k = 1, \dots, n. \end{cases}$$

Furthermore, $\chi^\ell(q, 1) = \underline{q}$ and $\chi^r(q, 1) = \overline{q}$.

Lemma 4.5. Let $h > 0$ and p_j be a positive non rational numbers such that $p_j \in [\underline{p}_j, \overline{p}_j]$ with $\underline{p}_j, \overline{p}_j$ rational numbers satisfying $\underline{p}_j \overline{p}_j > 0$, for $j = 1, \dots, m$. Consider the polynomial

$$\mathcal{U}(u_1, \dots, u_n) = \sum_{i_1 + \dots + i_n = 0}^M \left(\sum_{j=1}^m U_{j,i} p_j \right) u^i, \quad (23)$$

with $u^i = u_1^{i_1} \cdots u_n^{i_n}$, $i = (i_1, \dots, i_n)$, and $U_{j,i}$ rational numbers. Then,

$$U_i^\ell \leq \sum_{j=1}^m U_{j,i} p_j \leq U_i^r$$

with $U_i^\ell = \sum_{j=1}^m U_{j,i} \cdot \sigma^\ell(U_{j,i}, p_j)$ and $U_i^r = \sum_{j=1}^m U_{j,i} \cdot \sigma^r(U_{j,i}, p_j)$. Moreover, if $u_j \in [-h, h]$, for $j = 1, \dots, n$, and $U_i^\ell \cdot U_i^r > 0$ then

$$\underline{\mathcal{U}} = \sum_{i_1 + \dots + i_n = 0}^M \chi^\ell(U_i^\ell, u^i) \leq \mathcal{U}(u_1, \dots, u_n) \leq \sum_{i_1 + \dots + i_n = 0}^M \chi^r(U_i^r, u^i) = \overline{\mathcal{U}}.$$

The next example shows how the above two technical lemmas can be used to get rational lower and upper bounds for the values that a function takes in a given 3D-polyhedron. In this example the lower and upper bounds for the values of π and $\sqrt{3}$ are chosen from their continued fraction.

Example 4.6. Consider the polynomial

$$P(u, v, w) = P_0 + P_1 u + P_2 v^2 w^2 + P_3 u v^2 w^4, \quad (24)$$

with $P_0 = \pi - 5\sqrt{3} + 4$, $P_1 = -\pi^2 + 3\sqrt{3} - 3$, $P_2 = -2\pi^3 - \sqrt{3} + 70$, and $P_3 = 4\pi + \sqrt{3} + 7$. Consider the following intervals containing π and $\sqrt{3}$,

$$\pi \in [\underline{p}, \overline{p}] = \left[\frac{333}{106}, \frac{355}{113} \right] \quad \text{and} \quad \sqrt{3} \in [\underline{s}, \overline{s}] = \left[\frac{5}{3}, \frac{7}{4} \right].$$

Then, for $u, v, w \in [-1/9, 1/9]$, we have

$$P(u, v, w) \in \left[-\frac{8036904331130}{3236907751533}, -\frac{5753192708807927}{18184947748112394} \right].$$

Proof. Following the notation of Lemma 4.5, we take $p_1 = \pi$, $p_2 = \pi^2$, $p_3 = \pi^3$, and $p_4 = \sqrt{3}$. Then, the coefficients of (24) write as $P_0 = p_1 - 5p_4 + 4$, $P_1 = -p_2 + 3p_4 - 3$, $P_2 = -2p_3 - p_4 + 70$, and $P_3 = 4p_1 + p_4 + 7$.

The intervals given in the statement provide that $\{p_1, p_4, p_2\} \subset [\underline{p}^2, \bar{p}^2]$ and $p_3 \in [\underline{p}^3, \bar{p}^3]$. So, these new variables allow us to define the coefficients $U_{j,i}$ in (23). From Lemma 4.4(i) we have

$$\begin{aligned} -\frac{341}{212} &= \underline{p} - 5\underline{s} + 4 = U_{000}^\ell \leq P_0 \leq U_{000}^r = \bar{p} - 5\underline{s} + 4 = -\frac{404}{339}, \\ -\frac{100487}{12769} &= -\bar{p}^2 + 3\underline{s} - 3 = U_{100}^\ell \leq P_1 \leq U_{100}^r = -\underline{p}^2 + 3\underline{s} - 3 = -\frac{21402}{2809}, \\ \frac{35999881}{5771588} &= -2\underline{p}^3 - \underline{s} + 70 = U_{022}^\ell \leq P_2 \leq U_{022}^r = -2\underline{p}^3 - \underline{s} + 70 = \frac{11301029}{1786524}, \\ \frac{3376}{159} &= 4\underline{p} + \underline{s} + 7 = U_{124}^\ell \leq P_3 \leq U_{124}^r = 4\bar{p} + \bar{s} + 7 = \frac{9635}{452}. \end{aligned}$$

Finally, applying Lemma 4.4(ii) we get the following lower and upper bounds for P ,

$$U_{000}^\ell + U_{100}^\ell h - U_{124}^r h^7 = \underline{U} \leq P(u, v, w) \leq \bar{U} = U_{000}^r - U_{100}^\ell h + U_{022}^r h^4 + U_{124}^r h^7.$$

The proof follows by substituting $h = 1/2$ in the above expression. \square

The last proposition deals with second order perturbation of family S_2 , which exhibits the highest number of limit cycles found in this paper. Theorem 1.1 is a direct consequence of it. In Proposition 3.3 we have studied the zeros of the first averaged function for three different straight lines of discontinuity. The best result was obtained when $\Sigma = \{y + \sqrt{3}x = 0\}$. So, we shall perform the second order analysis only in this case.

Proposition 4.7. *For $|\varepsilon| > 0$ sufficiently small, the maximum number of limit cycles that $Z_{2,\varepsilon}$ can have in any neighborhood of the origin is at least 16 when the curve of discontinuity is $\{y + \sqrt{3}x = 0\}$.*

Proof. The proof follows basically the same steps as the proofs of Propositions 4.1 and 4.3. Nevertheless, in this case, some of the integrals of \mathcal{F}_2 cannot be explicitly obtained. Then, since the functions are analytic near the origin, we compute the Taylor series of the integrand before integrating. We recall that the first order analysis has been performed in Proposition 3.3. Again, an extra limit cycle can be obtained from a pseudo-Hopf bifurcation, so we may assume that $P^\pm(0, 0) = Q^\pm(0, 0) = 0$. Then, the proof will consist in applying the second order averaging method to get at least 15 limit cycles bifurcating from the origin.

Using (7) for $\alpha = -\pi/3$, the functions $r_1^+(\theta, R)$ and $r_1^-(\theta - \pi, R)$ write

$$r_1^+(\theta, R) = \int_{-\pi/3}^{-\pi/3+\theta} F_1^+(\theta, R) d\theta,$$

and

$$r_1^-(\theta - \pi, R) = \int_{-\pi/3}^{-\pi/3+\theta} F_1^-(\theta - \pi, -R) d\theta.$$

The above integrals can be computed using the expressions for $\{\mathcal{S}, \mathcal{C}\}_{k,\ell}^{\alpha=-\pi/3}$, for $\ell = 1$ and $k = 0, 1, 2, 3$, from Section 5. The second summand of the second averaged function, $\mathcal{F}_2^{(2)}$, follows from the integrals

$$\begin{aligned}\mathcal{G}^+(R) &= \int_{-\pi/3}^{2\pi/3} \left(\frac{\partial}{\partial R} F_1^+(\theta, R) r_1^+(\theta, R) \right) d\theta, \\ \mathcal{G}^-(R) &= \int_{-\pi/3}^{2\pi/3} \left(\frac{\partial}{\partial R} F_1^-(\theta - \pi, -R) r_1^-(\theta - \pi, -R) \right) d\theta.\end{aligned}$$

We point out that the integrands of the above integrals are rational functions with denominators $(1 + R \cos \theta)^2$ and numerators depending on

$$\{R, \theta, \cos \theta, \sin \theta, \lambda(R, -\pi/3 + \theta), \lambda(R, -4\pi/3 + \theta), \phi(R, -\pi/3 + \theta), \phi(R, -4\pi/3 + \theta)\}.$$

Computing the Taylor series around $R = 0$, integrating on the interval $[-\pi/3, 2\pi/3]$, and going back to the original variable r , we get the Taylor series around $r = 0$ of $\mathcal{F}_2^{(2)}(r)$. The Taylor series of $\mathcal{F}_2^{(1)}(r)$ is obtained analogously to the series of $\mathcal{F}_1(r)$ in Proposition 3.3. Accordingly, the second averaged function writes

$$\mathcal{F}_2(r) = \sum_{i=1}^n f_i r^i + O(r^{n+1}).$$

The coefficients f_i 's depend linearly on $\{p_{2,i,j}^\pm, q_{2,i,j}^\pm\}$ and quadratically on $\{p_{1,i,j}^\pm, q_{1,i,j}^\pm\}$.

The rest of the proof is devoted to show that there exists a transformation on the parameters such that the above function becomes

$$\mathcal{F}_2(r) = \sum_{i=1}^{16} d_i r^i + O(r^{n+1}),$$

where d_1, \dots, d_{16} are independent parameters. In fact, we shall prove the existence of a transversal curve of weak foci of order 16. The transversality also guarantees the unfolding of 15 simple zeros near the origin because our function, also the perturbed one, vanishes at zero. The existence of such curve is obtained in two steps. Firstly, we analyse the maximal rank (f_1, \dots, f_n) with respect to the linear parameters $\{p_{2,i,j}^\pm, q_{2,i,j}^\pm\}$. Secondly, proceeding with a change of parameters, which eliminates the linear terms, we study the quadratic terms regarding $\{p_{1,i,j}^\pm, q_{1,i,j}^\pm\}$ from (f_1, \dots, f_n) . We shall see that these quadratic terms are homogeneous and we show the existence of a transversal straight line such that these terms vanish on it. The described procedure is detailed in [9, Theorems. 2.1 and 3.1]. These ideas have been originally introduced in [6, 7] for quadratic vector fields and have also been employed in [16] for Liénard families.

Firstly, we see that the system of equations

$$\{f_1 = d_1, \dots, f_6 = d_6, f_8 = d_8, f_{10} = d_{10}\}$$

has a unique solution with respect to the variables $p_{2,1,0}^+, p_{2,0,2}^+, p_{2,1,1}^+, p_{2,2,0}^+, q_{2,2,0}^+, q_{2,0,1}^+, p_{2,1,1}^-,$ and $q_{2,2,0}^-$. Hence, after this change, the rest of coefficients remains quadratic.

Secondly, assuming $f_{16} \neq 0$ we shall obtain a transversal solution of the quadratic system

$$\mathcal{S} : \{f_7 = f_9 = f_{11} = f_{12} = f_{13} = f_{14} = f_{15} = 0\}.$$

Since there are more parameters than necessary, we impose that $\{p_{1,2,0}^- = p_{1,1,1}^+ = q_{1,0,1}^- = q_{1,0,2}^- = q_{1,1,0}^- = q_{1,1,1}^- = q_{1,2,0}^- = q_{1,0,1}^+ = q_{1,0,2}^+ = q_{1,1,0}^+ = q_{1,1,1}^+ = q_{1,2,0}^+ =$

$0, p_{1,2,0}^+ = 1\}$. Furthermore, it is not restrictive to assume that the first parameters $d_1, d_2, \dots, d_6, d_8$, and d_{10} vanish. For the sake of simplicity, we change the names of the remaining parameters $[p_{1,0,1}^-, p_{1,1,0}^-, p_{1,0,1}^+, p_{1,1,0}^+, p_{1,0,2}^-, p_{1,1,1}^-, p_{1,0,2}^+]$ to $[z_1, \dots, z_7]$. In order to solve the quadratic system \mathcal{S} we consider two quadratic subsystems, namely $\mathcal{S}_1 = \{f_7 = f_9 = f_{11} = f_{13} = f_{15} = 0\}$ and $\mathcal{S}_2 = \{f_{12} = f_{14} = f_{15} = 0\}$. Then, we study the intersection between their solutions.

Using the condition $f_{15} = 0$, the subsystem \mathcal{S}_1 can be rewritten in order that all the equations depend linearly on z_1, z_2, z_3 , and z_4 . So, solving \mathcal{S}_1 in these parameters we get

$$z_i = \frac{\zeta_i(z_5, z_6, z_7)}{\eta(z_5, z_6, z_7)}, \quad \text{for } i = 1, 2, 3, 4,$$

where ζ_i are polynomials of degree 5 and η is a polynomial of degree 4. Later on, we shall see that η does not vanish at the intersection point. Accordingly, the parameters f_{12}, f_{14} , and f_{15} write

$$f_{12} = \frac{\tilde{f}_{12}(z_5, z_6, z_7)}{(\eta(z_5, z_6, z_7))^2}, \quad f_{14} = \frac{\tilde{f}_{14}(z_5, z_6, z_7)}{(\eta(z_5, z_6, z_7))^2}, \quad f_{15} = \frac{\tilde{f}_{15}(z_5, z_6, z_7)}{(\eta(z_5, z_6, z_7))^2}.$$

where $\tilde{f}_{12}, \tilde{f}_{14}$, and \tilde{f}_{15} are polynomials of degree 10. So, on the variety provided by \mathcal{S}_1 , the subsystem \mathcal{S}_2 is equivalent to

$$\tilde{\mathcal{S}}_2 : \{\tilde{f}_{12}(z_5, z_6, z_7) = 0, \tilde{f}_{14}(z_5, z_6, z_7) = 0, \tilde{f}_{15}(z_5, z_6, z_7) = 0\},$$

provided that $\eta(z_5, z_6, z_7) \neq 0$. Consequently, the system \mathcal{S} is reduced to $\tilde{\mathcal{S}}_2$ whenever $\eta(z_5, z_6, z_7) \neq 0$. Although $\tilde{\mathcal{S}}_2$ has only 3 equations and 3 unknowns, the high degree of these equations is a barrier for solving the system. Furthermore, the algebraic varieties provided by each equation of $\tilde{\mathcal{S}}_2$ are numerically close to each other (see Figure 4), which adds an extra numerical difficulty.

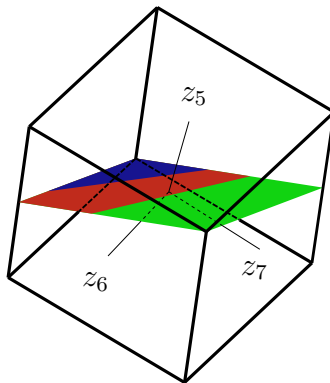


FIGURE 4. Plot of the varieties $\tilde{f}_{12} = 0$, $\tilde{f}_{14} = 0$, and $\tilde{f}_{15} = 0$ in a cube centered at (z_5^*, z_6^*, z_7^*) with edges of length 10^{-2} . They are depicted in red, blue, and green, respectively.

In order to overcome these difficulties, we shall first work with numerical approximations of the solutions. Then, using Lemmas 4.4 and 4.5, and Theorem 2.4 we prove analytically the numerical results. Working with enough precision we get the following numerical solution of the system $\tilde{\mathcal{S}}_2$,

$$(z_5^*, z_6^*, z_7^*) \approx (-0.260976000571, 0.111582119099, -0.84487667629841).$$

Moreover, we see that $\eta(z_5^*, z_6^*, z_7^*) \approx 1.85317498452382 \cdot 10^{-11}$ and $\tilde{f}_{16}(z_5^*, z_6^*, z_7^*) = 8.8767062451915 \cdot 10^{-26}$, so $f_{16} = \tilde{f}_{16}(z_5, z_6, z_7)/(\eta(z_5, z_6, z_7))^2 \neq 0$. Additionally, the intersection is transversal because the determinant of the Jacobian matrix of $f = (\tilde{f}_{12}, \tilde{f}_{14}, \tilde{f}_{15})$ with respect to (z_5, z_6, z_7) evaluated at the solution (z_5^*, z_6^*, z_7^*) does not vanish. In fact, $J_f(z_5^*, z_6^*, z_7^*) \approx -5.379835263496 \cdot 10^{-67}$. It is worthwhile to mention that although the values for η , \tilde{f}_{16} , and J_f are very small at (z_5^*, z_6^*, z_7^*) , we were able to observe that they remain fixed when we increase the precision of the computations, while the values for \tilde{f}_{12} , \tilde{f}_{14} , and \tilde{f}_{15} decrease to zero.

Finally, we shall prove analytically the existence of such transversal intersection point (z_5^*, z_6^*, z_7^*) . In order to apply Lemmas 4.4 and 4.5, we make the following change of variables,

$$\begin{aligned} z_5 &= -\frac{102563793961}{75692301} u_1 + \frac{93673471843}{117235838} u_2 - \frac{5228323783}{13687494949} u_3 - \frac{1104348344}{4231608813}, \\ z_6 &= -\frac{114951879798}{118751113} u_1 + \frac{66846520379}{116131808} u_2 + \frac{1728113446}{4218432187} u_3 + \frac{859801297}{7705547304}, \\ z_7 &= \frac{131538341646}{188147809} u_1 - \frac{23870722389}{57947275} u_2 + \frac{1387092713}{5464980203} u_3 - \frac{2790022856}{3302284149}. \end{aligned}$$

Then,

$$\begin{aligned} \tilde{f}_{12} &= \xi_1(u_1, u_2, u_3), & \tilde{f}_{14} &= \xi_2(u_1, u_2, u_3), & \tilde{f}_{15} &= \xi_3(u_1, u_2, u_3), \\ \tilde{f}_{16} &= \xi_4(u_1, u_2, u_3), & \eta &= \xi_5(u_1, u_2, u_3), & J_f &= \xi_6(u_1, u_2, u_3). \end{aligned}$$

Now, the problem is to check that the varieties defined by $\xi_1 = 0$, $\xi_2 = 0$, and $\xi_3 = 0$ intersect transversally near the origin (see Figure 5).

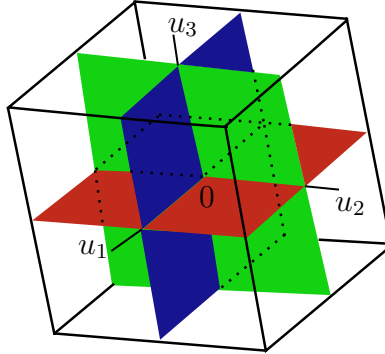


FIGURE 5. Plot of $\xi_1 = \xi_2 = \xi_3 = 0$ in a neighborhood of the origin. The varieties are drawn in green, blue, and red, respectively. The length of the edges of the 3d-cube is 10^{-10} .

We notice that the functions ξ_i , $i = 1, \dots, 6$, are polynomials in (u_1, u_2, u_3) . Moreover, ξ_i , $i = 1, \dots, 4$ have degree 10, ξ_5 has degree 4, and ξ_6 has degree 27. We see that the coefficients of the previous polynomials depend on the irrational numbers π and $\sqrt{3}$. More specifically, the coefficients of ξ_i , $i = 1, \dots, 4$, depend on π up to power 12, ξ_5 depends on π up to power 5, and ξ_6 depends on π up to power 37. The number $\sqrt{3}$ appears in these coefficients with no exponent. For each function ξ_i , let μ_i denote the degree with respect to π . Then, substituting $\sqrt{3}, \pi, \pi^2, \dots, \pi^{\mu_i}$ by $p_1, p_2, \dots, p_{\mu_i+1}$, respectively, all the coefficients of the polynomial $\xi_i(p_1, p_2, \dots, p_{\mu_i+1}, u_1, u_2, u_3)$ are

now rational numbers defined as the quotient of two big integers, around 1000 figures each. Moreover, they have 6292, 6292, 6006, 6292, 220 and 234668 monomials.

Finally, we can apply Theorem 2.4 to (ξ_1, ξ_2, ξ_3) in $B = [-h, h]^3$, with $h = 10^{-10}$, because

$$\begin{aligned}\xi_1(B_1^-) &\subset [a_1^-, b_1^-] \approx [-3.58842524 \cdot 10^{-32}, -3.20226086 \cdot 10^{-32}], \\ \xi_1(B_1^+) &\subset [a_1^+, b_1^+] \approx [3.28009013 \cdot 10^{-32}, 3.66625367 \cdot 10^{-32}], \\ \xi_2(B_2^-) &\subset [a_2^-, b_2^-] \approx [-3.70185379 \cdot 10^{-32}, -3.04215854 \cdot 10^{-32}], \\ \xi_2(B_2^+) &\subset [a_2^+, b_2^+] \approx [3.16666084 \cdot 10^{-32}, 3.82635712 \cdot 10^{-32}], \\ \xi_3(B_3^-) &\subset [a_3^-, b_3^-] \approx [-4.72369496 \cdot 10^{-32}, -1.41476503 \cdot 10^{-32}], \\ \xi_3(B_3^+) &\subset [a_3^+, b_3^+] \approx [2.14481860 \cdot 10^{-32}, 5.45375151 \cdot 10^{-32}].\end{aligned}$$

We notice that the values for a_i^\pm and b_i^\pm , for $i = 1, 2, 3$, are all rational numbers explicitly computed using Lemmas 4.4, 4.5, and

$$\pi \in \left[\frac{21053343141}{6701487259}, \frac{1783366216531}{567663097408} \right], \quad \sqrt{3} \in \left[\frac{716035}{413403}, \frac{978122}{564719} \right].$$

Hence, we conclude that there exists a point $(\xi_1^*, \xi_2^*, \xi_3^*)$ in B such that $\xi_1 = \xi_2 = \xi_3 = 0$. Additionally, ξ_4, ξ_5, ξ_6 do not vanish on B . Indeed,

$$\begin{aligned}\xi_4(B) &\subset [a_4^-, b_4^-] \approx [8.87669600 \cdot 10^{-26}, 8.87671664 \cdot 10^{-26}], \\ \xi_5(B) &\subset [a_5^-, b_5^-] \approx [1.85317477 \cdot 10^{-11}, 1.85317520 \cdot 10^{-11}], \\ \xi_6(B) &\subset [a_6^+, b_6^+] \approx [-5.37983643 \cdot 10^{-67}, -5.37983443 \cdot 10^{-67}],\end{aligned}$$

Again, the values for a_i^\pm and b_i^\pm , $i = 4, 5, 6$, are rational numbers explicitly computed using Lemmas 4.4 and 4.5. So, we conclude that the solution provided by Theorem 2.4 is a transversal solution of \mathcal{S} . \square

5. APPENDIX: EXPLICIT COMPUTATIONS OF THE INTEGRALS

This section is devoted to provide explicit expressions for some of the integrals necessary to compute the averaged functions. We also introduce new special integral functions as well as some of their properties and relations. The proofs follow closely the results from [30].

For each pair of natural numbers k and ℓ , we define the following functions:

$$\begin{aligned}\mathcal{S}_{k,\ell}^\alpha(r, \theta) &= \int_\alpha^{\alpha+\theta} \frac{\sin(k\psi)}{(1+r\cos\psi)^\ell} d\psi, & \mathcal{C}_{k,\ell}^\alpha(r, \theta) &= \int_\alpha^{\alpha+\theta} \frac{\cos(k\psi)}{(1+r\cos\psi)^\ell} d\psi, \\ s_{k,\ell}^\alpha(r) &= \int_\alpha^{\alpha+\pi} \frac{\sin(k\theta)}{(1+r\cos\theta)^\ell} d\theta, & c_{k,\ell}^\alpha(r) &= \int_\alpha^{\alpha+\pi} \frac{\cos(k\theta)}{(1+r\cos\theta)^\ell} d\theta, \\ s_{k,\ell}^\lambda(r) &= \int_0^\pi \frac{\sin(k\theta) \lambda(r, \theta)}{(1+r\cos\theta)^\ell} d\theta, & c_{k,\ell}^\lambda(r) &= \int_0^\pi \frac{\cos(k\theta) \lambda(r, \theta)}{(1+r\cos\theta)^\ell} d\theta, \\ s_{k,\ell}^\phi(r) &= \int_0^\pi \frac{\sin(k\theta) \phi(r, \theta)}{(1+r\cos\theta)^\ell} d\theta, & c_{k,\ell}^\phi(r) &= \int_0^\pi \frac{\cos(k\theta) \phi(r, \theta)}{(1+r\cos\theta)^\ell} d\theta, \\ s_{k,\ell}^\theta(r) &= \int_0^\pi \frac{\theta \sin(k\theta)}{(1+r\cos\theta)^\ell} d\theta, & c_{k,\ell}^\theta(r) &= \int_0^\pi \frac{\theta \cos(k\theta)}{(1+r\cos\theta)^\ell} d\theta.\end{aligned} \tag{25}$$

Here, $r \in (-1, 1)$, $\theta \in [-\pi, \pi]$, and ϕ , λ are the two periodic functions

$$\begin{aligned}\phi(r, \theta) &= \frac{1}{\sqrt{1-r^2}} \left(\theta - 2 \arctan \left(\sqrt{\frac{1-r}{1+r}} \tan \left(\frac{\theta}{2} \right) \right) \right), \\ \lambda(r, \theta) &= \log(1 + r \cos \theta).\end{aligned}\tag{26}$$

Lemma 5.1. *Let ϕ, λ be the functions defined by (26). Then, $\phi(r, 0) = \phi(-r, 0) = 0$, $\phi(r, \pi) = \phi(r, -\pi) = 0$, and $\phi(-r, t + \pi) = \phi(r, t)$. Moreover,*

$$\begin{aligned}\frac{\partial}{\partial r} \phi(r, \theta) &= \frac{r\phi(r, \theta)}{1-r^2} + \frac{\sin \theta}{(1-r^2)(1+r \cos \theta)}, \\ \frac{\partial}{\partial r} \lambda(r, \theta) &= \frac{\cos \theta}{1+r \cos \theta}.\end{aligned}$$

Proof. The first properties follow simply by substituting $\theta = 0, \pi$ in the definition of ϕ . The last is satisfied because $F(r, 0) = 0$ and the derivative, with respect to θ , of $F(r, \theta) = \phi(-r, \theta + \pi) - \phi(r, \theta)$ vanishes identically. The expressions of their derivatives are easily to be checked. \square

We notice that not all the above integrals can be explicitly obtained. So, next lemma introduce some new functions. They, together with their derivatives, are useful for the proofs of the results.

Lemma 5.2. *Let $\Phi_0^0(r)$, $\Lambda_0^0(r)$, and $\Lambda_1^0(r)$ be the functions*

$$\begin{aligned}\Phi_0^0(r) &= \int_0^\pi \phi(r, \theta) d\theta, \\ \Lambda_0^0(r) &= \int_0^\pi \lambda(r, \theta) d\theta = -\pi \log \left(\frac{2(1 - \sqrt{1-r^2})}{r^2} \right), \\ \Lambda_1^0(r) &= \int_0^\pi \frac{\lambda(r, \theta)}{1+r \cos \theta} d\theta = \frac{\pi \log(1-r^2) - \Lambda_0^0(r)}{\sqrt{1-r^2}}.\end{aligned}\tag{27}$$

Then, their derivatives can be expressed explicitly as functions of (26) and (27),

$$\begin{aligned}\frac{d}{dr} \Phi_0^0(r) &= \frac{r^2 \Phi_0^0(r) + \lambda(r, 0) - \lambda(-r, 0)}{r(1-r^2)} \\ \frac{d}{dr} \Lambda_0^0(r) &= -\frac{\pi(1 - \sqrt{1-r^2})}{r\sqrt{1-r^2}}, \\ \frac{d}{dr} \Lambda_1^0(r) &= \frac{1}{1-r^2} \left(r \Lambda_1^0(r) - \frac{(1+r^2 - \sqrt{1-r^2})\pi}{r\sqrt{1-r^2}} \right).\end{aligned}$$

The following results give recurrent formulas in terms of k and ℓ for all the functions defined at the beginning of this section.

Proposition 5.3. *The functions $\mathcal{S}_{k,\ell}^\alpha$ and $\mathcal{C}_{k,\ell}^\alpha$, defined in (25), write as*

$$\mathcal{S}_{k,\ell}^\alpha(r, \theta) = \begin{cases} 0 & k = 0, \ell \geq 0, \\ (\cos(k\alpha) - \cos(k(\theta + \alpha)))k^{-1} & k \geq 1, \ell = 0, \\ (\lambda(r, \alpha) - \lambda(r, \theta + \alpha))r^{-1} & k = 1, \ell = 1, \\ \frac{(1+r \cos(\theta + \alpha))^{1-\ell} - (1+r \cos \alpha)^{1-\ell}}{r(\ell-1)} & k = 1, \ell \geq 2, \\ 2(\mathcal{S}_{k-1,\ell-1}^\alpha(r, \theta) - \mathcal{S}_{k-1,\ell}^\alpha(r, \theta))r^{-1} - \mathcal{S}_{k-2,\ell}^\alpha(r, \theta) & k \geq 2, \ell \geq 1, \end{cases}$$

$$\mathcal{C}_{k,\ell}^\alpha(r, \theta) = \begin{cases} \theta & k = 0, \ell = 0, \\ (\sin(k(\theta + \alpha)) - \sin(k\alpha))k^{-1} & k \geq 1, \ell = 0, \\ \phi(r, \alpha) - \phi(r, \theta + \alpha) + \theta(1 - r^2)^{-\frac{1}{2}} & k = 0, \ell = 1, \\ \mathcal{C}_{0,\ell-1}^\alpha(r, \theta) + \frac{r}{\ell-1} \frac{\partial}{\partial r} \mathcal{C}_{0,\ell-1}^\alpha(r, \theta) & k = 0, \ell \geq 2, \\ (\mathcal{C}_{0,\ell-1}^\alpha(r, \theta) - \mathcal{C}_{0,\ell}^\alpha(r, \theta)) r^{-1} & k = 1, \ell \geq 1, \\ 2(\mathcal{C}_{k-1,\ell-1}^\alpha(r, \theta) - \mathcal{C}_{k-1,\ell}^\alpha(r, \theta)) r^{-1} - \mathcal{C}_{k-2,\ell}^\alpha(r, \theta) & k \geq 2, \ell \geq 1, \end{cases}$$

when $r \neq 0$. Furthermore, $\mathcal{S}_{k,\ell}^\alpha(0, \theta) = \mathcal{S}_{k,0}^\alpha(r, \theta)$ and $\mathcal{C}_{k,\ell}^\alpha(0, \theta) = \mathcal{C}_{k,0}^\alpha(r, \theta)$.

Proof. The expressions of $\mathcal{S}_{k,\ell}^\alpha(r, \theta)$ for $k = 0, 1$ follow by direct integration. When $k \geq 2$ and $\ell \geq 2$, from its definition and by using elementary transformations, we get

$$\begin{aligned} \mathcal{S}_{k-1,\ell-1}^\alpha(r, \theta) &= \int_\alpha^{\alpha+\theta} \frac{\sin((k-1)\psi)}{(1+r \cos \psi)^{\ell-1}} d\psi = \int_\alpha^{\alpha+\theta} \frac{\sin((k-1)\psi)(1+\cos \psi)}{(1+r \cos \psi)^\ell} d\psi \\ &\quad \int_\alpha^{\alpha+\theta} \frac{\sin((k-1)\psi)}{(1+r \cos \psi)^\ell} d\psi + r \int_\alpha^{\alpha+\theta} \frac{\sin((k-1)\psi) \cos \psi}{(1+r \cos \psi)^\ell} d\psi. \end{aligned} \quad (28)$$

Using the identity $2 \sin((k-1)\psi) \cos \psi = \sin(k\psi) + \sin((k-2)\psi)$ the above expression writes

$$\mathcal{S}_{k-1,\ell-1}^\alpha(r, \theta) = \mathcal{S}_{k-1,\ell}^\alpha(r, \theta) + \frac{1}{2}r \mathcal{S}_{k,\ell}^\alpha(r, \theta) + \frac{1}{2}r \mathcal{S}_{k-2,\ell}^\alpha(r, \theta).$$

Then, solving $\mathcal{S}_{k,\ell}^\alpha(r, \theta)$ in this expression we recover the one appearing in the statement.

The expression for $\mathcal{C}_{k,0}^\alpha(r, \theta)$ follows by a direct integration, whereas $\mathcal{C}_{0,1}^\alpha$ follows from the definition of ϕ given in (26) and the change of variables $\tan(\psi/2) = \varphi$. The expression of $\mathcal{C}_{0,\ell}^\alpha(r, \theta)$, for $\ell \geq 2$, follows deriving with respect to r . The expression for $\mathcal{C}_{k,\ell}^\alpha(r, \theta)$ can be obtained analogously to $\mathcal{S}_{k,\ell}^\alpha(r, \theta)$. \square

The next corollary follows straightaway by evaluating $\theta = \pi$ in the last result.

Corollary 5.4. *The functions $s_{k,\ell}^\alpha$ and $c_{k,\ell}^\alpha$ defined in (25) write as*

$$s_{k,\ell}^\alpha(r) = \begin{cases} 0 & k = 0, \ell \geq 0, \\ (\lambda(r, \alpha) - \lambda(r, \alpha + \pi)) r^{-1} & k = 1, \ell = 1, \\ \frac{(1 - (-1)^k)}{k} \cos(k\alpha) & k \geq 1, \ell = 0, \\ \frac{(1 + r \cos \alpha)^{1-\ell} - (1 - r \cos \alpha)^{1-\ell}}{r(1-\ell)} & k = 1, \ell \geq 2, \\ 2(s_{k-1,\ell-1}^\alpha(r) - s_{k-1,\ell}^\alpha(r)) r^{-1} - s_{k-2,\ell}^\alpha(r) & k \geq 2, \ell \geq 1, \end{cases}$$

$$c_{k,\ell}^\alpha(r) = \begin{cases} \pi & k = 0, \ell = 0, \\ \phi(r, \alpha) - \phi(r, \alpha + \pi) + \frac{\pi}{\sqrt{1-r^2}} & k = 0, \ell = 1, \\ \frac{r}{\ell-1} \frac{d}{dr} c_{0,\ell-1}^\alpha(r) + c_{0,\ell-1}^\alpha(r) & k = 0, \ell \geq 2, \\ \frac{-1 + (-1)^k}{k} \sin(k\alpha) & k \geq 1, \ell = 0, \\ (c_{0,\ell-1}^\alpha(r) - c_{0,\ell}^\alpha(r)) r^{-1} & k = 1, \ell \geq 1, \\ 2(c_{k-1,\ell-1}^\alpha(r) - c_{k-1,\ell}^\alpha(r)) r^{-1} - c_{k-2,\ell}^\alpha(r) & k \geq 2, \ell \geq 1, \end{cases}$$

when $r \neq 0$. Furthermore, $s_{0,\ell}^\alpha(0) = 0$, $c_{0,\ell}^\alpha(0) = \pi$ for $\ell \geq 0$ and

$$s_{k,\ell}^\alpha(0) = \frac{1 - (-1)^k}{k} \cos(k\alpha), \quad c_{k,\ell}^\alpha(0) = \frac{-1 + (-1)^k}{k} \sin(k\alpha),$$

for $k \geq 1$ and $\ell \geq 0$.

Proposition 5.5. *The functions $s_{k,\ell}^\lambda$ and $c_{k,\ell}^\lambda$, defined in (25), write as*

$$s_{k,\ell}^\lambda(r) = \begin{cases} 0 & k = 0, \ell \geq 0, \\ \frac{\lambda(r, 0) - (-1)^k \lambda(-r, 0)}{k} + \frac{r}{2k} (s_{k-1,1}(r) - s_{k+1,1}(r)) & k \geq 1, \ell = 0, \\ \frac{\lambda(r, 0)^2 - \lambda(-r, 0)^2}{2r} & k = 1, \ell = 1, \\ \frac{1}{r(\ell-1)} \left(\frac{\lambda(-r, 0) + (\ell-1)^{-1}}{(1-r)^{\ell-1}} - \frac{\lambda(r, 0) + (\ell-1)^{-1}}{(1+r)^{\ell-1}} \right) & k = 1, \ell \geq 2, \\ 2(s_{k-1,\ell-1}^\lambda(r) - s_{k-1,\ell}^\lambda(r)) r^{-1} - s_{k-2,\ell}^\lambda(r) & k \geq 2, \ell \geq 1, \end{cases}$$

$$c_{k,\ell}^\lambda(r) = \begin{cases} \Lambda_\ell^0(r) & k = 0, \ell = 0, 1, \\ \frac{r}{\ell-1} \left(\frac{d}{dr} c_{0,\ell-1}^\lambda(r) - c_{1,\ell}(r) \right) + c_{0,\ell-1}^\lambda(r) & k = 0, \ell \geq 2, \\ \frac{r}{2k} (c_{k-1,1}(r) - c_{k+1,1}(r)) & k \geq 1, \ell = 0, \\ (c_{0,\ell-1}^\lambda(r) - c_{0,\ell}^\lambda(r)) r^{-1} & k = 1, \ell \geq 1, \\ 2(c_{k-1,\ell-1}^\lambda(r) - c_{k-1,\ell}^\lambda(r)) r^{-1} - c_{k-2,\ell}^\lambda(r) & k \geq 2, \ell \geq 1, \end{cases}$$

when $r \neq 0$. Furthermore, $s_{k,\ell}^\lambda(0) = s_{k,\ell}(0)$ and $c_{k,\ell}^\lambda(0) = 0$ for $k \geq 0$ and $\ell \geq 0$.

Proof. The general expression for $k \geq 1, \ell \geq 1$ follows similarly to (28). The other cases follow straightforward. In some of them, the integration by parts rule is necessary and also the fact that $c_{k,\ell}^\lambda$ is an even function. \square

Proposition 5.6. *The functions $s_{k,\ell}^\phi$ and $c_{k,\ell}^\phi$, defined in (25), write as*

$$s_{k,\ell}^\phi(r) = \begin{cases} 0 & k = 0, \ell \geq 0, \\ -\frac{c_{k,1}(r)}{k} & k \geq 1, \ell = 0, \\ \frac{\Lambda_0^0(r)}{r\sqrt{1-r^2}} - \frac{\Lambda_1^0(r)}{r} & k = 1, \ell = 1, \\ \frac{1}{r(\ell-1)} \left(c_{0,\ell} - \frac{c_{0,\ell-1}}{\sqrt{1-r^2}} \right) & k = 1, \ell \geq 2, \\ 2 \left(s_{k-1,\ell-1}^\phi(r) - s_{k-1,\ell}^\phi(r) \right) r^{-1} - s_{k-2,\ell}^\phi(r) & k \geq 2, \ell \geq 1, \end{cases}$$

$$c_{k,\ell}^\phi(r) = \begin{cases} \Phi_0^0(r) & k = 0, \ell = 0, \\ \frac{c_{0,1}^\phi(r)}{\sqrt{1-r^2}} - \frac{1}{2} \left(\frac{\pi}{\sqrt{1-r^2}} \right)^2 & k = 0, \ell = 1, \\ \frac{r}{\ell-1} \left(\frac{d}{dr} c_{0,\ell-1}^\phi(r) - \frac{s_{1,\ell}^\phi(r)}{1-r^2} \right) + \frac{\ell(1-r^2)-1}{(\ell-1)(1-r^2)} c_{0,\ell-1}^\phi(r) & k = 0, \ell \geq 2, \\ \frac{s_{k,1}(r)}{k} + \frac{(-1)^k - 1}{k^2\sqrt{1-r^2}} & k \geq 1, \ell = 0, \\ \left(c_{0,\ell-1}^\phi(r) - c_{0,\ell}^\phi(r) \right) r^{-1} & k = 1, \ell \geq 1, \\ 2 \left(c_{k-1,\ell-1}^\phi(r) - c_{k-1,\ell}^\phi(r) \right) r^{-1} - c_{k-2,\ell}^\phi(r) & k \geq 2, \ell \geq 1, \end{cases}$$

when $r \neq 0$. Furthermore, $s_{k,\ell}^\phi(0) = c_{k,\ell}^\phi(0) = 0$.

Proof. The expression of $s_{k,\ell}^\phi$ follows from the fact that it is an even function and the results in [30]. For $k = 1, \ell \geq 1$ and $k \geq 2, \ell \geq 1$, proceeding analogously to the previous proofs, we get $c_{k,\ell}^\phi$. Finally, for $k = 0, \ell \geq 2$, we compute directly the derivative, with respect to r , of $c_{0,\ell-1}^\phi$. The other cases follow using the integration by parts rule. \square

The last result follows similarly as all the previous results.

Proposition 5.7. *The functions $s_{k,\ell}^\theta$ and $c_{k,\ell}^\theta$ defined in (25) write as*

$$s_{k,\ell}^\theta(r) = \begin{cases} 0 & k = 0, \ell \geq 0, \\ \frac{-(-1)^k \pi}{k} & k \geq 1, \ell = 0, \\ \left(-\pi \lambda(-r, 0) + \Lambda_0^0(r) \right) r^{-1} & k = 1, \ell = 1, \\ \frac{1}{r(\ell-1)} \left(\frac{\pi}{(1-r)^{\ell-1}} - c_{0,\ell-1}(r) \right) & k = 1, \ell \geq 2, \\ 2 \left(s_{k-1,\ell-1}^\theta(r) - s_{k-1,\ell}^\theta(r) \right) r^{-1} - s_{k-2,\ell}^\theta(r) & k \geq 2, \ell \geq 1, \end{cases}$$

$$c_{k,\ell}^\theta(r) = \begin{cases} \frac{\pi^2}{2} & k = 0, \ell = 0, \\ \Phi_0^\theta(r) + \frac{\pi^2}{2\sqrt{1-r^2}} & k = 0, \ell = 1, \\ \frac{r}{\ell-1} \frac{d}{dr} c_{0,\ell-1}^\theta(r) + c_{0,\ell-1}^\theta(r) & k = 0, \ell \geq 2, \\ \frac{-1 + (-1)^k}{k^2} & k \geq 1, \ell = 0, \\ \left(c_{0,\ell-1}^\theta(r) - c_{0,\ell}^\theta(r) \right) r^{-1} & k = 1, \ell \geq 1, \\ 2 \left(c_{k-1,\ell-1}^\theta(r) - c_{k-1,\ell}^\theta(r) \right) r^{-1} - c_{k-2,\ell}^\theta(r) & k \geq 2, \ell \geq 1, \end{cases}$$

when $r \neq 0$. Furthermore, $s_{0,\ell}^\theta(0) = 0$, $c_{0,\ell}^\theta(0) = \pi^2/2$ for $\ell \geq 0$ and

$$s_{k,\ell}^\theta(0) = \frac{-(-1)^k \pi}{k}, \quad c_{k,\ell}^\theta(0) = \frac{-1 + (-1)^k}{k^2}.$$

for $k \geq 1$ and $\ell \geq 0$.

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