Level raising for automorphic representations of $\mathrm{GL}(2n)$

Christos Anastassiades

Department of Pure Mathematics and Mathematical Statistics

University of Cambridge

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Summary

To each regular algebraic, conjugate self-dual, cuspidal automorphic representation Π of $\mathrm{GL}(N)$ over a CM number field E (or, more generally, to a regular algebraic isobaric sum of conjugate self-dual, cuspidal representations), we can attach a continuous ℓ-adic Galois representation $r(\Pi)$ of the absolute Galois group of E. The residual Galois representation

$$\overline{r}(\Pi) : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_N(\overline{\mathbb{F}}_\ell)$$

of π is defined to be the semisimplification of the reduction of $r(\Pi)$ (modulo the maximal ideal of $\overline{\mathbb{Z}}_\ell$), with respect to any invariant $\overline{\mathbb{Z}}_\ell$-lattice. The aim of this thesis is to prove a level raising theorem for automorphic representations of $\mathrm{GL}(2n)$. More precisely, given a regular algebraic automorphic representation Π of $\mathrm{GL}(2n)$ over E, which is either unitary, conjugate self-dual and cuspidal or an isobaric sum $\Pi_1 \boxplus \Pi_2$ of two unitary, conjugate self-dual cuspidal representations of $\mathrm{GL}(n)$, we want to construct a unitary, conjugate self-dual cuspidal representation Π' of $\mathrm{GL}(2n)$ that has the same residual Galois representation as Π and whose component at a finite place w of E is an unramified twist of the Steinberg representation.

We prove that this is possible, after replacing Π with its base change along a CM biquadratic extension, under certain assumptions on Π (including a local obstruction at the place w). Our proof uses the results of Kaletha, Minguez, Shin and White on the endoscopic classification of representations of (inner forms of) unitary groups to descend Π to an automorphic representation of a totally definite unitary group G over the maximal totally real subfield of E. We then prove a level raising theorem for the group G; we do this by proving an analogue of "Ihara's lemma" for G, using the strong approximation theorem for the derived subgroup of G.

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text.

It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text.

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Contents

Introduction

The problem of "level raising" was first considered by Ribet in [**Rib84**], in the context of (classical) modular forms. Concretely, Ribet addresses the following question: Let f be a cusp form of weight 2 and level $\Gamma_0(N)$, which is an eigenform for the Hecke operators. If ℓ is a rational prime and $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ is an abstract field isomorphism, there exists a continuous semisimple Galois representation

$$r_\iota(f) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell),$$

unramified outside $N\ell$, determined (up to isomorphism) by the fact that the characteristic polynomial of $r_\iota(f)(\mathrm{Frob}_p)$ is

$$x^2 - \iota^{-1}(a_p)x + p$$

for each prime $p \nmid N\ell$, where a_p is the eigenvalue of the Hecke operator T_p (and Frob_p denotes the arithmetic Frobenius at p). After conjugation, we may assume that $r_\iota(f)$ takes values in $\mathrm{GL}_2(\overline{\mathbb{Z}}_\ell)$ and consider its reduction modulo the maximal ideal $\mathfrak{m}_{\overline{\mathbb{Z}}_\ell}$ of $\overline{\mathbb{Z}}_\ell$; we write

$$\overline{r_\iota}(f) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$$

for the *semisimplification* of $r_\iota(f) \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_\ell}$. One can then ask whether there exists a cuspidal Hecke eigenform g of weight 2 and level $\Gamma_0(Np)$ (where p is a fixed prime not dividing $N\ell$), which is new at p, such that $\overline{r_\iota}(g) \cong \overline{r_\iota}(f)$. Ribet proves that the answer is affirmative, provided that f satisfies certain conditions; in particular, f must satisfy

$$\iota^{-1}(a_p) \equiv \pm(p+1) \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_\ell}.$$

It's easy to see that the latter condition is necessary, because $r_\iota(g)|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is isomorphic to a representation of the form

$$\begin{pmatrix} \varepsilon_\ell & * \\ & 1 \end{pmatrix} \otimes \chi,$$

where $\varepsilon_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathbb{Z}_\ell^\times$ is the ℓ-adic cyclotomic character and

$$\chi : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \overline{\mathbb{Z}}_\ell^\times$$

is an unramified quadratic character. We note that this is not a sufficient condition; an additional assumption is needed for Ribet's theorem. A key step in Ribet's proof was a lemma of Ihara.

In [**DT94**], Diamond and Taylor proved more general level raising results for modular forms, by studying automorphic forms on quaternion algebras. In the case of definite quaternion algebras, the analogue of Ihara's lemma is a simple consequence of the strong approximation theorem. Moreover, in Section 5.3 of [**CHT08**], Clozel, Harris and Taylor prove a level raising result for automorphic representations of GL_n over a CM number field, which is conditional on an analogue of Ihara's lemma for automorphic forms on a totally definite unitary group.

Level raising theorems are a useful tool in the study of automorphic forms and the automorphy of Galois representations. For example, Ribet's level raising theorem was used as an ingredient in the proof of his level lowering theorem in [**Rib90**], which showed that Fermat's last theorem follows from the Taniyama–Shimura conjecture. Moreover, level raising theorems often play a part in proofs of automorphy lifting theorems; e.g. in [**Kis09**], Kisin uses a level raising theorem (Corollary 3.1.11) as an ingredient in the proof of a modularity lifting theorem for 2-dimensional Galois representations.

Our main theorem is a new level raising theorem for (regular C-algebraic) automorphic representations Π of GL_{2m} over a totally imaginary CM number field E. The representation Π is assumed to be either a conjugate self-dual, unitary cuspidal representation of $\mathrm{GL}_{2m}(\mathbb{A}_E)$ or of the form $\Pi_1 \boxplus \Pi_2$, where each Π_i is a conjugate self-dual, unitary cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_E)$. The aim is to construct a regular C-algebraic, conjugate self-dual, unitary cuspidal representation of GL_{2m} (possibly over a solvable CM extension of E), which has the same residual Galois representation as Π and whose local component at finite place w is isomorphic to an unramified twist of the Steinberg representation. Unlike most level raising results in the literature, the mod ℓ residual Galois representation

$$\overline{r_\iota}(\Pi) : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_{2m}(\overline{\mathbb{Q}}_\ell)$$

of Π is allowed to reducible; the only condition on $\overline{r_\iota}(\Pi)$ is that it's not allowed to be isomorphic to a twist of $1 \oplus \overline{\varepsilon}_\ell \oplus \cdots \overline{\varepsilon}_\ell^{1-N}$. We must also assume, as in the case of modular forms studied by Ribet, that Π satisfies the obvious necessary local congruence condition at the finite place w. We refer the reader to Chapter 3 for a precise statement of the theorems. These level raising theorems are conditional on the results of [**KMSW14**] on the endoscopic classification of representations of (inner forms of) unitary groups.

Let us briefly discuss the strategy of the proof: The first step is to transfer the automorphic representation Π to an automorphic representation π of a suitable unitary group G over F. The group G is an inner form of the quasi-split unitary group $U_{E/F}(2m)$ (where F denotes the maximal totally real subfield of E) and is chosen to be compact at infinity and isomorphic to $\mathrm{GL}(2, D)$ at the place w, where D is a central division F_w-algebra. The existence of this transfer follows from the theorems stated in [**KMSW14**]. One can then construct a space of automorphic forms on G, with $\overline{\mathbb{Z}}_\ell$ or $\overline{\mathbb{F}}_\ell$ coefficients, and consider the action of a Hecke algebra on these spaces. One can state and prove the analogue of Ihara's lemma in this setting, using the strong approximation theorem, and hence prove a level raising theorem for automorphic representations on G. Then one can construct an automorphic representation of GL_{2m} with the desired properties, using the results of [**KMSW14**] once again.

Using an inductive argument, we can generalise this main theorem to a level raising theorem (Theorem 3.6) for regular C-algebraic representations of $\mathrm{GL}_{2^r m}(\mathbb{A}_E)$ of the form

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_{2^r},$$

where each Π_i is a conjugate self-dual, unitary cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_E)$. In the special case $m = 1$, the theorem is the following:

THEOREM. *Let $\chi_1, \cdots, \chi_{2^k} : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$ be conjugate self-dual, unitary characters and assume that*

$$\Pi = \chi_1 \boxplus \cdots \boxplus \chi_{2^k}$$

is regular C-algebraic. Assume, in addition, that there is a finite place $\mathfrak{P}$ of E such that all the χ_i are unramified at $\mathfrak{P}$ and

$$\iota^{-1}(\chi_i(\varpi)/\chi_{i+1}(\varpi)) \equiv (\mathbb{N}\mathfrak{P})^{-1} \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_\ell} \quad \textit{for all } 1 \le i < 2^k,$$

where ϖ is a uniformiser for $E_{\mathfrak{P}}$ (and $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ is a fixed field isomorphism). Then there is a CM biquadratic extension E_1/E and, for each choice of a place $\mathfrak{P}_1$ of E_1 above $\mathfrak{P}$, a regular C-algebraic, conjugate self-dual, unitary cuspidal automorphic representation Π' of $\mathrm{GL}_{2^k}(\mathbb{A}_{E_1})$, such that

$$\overline{r_\iota}(\Pi') \cong \overline{r_\iota}(\Pi)|_{\mathrm{Gal}(\overline{E_1}/E_1)}$$

and $\Pi'_{\mathfrak{P}_1}$ is isomorphic to an unramified twist of the Steinberg representation of $\mathrm{GL}_{2^k}(E_{\mathfrak{P}_1})$.

We remark that the above theorem implies, in particular, the following result (Corollary 3.7):

Proposition. *Let $K/\mathbb{Q}$ be an imaginary quadratic field, let $\ell > 3$ be a rational prime and let $\overline{\psi} : \mathrm{Gal}(\overline{K}/K) \to \overline{\mathbb{F}}_\ell^\times$ be a continuous character. Let*

$$\bar{r} := \overline{\psi} \oplus \overline{\psi}^c : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell).$$

Then, for every integer $k \geq 1$, there exists a CM biquadratic extension E/K, and a regular C-algebraic, essentially conjugate self-dual, unitary cuspidal automorphic representation Π of $\mathrm{GL}_{2^k}(\mathbb{A}_E)$, such that

(i) $\overline{r_\iota}(\Pi) \cong (\mathrm{Sym}^{2^k-1}\bar{r})|_{\mathrm{Gal}(\overline{E}/E)}$, and

(ii) There exists a finite place w of E such that Π_w is isomorphic to an unramified twist of the Steinberg representation of $\mathrm{GL}_{2^k}(E_w)$.

One can try to combine this result with a suitable automorphy lifting theorem (for residually reducible Galois representations) to prove some cases of symmetric power functoriality for modular forms with dihedral residual Galois representation, as in the work of Clozel and Thorne [**CT14**].

The overview of this thesis is as follows: In Chapter 1, we recall the definition and some of the basic properties of unitary groups (including the classification of unitary groups over a number field) and give a summary of the results of [**KMSW14**] on the endoscopic classification of representations of unitary groups, which will be important for us in the following chapters. In Chapter 2, we explain why the results of [**KMSW14**] imply that the "descent" of Π to the unitary group G exists (after a biquadratic base chage) and the existence of Galois representations attached to automorphic representations of G, and we define spaces of ℓ-adic automorphic forms on G and the Hecke algebra acting on them. In Chapter 3, we state and prove our main theorems (i.e. the level raising theorems discussed above), by proving a version of Ihara's lemma in the setting of automorphic representations of the unitary group G, and explain how the proposition above follows from these theorems.

CHAPTER 1

Endoscopic classification of representations of unitary groups

In this chapter, we review the results of [**Mok15**] and [**KMSW14**] on the endoscopic classification of representations of unitary groups, which will be important for us in the following chapters; these papers follow the strategy of Arthur [**Art13**], who established the endoscopic classification for quasi-split orthogonal and symplectic groups. We begin this first chapter by recalling the classification of unitary groups over local and global fields (of characteristic 0). We then summarise the properties of local A-packets of representations of unitary groups and inner forms of the general linear group. Finally, we state the main global theorem of [**Mok15**] and [**KMSW14**], which gives a decomposition of the space of L^2-automorphic forms on a unitary group over a number field into automorphic A-packets.

1. Unitary groups

Let F be a field (we'll only be interested in the case where F is a local or global field of characteristic 0) and let E be a quadratic étale F-algebra. Thus, E is either a quadratic field extension of F or isomorphic (as an F-algebra) to $F \times F$. We let c be the non-trivial F-automorphism of E. In the first case, c is the non-trivial element of the Galois group $\mathrm{Gal}(E/F)$, while in the second case, it is the automorphism $(x, y) \mapsto (y, x)$. The map c induces an algebraic automorphism $\widetilde{c}$ of the group $G_{E/F}(N) := \mathrm{Res}_{E/F}\mathrm{GL}(N, E)$. We define $G^* := U_{E/F}(N)$ to be the subgroup of fixed points of the automorphism $\theta : x \mapsto J_N \widetilde{c}(x)^{-t} J_N^{-1}$, where

$$J_N := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{N-1} & & & \end{pmatrix} \in \mathrm{GL}_N(E).$$

Hence, G^* is the reductive group over F with functor of points

$$R \mapsto \{x \in \mathrm{GL}_N(E \otimes_F R) : (c \otimes \mathrm{id}_R)(x)^t \, J_N \, x = J_N\}.$$

If E is a field, then G^* is the quasi-split unitary group of rank N with respect to E/F. If $E \cong F \times F$, then $G^* \cong \mathrm{GL}(N,F)$ and each map $E \to F$ induces a canonical isomorphism between $U_{E/F}(N)$ and $\mathrm{GL}(N,F)$ (note that these two isomorphisms differ by the automorphism $x \mapsto J_N x^{-t} J_N^{-1}$ of $\mathrm{GL}(N,F)$).

DEFINITION 1.1. *A* unitary group *over a field F is an inner form of $U_{E/F}(N)$, where E/F is a quadratic field extension.*

1.1. The dual group of $U_{E/F}(N)$. Assume that E/F is a quadratic field extension. $G^* = U_{E/F}(N)$ is a quasi-split group with Borel pair (B,T), where B consists of the upper-triangular matrices in $U_{E/F}(N)$ and T consists of the diagonal matrices. The group $G^*_E = U_{E\otimes_F E/E}$ is isomorphic to $\mathrm{GL}(N,E)$, so $\widehat{G^*} = \mathrm{GL}(N,\mathbb{C})$. Fixing the standard (upper-triangular) pinning on $\widehat{G^*} = \mathrm{GL}(N,\mathbb{C})$, we get an algebraic action of $\mathrm{Gal}(E/F) = \langle c\rangle$ on $\widehat{G^*}$, given by $x \mapsto J_N x^{-t} J_N^{-1}$ (where we now think of J_N as an element of $\mathrm{GL}_N(\mathbb{C})$). We have ${}^L G = \widehat{G}(\mathbb{C}) \rtimes W_F$, where W_F acts on $\widehat{G}(\mathbb{C})$ through $W_F \to \mathrm{Gal}(E/F)$.

1.2. Classification of inner forms of $U_{E/F}(N)$. The (isomorphism classes of) inner forms of $U_{E/F}(N)$ are in bijection with the image of the Galois cohomology set $H^1(F, U_{E/F}(N)_{\mathrm{ad}})$ in $H^1(F, \mathrm{Aut}(U_{E/F}(N)))$. The pointed set $H^1(F, U_{E/F}(N)_{\mathrm{ad}})$ classifies equivalence classes of inner twists of $U_{E/F}(N)$.

The classification of inner forms of $U_{E/F}(N)$ follows from the classification of Hermitian forms over local and global fields, which is explained in [**Sch85**].

We first consider the case where F is a local field of characteristic 0.

1.2.1. *Assume* $E \cong F \times F$. Then $U_{E/F}(N) \cong \mathrm{GL}(N)$ and we need to classify inner forms of $\mathrm{GL}(N)$. We have

$$H^1(F, \mathrm{PGL}(N)) = \begin{cases} \frac{1}{N}\mathbb{Z}/\mathbb{Z} & \text{if } F \text{ is non-archimedean} \\ \frac{1}{2}\mathbb{Z}/\mathbb{Z} & \text{if } F = \mathbb{R} \text{ and } N \text{ is even} \\ \text{trivial} & \text{if } F = \mathbb{C}, \text{ or } F = \mathbb{R} \text{ and } N \text{ is odd} \end{cases}$$

The inner form of $\mathrm{GL}(N)$ corresponding to $r/s \in H^1(F, \mathrm{PGL}(N)) \subset \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ (where $s \mid N$, $r \in (\mathbb{Z}/s\mathbb{Z})^\times$) is $\mathrm{GL}(m,D)$, where $m = N/s$ and D is the central division F-algebra with invariant r/s (in particular, $\dim_F D = s^2$).

REMARK. The isomorphism $H^1(F, U_{E/F}(N)_{\mathrm{ad}}) \xrightarrow{\sim} H^1(F, \mathrm{PGL}(N)) \cong \mathbb{Z}/N\mathbb{Z}$ (when F is a finite extension of $\mathbb{Q}_p$) depends on the choice of map $E \to F$. Indeed, we have a short exact sequence

$$1 \to \mathbb{G}_m \to \mathrm{GL}(N) \to \mathrm{PGL}(N) \to 1$$

and $H^1(F, \mathrm{GL}(N))$ is trivial (by Hilbert's Theorem 90), so we get an injective map

$$H^1(F, \mathrm{PGL}(N)) \hookrightarrow H^2(F, \mathbb{G}_m).$$

The Brauer group of F is $H^2(F, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$ in and the image of $H^1(F, \mathrm{PGL}(N))$ in $H^2(F, \mathbb{G}_m)$ is the subgroup of N torsion points. The two isomorphisms

$$U_{E/F}(N) \xrightarrow{\sim} \mathrm{GL}(N)$$

(corresponding to the two maps $E \to F$) differ by the automorphism $x \mapsto J_N x^{-t} J_N^{-1}$ of $\mathrm{GL}(N)$. We have a commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{GL}(N) & \longrightarrow & \mathrm{PGL}(N) & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle x \mapsto x^{-1}} & & \big\downarrow{\scriptstyle x \mapsto J_N x^{-t} J_N^{-1}} & & \big\downarrow{\scriptstyle x \mapsto J_N x^{-t} J_N^{-1}} & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{GL}(N) & \longrightarrow & \mathrm{PGL}(N) & \longrightarrow & 1
\end{array}$$

This gives a commutative diagram of Galois cohomology groups

$$\begin{array}{ccc}
H^1(F, \mathrm{PGL}(N)) & \hookrightarrow & H^2(F, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z} \\
\big\downarrow{\scriptstyle x \mapsto J_N x^{-t} J_N^{-1}} & & \big\downarrow{\scriptstyle x \mapsto -x} \\
H^1(F, \mathrm{PGL}(N)) & \hookrightarrow & H^2(F, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}
\end{array}$$

so the two isomorphisms $H^1(F, U_{E/F}(N)_{\mathrm{ad}}) \xrightarrow{\sim} H^1(F, \mathrm{PGL}(N)) \cong \mathbb{Z}/N\mathbb{Z}$, corresponding two the maps $E \to F$, differ by the automorphism $x \mapsto -x$ of $\mathbb{Z}/N\mathbb{Z}$.

1.2.2. *Assume E/F is a quadratic field extension.* If F is non-archimedean, then

$$H^1(F, U_{E/F}(N)_{\mathrm{ad}}) = \begin{cases} \frac{1}{2}\mathbb{Z}/\mathbb{Z} & \text{if } N \text{ is even} \\ \text{trivial} & \text{if } N \text{ is odd} \end{cases}$$

i.e. there is a unique non-quasi-split inner form of $U_{E/F}(N)$ if N is even, and no non-quasi-split inner forms if N is odd.

If $F = \mathbb{R}$, then

$$H^1(F, U_{E/F}(N)_{\mathrm{ad}}) = \left\{(p,q) \in (\mathbb{Z}_{\geq 0})^2 : p + q = N\right\} / (p,q) \sim (q,p),$$

where (p,q) corresponds to the inner form

$$U(p,q)(R) = \left\{ x \in \mathrm{GL}_N(\mathbb{C} \otimes_{\mathbb{R}} R) : (c \otimes \mathrm{id}_R)(x)^t \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix} x = \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix} \right\}.$$

Note that $U_{\mathbb{C}/\mathbb{R}}(N) \cong U(\lceil \frac{N}{2} \rceil, \lfloor \frac{N}{2} \rfloor)$ and $U(p,q) \cong U(q,p)$. Moreover, $U(N) := U(N,0)$ is the unique inner form of $U_{\mathbb{C}/\mathbb{R}}(N)$ whose $\mathbb{R}$-points form a compact group.

We now turn to the global case:

1.2.3. *Assume E/F is a quadratic extension of number fields.* We have a localisation map

$$H^1(F, U_{E/F}(N)_{\mathrm{ad}}) \to \bigoplus_v H^1(F_v, U_{E_v/F_v}(N)_{\mathrm{ad}}),$$

where v runs through all places of F. We write E_v for $E \otimes_F F_v$; E_v is a quadratic field extension of F_v if v splits in E, and $E_v \cong F_v \times F_v$ (non-canonically) if v is non-split in E.

Notation. $\bigoplus_v H_v$ (where H_v are pointed sets) denotes the subset of $\prod_v H_v$ consisting of all $(x_v)_v$ such that $x_v = 0$ (the distinguished element of H_v) for all but finitely many v.

The localisation map is injective. If N is odd, it is also surjective. If N is even, the image of the localisation map is equal to the kernel of

$$\bigoplus_v H^1(F_v, U_{E_v/F_v}(N)_{\mathrm{ad}}) \xrightarrow{\sum_v a_v} \mathbb{Z}/2\mathbb{Z},$$

where the maps a_v are defined as follows:

(i) If v is finite and non-split in E, then $a_v(0) = 0$ and $a_v(1) = 1$.
(ii) If v is real and non-split in E, then $a_v : (p,q) \mapsto \frac{1}{2}(p-q) \bmod 2$.
(iii) If v is split in E, then a_v maps

$$r/s \in H^1(F, U_{E_v/F_v}(N)_{\mathrm{ad}}) \subset \frac{1}{N}\mathbb{Z}/\mathbb{Z}$$

to $Nr/s \bmod 2$.

1.3. Determinant map. Let E/F be a quadratic field extension of characteristic 0 and (G,ξ) be an inner twist of $U_{E/F}(N)$. Thus, G is an inner form of $U_{E/F}(N)$ and ξ is an isomorphism

$$U_{E/F}(N)_{\overline{F}} \xrightarrow{\sim} G_{\overline{F}}$$

of $\overline{F}$-groups, such that $\xi^{-1} \circ {}^\sigma\xi$ is an *inner* automorphism of the group $U_{E/F}(N)_{\overline{F}}$, for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$. Let $\xi^{-1} \circ {}^\sigma\xi = \mathrm{Ad}(a_\sigma)$, where $a_\sigma \in U_{E/F}(N)(\overline{F})$. Let

$$\det : U_{E/F}(N) \to U_{E/F}(1)$$

be the determinant map on the quasi-split unitary group, and define

$$\det_\xi := \det_{\overline{F}} \circ \xi^{-1} : G_{\overline{F}} \to U_{E/F}(1)_{\overline{F}}.$$

For any $\sigma \in \mathrm{Gal}(\overline{F}/F)$, we have

$$^\sigma(\det_\xi) = {}^\sigma(\det_{\overline{F}}) \circ ({}^\sigma\xi)^{-1} = \det_{\overline{F}} \circ \mathrm{Ad}(a_\sigma^{-1}) \circ \xi^{-1} = \det_{\overline{F}} \circ \xi^{-1} = \det_\xi,$$

so $\det_\xi$ descends to a map $G \to U_{E/F}(1)$ of F-groups.

The kernel of $\det_\xi$ is the derived subgroup G_1 of G. To simplify notation, we will often write det instead of $\det_\xi$, but note that $\det_\xi$ does depend on (the conjugacy class of) ξ.

Proposition 1.2. *Let E/F be a quadratic extension of local or global fields of characteristic 0. Let (G, ξ) be an inner twist of $U_{E/F}(N)$. Then the determinant map induces a* surjective *map on F-points:*

$$\det : G(F) \to U_{E/F}(1)(F) = \{x \in E^\times : \mathbb{N}_{E/F}(x) = 1\}.$$

If F is a number field, then det *is also surjective on $\mathbb{A}_F^\infty$-points.*

Proof. We have a short exact sequence of linear algebraic groups

$$1 \to G_1 \to G \to U_{E/F}(1) \to 1,$$

which gives rise to a long exact sequence

$$G(F) \to U_{E/F}(1)(F) \to H^1(F, G_1) \to H^1(F, G)$$

of pointed Galois cohomology sets. It suffices to show that $H^1(F, G_1) \to H^1(F, G)$ has trivial kernel. If F is a finite extension of $\mathbb{Q}_p$, then $H^1(F, G_1)$ is trivial (since G_1 is semisimple and simply-connected). If $F = \mathbb{R}$, then one can easily see that the map $H^1(F, G_1) \to H^1(F, G)$ is injective, by direct computation.

If F is a number field, then we have a commutative diagram

$$\begin{array}{ccc} H^1(F, G_1) & \longrightarrow & H^1(F, G) \\ \downarrow & & \downarrow \\ \prod_{v|\infty} H^1(F_v, G_1) & \longrightarrow & \prod_{v|\infty} H^1(F_v, G) \end{array}$$

and the left vertical arrow is bijective. It follows that the map $H^1(F, G_1) \to H^1(F, G)$ is injective. To prove the surjectivity of det on $\mathbb{A}_F^\infty$-points, observe that, if v is a finite place of F which is inert in E, then the image of

$$U_{E_v/F_v}(N)(\mathcal{O}_{F_v}) := \{x \in \mathrm{GL}_N(\mathcal{O}_{E_v}) : c(x)^t\, J_N\, x = J_N\}$$

under det is $U_{E_v/F_v}(1)(\mathcal{O}_{F_v})$. □

1.4. Extended pure inner twists. We will need the notion of an extended pure inner twist; this was defined in [**KMSW14**] §0.3. In this subsection, we briefly recall the definition of an extended pure inner twist. In order to do this, we need to introduce Kottwitz's cohomology set $B(F, G)$, which was constructed in [**Kot14**].

Let F be a local or global field. We will only be interested in the case where F has characterestic 0, but the results of Kottwitz are also true for (local or global) fields of positive characteristic. In [**Kot14**], Kottwitz defines a functor

$$G \mapsto B(F, G)$$

from the category of linear algebraic groups over F to the category of pointed sets. The pointed set $B(F, G)$ is defined as a direct limit

$$B(F, G) = \varinjlim_{K} H^1_{\mathrm{alg}}(\mathcal{E}(K/F), G),$$

where the limit is taken over all finite Galois extensions K/F inside a fixed separable closure $\overline{F}$ of F.

Definition 1.3. *A* Galois gerb $\mathcal{E}$ *for a finite Galois extension K/F, bound by an F-group D of multiplicative type, is an extension*

$$1 \to D(K) \to \mathcal{E} \to \mathrm{Gal}(K/F) \to 1$$

of $\mathrm{Gal}(K/F)$ *by $D(K)$. The isomorphism classes of such extensions are classified by the Galois cohomology group $H^2(K/F, D)$.*

The group $\mathcal{E}(K/F)$ is an example of a Galois gerb for K/F, bound by the F-group $\mathbb{D}_{K/F}$. If F is local, then $\mathbb{D}_{K/F}$ is the multiplicative group $\mathbb{G}_m$ and $\mathcal{E}(K/F)$ is taken to be the relative Weil group of K/F. If F is global, then $\mathbb{D}_{K/F}$ is a protorus over F, whose group of K-characters is the $\mathrm{Gal}(K/F)$-module $\mathbb{Z}[V_K]_0$, where $\mathbb{Z}[V_K]$ is the free abelian group generated by the set V_K of places of K and $\mathbb{Z}[V_K]_0$ is the kernel of the map

$$\mathbb{Z}[V_K] \to \mathbb{Z}, \quad \sum_{v \in V_K} n_v v \mapsto \sum_{v \in V_K} n_v.$$

Given a Galois gerb $\mathcal{E}$, as above, and a linear algebraic group G over F, we can define the (pointed) set $Z^1_{\mathrm{alg}}(\mathcal{E}, G)$ of algebraic 1-cocycles:

Definition 1.4. *An algebraic* 1*-cocycle of* $\mathcal{E}$ *in* $G(K)$ *is a pair* (ν, x) *consisting of a homomorphism* $\nu : D \to G$ *of algebraic groups over* K *and an abstract* 1*-cocycle*

$$x : \mathcal{E} \to G(K)$$

(where $\mathcal{E}$ *acts on* $G(K)$ *via* $\mathcal{E} \to \mathrm{Gal}(K/F)$*), satisfying the following two conditions:*

(i) $x_d = \nu(d)$ *for all* $d \in D(K)$*;*

(ii) $\mathrm{Ad}(x_w) \circ {}^{\sigma}\nu = \nu$ *for all* $w \in \mathcal{E}$ *that map to* $\sigma \in \mathrm{Gal}(K/F)$*.*

The group $G(K)$ *of* K*-points of* G *acts on the set* $Z^1_{\mathrm{alg}}(\mathcal{E}, G)$ *of algebraic* 1*-cocycles by*

$$g : (\nu, x) \mapsto (\mathrm{Ad}(g) \circ \nu, w \mapsto g x_w w(g)^{-1});$$

we define $H^1_{\mathrm{alg}}(\mathcal{E}, G)$ *to be the quotient of* $Z^1_{\mathrm{alg}}(\mathcal{E}, G)$ *by the action of* $G(K)$*.*

We refer the reader to [**Kot14**] for the definition of $\mathcal{E}(K/F)$ in the global case and the definition of the inflation maps $H^1_{\mathrm{alg}}(\mathcal{E}(K/F), G) \to H^1_{\mathrm{alg}}(\mathcal{E}(L/F), G)$ (when $L \supset K$ is another finite Galois extension of F inside the fixed separable closure of F).

We observe that the map $(\nu, x) \mapsto \nu$ induces a "Newton map"

$$H^1_{\mathrm{alg}}(\mathcal{E}, G) \to [\mathrm{Hom}_K(D, G)/G(K)]^{\mathrm{Gal}(K/F)};$$

we write $H^1_{\mathrm{alg}}(\mathcal{E}, G)_{\mathrm{bsc}}$ and $Z^1_{\mathrm{alg}}(\mathcal{E}, G)_{\mathrm{bsc}}$ for the preimage of $\mathrm{Hom}_F(D, Z(G))$ under the Newton map in $H^1_{\mathrm{alg}}(\mathcal{E}, G)$ and $Z^1_{\mathrm{alg}}(\mathcal{E}, G)$, respectively. The elements of $Z^1_{\mathrm{alg}}(\mathcal{E}, G)_{\mathrm{bsc}}$ are called "basic cocycles". Similarly, we have a Newton map

$$B(F, G) \to \left[\mathrm{Hom}_{\overline{F}}(\mathbb{D}_F, G)/G(\overline{F})\right]^{\mathrm{Gal}(\overline{F}/F)},$$

where $\mathbb{D}_F := \varprojlim_K \mathbb{D}_{K/F}$ (with transition maps defined as in [**Kot14**], §8.3 and §10.3), and we define $B(F, G)_{\mathrm{bsc}}$ to be the preimage of $\mathrm{Hom}_F(\mathbb{D}_F, Z(G))$ under this map.

We mention some of the properties of the cohomology set $B(F, G)$ (proved in [**Kot14**]):

(i) There is a natural inclusion $H^1(F, G) \hookrightarrow B(F, G)_{\mathrm{bsc}}$, whose image is the kernel of the Newton map. This map is a bijection when the centre of G is trivial.

(ii) Given a finite separable extension E/F, there is a restriction map

$$B(F, G) \to B(E, G)$$

and a Shapiro isomorphism $B(F, \mathrm{Res}_{E/F} H) = B(E, H)$ (where H denotes a linear algebraic group over E).

(iii) If F is a local field and G is a connected reductive group over F, then there is a canonical map

$$\kappa_G : B(F, G)_{\mathrm{bsc}} \to X^*(Z(\widehat{G})^{\mathrm{Gal}(\overline{F}/F)}).$$

(iv) If F is a global field, then, for each place v of F, there is a localisation map

$$B(F, G) \to B(F_v, G),$$

sending $B(F, G)_{\mathrm{bsc}}$ to $B(F_v, G)_{\mathrm{bsc}}$. When G is connected, the image of an element $b \in B(F, G)$ in $B(F_v, G)$ is trivial for all but finitely many places v.

(v) Let F be a global field and let G be a connected reductive group over F. Then the image of the map

$$B(F, G)_{\mathrm{bsc}} \to \bigoplus_v B(F_v, G)_{\mathrm{bsc}}$$

is equal to the kernel of the composition

$$\bigoplus_v B(F_v, G)_{\mathrm{bsc}} \to \bigoplus_v X^*(Z(\widehat{G})^{\mathrm{Gal}(\overline{F_v}/F_v)}) \xrightarrow{\Sigma} X^*(Z(\widehat{G})^{\mathrm{Gal}(\overline{F}/F)}).$$

We can now give the definition of an extended pure inner twist:

Definition 1.5. *Let G^* be a connected reductive group over a local or global field F. An* extended pure inner twist *of G^* is a triple (G, ξ, z) consisting of*

(a) a connected reductive group G over F, which is an inner form of G^*;

(b) an isomorphism $\xi : G^*_{\overline{F}} \xrightarrow{\sim} G_{\overline{F}}$ such that $\xi^{-1} \circ {}^\sigma\xi$ is an inner automorphism of $G^*_{\overline{F}}$ for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$;

(c) an element $z \in Z^1_{\mathrm{alg}}(\mathcal{E}(K/F), G^*)_{\mathrm{bsc}}$ (for some finite Galois extension K/F) whose image in $Z^1_{\mathrm{alg}}(\mathcal{E}(K/F), G^*_{\mathrm{ad}})_{\mathrm{bsc}} = Z^1(K/F, G^*_{\mathrm{ad}})$ is equal to $\sigma \mapsto \xi^{-1} \circ {}^\sigma\xi$.

Remark. Proposition 10.4 of [**Kot14**] implies that if $Z(G^*)$ is connected, then the map $B(F, G^*)_{\mathrm{bsc}} \to B(F, G^*_{\mathrm{ad}})_{\mathrm{bsc}} = H^1(F, G^*_{\mathrm{ad}})$ is surjective. In particular, assuming that $Z(G^*)$ is connected, any inner twist of G^* can be made into an extended pure inner twist.

2. Local classification

Throughout this section, F denotes a local field of characteristic 0.

2.1. Generalities on A-parameters. Let G be a connected reductive group over F. Fixing a pinning on the dual group $\widehat{G}/\mathbb{C}$, we get an algebraic action of the Galois group $\mathrm{Gal}(\overline{F}/F)$ on $\widehat{G}$ and we can form (the Weil form of) the L-group

$$^LG := \widehat{G}(\mathbb{C}) \rtimes W_F,$$

where the Weil group W_F of F acts on $\widehat{G}(\mathbb{C})$ through $W_F \to \mathrm{Gal}(\overline{F}/F)$.

Let L_F be the Langlands group of F:

$$L_F := \begin{cases} W_F & \text{if } F \text{ is archimedean,} \\ W_F \times \mathrm{SL}_2(\mathbb{C}) & \text{if } F \text{ is non-archimedean.} \end{cases}$$

An L-parameter for G is a continuous homomorphism $\phi : L_F \to {}^LG$ which

(i) commutes with the projections $L_F \to W_F$ and ${}^LG \to W_F$,

(ii) is such that, for each $w \in W_F$, the image of $\phi(w)$ in $\widehat{G}(\mathbb{C})$ is a semisimple element,

(iii) is algebraic on $\mathrm{SL}_2(\mathbb{C})$ (in the non-archimedean case).

Two L-parameters are equivalent if they are $\widehat{G}(\mathbb{C})$-conjugate. We denote the set of equivalence classes of L-parameters by $\Phi(G)$. An L-parameter ϕ is called bounded (or tempered) if $\phi(W_F)$ projects to a relatively compact subset of $\widehat{G}(\mathbb{C})$. We write $\Phi_{\mathrm{bdd}}(G) \subset \Phi(G)$ for the subset of (equivalence classes of) bounded L-parameters.

The local endoscopic classification is given in terms of A-parameters: An A-parameter is a homomorphism

$$\psi : L_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$$

such $\psi|_{L_F}$ is a bounded L-parameter and $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic. The set of equivalence classes of A-parameters is denoted by $\Psi(G)$ (where two parameters are equivalent if they are $\widehat{G}(\mathbb{C})$-conjugate). We also introduce the larger set $\Psi^+(G) \supset \Psi(G)$ of $\widehat{G}(\mathbb{C})$-conjugacy classes of homomorphisms

$$\psi : L_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$$

which are algebraic on $\mathrm{SL}_2(\mathbb{C})$ and whose restriction to L_F is a (not necessarily bounded) L-parameter. A parameter $\psi \in \Psi^+(G)$ is called generic if $\psi|_{\mathrm{SL}_2(\mathbb{C})} = 1$. We can identify the set of generic parameters in $\Psi(G)$ (resp. $\Psi^+(G)$) with the set

$\Phi_{\mathrm{bdd}}(G)$ (resp. $\Phi(G)$). We note that there is a map $\Psi^+(G) \to \Phi(G)$, $\psi \mapsto \phi_\psi$, defined by

$$\phi_\psi(w) := \psi\left(w, \begin{pmatrix} |w|_F^{1/2} & \\ & |w|_F^{-1/2} \end{pmatrix}\right).$$

We also need the notion of a relevant parameter: Recall ([**Bor79**] §3, [**KMSW14**] §0.4) that there is a bijection between the set $\mathscr{P}(G)^{\mathrm{Gal}(\overline{F}/F)}$ of $G(\overline{F})$-conjugacy classes of parabolic subgroups of $G_{\overline{F}}$ which are $\mathrm{Gal}(\overline{F}/F)$-invariant and the set $\mathscr{P}({}^LG)$ of ${}^LG$-conjugacy classes of parabolic subgroups of ${}^LG$. A parabolic subgroup of ${}^LG$ is a full subgroup $\mathbf{P} \subset {}^LG$ such that $\mathbf{P} \cap \widehat{G}(\mathbb{C})$ is (the group of $\mathbb{C}$-points of) a parabolic subgroup P of $\widehat{G}$. (We say that a subgroup $\mathbf{A}$ of ${}^LG$ is *full* if the restriction of the projection map ${}^LG \to W_F$ to $\mathbf{A}$ is surjective). A Levi factor of the parabolic subgroup $\mathbf{P}$ is a full subgroup $\mathbf{M} \subseteq \mathbf{P}$ such that $\mathbf{M} \cap \widehat{G}(\mathbb{C})$ is (the group of $\mathbb{C}$-points of) a Levi factor of P. We write $\mathscr{P}(G/F) \subset \mathscr{P}(G)^{\mathrm{Gal}(\overline{F}/F)}$ for the subset of conjugacy classes containing a parabolic subgroup defined over F. A parabolic subgroup $\mathbf{P}$ of ${}^LG$ is called (G-)relevant if its conjugacy class corresponds to a conjugacy class in $\mathscr{P}(G/F)$. Note that if G is quasi-split all parabolic subgroups of ${}^LG$ are relevant for G.

Definition 1.6. *A parameter $\psi \in \Psi(G)$ is G-relevant if every Levi subgroup $\mathbf{M} \subset {}^LG$ containing $\mathrm{Im}\,\psi$ is a Levi component of a relevant parabolic. We write $\Psi(G)_{G\text{-rel}} \subset \Psi(G)$ for the subset of G-relevant parameters.*[1]

Finally, given a parameter $\psi \in \Psi^+(G)$, we introduce the groups

$$S_\psi := \mathrm{Cent}(\mathrm{Im}\,\psi, \widehat{G}(\mathbb{C}))$$

and $S_\psi^\natural := S_\psi/(S_\psi \cap \widehat{G}_{\mathrm{der}}(\mathbb{C}))^\circ$. We remark that $S_\psi \supset Z(\widehat{G}(\mathbb{C}))^{\mathrm{Gal}(\overline{F}/F)}$. These groups will be important for the local endoscopic classification of representations.

Lemma 1.7. *Let $\psi \in \Psi^+(G)$. We have inclusion-preserving maps:*

$$\left\{ \begin{array}{c} \textit{Levi subgroups of } {}^LG \\ \textit{containing } \mathrm{Im}\,\psi \end{array} \right\} \underset{\mathrm{infl}}{\overset{\mathrm{cntr}}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{Levi subgroups} \\ \textit{of } S_\psi^\circ \end{array} \right\}$$

[1]Note the difference between our notation and Borel's [**Bor79**].

defined by $\mathrm{cntr}(\mathbf{M}) := \mathbf{M} \cap S_\psi^\circ$, $\mathrm{infl}(M) := \mathrm{Cent}(Z(M)^\circ, {}^LG)$. *These maps satisfy* $\mathrm{cntr}(\mathrm{infl}(M)) = M$ *and* $\mathrm{infl}(\mathrm{cent}(\mathbf{M})) \subseteq \mathbf{M}$. *In particular, the minimal Levi subgroups containing* $\mathrm{Im}\,\psi$ *are precisely the centralisers in* ${}^LG$ *of the maximal tori of* S_ψ.

PROOF. This is [**KMSW14**] Lemma 0.4.12. □

2.2. Parameters of unitary groups. Let E be a quadratic étale F-algebra and set $G^* := U_{E/F}(N)$ (as in Section 1). Let G be an inner form of G^*, and fix a choice of an extended pure inner twist $(\xi, z) : G^* \to G$ (see Section 1.4; note that any inner form of G^* admits a choice of an extended pure inner twist). We have ${}^LG = {}^LG^*$, and thus, $\Psi(G) = \Psi(G^*)$.

Assume that $E \cong F \times F$ and fix a map $E \to F$. We get an F-isomorphism $G^* \to \mathrm{GL}(N)$ and we have ${}^LG = \mathrm{GL}_N(\mathbb{C}) \times W_F$. We can think of a parameter $\psi \in \Psi^+(G)$ as a continuous semisimple representation

$$\psi : L_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(V)$$

of the group $L_F \times \mathrm{SL}_2(\mathbb{C})$ on an N-dimensional complex vector space V. We can decompose ψ as

$$\psi = \ell_1\psi_1 \oplus \ldots \oplus \ell_r\psi_r,$$

where the ℓ_i's are positive integers and the ψ_i's are simple (i.e. irreducible) and pairwise non-isomorphic constituents of ψ. Schur's lemma implies that

$$S_\psi \cong \mathrm{GL}_{\ell_1}(\mathbb{C}) \times \ldots \times \mathrm{GL}_{\ell_r}(\mathbb{C}).$$

Assume now that E/F is a quadratic field extension. Given a parameter

$$\psi \in \Psi^+(\mathrm{GL}(N, E)),$$

we define its conjugate parameter ψ^c by

$$\psi^c(x) := \psi(w_c x w_c^{-1}),$$

where w_c is a fixed choice of an element of $W_F \backslash W_E$. Note that the equivalence class of ψ^c does not depend on the choice of w_c. We say that a parameter $\psi \in \Psi^+(\mathrm{GL}(N, E))$ is conjugate self-dual if ψ^c is equivalent to its dual parameter $\psi^\vee : x \mapsto \psi(x)^{-t}$. We write $\widetilde{\Psi}(N) \subset \Psi(\mathrm{GL}(N, E))$ for the subset of conjugate self-dual parameters. We similarly define $\widetilde{\Psi}^+(N)$, $\widetilde{\Phi}(N)$, $\widetilde{\Phi}_{\mathrm{bdd}}(N)$ to be the subsets of conjugate self-dual parameters in $\Psi^+(\mathrm{GL}(N, E))$, $\Phi(\mathrm{GL}(N, E))$, $\Phi_{\mathrm{bdd}}(\mathrm{GL}(N, E))$, respectively.

Let $\chi : W_E \to \mathbb{C}^\times$ be a conjugate self-dual character (i.e. $\chi^c = \chi^{-1}$). We have a "base change" map

$$\eta^*_\chi : \Psi^+(G) \to \widetilde{\Psi}^+(N)$$

defined by $\psi \mapsto \chi \otimes \psi|_{L_E \times \mathrm{SL}_2(\mathbb{C})}$. We have $\eta^{*-1}_\chi(\widetilde{\Psi}(N)) = \Psi(U_{E/F}(N))$, and the same is true if we replace by Ψ with Φ or Φ_{bdd}.

Lemma 1.8. *The map η^*_χ is injective. An element $\psi^N \in \widetilde{\Psi}^+(N)$ is in the image of η^*_χ if and only if there exists $P \in \mathrm{GL}_N(\mathbb{C})$ such that*

(i) $\psi^N(w_c x w_c^{-1}) = P\psi^N(x)^{-t}P^{-1}$ for all $x \in L_E \times \mathrm{SL}_2(\mathbb{C})$, and

(ii) $\psi^N(w_c^2) = \kappa(\chi)(-1)^{N-1}PP^{-t}$, where $\kappa(\chi) := \chi(w_c^2) \in \{\pm 1\}$.

If such a P exists, the preimage ψ of ψ^N is given by

$$\psi(x) = \begin{cases} (\chi^{-1} \otimes \psi^N)(x) & \text{if } x \in L_E \times \mathrm{SL}_2(\mathbb{C}), \\ PJ_N \rtimes w_c & \text{if } x = w_c. \end{cases}$$

Proof. This is an easy calculation: If $\psi^N = \eta^*_\chi \psi$ for some $\psi \in \Psi^+(G)$, then properties (i) and (ii) are satisfied for P given by $\psi(w_c) = PJ_N \rtimes w_c$. Conversely, if (i) and (ii) are satisfied for some $P \in \mathrm{GL}_N(\mathbb{C})$, the formula above defines an element $\psi \in \Psi^+(G)$ such that $\eta^*_\chi \psi = \psi^N$. □

Given a parameter $\psi \in \Psi^+(G)$, we can decompose $\psi^N = \eta^*_\chi \psi$ into simple parameters, as above:

$$\psi^N = \ell_1\psi_1 \oplus \ldots \oplus \ell_r\psi_r.$$

Let $I \subset \{1, \ldots, r\}$ be the set of indices i such that $\psi_i^* := \psi_i^{\vee,c}$ is isomorphic to ψ_i. The set $\{1, \ldots, r\} \setminus I$ can be partitioned into pairs $\{j, j^*\}$ such that

$$\psi_j^* \cong \psi_{j^*} \quad \text{and} \quad \ell_j = \ell_{j^*}.$$

Let J be a subset of $I \setminus \{1, \ldots, r\}$ consisting of one representative from each pair $\{j, j^*\}$. We can thus write

$$\psi^N = \bigoplus_{i \in I} \ell_i\psi_i \oplus \bigoplus_{j \in J} \ell_j(\psi_j \oplus \psi_j^*).$$

Definition 1.9. *Let $\widetilde{\psi} \in \widetilde{\Psi}^+(m)$ be simple. The parity $\kappa(\widetilde{\psi}) \in \{\pm 1\}$ of $\widetilde{\psi}$ is defined by $\widetilde{\psi}(w_c^2) = \kappa(\widetilde{\psi})PP^{-t}$, where $P \in \mathrm{GL}_m(\mathbb{C})$ is any element such that $\widetilde{\psi}^c = \mathrm{Ad}(P) \circ \widetilde{\psi}^\vee$.*

Let us explain why $\kappa(\widetilde{\psi})$ is well-defined:

(a) Since $\widetilde{\psi}$ is conjugate self-dual, there exists some $P \in \mathrm{GL}_m(\mathbb{C})$ such that $\widetilde{\psi}^c = \mathrm{Ad}(P) \circ \widetilde{\psi}^\vee$. By Schur's lemma, any two choices for P differ by a scalar and, hence, give the same value for PP^{-t}.

(b) We have $\mathrm{Ad}(\widetilde{\psi}(w_c^2)) \circ \widetilde{\psi} = \mathrm{Ad}(PP^{-t}) \circ \widetilde{\psi}$, so $\widetilde{\psi}(w_c^2)$ and PP^{-t} differ by a scalar. Say $\widetilde{\psi}(w_c^2) = \lambda PP^{-t}$, where $\lambda \in \mathbb{C}^\times$. Since $\widetilde{\psi}(w_c^2) = P\widetilde{\psi}(w_c^2)^{-t}P^{-1} = \lambda^{-1}PP^{-t} = \lambda^{-2}\widetilde{\psi}(w_c^2)$, we conclude that $\lambda \in \{\pm 1\}$.

(c) It's not hard to check that $\kappa(\widetilde{\psi})$ is independent of the choice of w_c and the choice of $\widetilde{\psi}$ in its equivalence class.

We can now partition I as $I^+ \sqcup I^-$, where

$$I^\varepsilon := \{i \in I : \kappa(\psi_i) = \varepsilon\, \kappa(\chi)\, (-1)^{N-1}\}, \quad \varepsilon \in \{\pm 1\}.$$

Remark. For each $1 \le i \le r$, we have $\psi_i = \mu_i \boxtimes \nu_i$, where μ_i is an irreducible representation of L_F and ν_i is an irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. Let $m_i := \dim \mu_i$ and $n_i := \dim \nu_i$. For $i \in I$, $\mu_i \in \widetilde{\Phi}(m_i)$ and its parity satisfies $\kappa(\psi_i) = (-1)^{n_i - 1}\kappa(\mu_i)$.

Lemma 1.10. *Let $\psi \in \Psi^+(U_{E/F}(N))$. With notation as above, we have that ℓ_i is even for all $i \in I^-$ and*

$$S_\psi \cong \prod_{i \in I^+} \mathrm{O}_{\ell_i}(\mathbb{C}) \times \prod_{i \in I^-} \mathrm{Sp}_{\ell_i}(\mathbb{C}) \times \prod_{j \in J} \mathrm{GL}_{\ell_j}(\mathbb{C}).$$

Moreover, $S_\psi^\natural \cong (\mathbb{Z}/2\mathbb{Z})^{I^+}$.

Proof. This lemma is proven in [**GGP12**] (the proof is given for L-parameters, but the general case is the same). If S_ψ is as above, then

$$(S_\psi \cap G_{\mathrm{der}}(\mathbb{C}))^\circ = \prod_{i \in I^+} \mathrm{SO}_{\ell_i}(\mathbb{C}) \times \prod_{i \in I^-} \mathrm{Sp}_{\ell_i}(\mathbb{C}) \times \prod_{j \in J} \mathrm{GL}_{\ell_j}(\mathbb{C}),$$

and, hence, $S_\psi^\natural \cong (\mathbb{Z}/2\mathbb{Z})^{I^+}$. □

2.3. The main local theorem. Recall that G is an inner form of $G^* := U_{E/F}(N)$ (which is either a unitary group or a general linear group) and that we have fixed a choice of an extended pure inner twist $(\xi, z) : G^* \to G$. The choice of (ξ, z) determines a character

$$\chi_z : Z(\widehat{G}(\mathbb{C}))^{\mathrm{Gal}(\overline{F}/F)} \to \mathbb{C}^\times$$

via the Kottwitz map $\kappa_G : B(F, G^*)_{\mathrm{bsc}} \to X^*(Z(\widehat{G})^{\mathrm{Gal}(\overline{F}/F)})$. We write $\mathrm{Irr}(S_\psi^\natural, \chi_z)$ for the set of characters of the (abelian) group $S_\psi^\natural$ which restrict to χ_z under

$$Z(\widehat{G}(\mathbb{C}))^{\mathrm{Gal}(\overline{F}/F)} \hookrightarrow S_\psi \to S_\psi^\natural.$$

Finally, we fix a choice of a non-trivial additive character $\psi_F : F \to \mathbb{C}^\times$ (the map $\pi \mapsto \langle \cdot, \pi \rangle_{\xi,z}$ below also depends on the choice of ψ_F).

Theorem 1.11. *For each $\psi \in \Psi(G)$, there exists a finite multiset $\Pi_\psi(G, \xi)$ of (equivalence classes of) irreducible unitary representations of $G(F)$ and a map*

$$\Pi_\psi(G, \xi) \to \mathrm{Irr}(S_\psi^\natural, \chi_z), \quad \pi \mapsto \langle \cdot, \pi \rangle_{\xi,z}$$

such that

- *(i) The multiset $\Pi_\psi(G, \xi)$ is independent of the choice of z.*
- *(ii) If ψ is not G-relevant, then $\Pi_\psi(G, \xi)$ is empty. If ψ is generic, then $\Pi_\psi(G, \xi)$ is non-empty if and only if ψ is G-relevant.*
- *(iii) All representations in $\Pi_\psi(G, \xi)$ have the same central character.*
- *(iv) If ψ is generic, all representations in $\Pi_\psi(G, \xi)$ are tempered and only appear once in $\Pi_\psi(G, \xi)$. Moreover, the map $\pi \mapsto \langle \cdot, \pi \rangle_{\xi,z}$ is injective; if F is non-archimedean, it is also surjective.*
- *(v) The packets $(\Pi_\psi : \psi \in \Phi_{\mathrm{bdd}}(G))$ are disjoint and exhaust the tempered representations of $G(F)$.*

Proof. This is [**Mok15**] Theorem 2.5.1, [**KMSW14**] Theorem 1.6.1. □

Remark. The theorem in the quasi-split case $(\xi, z) = (\mathrm{id}, 1)$ is due to Mok. The non-quasi-split case is due to Kaletha, Minguez, Shin and White. At the moment, the theorem in the non-quasi-split case has only been established for generic parameters ψ. A later paper by Kaletha, Minguez and Shin is expected to complete the proof of the theorem for all parameters $\psi \in \Psi(G)$. In this thesis, we only use the theorem for generic parameters ψ.

In the remainder of this section, we descibe the A-packets in some special cases.

2.4. A-packets for inner forms of general linear groups. We consider the A-packets in the split case: $E \cong F \times F$. Fixing a projection map $E \to F$, we get an F-isomorphism $\iota : G^* \cong \mathrm{GL}(N)$. Let $\psi \in \Psi(G)$. Composing ψ with ${}^L\iota$, we can identify ψ with a parameter in $\Psi(\mathrm{GL}(N,F))$. Recall that we have an L-parameter $\phi_\psi \in \Phi(\mathrm{GL}(N,F))$ defined by

$$\phi_\psi(w) := \psi\left(w, \begin{pmatrix} |w|_F^{1/2} & \\ & |w|_F^{-1/2} \end{pmatrix}\right).$$

The local Langlands correspondence for $\mathrm{GL}(N)$ (normalised so that uniformisers correspond to geometric Frobenii) associates an irreducible unitary representation $\pi_\psi^* := \mathrm{rec}_F^{-1}(\phi_\psi)$ of $\mathrm{GL}_N(F)$ to ϕ_ψ.

We say that a unitary smooth irreducible representation π of $\mathrm{GL}_N(F)$ is d-compatible if there exists a regular semisimple conjugacy class x of $\mathrm{GL}_N(F)$ such that:

(a) Any irreducible factor $g \in F[X]$ of the characteristic polynomial of x has degree $\deg g$ divisible by d.
(b) The character Θ_π of π does not vanish on x:

$$\Theta_\pi(x) \neq 0.$$

We have a Jacquet-Langlands map $|\mathbf{LJ}| := |\mathbf{LJ}|_\xi$ (cf. [**DKV84**], [**Bad08**], [**BR10**]) between the set of (equivalence classes of) all d-compatible irreducible unitary representations π of $\mathrm{GL}_N(F)$ and the set of (equivalence classes of) all irreducible unitary representations π' of the inner form $G(F) \cong \mathrm{GL}_m(D)$ of $\mathrm{GL}_N(F)$ (where D is a central division F-algebra of degree d and m is an integer such that $md = N$). The map is characterised by the following property: there exists a (necessarily unique) sign $\varepsilon \in \{\pm 1\}$ such that the characters Θ_π and $\Theta_{\pi'}$, of π and π' respectively, satisfy the relation

$$\Theta_\pi(x) = \varepsilon\, \Theta_{\pi'}(x'),$$

for all $x \in \mathrm{GL}_N(F)$ and $x' \in G(F)$ such that x and $\iota \circ \xi^{-1}(x')$ lie in the same regular semisimple $\mathrm{GL}_N(\overline{F})$-conjugacy class. In general, the map $|\mathbf{LJ}|$ is neither injective nor surjective.

Write $\psi = \ell_1\psi_1 \oplus \ldots \oplus \ell_r\psi_r$, where the parameters $\psi_i \in \Psi(\mathrm{GL}(m_i)_F)$ are simple and pairwise non-isomorphic. If F is non-archimedean, let

$$\phi_i := \psi_i|_{W_F \times \mathrm{SL}_2(\mathbb{C}) \times 1} \quad \text{and} \quad \widehat{\phi}_i := \psi_i|_{W_F \times 1 \times \mathrm{SL}_2(\mathbb{C})} \quad \in \Phi(\mathrm{GL}(m_i, F)).$$

If F is archimedean, set $\phi_i = \widehat{\phi}_i := \psi_i|_{W_F} \in \Phi(\mathrm{GL}(m_i, F))$.

We say that a parameter $\phi \in \Phi(\mathrm{GL}(M))$ is d-relevant if any Levi subgroup $\mathbf{M}$ of $\mathrm{GL}_M(\mathbb{C})$ containing the image of ϕ is conjugate to a standard (block-diagonal) Levi subgroup $\mathrm{GL}_{m_1}(\mathbb{C})\times\ldots\times\mathrm{GL}_{m_k}(\mathbb{C})$ such that m_i is divisible by d for all $1 \leq i \leq k$ (in particular, we must have $d \mid M$). Hence, ϕ is d-relevant if and only if it is relevant for $\mathrm{GL}_m(D)$, where D is a central division F-algebra of degree d and $M = md$ (provided that such a division algebra exists).

Theorem 1.12. *If there exists some $i \in \{1, \ldots, r\}$ such that neither of ϕ_i and $\widehat{\phi}_i$ is d-relevant, then $\Pi(G, \xi)$ is empty. Otherwise (i.e. if, for all $i \in \{1, \ldots, r\}$, either ϕ_i or $\widehat{\phi}_i$ is d-relevant), π^*_ψ is d-compatible and $\Pi_\psi(G, \xi) = \{\pi_\psi\}$, where $\pi_\psi := |\mathbf{LJ}|(\pi^*_\psi)$.*

Proof. This is [**KMSW14**] Theorem 1.6.4. ☐

Remark. The packet $\Pi_\psi(G, \xi)$ is independent of the choice of map $E \to F$.

2.5. Unramified A-packets. Let E/F be an unramified quadratic extension of non-archimedean local fields and let $G := U_{E/F}(N)$ (thus, G is an unramified group over F). We take $(\xi, z) = (\mathrm{id}, 1)$ and assume that the character $\chi : W_E \to \mathbb{C}^\times$ is unramified. Let $K := U_{E/F}(N)(\mathcal{O}_F)$ be the "standard" hyperspecial maximal compact subgroup of $G(F)$.[2] We have the following result of Mok ([**Mok15**] Theorem 2.51(a) and Lemma 3.2.1):

Theorem 1.13. *Let (G, ξ, z) be as above and $\psi \in \Psi(G)$. Assume, in addition, that our fixed additive character $\psi_F : F \to \mathbb{C}^\times$ is trivial on $\mathcal{O}_F$, but not on any larger fractional ideal of $\mathcal{O}_F$.*

(i) If $\pi \in \Pi_\psi(G, \xi)$ has K-fixed vectors (i.e. $\pi^K \neq 0$), then $\langle \cdot, \pi\rangle = 1$.

(ii) For any $f \in \mathcal{H}(G(F))$, $f_N \in \mathcal{H}(\mathrm{GL}_N(E))$ with $\Delta[\mathfrak{e}, \xi, z]$-matching orbital integrals, we have

$$\sum_{\pi\in\Pi_\psi(G,\xi)} \langle s_\psi, \pi\rangle f(\pi) = \widetilde{f_N}(\eta^*_\chi\psi).$$

(iii) The map $f \mapsto \sum_{\pi\in\Pi_\psi(G,\xi)}\langle s_\psi, \pi\rangle f(\pi)$ is a stable *character $\mathcal{H}(G(F)) \to \mathbb{C}$.*

[2]Here, $U_{E/F}(N)$ denotes the reductive group scheme over $\mathcal{O}_F$ with functor of points

$$U_{E/F}(N)(R) = \{x \in \mathrm{GL}_N(\mathcal{O}_E \otimes_{\mathcal{O}_F} R) : (c \otimes \mathrm{id}_R)(x)^t\, J_N\, x = J_N\}.$$

NOTATION. (a) G is an endoscopic group of the twisted group $(G_{E/F}(N), \theta)$. Throughout this subsection, we write $\mathfrak{e}$ for the endoscopic triple $(G, 1, \eta_\chi)$, where η_χ is the L-homomorphism

$$^LG \to {}^LG_{E/F}(N) = (\mathrm{GL}_N(\mathbb{C}) \times \mathrm{GL}_N(\mathbb{C})) \rtimes W_F$$

defined by $g \rtimes w \mapsto (\chi(w)g, \chi(w)^{-1}J_N g^{-t} J_N^{-1})$ for all $(g,w) \in \widehat{G}(\mathbb{C}) \times W_E$, and $1 \rtimes w_c \mapsto (1, \kappa(\chi)1) \rtimes w_c$ (recall that w_c is a fixed choice of an element in $W_F \backslash W_E$ and $\kappa(\chi) := \chi(w_c^2)$ is the parity of χ; the $\widehat{G}_{E/F}(N)(\mathbb{C})$-conjugacy class of η_χ is independent of the choice of w_c).

(b) For the definition of transfer factors $\Delta[\mathfrak{e}, \xi, z](\gamma, \delta)$ and what it means for f, f_N to have $\Delta[\mathfrak{e}, \xi, z]$-matching orbital integrals, see [**KMSW14**] Section 1.1.2 and the papers of Kottwitz and Shelstad [**KS99**], [**KS12**].

(c) $s_\psi \in S_\psi$ denotes the element

$$s_\psi := \psi\left(1, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}\right)$$

and $f(\pi)$ denotes the trace of $\pi(f)$.

(d) $\widetilde{f_N} \in \mathcal{H}(\mathrm{GL}_N(E) \rtimes \langle\theta\rangle)$ is the function supported on $\mathrm{GL}_N(E) \rtimes \theta$ that satisfies $\widetilde{f_N}(g \rtimes \theta) = f_N(g)$ for all $g \in \mathrm{GL}_N(E)$. The local Langlands correspondence, associates to the L-parameter $\phi_{\eta_\chi^*\psi}$, a representation $\Pi_{\eta_\chi^*\psi}$ of $\mathrm{GL}_N(E)$. This representation admits an extension $\widetilde{\Pi}_{\eta_\chi^*\psi}$ to a representation of the twisted group $\mathrm{GL}_N(E) \rtimes \langle\theta\rangle$.[3] Then $\widetilde{f_N}(\eta_\chi^*\psi)$ is defined as $\widetilde{f_N}(\widetilde{\Pi}_{\eta_\chi^*\psi}) := \mathrm{tr}\, \widetilde{\Pi}_{\eta_\chi^*\psi}(\widetilde{f_N})$.

Let $K_N := \mathrm{GL}_N(\mathcal{O}_E)$, a θ-stable hyperspecial subgroup of $\mathrm{GL}_N(E)$. The map $\eta_\chi : {}^LG \to {}^LG_{E/F}(N)$ induces, via the Satake isomorphism, a map b between the (unramified) Hecke algebras of $G_{E/F}(N)$ and G:

$$b : \mathcal{H}(\mathrm{GL}_N(E); K_N) \to \mathcal{H}(G(F); K).$$

[3] There are two such extensions; we should choose the one which is compatible with the Whittaker datum $(B(N), \lambda)$ induced by the standard pinning on $\mathrm{GL}(N)$ and our fixed choice of additive character $\psi_F : F \to \mathbb{C}^\times$: see [**Mok15**] Section 3.2 for more details.

The map b gives rise to a "base change" map

$$B : \left\{ \begin{array}{c} \text{(isomorphism classes of)} \\ \text{irreducible admissible} \\ \text{representations } \pi \text{ of } G(F) \\ \text{such that } \pi^K \neq 0 \end{array} \right\} \to \left\{ \begin{array}{c} \text{(isomorphism classes of)} \\ \text{irreducible admissible} \\ \text{representations } \Pi \text{ of } \mathrm{GL}_N(E) \\ \text{such that } \Pi^{K_N} \neq 0 \end{array} \right\}$$

defined by $(B(\pi))(f_N) = \pi(b(f_N))$ for all $f_N \in \mathcal{H}(\mathrm{GL}_N(E); K_N)$.

Let $\widetilde{K}_N := K_N \rtimes \theta \subset \mathrm{GL}_N(E) \rtimes \langle\theta\rangle$. The following (twisted) version of the fundamental lemma is due to Lemaire, Mœglin and Waldspurger [**LMW15**]:

THEOREM 1.14. *For any $f \in \mathcal{H}(\mathrm{GL}_N(E); K_N)$, $f_N * 1_{\widetilde{K}_N}$ and $f := b(f_N)$ have matching $\Delta[\mathfrak{e}, \xi, z]$-matching orbital integrals.*

We use Theorems 1.13 and 1.14 to deduce the following lemma:

LEMMA 1.15. *(i) If $\Pi_\psi(G, \mathrm{id})$ contains a representation π of $G(F)$ such that $\pi^K \neq 0$, then ψ is unramified (i.e. $\psi|_{L_F}$ is unramified).*

(ii) If ψ is unramified, then $\Pi_\psi(G, \mathrm{id})$ contains a unique representation π such that $\pi^K \neq 0$ and π is such that $B(\pi)$ has L-parameter $\phi_{\eta_\chi^*\psi}$.

PROOF. Let $f_N \in \mathcal{H}(\mathrm{GL}_N(E); K_N)$, $f := b(f_N) \in \mathcal{H}(G(F); K)$. We have

$$\langle s_\psi, \pi\rangle f(\pi) = \begin{cases} f(\pi) = f_N(B(\pi)) & \text{if } \pi^K \neq 0 \\ 0 & \text{if } \pi^K = 0 \end{cases}$$

and $\widetilde{f_N}(\eta_\chi^*\psi) := \widetilde{f_N}(\widetilde{\Pi}_{\eta_\chi^*\psi}) = f_N(\Pi_{\eta_\chi^*\psi})$. Theorem 1.13 (ii) then implies that:

$$\sum_{\substack{\pi \in \Pi_\psi(G, \mathrm{id}) \\ \pi^K \neq 0}} f_N(B(\pi)) = f_N(\Pi_{\eta_\chi^*\psi}) \quad \text{for all } f_N \in \mathcal{H}(\mathrm{GL}_N(E); K_N).$$

Assume that $\Pi_\psi(G, \mathrm{id})$ contains a representation π with $\pi^K \neq 0$. Then the LHS is a non-zero linear map $\mathcal{H}(\mathrm{GL}_N(E); K_N) \to \mathbb{C}$, so $\Pi_{\eta_\chi^*\psi}$ must have K_N-fixed vectors. Hence, $\eta_\chi^*\psi$ is unramified, which implies that ψ is also unramified.

Conversely, assume that ψ is unramified. Then the identity above implies that there is a unique $\pi \in \Pi_\psi(G, \mathrm{id})$ with $\pi^K \neq 0$, and π satisfies $f_N(B(\pi)) = f_N(\Pi_{\eta_\chi^*\psi})$ for all $f_N \in \mathcal{H}(\mathrm{GL}_N(E); K_N)$, i.e. $B(\pi) = \Pi_{\eta_\chi^*\psi}$. ◻

PROPOSITION 1.16. *If ψ is unramified and generic, then all representations in $\Pi_\psi(G, \mathrm{id})$ are unramified.*

PROOF. This follows from the construction of the A-packet in this case; see [**Mok15**] Section 7.1. If N is odd, then the packet $\Pi_\psi(G, \mathrm{id})$ consists of a single representation. If N is even, then $\Pi_\psi(G, \mathrm{id})$ has order at most 2.

In both cases, there is a unique element π in the packet $\Pi_\psi(G, \mathrm{id})$ with K-fixed vectors. In the case where N is even, there are two $U_{E/F}(N)(F)$-conjugacy classes of hyperspecial subgroups of $U_{E/F}(N)$. Let K' be a hyperspecial subgroup which is not conjugate to K. If $\pi^{K'} = 0$, then $\Pi_\psi(G, \mathrm{id})$ contains two elements π and π', where π' has K'-fixed vectors and $(\pi')^K = 0$. □

2.6. Tempered archimedean A-packets. Let $F = \mathbb{R}$ and let $\phi \in \Phi(U_{E/F}(N))_{\mathrm{bdd}}$ be a generic A-parameter (where E/F is a quadratic field extension). In this case, the packet $\Pi_\phi(G, \xi)$ was constructed by Langlands and Shelstad.

NOTATION. σ is a choice of an $\mathbb{R}$-isomorphism $E \cong \mathbb{C}$ and $[\cdot] : \mathbb{C}^\times \to U(1)$ denotes the unitary character $z \mapsto z/|z|$. Also, $|z|_E = \sigma(z)\,\overline{\sigma(z)} = |\sigma(z)|^2$.

All representations in $\Pi_\phi(G, \xi)$ have the same infinitesimal character $\mu(\phi)$. One can recover $\mu(\phi)$ from the parameter ϕ as follows: The restriction

$$\phi|_{W_E} : W_E \to \mathrm{GL}_N(\mathbb{C})$$

of ϕ to $W_E = E^\times$ is a sum of characters

$$\eta_1 \oplus \ldots \oplus \eta_N,$$

where each $\eta_j : E^\times \to \mathbb{C}^\times$ is of the form $z \mapsto [\sigma(z)]^{2a_j}\, |z|_E^{i t_j}$ with $a_j \in \frac{1}{2}\mathbb{Z}$, $t_j \in \mathbb{R}$.

Then the infinitesimal character $\mu(\phi)$ corresponds to the unordered N-tuple

$$(a_1 + i\,t_1, \cdots, a_N + i\,t_N).$$

Note that $\mu(\phi)$ is also equal to the *unordered* N-tuple $(a_1 - i\,t_1, \cdots, a_N - i\,t_N)$, since $\phi|_{W_E}$ is a conjugate self-dual parameter.

NOTATION. We have

$$G_{\mathbb{C}} \xrightarrow[\xi]{\sim} U_{E/F}(N)_{\mathbb{C}} \xrightarrow[i_\sigma]{\sim} \mathrm{GL}(N, \mathbb{C})$$

(the second isomorphism is the one induced by σ). We can thus identify the infinitesimal character of an irreducible admissible representation π of $G(\mathbb{R})$ with an element of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}/W(T) = \mathbb{C}^N/S_N$, via the Harish-Chandra isomorphism.

Here, T denotes the maximal torus of $\mathrm{GL}(N, \mathbb{C})$ consisting of all diagonal matrices and $W(T)$ denotes its Weyl group.

3. Global classification

We now describe the endoscopic classification of automorphic representations of unitary groups. In this section, E/F denotes a quadratic extension of number fields.

3.1. Global parameters and localisation. Let G be a connected reductive group over a number field K. Define

$$G(\mathbb{A}_K)^1 := \{x \in G(\mathbb{A}_K) : \|\chi(x)\|_{\mathbb{A}_K} = 1 \text{ for all } \chi \in \mathrm{Hom}_K(G, \mathbb{G}_m)\}.$$

DEFINITION 1.17. *A unitary cuspidal automorphic representation of G is a unitary irreducible admissible representation of $G(\mathbb{A}_K)$ whose restriction to $G(\mathbb{A}_K)^1$ is a constituent of*

$$L^2_{\mathrm{cusp}}(G(K)\backslash G(\mathbb{A}_K)^1).$$

We write $\mathcal{A}_{\mathrm{cusp}}(G)$ for the set of equivalence classes of all unitary cuspidal automorphic representations of G.

The set of global parameters $\Psi(\mathrm{GL}(N, E))$ of $\mathrm{GL}(N, E)$ is defined to be the set of formal (unordered) sums

$$\psi = (\mu_1 \boxtimes \nu_1) \boxplus \cdots \boxplus (\mu_r \boxtimes \nu_r),$$

where, for each $1 \leq i \leq r$, $\mu_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}(m_i, E))$ and ν_i is an algebraic irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension n_i, such that $m_1 n_1 + \ldots + m_r n_r = N$.

For each place w of E, we have a localisation map

$$\Psi(\mathrm{GL}(N, E)) \to \Psi^+(\mathrm{GL}(N, E_v)), \quad \psi \mapsto \psi_w,$$

where

$$\psi_w := (\phi_{1,w} \boxtimes \nu_1) \oplus \cdots \oplus (\phi_{r,w} \boxtimes \nu_r)$$

and $\phi_{i,w} := \mathrm{rec}_{E_w}(\mu_{i,w})$ is the L-parameter of the w-component of μ_i. It is expected that the localisations ψ_w should always lie in the set $\Psi(\mathrm{GL}(N, E_w))$ of A-parameters (this is the Ramanujan conjecture for $\mathrm{GL}(N)$). Note that we can attach to each $\psi \in \Psi(\mathrm{GL}(N, E))$ an irreducible unitary representation

$$\Pi_\psi = \otimes_w \Pi_w$$

of $\mathrm{GL}_N(\mathbb{A}_E)$, where Π_w is the irreducible representation of $\mathrm{GL}_N(E_w)$ with L-parameter ϕ_{ψ_w} (for all places w of E).

For each sign $\kappa \in \{\pm 1\}$, fix a conjugate self-dual character $\chi_\kappa : W_E \to \mathbb{C}^\times$ of parity κ (the parity $\kappa(\chi)$ of a character χ is defined as in the local case: $\kappa(\chi) := \chi(w_c^2)$, where w_c is any element of $W_F \setminus W_E$). We define the set $\Psi_2(U_{E/F}(N), \eta_{\chi_\kappa})$ of global parameters of $U_{E/F}(N)$ that appear in the main global theorem of [**Mok15**] and [**KMSW14**]:

Definition 1.18. *$\Psi_2(U_{E/F}(N), \eta_{\chi_\kappa})$ is the set of all pairs $(\psi^N, \widetilde{\psi})$, where*

(i) ψ^N is an element of $\Psi(\mathrm{GL}(N,E))$, as above, such the $\mu_i \boxtimes \nu_i$ are pairwise distinct, and all the μ_i are conjugate self-dual of parity $\eta_i = \kappa\,(-1)^{N-n_i}$ (the parity η_i of μ_i is defined to be $\eta_i := 1$ if μ_i is conjugate orthogonal and $\eta_i := -1$ if μ_i is conjugate symplectic; cf. [**Mok15**] *Remark 2.5.3 or* [**KMSW14**] *§1.3.4).*

(ii) $\widetilde{\psi}$ is a $\widehat{U}_{E/F}(N)$-conjugacy class of L-homomorphisms

$$\mathcal{L}_{\psi^N} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L U_{E/F}(N)$$

such that $\widetilde{\psi}^N = \eta_{\chi_\kappa} \circ \widetilde{\psi}$. The group $\mathcal{L}_{\psi^N}$ is defined to be

$$\mathcal{L}_{\psi^N} := \Big(\prod_{i=1}^{r} \widehat{U}_{E/F}(m_i) \Big) \rtimes W_F;$$

we refer the reader to [**Mok15**] *§2.4 or* [**KMSW14**] *§1.3.4 for the definition of the map $\widetilde{\psi}^N : \mathcal{L}_{\psi^N} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G_{E/F}(N)$.*

Remark. The natural analogue of Lemma 1.8 holds. In particular, if $\widetilde{\psi}$ exists, it is uniquely determined by ψ^N.

Let v be a place of F which splits as ww^c in E. The localisation ψ_v of a parameter $\psi \in \Psi_2(U_{E/F}(N), \eta_{\chi_\kappa})$ is defined as the composition

$$L_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow[\psi_w^N]{} {}^L\mathrm{GL}(N, E_w) \xrightarrow[{}^L i_w]{\sim} {}^L U_{E_v/F_v}(N),$$

where $E_v := E \otimes_F F_v \cong E_w \times E_{w^c}$ and i_w is the isomorphism corresponding to the projection $E_v \to E_w$; ψ_v does not depend on the choice of the place w above v. In the case of non-split places, we have the following result of Mok:

Proposition 1.19 (Second seed theorem). *Let $\psi = (\psi^N, \widetilde{\psi}) \in \Psi_2(U_{E/F}(N), \eta_{\chi_\kappa})$ and let v be a place of F which does not split in E. Then the localisation $\psi_v^N \in \Psi^+(\mathrm{GL}(N, E_v))$ lies in the image $\eta_{\chi_\kappa}^*$. We define $\psi_v := \eta_{\chi_\kappa}^{*-1}(\psi_v^N)$ to be the preimage of ψ_v^N under the (injective) map $\eta_{\chi_\kappa}^*$.*

We have thus defined a localisation map

$$\Psi_2(U_{E/F}(N), \eta_{\chi_\kappa}) \to \Psi^+(U_{E_v/F_v}(N)), \quad \psi \mapsto \psi_v$$

for parameters of unitary groups. We also get natural maps

$$S_\psi \to S_{\psi_v}, \qquad S_\psi^\natural \to S_{\psi_v}^\natural,$$

where $S_\psi := \mathrm{Cent}(\mathrm{Im}\,\widetilde{\psi}, \widehat{U}_{E/F}(N)(\mathbb{C}))$, $S_\psi^\natural := S_\psi / (S_\psi \cap \widehat{U}_{E/F}(N)_{\mathrm{der}}(\mathbb{C}))^\circ$.

3.2. The main global theorem. Let G be an inner form of $U_{E/F}(N)$ and let $(\xi, z) : U_{E/F} \to G$ be an extended pure inner twist. The main global theorem will be independent of the choice of cocycle z. We define the global A-packet: Let

$$\Pi_\psi(G,\xi) := \left\{ \bigotimes_v \pi_v : \pi_v \in \Pi_{\psi_v}(G_{F_v}, \xi_v),\ \langle \cdot\,, \pi_v\rangle_{\xi_v, z_v} = 1 \text{ for almost all } v \right\}$$

and let $\Pi_\psi(G,\xi) \to \mathrm{Irr}(S_\psi^\natural)$, $\pi \mapsto \langle \cdot\,, \pi\rangle_\xi$ be given by

$$\langle s, \pi\rangle_\xi = \prod_v \langle s, \pi_v\rangle_{\xi_v, z_v}.$$

The map is independent of the choice of z and its pullback via

$$Z(\widehat{G}(\mathbb{C}))^{\mathrm{Gal}(\overline{F}/F)} \to S_\psi^\natural$$

is the trivial character. Let $\overline{\mathcal{S}}_\psi$ be the cokernel of $Z(\widehat{G}(\mathbb{C}))^{\mathrm{Gal}(\overline{F}/F)} \to S_\psi^\natural$. Let

$$\epsilon_\psi : \overline{\mathcal{S}}_\psi \to \{\pm 1\}$$

be the character defined in [**Mok15**] §2.5; this character is trivial if ψ is generic (i.e. if all the ν_i are trivial).

Finally, we define the automorphic A-packet of ψ to be

$$\Pi_\psi(G, \xi, \epsilon_\psi) := \{\pi \in \Pi_\psi(G,\xi) : \langle \cdot\,, \pi\rangle_\xi = \epsilon_\psi\}.$$

We can now state the main global theorem of endoscopic classification of representations of unitary groups:

THEOREM 1.20. *We have a decomposition of $G(\mathbb{A}_F)$-modules*

$$L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}_F)) = \bigoplus_{\psi \in \Psi_2(U_{E/F}(N), \eta_{\chi_\kappa})} \ \bigoplus_{\pi \in \Pi_\psi(G,\xi,\epsilon_\psi)} \pi$$

(for any choice of conjugate self-dual character χ_κ).

PROOF. This is [**Mok15**] Theorem 1.5.2, [**KMSW14**] Theorem 1.7.1. □

REMARKS. (i) The theorem in the general case (i.e. when G is not quasi-split) is still conditional on certain hypotheses, which are expected to be resolved in future papers. See the discussion below Theorem 1.7.1 in [**KMSW14**].

(ii) If π is an everywhere tempered automorphic representation in the discrete spectrum of $G(\mathbb{A}_F)$, then π occurs in the packet $\Pi_\psi(G, \xi, \epsilon_\psi)$ of some generic parameter $\psi \in \Psi_2(U_{E/F}(N), \eta_{\chi_\kappa})$. Conversely, a proof of the Ramanujan conjecture would imply that all representations in the A-packet of a generic parameter are everywhere tempered.

(iii) One should define local A-packets $\Pi_{\psi_v}(G_{F_v}, \xi_v)$, $\pi_v \mapsto \langle \cdot , \pi_v \rangle_{\xi_v, z_v}$ for parameters $\psi_v \in \Psi^+(G) \setminus \Psi(G)$ to account for the possible failure of the Ramanujan conjecture. This is done in [**KMSW14**] §1.6.4. The representations in the packet $\Pi_{\psi_v}(G_{F_v}, \xi_v)$ of a parameter $\psi_v \notin \Psi(G)$ need not be irreducible or unitary; the packet is empty if ψ_v is not G_{F_v}-relevant.

CHAPTER 2

Automorphic representations on definite unitary groups

In this chapter, we study automorphic forms on totally definite unitary groups and their associated Galois representations. More precisely, we use the theory of endoscopic classification (summarised in the previous chapter) to attach Galois representations to automorphic representations of a totally definite unitary group G and to prove descent theorems allowing us to transfer conjugate self-dual automorphic representations of $\mathrm{GL}(N)$ over a CM number field to automorphic representations of G. We end the chapter by defining spaces of automorphic forms on G with coefficients in a $\mathbb{Z}_\ell$-algebra A (e.g. $\overline{\mathbb{Z}}_\ell$ or $\overline{\mathbb{F}}_\ell$).

1. Galois representations

Let E be a CM number field and let F be the maximal totally real subfield of E. We write c for the non-trivial element of $\mathrm{Gal}(E/F)$. Let G denote an inner form of $U_{E/F}(N)$ such that $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact (i.e. $G_{\mathbb{R}} \cong U(N)$ for every embedding $F \hookrightarrow \mathbb{R}$), and fix an inner twist

$$\xi : U_{E/F}(N)_{\overline{F}} \xrightarrow{\sim} G_{\overline{F}}$$

(i.e. an isomorphism ξ of $\overline{F}$-groups such that $\xi^{-1} \circ {}^{\sigma}\xi$ is an *inner* automorphism of $U_{E/F}(N)_{\overline{F}}$ for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$).

We recall the following theorem, which associates a Galois representation to regular algebraic, conjugate self-dual cuspidal automorphic representations:

Theorem 2.1. *Let* Π *be a cuspidal automorphic representaton of* $\mathrm{GL}_n(\mathbb{A}_E)$*, such that*

(a) Π *is conjugate self-dual:* $\Pi^c \cong \Pi^\vee$.

(b) For some integer k*,* $\Pi(\frac{k}{2}) := \Pi\|\det\|^{\frac{k}{2}}$ *is regular* L*-algebraic: For every infinite place* w *of* E*,* $\mathrm{rec}_{E_w}(\Pi(\frac{k}{2})_w)$ *is a direct sum of pairwise distinct algebraic characters of* $E_w^\times$.

Then

(i) Π *is everywhere tempered.*

(ii) For any prime number ℓ and any field isomorphism $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, there exists a continuous semisimple ℓ-adic Galois representation

$$\rho_\iota(\Pi(\tfrac{k}{2})) : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

characterised by

$$\mathrm{WD}(\rho_\iota(\Pi(\tfrac{k}{2}))|_{\mathrm{Gal}(\overline{E}_w/E_w)})^{\mathrm{F-ss}} \cong \iota^{-1}\mathrm{rec}_{E_w}(\Pi(\tfrac{k}{2})_w)$$

for all finite places w of E, not dividing ℓ, where $\mathrm{WD}(\sigma)^{\mathrm{F-ss}}$ denotes the Frobenious semisimplification of the Weil-Deligne representation associated to the representation $\sigma : \mathrm{Gal}(\overline{E}_w/E_w) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$.

Proof. This theorem is due to many people: Kottwitz [**Kot92**], Clozel [**Clo91**], Harris–Taylor [**HT01**], Taylor–Yoshida [**TY07**], Clozel–Harris–Labesse [**CHL11**], Shin [**Shi11**], Chenevier–Harris [**CH13**], Caraiani [**Car12**]. □

Notation. If Π_1 (resp. Π_2) is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_E)$ (resp. of $\mathrm{GL}_m(\mathbb{A}_E)$), we write $\Pi_1 \boxplus \Pi_2$ for the representation of $\mathrm{GL}_{n+m}(\mathbb{A}_E)$ satisfying $\mathrm{rec}_{E_w}((\Pi_1 \boxplus \Pi_2)_w) = \mathrm{rec}_{E_w}(\Pi_{1,w}) \oplus \mathrm{rec}_{E_w}(\Pi_{2,w})$ for all places w of E.

The main global theorem of endoscopic classification (Theorem 1.20) implies the following theorem:

Theorem 2.2. *Let π be an irreducible subrepresentation of*

$$L^2(G(F)\backslash G(\mathbb{A}_F)).$$

There exist pairwise distinct automorphic representations $\Pi_1, \dots, \Pi_r$ such that

(i) Each Π_i is an automorphic representation of $\mathrm{GL}_{m_i n_i}(\mathbb{A}_E)$ of the form

$$\Pi_i = \mu_i|\det|^{\frac{n_i-1}{2}} \boxplus \mu_i|\det|^{\frac{n_i-3}{2}} \boxplus \cdots \boxplus \mu_i|\det|^{\frac{1-n_i}{2}},$$

where μ_i is a unitary, conjugate self-dual, cuspidal automorphic representation of $\mathrm{GL}_{m_i}(\mathbb{A}_E)$ and $m_1n_1 + \ldots + m_rn_r = N$.

(ii) For all $i \in \{1, \dots, r\}$ and every complex place w of E,

$$\mu_{i,w}\ |\det|_w^{\frac{n_i-N}{2}}$$

is regular L-algebraic.

(iii) $\Pi := \Pi_1 \boxplus \cdots \boxplus \Pi_r$ is the base change of π, i.e.

(a) *For every finite place v of F which splits as ww^c in E,*

$$|\mathbf{LJ}|_{i_w \circ \xi^{-1}}(\Pi_w) = \pi_v \quad \textit{and} \quad \Pi_{w^c} = \Pi_w^\vee.$$

(b) *For every finite place w of E which is non-split and unramified in E and such that G_{F_v} is quasi-split and π_v is unramified,*

$$\Pi_v = B_\xi(\pi_v),$$

where B_ξ is the base change map induced by $\eta_1 \circ {}^L\xi$ via the Satake isomorphism, similarly to Chapter 1, Section 2.5.

Proof. Fix a basic cocycle z corresponding to $\xi^{-1} \circ {}^\sigma\xi \in Z^1(F, U_{E/F}(N)_{\mathrm{ad}})$. By the main global theorem (with $\chi_\kappa = 1$), we have $\pi \in \Pi_\psi(G, \xi)$, for some global parameter $\psi = (\psi^N, \widetilde{\psi}) \in \Psi_2(U_{E/F}(N), \eta_1)$. Let Π be the automorphic representation of $\mathrm{GL}_N(\mathbb{A}_E)$ corresponding to ψ^N. We have

$$\psi^N = \psi_1 \boxplus \cdots \boxplus \psi_r,$$

where $\psi_i = \mu_i \boxtimes \nu_i$ are pairwise distinct. Let $m_i = \dim \mu_i$, $n_i = \dim \nu_i$. By definition, $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$, where the Π_i are as in (i).

Let w be an infinite place of E. Let ψ_v be the localisation of ψ at the place $v = w|_F$. We have $\psi_v^N = \eta_1^* \psi_v$, where $\psi_v^N \in \widetilde{\Psi}^+(N)$ is the localisation of ψ^N. Since G_{F_v} is isomorphic to the compact unitary group, ${}^L U_{E_v/F_v}(N)$ has no proper G_{F_v}-relevant parabolic subgroups. We conclude (cf. [**KMSW14**] §1.6.4) that $\psi_v \in \Psi(U_{E_v/F_v}(N))$. We can write

$$\psi_v^N = \bigoplus_{i \in I^+} \ell_i \psi_i^{(N)} \oplus \bigoplus_{i \in I^-} \ell_i \psi_i^{(N)} \oplus \bigoplus_{j \in J} \ell_j (\psi_j^{(N)} \otimes \psi_j^{(N)*}),$$

where the $\psi_i^{(N)} := \chi_i \boxtimes \nu_i^{(N)}$ are simple (in particular, the χ_i are characters). The set I^+ (resp. I^-) consists of all indices i such that χ_i is conjugate self-dual and $\chi_i(-1) = (-1)^{N - \dim \nu_i^{(N)}}$ (resp. $(-1)^{N - \dim \nu_i^{(N)} + 1}$). Then, by Lemma 1.10,

$$S_{\psi_v} \cong \prod_{i \in I^+} \mathrm{O}_{\ell_i}(\mathbb{C}) \times \prod_{i \in I^-} \mathrm{Sp}_{\ell_i}(\mathbb{C}) \times \prod_{j \in J} \mathrm{GL}_{\ell_j}(\mathbb{C}),$$

and all $(\ell_i : i \in I^-)$ are even. We use Lemma 1.7 to deduce that $S_{\psi_v}^\circ$ is a torus and that $\mathrm{Cent}(S_{\psi_v}^\circ, {}^L U_{E_v/F_v}(N)) = {}^L U_{E_v/F_v}(N)$. Equivalently:

$$S_{\psi_v}^\circ \subset Z({}^L U_{E_v/F_v}(N)) \cap \widehat{U}_{E_v/F_v}(N) = Z(\widehat{U}_{E_v/F_v}(N))^{\mathrm{Gal}(\overline{F}/F)} = \{\pm 1\}.$$

Hence, $\ell_i = 1$ for all $i \in I^+$ and $I^- = J = \varnothing$. Thus, part (ii) follows, as this shows that, for each $1 \leq i \leq r$, $\mathrm{rec}_{E_w}(\mu_{i,w})$ is a direct sum of distinct conjugate-self dual characters of $E_w^\times \cong \mathbb{C}^\times$, each having parity $(-1)^{N-n_i}$. Moreover, we can conclude that the μ_i are everywhere tempered (Theorem 2.1 (i)), i.e. $\psi_v \in \Psi(U_{E_v/F_v}(N))$ for all places v.

The remaining parts follow from the description of the local A-packets in the split and unramified case (cf. Chapter 1, Sections 2.4 and 2.5). □

Suppose that π is an irreducible subrepresentation of $L^2(G(F)\backslash G(\mathbb{A}_F))$, as above. Let v be a place of F and w be a place of E diving v. We define an irreducible admissible representation $\mathbf{BC}_w(\pi_v)$ of $\mathrm{GL}_N(E_w)$, in the following two cases:

(i) $\mathbf{BC}_w(\pi_v) := |\mathbf{LJ}|^{-1}_{i_w \circ \xi^{-1}}(\pi_v)$, if v splits in E.

(ii) $\mathbf{BC}_w(\pi_v) := B_\xi(\pi_v)$, if v is inert in E, G_{F_v} is quasi-split and π_v is an unramified representation of $G(F_v)$.

Remark. Let K be a nonarchimedean local field and let D be a central division K-algebra of dimension d^2. $|\mathbf{LJ}|$ restricts to a bijection between the set of isomorphism classes of tempered representations of $\mathrm{GL}_{rd}(K)$ which are $\mathrm{GL}_r(D)$-relevant and the set of isomorphism classes of tempered representations of $\mathrm{GL}_r(D)$.

We can now attach a Galois representation to any discrete automorphic representation π of G. More precisely, we have the following corollary:

Corollary 2.3. *Let π be an irreducible subrepresentation of $L^2(G(F)\backslash G(\mathbb{A}_F))$ and fix a field isomorphism $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. Then there exists a continuous semisimple representation*

$$r_\iota(\pi) : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$$

(which also depends on ξ) such that

$$r_\iota(\pi)^{\vee,c} \cong r_\iota(\pi) \otimes \varepsilon_\ell^{N-1}$$

and

$$\mathrm{WD}\big(r_\iota(\pi)|_{\mathrm{Gal}(\overline{E}_w/E_w)}\big)^{\mathrm{F-ss}} \cong \iota^{-1}\mathrm{rec}_{E_w}\big(\mathbf{BC}_w(\pi_v)|\det|^{\frac{1-N}{2}}\big)$$

for every finite place $w \nmid \ell$ of E (lying over the place v of F) satisfying one of the following two conditions:

(a) w is split over F;

(b) w is inert over F, G_{F_v} is quasi-split and π_v is an unramified representation of $G(F_v)$.

Moreover, π is everywhere tempered if and only if there are no subrepresentations r_1 and r_2 of $r_\iota(\pi)$ such that $r_1 \cong r_2 \otimes \varepsilon_\ell$.

NOTATION. $\varepsilon_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^\times$ denotes the ℓ-adic cyclotomic character.

PROOF. We define $r_\iota(\pi)$ to be

$$r_\iota(\pi) := \bigoplus_{i=1}^{r} \bigoplus_{k=1}^{n_i} \rho_\iota(\mu_i \| \det \|^{\frac{n_i - N}{2}}) \otimes \varepsilon_\ell^{1-k},$$

where the notation is as in theorems 2.1 and 2.2. This satisfies the desired local-global compatibility. Assume that there are no subrepresentations r_1 and r_2 of $r_\iota(\pi)$ such that $r_1 \cong r_2 \otimes \varepsilon_\ell$; then $n_1 = \cdots = n_r = 1$ and, hence, π is everywhere tempered. Conversely, suppose that there are subrepresentations r_1 and r_2 of $r_\iota(\pi)$ such that $r_1 \cong r_2 \otimes \varepsilon_\ell$. We may assume that r_1 and r_2 are irreducible; then

$$r_\iota(\pi) \cong r_1 \oplus (r_1 \otimes \varepsilon_\ell^{-1}) \oplus r'.$$

Let v be a place of F which splits in E and is such that G_{F_v} is quasi-split (i.e. isomorphic to $\mathrm{GL}(N)$), and let w be a place of E above v. Then

$$\mathrm{rec}_{E_w}(\pi_v \circ i_w^{-1}) \otimes |\mathrm{Art}_{E_w}^{-1}|^{\frac{1-N}{2}} \cong \phi_1 \oplus (\phi_1 \otimes |\mathrm{Art}_{E_w}^{-1}|^{-1}) \oplus \phi',$$

where $\phi_1 := \iota\mathrm{WD}(r_1|_{\mathrm{Gal}(\overline{E}_w/E_w)})^{\mathrm{F-ss}}$ and $\phi' := \iota\mathrm{WD}(r'|_{\mathrm{Gal}(\overline{E}_w/E_w)})^{\mathrm{F-ss}}$. If π_v is tempered, then $\mathrm{rec}_{E_w}(\pi_v \circ i_w^{-1})$ is bounded and, hence, $\phi_1 \otimes |\mathrm{Art}_{E_w}^{-1}|^{\frac{N-1}{2}}$ and $\phi_1 \otimes |\mathrm{Art}_{E_w}^{-1}|^{\frac{N-3}{2}}$ are both bounded. This is not possible, so the last part follows. □

2. Descent to a definite unitary group

Let Π_1, Π_2 be unitary, conjugate self-dual, cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A}_E)$, and assume that

$$\Pi = \Pi_1 \boxplus \Pi_2$$

is a regular C-algebraic[4] representation of $\mathrm{GL}_N(\mathbb{A}_E)$ (where $N = 2m$); in particular, Π_1 and Π_2 are non-isomorphic. We remark that there is a Galois representation

$$r_\iota(\Pi) : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$$

[4]This means that $\Pi \| \det \|^{\frac{1-N}{2}}$ is regular L-algebraic.

associated to Π (where ι is a field isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$); in the notation of Theorem 2.1, $r_\iota(\Pi)$ is equal to

$$\bigoplus_{i\in\{1,2\}} \rho\big(\Pi_i \,\|\det\|^{\frac{1-N}{2}}\big)$$

and, thus,

$$\mathrm{WD}\big(r_\iota(\Pi)|_{\mathrm{Gal}(\overline{E}_w/E_w)}\big)^{\mathrm{F-ss}} \cong \iota^{-1}\mathrm{rec}_{E_w}\big(\Pi_w|\det|^{\frac{1-N}{2}}\big)$$

for all finite place $w \nmid \ell$ of E.

Let us, moreover, assume that there is a place $\mathfrak{P}$ of E such that, for each $i \in \{1,2\}$, $\Pi_{i,\mathfrak{P}}$ is a twist of the Steinberg representation St_m of $\mathrm{GL}_m(E_\mathfrak{P})$ by an unramified character $\xi_i : E_\mathfrak{P}^\times \to \mathbb{C}^\times$.

Lemma 2.4. *There is a biquadratic extension F_1/F of totally real number fields, an inner twist (G,ξ) of $U_{E_1/F_1}(N)$ (where $E_1 = E\cdot F_1$), and an irreducible subrepresentation π of $L^2(G(F_1)\backslash G(\mathbb{A}_{F_1}))$ such that:*

- *(i) E_1/F_1 is unramified at all finite places. If Π_w is ramified at a place w of E, then any place w_1 of E_1 above w is split over F_1.*
- *(ii) $G(F_1\otimes_\mathbb{Q}\mathbb{R})$ is compact, i.e. G is a (totally) definite unitary group.*
- *(iii) For any place $\mathfrak{p}_1$ of F_1 above $\mathfrak{p} := \mathfrak{P}\cap F$, the group $G_{F_{\mathfrak{p}_1}}$ is isomorphic to $\mathrm{GL}(2,D_{\mathfrak{p}_1})$ and $\pi_{\mathfrak{p}_1}$ has $\mathrm{GL}_2(\mathcal{O}_{D_{\mathfrak{p}_1}})$-fixed vectors, where $D_{\mathfrak{p}_1}$ is a central division $F_{1,\mathfrak{p}_1}$-algebra with integer ring $\mathcal{O}_{D_{\mathfrak{p}_1}}$.*
- *(iv) $G_{F_{v_1}}$ is quasi-split for all finite places v_1 of F_1 prime to $\mathfrak{p}$.*
- *(v) $r_\iota(\pi) \cong r_\iota(\Pi)|_{\mathrm{Gal}(\overline{E_1}/E_1)}$.*

Moreover, given a finite Galois extension K/E, we can choose F_1/F such that the map $\mathrm{Gal}(K\cdot E_1/E_1) \to \mathrm{Gal}(K/E)$ is an isomorphism (and, in particular, $K\cap E_1 = E$).

Proof. Write $E = F(\sqrt{\alpha})$, where $\alpha \in F^\times \setminus (F^\times)^2$. Let S be a finite set of non-archimedean places of F containing $\mathfrak{p}$, all places that ramify in E, and all places v such that Π_w is ramified for some place $w \mid v$ of E. Moreover, let T be a set of places of F, disjoint from S, such that each place $w \mid T$ of E is unramified in K and

$$\mathrm{Gal}(K/E) = \bigcup_{w|T} \mathrm{Frob}_w,$$

where Frob_w denotes the Frobenius conjugacy class in $\mathrm{Gal}(K/E)$ at the place w. There exists (by weak approximation) an element $\beta \in F^\times \setminus (F^\times)^2$ such that

(i) $\beta \in \alpha(F_v^\times)^2$ for all $v \in S$

(ii) $\beta \in (F_v^\times)^2$ for all $v \in T$, and

(iii) $\sigma(\beta) \in \mathbb{R}_{>0}$ for all real embeddings $\sigma : F \hookrightarrow \mathbb{R}$.

Let $F_0 := F(\sqrt{\beta})$:

(a) F_0 is a totally real quadratic extension of F.

(b) If v_0 is a place of F_0 above S, then v_0 splits in $E_0 = E \cdot F_0$.

(c) If v_0 is a place of F_0 not dividing S, then v_0 is unramified in E_0 (as $v_0|_F$ is unramified in E). Hence, E_0/F_0 is everywere unramified.

(d) Any place $v \in T$ splits in F_0. Hence, any place $w \mid T$ of E splits in E_0.

There exists an inner twist (G_0, ξ_0) of $U_{E_0/F_0}(N)$ (cf. Chapter 1, Section 1.2) such that

(a) $G(F_0 \otimes_{\mathbb{Q}} \mathbb{R})$ is compact (equivalently, $G_{F_{0,v_0}}$ for any real place of v_0 of F_0),

(b) $G_{F_{0,\mathfrak{p}_0}}$ is isomorphic to $\mathrm{GL}(2, D_{\mathfrak{p}_0})$ for any place $\mathfrak{p}_0 \mid \mathfrak{p}$ of F_0, where $D_{\mathfrak{p}_0}$ is a central division $F_{\mathfrak{p}_0}$-algebra, and

(c) $G_{F_{0,v_0}}$ is quasi-split for any finite place $v_0 \nmid \mathfrak{p}$ of F_0.

Fix a basic cocycle z_0 corresponding to $\xi_0^{-1} \circ {}^\sigma\xi_0 \in Z^1(F_0, U_{E_0/F_0}(N)_{\mathrm{ad}})$. We write Σ for the set of finite places v_0 of F_0 such that the image of z_0 under the localisation map $B(F_0, G_0)_{\mathrm{bsc}} \to B(F_{0,v_0}, G_0)_{\mathrm{bsc}}$ is non-trivial. Let $\Sigma_{\mathbb{Q}}$ be a finite set of rational primes containing all primes lying below Σ and all primes lying below $T \cup \mathfrak{p}$. Let $d > 1$ be a square-free integer such that every prime in $\Sigma_{\mathbb{Q}}$ splits in $\mathbb{Q}(\sqrt{d})$. Set

$$F_1 := F_0(\sqrt{d}) = F(\sqrt{\beta}, \sqrt{d}), \qquad E_1 := E \cdot F_1.$$

Thus,

(a) F_1 is a totally real biquadratic extension of F.

(b) If v_1 is a place of F_1 above S, then v_1 splits in E_1.

(c) E_1/F_1 is everywere unramified.

(d) If v_0 is a place of F_0 lying over $\Sigma_{\mathbb{Q}}$, then v_0 splits in F_1.

(e) Any place $v \in T$ splits completely in F_1. Hence, any place $w \mid T$ of E splits completely in E_1; in particular, $\mathrm{Gal}(K \cdot E_1/E_1) \to \mathrm{Gal}(K/E)$ is surjective.

Let (G, ξ, z) be the base change of the extended pure inner twist (G_0, ξ_0, z_0) to F_1. Moreover, let $\Pi_b = \Pi_{1,b} \boxplus \Pi_{2,b}$ be the base change of $\Pi = \Pi_1 \boxplus \Pi_2$ to E_1:

Definition 2.5. *A representation* Π *of* $\mathrm{GL}_N(\mathbb{A}_E)$ *is called* isobaric *if*

$$\Pi \cong \Pi_1 \boxplus \cdots \boxplus \Pi_r,$$

for some integer $r \geq 1$ *and some cuspidal automorphic representations* Π_i *of* $\mathrm{GL}_{m_i}(\mathbb{A}_E)$ $(i = 1, \cdots, r)$ *with* $m_1 + \cdots + m_r = N$.

Let Π *be an isobaric representation of* $\mathrm{GL}_N(\mathbb{A}_E)$ *and let* Π_b *be an isobaric representation of* $\mathrm{GL}_N(\mathbb{A}_{E_1})$, *where* E_1/E *is a Galois extension of number fields. We say that* Π_b *is the* base change *of* Π *to* E_1 *if*

$$\mathrm{rec}_{E_{1,w_1}}(\Pi_{b,w_1}) \cong \mathrm{rec}_{E_w}(\Pi_w)|_{L_{E_{1,w_1}}}$$

for every place w_1 *of* E_1 *lying over a place* w *of* E.

THEOREM 2.6. *Let* Π *be an isobaric representation of* $\mathrm{GL}_N(\mathbb{A}_E)$ *and let* E_1/E *be a* solvable *Galois extension of number fields. Then the base change* Π_b *of* Π *to* E_1 *exists. If* Π *is an isobaric sum of unitary cuspidal representations, then so is* Π_b.

PROOF. The theorem is due to Arthur and Clozel. It suffices to prove the theorem when Π is a unitary cuspidal representation and E_1/E is a cyclic extension of prime degree: This is [**AC89**] Chapter 3, Theorem 4.2 (weak lifting) and Theorem 5.1 (strong lifting). The "strong lifting" of Arthur-Clozel corresponds to restriction of local L-parameters under the local Langlands correspondence (cf. [**HT01**] Lemma VII.2.6 and [**AC89**] Chapter 1, Section 6). Indeed, let L/K be a cyclic extension of non-archimedean local fields of characteristic 0 (of prime degree $[L : K] = \ell$), let π be a smooth irreducible representation of $\mathrm{GL}_N(K)$, and let π_L be its strong base change to L, in the sense of Arthur-Clozel. We want to see that

$$\mathrm{rec}_K(\pi)|_{L_L} = \mathrm{rec}_L(\pi_L).$$

If π is supercuspidal, this is Lemma VII.2.6, part (5), of [**HT01**].

If π is square-integrable, then $\pi = \mathrm{St}(\omega, a)$ for some unitary cuspidal representation ω of $\mathrm{GL}_{N/s}(K)$, where $\mathrm{St}(\omega, a)$ denotes the generalised Steinberg representation, i.e. the unique quotient of the normalised parabolic induction of

$$\omega|\det|^{\frac{1-a}{2}} \boxtimes \cdots \boxtimes \omega|\det|^{\frac{a-1}{2}}$$

(in the notation of [**HT01**], $\mathrm{St}(\omega, a) = \mathrm{Sp}_a(\omega|\det|^{\frac{1-a}{2}})$), and we have $\mathrm{rec}_K(\mathrm{St}(\omega, a)) = \mathrm{rec}_K(\omega) \boxtimes \nu_a$, where ν_a is the a-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. We write Ω for the base change lift of ω (so $\mathrm{rec}_K(\omega)|_{L_L} = \mathrm{rec}_L(\Omega)$). By Lemma 6.10 and Lemma 6.12 of [**AC89**], we have the following two possibilities:

(i) Ω is cuspidal, and $\pi_L = \mathrm{St}(\Omega, a)$.

(ii) $\Omega = \Gamma \times \cdots \times \Gamma^{\sigma^{\ell-1}}$, where σ is a generator of $\mathrm{Gal}(L/K)$, with $\Gamma \not\cong \Gamma^\sigma$ cuspidal, and

$$\pi_L = \mathrm{St}(\Omega, a) = \mathrm{St}(\Gamma, a) \times \cdots \times \mathrm{St}(\Gamma^{\sigma^{\ell-1}}, a).$$

We thus have $\mathrm{rec}_L(\Omega) = \bigoplus_{i=0}^{\ell-1} \mathrm{rec}_L(\Gamma^{\sigma^i})$ and $\mathrm{rec}_L(\pi_L) = \bigoplus_{i=0}^{\ell-1} \mathrm{rec}_L(\Gamma^{\sigma^i}) \boxtimes \nu_a = \mathrm{rec}_K(\Omega) \boxtimes \nu_a$.

In both cases, we have $\mathrm{rec}_K(\pi)|_{L_L} = \mathrm{rec}_L(\pi_L)$.

The case where π is (essentially) tempered follows from the discussion in [**AC89**] Section 6.2 (and compatibility of strong lifting and rec_K with twists).

In general, we can write π as the unique quotient of the normalised parabolic induction of an essentially tempered and dominant representation

$$\pi_M = \pi_1 \boxtimes \cdots \boxtimes \pi_r$$

of a Levi subgroup $M(F) = \mathrm{GL}_{m_1}(F) \times \cdots \times \mathrm{GL}_{m_r}(F)$, and $\mathrm{rec}_K(\pi) = \bigoplus_{i=1}^{r} \mathrm{rec}_K(\pi_i)$. The strong base change lifting

$$\pi'_M = \pi'_1 \boxtimes \cdots \boxtimes \pi'_r$$

of π_M is also essentially tempered and dominant, and π_L is the unique quotient of the normalised parabolic induction of π'_M to $\mathrm{GL}_N(E)$. We have $\mathrm{rec}_K(\pi_i)|_{L_L} = \mathrm{rec}_L(\pi'_i)$ for all $1 \le i \le r$, so the relation $\mathrm{rec}_K(\pi)|_{L_L} = \mathrm{rec}_L(\pi_L)$ holds in general. $\square$

Remark. Let K_1/K be a Galois extension of local non-archimedean fields of characteristic 0. Then, for any character $\xi : K^\times \to \mathbb{C}^\times$, we have

$$\mathrm{rec}_K(\mathrm{St}_m \otimes \xi(\det))|_{L_{K_1}} \cong \mathrm{rec}_{K_1}(\mathrm{St}_m \otimes (\xi \circ \mathbb{N}_{K_1/K})(\det)).$$

In particular, if Π is a cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_E)$ and w is a finite place of the number field E such that Π_w is an unramified twist of the Steinberg representation of $\mathrm{GL}_m(E_w)$, then the same is true for the base change Π_b of Π to E_1 (if it exists) and any place w_1 of E over w (for any finite Galois extension E_1/E).

Let $\psi^N = \psi_1^N \boxplus \psi_2^N \in \Psi(\mathrm{GL}(N, E_1))$, where $\psi_i^N \in \Psi(\mathrm{GL}(m, E_1))$ is the formal global parameter corresponding to $\Pi_{i,b}$. We compute the parity of $\Pi_{i,b}$:

Let w_1 be an archimedean place of E_1 and fix an $\mathbb{R}$-isomorphism $\sigma : E_{1,W_1} \cong \mathbb{C}$. Since Π_b is regular C-algebraic and everywhere tempered, $\psi_{i,w_1}^N = \mathrm{rec}_{E_{1,w_1}}(\Pi_{i,b,w_1})$ is of the form

$$\eta_1 \oplus \cdots \oplus \eta_m,$$

where $\eta_j : E_{1,w_1}^\times \to \mathbb{C}^\times$, $z \mapsto [\sigma(z)]^{2a_j}$, and $a_1, \cdots, a_m \in \frac{N-1}{2} + \mathbb{Z}$ are distinct (recall that we write $[s] := s/|s|$). Applying Lemma 1.8, we conclude that ψ_{i,w_1}^N is in the image of

$$\eta_\chi^* : \Phi_{\mathrm{bdd}}(U_{E_1/F_1}(m)) \to \widetilde{\Phi}_{\mathrm{bdd}}(m)$$

if and only if $\kappa(\chi) = (-1)^{N-m} = (-1)^m$. We observe that ψ_i^N extends to a parameter

$$(\psi_i^N, \widetilde{\psi}_i) \in \Psi_2(U_{E_1/F_1}(m), \eta_{\chi\kappa}),$$

where $(-1)^{m-1}\kappa$ is equal to the parity of $\Pi_{i,b}$ (we simply take $\widetilde{\psi}_i$ to be the identity map on $\mathcal{L}_{\psi_i^N} = {}^L U_{E_1/F_1}(m)$ and trivial on $\mathrm{SL}_2(\mathbb{C})$). Now the "second seed theorem" (Proposition 1.19) implies that $\kappa = (-1)^m$. Hence, $\Pi_{i,b}$ has parity -1.

We claim that ψ^N extends to a parameter $\psi = (\psi^N, \widetilde{\psi}) \in \Psi(U_{E_1/F_1}(N), \eta_1)$. This follows from the global analogue of Lemma 1.8. Explictly, we can take

$$\widetilde{\psi} : \mathcal{L}_{\psi^N} \times \mathrm{SL}_2(\mathbb{C}) = {}^L\big(U_{E_1/F_1}(m) \times U_{E_1/F_1}(m)\big) \times \mathrm{SL}_2(\mathbb{C}) \to {}^L U_{E_1/F_1}(N)$$

to be given by

$$\begin{aligned}
(x_1, x_2) &\mapsto \begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix}, && \text{for any } x_1, x_2 \in \widehat{U}_{E_1/F_1}(m)\\
1 \rtimes w &\mapsto \begin{pmatrix} \chi_{(-1)^m}(w) & \\ & \chi_{(-1)^m}(w)^{-1} \end{pmatrix} \rtimes w, && \text{for any } w \in W_{E_1}\\
1 \rtimes w_c &\mapsto \begin{pmatrix} J_m & \\ & J_m \end{pmatrix} J_N \rtimes w_c
\end{aligned}$$

(where $\chi_{(-1)^m} : W_{E_1} \to \mathbb{C}^\times$ denotes a conjugate self-dual character of parity $(-1)^m$ and w_c denotes a fixed element of $W_{F_1} \setminus W_{E_1}$); $\widetilde{\psi}$ is defined to be trivial on the $\mathrm{SL}_2(\mathbb{C})$ factor.

We can now consider the A-packet $\Pi_\psi(G, \xi)$. We know that the localisation ψ_v of ψ is a generic A-parameter for every place v of F_1 (since Π is everywhere tempered). Since ψ_v is $G_{F_{1,v}}$-relevant for all places v of F_1 (by our assumptions on $\Pi_\mathfrak{p}$ and Π_v for $v \nmid \infty$), $\Pi_\psi(G, \xi)$ is non-empty. Let $\pi \in \Pi_\psi(G, \xi)$:

(i) If $\mathfrak{p}_1 \mid \mathfrak{p}$ and $\mathfrak{P}_1$ is a place of E_1 above $\mathfrak{p}_1$, then

$$|\mathbf{LJ}|_{i_{\mathfrak{P}_1} \circ \xi^{-1}}(\pi_{\mathfrak{p}_1}) = \Pi_{b,\mathfrak{P}_1} = \boxplus_{i \in \{1,2\}} \mathrm{St} \otimes \xi_i'(\det),$$

where $\xi_i' := \xi_i \circ \mathbb{N}_{E_{1,\mathfrak{P}_1}/E_{\mathfrak{P}_1}}$ (an unramified, unitary character of $E_{1,\mathfrak{P}_1}^\times$). Therefore, π_v is isomorphic to the representation $\mathrm{Ind}_{D_{\mathfrak{p}_0}^\times \times D_{\mathfrak{p}_0}^\times}^{\mathrm{GL}_2(D_{\mathfrak{p}_0})}\big((\xi_1' \circ \mathbb{N}\mathrm{rd}) \boxtimes (\xi_2' \circ \mathbb{N}\mathrm{rd})\big)$ of $\mathrm{GL}_2(D_{\mathfrak{p}_0})$, which is irreducible and has $\mathrm{GL}_2(\mathcal{O}_{D_{\mathfrak{p}_0}})$-fixed vectors.

Notation. $\mathfrak{p}_0 = \mathfrak{p}_1 \cap F_0$. Ind denotes unitary induction. $\mathbb{N}\mathrm{rd}$ denotes the reduced norm map $D_{\mathfrak{p}_0}^\times \to F_{0,\mathfrak{p}_0}^\times$ (and we have canonical isomorpisms $F_{0,\mathfrak{p}_0} \cong F_{1,\mathfrak{p}_1} \cong E_{1,\mathfrak{P}_1}$).

(ii) If v is a finite place of F_1 not dividing $\mathfrak{p}$ and v splits in E_1, choose a place w of E_1 above v and an F_v-isomorphism $\xi_v = \xi_{F_v} \circ \mathrm{Ad}(\alpha)$ (where $\alpha \in U_{E_1/F_1}(N)(\overline{F_{1,v_1}})$) between $U_{E_1/F_1}(N)_{F_{1,v}}$ and $G_{F_{1,v}}$. We then get an isomorphism

$$i_w \circ \xi_v^{-1} : G(F_{1,v}) \xrightarrow{\sim} \mathrm{GL}_N(E_w)$$

and $\pi_v \cong \Pi_{b,w} \circ (i_w \circ \xi_v^{-1})$ (the isomorphism class of $\Pi_{b,w} \circ (i_w \circ \xi_v^{-1})$ is independent of the choice of w and ξ_v).

(iii) If v is a finite place of F_1 not dividing $\mathfrak{p}$ and v does not split in E_1, then $\Pi_{b,v}$ is unramified and we deduce that π_v is unramified (see Chapter 1, Section 2.5).

(iv) If v is an archimedean place of F_1, then, because $G(F_{1,v})$ is compact, π_v is determined by its infinitesimal character. Let w be the complex place lying over v. The infinitesimal character of π_v can be described in terms of ψ_w^N, as we've seen in Chapter 1, Section 2.6.

All that's left to show is that the automorphic A-packet $\Pi_\psi(G, \xi, \epsilon_\psi)$ is non-trivial. In this case, ψ is generic, so $\epsilon_\psi = 1$. Given a finite place v of F_1 such that the image z_v of the basic cocycle z under

$$B(F_1, G)_{\mathrm{bsc}} \to B(F_{1,v}, G)_{\mathrm{bsc}}$$

is trivial, let π_v be the element of $\Pi_{\psi_v}(G, \xi)$ corresponding to the trivial character of $S_{\psi_v}^\natural$ under the pairing $\pi_v \mapsto \langle \cdot\,, \pi_v \rangle_{\xi, z_v}$. Let v be a place of F_1 such that the image z_v of z in $B(F_{1,v}, G)_{\mathrm{bsc}}$ is non-trivial. Then, by construction, v is split over $v_0 := v|_{F_0}$. Let v' be the $\mathrm{Gal}(F_1/F_0)$-conjugate of v. We have isomorphisms of pointed sets:

$$B(F_{1,v}, G)_{\mathrm{bsc}} \xleftarrow{\sim} B(F_{0,v_0}, G)_{\mathrm{bsc}} \xrightarrow{\sim} B(F_{1,v'}, G)_{\mathrm{bsc}}$$

Let π_v (resp. $\pi_{v'}$) be an element of $\Pi_{\psi_v}(G,\xi)$ (resp. $\Pi_{\psi_{v'}}(G,\xi)$) such that $\langle\,\cdot\,,\pi_v\rangle_{\xi,z_v} = \langle\,\cdot\,,\pi_{v'}\rangle_{\xi,z_{v'}} \in \mathrm{Irr}(S^{\natural}_{\psi_v},\chi_{z_v}) = \mathrm{Irr}(S^{\natural}_{\psi_{v'}},\chi_{z_{v'}})$. Define

$$\pi := \bigotimes_v \pi_v \in \Pi_\psi(G,\xi).$$

Since $S^{\natural}_{\psi}$ is an elementary abelian 2-group, we have $\langle\,\cdot\,,\pi\rangle_\xi = \prod_v \langle\,\cdot\,,\pi_v\rangle_{\xi,z_v}|_{S^{\natural}_{\psi}} = 1$. Hence, $\pi \in \Pi_\psi(G,\xi,\epsilon_\psi)$, i.e. π is an irreducible subrepresentation of $L^2(G(F_1)\backslash G(\mathbb{A}_{F_1}))$. □

We end this section by proving a different "descent" theorem. This theorem allows us, under stronger assumptions on Π, to descend to a totally define unitary group without the need to replace Π by its base change first (with respect to an extension E_1/E). In the following chapter, we will only make use of Lemma 2.4.

THEOREM 2.7. *Let Π_i $(i = 1,\cdots,r)$ be unitary, conjugate self-dual cuspidal automorphic representations of $\mathrm{GL}_{m_i}(\mathbb{A}_E)$ and let*

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r.$$

Assume that Π is a regular C-algebraic representation of $\mathrm{GL}_N(\mathbb{A}_E)$, where $N = m_1 + \cdots + m_r$. Let (G,ξ) be an inner twist of $U_{E/F}(N)$ such that

(i) $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact.

(ii) If v is a finite place of F that does not split in E, then G_{F_v} is quasi-split.

(iii) If v is a finite place of F that splits as ww^c in E, then the L-parameter

$$\mathrm{rec}_{E_w}(\Pi_w) : W_{E_w} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$$

is G_{E_w}-relevant.

Let v be a finite place of F which is inert in E. Let $\psi_v^N := \mathrm{rec}_{E_v}(\Pi_v)$. We can decompose ψ_v^N into a sum of simple parameters

$$\psi_v^N = \bigoplus_{j\in I_v^+} \ell_j\psi_j \oplus \bigoplus_{j\in I_v^-} \ell_j\psi_j \oplus \bigoplus_{j\in J_v} \ell_j(\psi_j \oplus \psi_j^*),$$

as in Chapter 1, Section 2.2. For each $1 \le i \le r$ and each $j \in I_v^+$, let $\ell_{j,i}$ be the multiplicity of ψ_j in $\mathrm{rec}_{E_v}(\Pi_{i,v})$; thus, $\ell_{j,1} + \cdots + \ell_{j,r} = \ell_j$ for all $j \in I_v^+$. We define an $\mathbb{F}_2$-linear map

$$\theta_v : \mathbb{F}_2^r \to \mathbb{F}_2^{I_v^+}$$

by $e_i \mapsto (\ell_{j,i} \bmod 2)_{j\in I_v^+}$ (for all $1 \le i \le r$), where e_i is the vector whose i-th coordinate is 1 and whose other coordinates are 0.

Assume that, for some finite place v of F which is inert in E, the linear map θ_v is injective. Then there exists an irreducible subrepresentation π of $L^2(G(F)\backslash G(\mathbb{A}_F))$ such that $r_\iota(\pi) \cong r_\iota(\Pi)$.

REMARK. The assumption that the map θ_v is injective is satisfied, for example, if $\mathrm{rec}_{E_w}(\Pi_v)$ is the sum of N distinct conjugate self-dual characters $W_{E_v}^\times \to \mathbb{C}^\times$ of parity $(-1)^{N-1}$ (then I_v^+ contains N elements and $\ell_j = 1$ for all $j \in I_v^+$).

PROOF. Let ψ_i^N be the formal global parameter corresponding to Π_i (for $1 \leq i \leq r$) and let

$$\psi^N = \psi_1^N \boxplus \cdots \boxplus \psi_r^N$$

be the formal parameter corresponding to Π.

We compute the parity of Π. We do this by computing the parity of the local component of Π_i at an archimedean place w, in the same way as in Lemma 2.4: Since Π is regular C-algebraic and everywhere tempered, we have

$$\psi_{i,w}^N = \mathrm{rec}_{E_w}(\Pi_{i,w}) = \eta_1 \oplus \cdots \oplus \eta_{m_i},$$

where $\eta_j : E_w^\times \to \mathbb{C}^\times$, $z \mapsto [\sigma(z)]^{2a_j}$ and $a_1, \cdots, a_{m_i} \in \frac{N-1}{2} + \mathbb{Z}$ are distinct. As before, σ denotes a fixed choice of $\mathbb{R}$-isomorphism $E_w \cong \mathbb{C}$ and $[x]$ denotes $x/|x|$. Lemma 1.8 implies that $\psi_{i,w}^N$ is in the image of

$$\eta_\chi^* : \Phi_{\mathrm{bdd}}(U_{E/F}(m_i)) \to \widetilde{\Phi}_{\mathrm{bdd}}(m_i)$$

if and only if the character χ has parity $\kappa(\chi) = (-1)^{N-m_i}$. We now use the second seed theorem (Proposition 1.19); this tells us that (for any place w) $\psi_{i,w}^N$ is in the image of η_χ^* for any character χ with $\kappa(\chi) = (-1)^{m_i-1}\eta_i$, where η_i is the parity of Π_i. Hence, we have $\eta_i = (-1)^{N-1}$.

As in Lemma 2.4, we observe that ψ^N extends to a global parameter

$$\psi = (\psi^N, \widetilde{\psi}) \in \Psi(U_{E/F}(N), \eta_1).$$

This follows from (the natural analogue of) Lemma 1.8. Explicitly,

$$\widetilde{\psi} : \mathcal{L}_{\psi^N} \times \mathrm{SL}_2(\mathbb{C}) = {}^L\left(\prod_{i=1}^r U_{E/F}(m_i)\right) \times \mathrm{SL}_2(\mathbb{C}) \to {}^L U_{E/F}(N)$$

is given by $\widetilde{\psi}(\mathrm{SL}_2(\mathbb{C})) = \{1\}$ and

$$(x_1, \cdots, x_r) \rtimes w \mapsto \begin{pmatrix} \chi_1(w)x_1 & & \\ & \ddots & \\ & & \chi_r(w)x_r \end{pmatrix} \rtimes w, \quad \text{for } x_i \in \widehat{U}_{E/F}(m_i), w \in W_E,$$

$$1 \rtimes w_c \mapsto \begin{pmatrix} J_{m_1} & & \\ & \ddots & \\ & & J_{m_r} \end{pmatrix} J_N \rtimes w_c,$$

where $\chi_i = \chi_{(-1)^{N-m_i}}$ and χ_κ denotes a conjugate self-dual chracter $W_E \to \mathbb{C}^\times$ of parity κ (for $\kappa \in \{\pm 1\}$); w_c denotes a fixed element of $W_F \setminus W_E$.

Since Π is everywhere tempered and by our assumptions on Π and G, we deduce that the localisation ψ_v of ψ is generic and G_{F_v}-relevant, for all places v of F. Hence, the A-packet $\Pi_\psi(G, \xi)$ is non-empty. We need to show that the automorphic A-packet $\Pi_\psi(G, \xi, \epsilon_\psi)$ is non-empty. Let (G, ξ, z) be an extended pure inner twist, extending (G, ξ). We have $S_\psi^\natural \cong (\mathbb{Z}/2\mathbb{Z})^r$ and, given a finite place v of F that is inert in E, we have $S_{\psi_v}^\natural \cong (\mathbb{Z}/2\mathbb{Z})^{I_v^+}$ (by Lemma 1.10). The map $S_\psi^\natural \to S_{\psi_v}^\natural$ is then the map θ_v defined in the statement of the theorem. Thus, there is a finite inert place v_0 of F such that $S_\psi^\natural$ injects into $S_{\psi_{v_0}}^\natural$. For each place $v \neq v_0$ of F, choose some $\pi_v \in \Pi_\psi(G, \xi)$ arbitrarily. Then choose a character $\lambda \in \mathrm{Irr}(S_{\psi_{v_0}}^\natural, \chi_{z_{v_0}})$ extending the character

$$\left(\prod_{v \neq v_0} \langle \cdot, \pi_v \rangle_{\xi, z_v} |_{S_\psi^\natural} \right)^{-1}$$

on $S_\psi^\natural$. Since the pairing $\Pi_{\psi_{v_0}}(G, \xi) \to \mathrm{Irr}(S_{\psi_{v_0}}^\natural, \chi_{z_{v_0}})$ is bijective, there is an element π_{v_0} in the local A-packet $\Pi_{\psi_{v_0}}(G, \xi)$ such that $\langle \cdot, \pi_{v_0} \rangle_{\xi, z_{v_0}} = \lambda$. Let

$$\pi := \bigotimes_v \pi_v \in \Pi_\psi(G, \xi).$$

We have

$$\prod_v \langle \cdot, \pi_v \rangle_{\xi, z_v} |_{S_\psi^\natural} = 1 = \epsilon_\psi,$$

so π lies in the automorphic A-packet $\Pi_\psi(G, \xi, \epsilon_\psi)$. This completes the proof of the theorem. □

3. ℓ-adic automorphic forms on a definite unitary group

Let (G, ξ) be an inner twist of $U_{E/F}(N)$, where E is a CM number field with maximal totally real subfield F, such that $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact. Let ℓ be a rational prime such that G_{F_λ} is quasi-split for all places $\lambda \mid \ell$.

For each place $\lambda \mid \ell$ of F, choose a smooth integral model $\mathcal{G}_\lambda$ over $\mathcal{O}_{F_\lambda}$ for G_{F_λ} (if λ is unramified in E, choose $\mathcal{G}_\lambda$ to be a reductive model, so that $\mathcal{G}_\lambda(\mathcal{O}_{F_\lambda})$ is a hyperspecial maximal compact subgroup of $G(F_\lambda)$). Let

$$\mathcal{G} := \prod_{\lambda \mid \ell} \operatorname{Res}_{\mathcal{O}_{F_\lambda}/\mathbb{Z}_\ell} \mathcal{G}_\lambda,$$

a group scheme over $\mathbb{Z}_\ell$ with generic fibre $\mathcal{G}_{\mathbb{Q}_\ell} \cong (\operatorname{Res}_{F/\mathbb{Q}} G)_{\mathbb{Q}_\ell}$.

Let $K/\mathbb{Q}_\ell$ be a finite extension containing all embeddings $E \hookrightarrow \overline{K}$, with integer ring $\mathcal{O} = \mathcal{O}_K$, and residue field k. Let M be a finite free $\mathcal{O}$-module and let

$$\rho : \mathcal{G}_{\mathcal{O}} \to \mathrm{GL}(M)$$

be a representation defined over $\mathcal{O}$.

Definition 2.8. *Let A be an $\mathcal{O}$-algebra and let U be an open compact subgroup of $G(\mathbb{A}_F^\infty)$. Assume that either*

(a) A is a K-algebra, or

(b) the image of U under the projection $G(\mathbb{A}_F^\infty) \to G(F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) = \mathcal{G}(\mathbb{Q}_\ell)$ is contained in

$$\mathcal{G}(\mathbb{Z}_\ell) = \prod_{\lambda \mid \ell} \mathcal{G}_\lambda(\mathcal{O}_{F_\lambda}).$$

Define $S_M(U, A)$ to be the space of all functions

$$f : G(F)\backslash G(\mathbb{A}_F^\infty) \to M \otimes_{\mathcal{O}} A$$

such that $f(xu) = \rho(u_\ell)^{-1} f(x)$ for all $x \in G(\mathbb{A}_F^\infty)$, $u \in U$, where u_ℓ denotes the image of u under the projection $G(\mathbb{A}_F^\infty) \to G(F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$.

Given an open compact subgroup $U \subset G(\mathbb{A}_F^\infty)$ and an element $x \in G(\mathbb{A}_F^\infty)$, we write $\Gamma_{U,x}$ for the finite subgroup $xUx^{-1} \cap G(F)$ of $G(\mathbb{A}_F^\infty)$. The order $|\Gamma_{U,x}|$ of this group only depends on the class of x in the finite double quotient $X_U := G(F)\backslash G(\mathbb{A}_F^\infty)/U$.

Definition 2.9. *We say that the open compact subgroup U is* sufficiently small *if ℓ does not divide $|\Gamma_{U,x}|$ for all $x \in G(\mathbb{A}_F^\infty)$.*

PROPOSITION 2.10. *Let A be an $\mathcal{O}$-algebra and let U be an open compact subgroup of $G(\mathbb{A}_F^\infty)$ (contained in $\mathcal{G}(\mathbb{Z}_\ell) \times G(\mathbb{A}_F^{\infty,\ell})$, if A is not a K-algebra). Let $\{x_\alpha : \alpha \in X_U\} \subset G(\mathbb{A}_F^\infty)$ be a set of representatives for the double quotient $X_U := G(F)\backslash G(\mathbb{A}_F^\infty)/U$. Then:*

(i) The map $f \mapsto (f(x_\alpha) : \alpha \in X_U)$ defines an isomorphism of A-modules

$$S_M(U, A) \cong \bigoplus_{\alpha \in X_U} (M \otimes_{\mathcal{O}} A)^{H_\alpha},$$

where H_α is the image of the group $x_\alpha^{-1} G(F) x_\alpha \cap U$ under $G(\mathbb{A}_F^\infty) \to G(F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$ (H_α has order dividing $|\Gamma_{U,x_\alpha}|$ and acts on $M \otimes_{\mathcal{O}} A$ by ρ).

(ii) If U is sufficiently small and $U \subset \mathcal{G}(\mathbb{Z}_\ell) \times G(\mathbb{A}_F^{\infty,\ell})$, then $A \mapsto S_M(U, A)$ is an exact functor from the category of $\mathcal{O}$-algebras to the category of $\mathcal{O}$-modules.

(iii) If $A \to B$ is flat $\mathcal{O}$-homomorphism, then $S_M(U, B) = S_M(U, A) \otimes_A B$.

PROOF. (i) The map $f \mapsto (f(x_\alpha) : \alpha \in X_U)$ admits a well-defined inverse map $m = (m_\alpha) \mapsto f_m$, where

$$f_m(\gamma x_\alpha u) = \rho(u_\ell)^{-1} m_\alpha,$$

for all $\gamma \in G(F)$, $u \in U$.

(ii) For any $\mathcal{O}$-algebra A, we have $H^1(H_\alpha, M \otimes_{\mathcal{O}} A) = 0$, since this is an $\mathcal{O}$-module killed by $|H_\alpha| = |\Gamma_{U,x_\alpha}| \in \mathcal{O}^\times$.

(iii) We have an exact sequence

$$0 \to (M \otimes_{\mathcal{O}} A)^{H_\alpha} \to M \otimes_{\mathcal{O}} A \xrightarrow{a \mapsto (ha - a)_h} \prod_{h \in H_\alpha} M \otimes_{\mathcal{O}} A.$$

The flatness of $A \to B$ implies that

$$0 \to (M \otimes_{\mathcal{O}} A)^{H_\alpha} \otimes_A B \to M \otimes_{\mathcal{O}} B \xrightarrow{a \mapsto (ha - a)_h} \prod_{h \in H_\alpha} M \otimes_{\mathcal{O}} B$$

is exact, i.e. that $(M \otimes_{\mathcal{O}} A)^{H_\alpha} \otimes_A B = (M \otimes_{\mathcal{O}} B)^{H_\alpha}$.

□

3.1. Hecke operators. Let U be an open compact subgroup of $G(\mathbb{A}_F^\infty)$ (contained in $\mathcal{G}(\mathbb{Z}_\ell) \times G(\mathbb{A}_F^{\infty,\ell})$, if A is not a K-algebra). For every $x \in G(\mathbb{A}_F^\infty)$ (with $x_\ell \in \mathcal{G}(\mathbb{Z}_\ell)$, if A is not a K-algebra), we have a natural map

$$S_M(U, A) \xrightarrow{\sim} S_M(xUx^{-1}, A), \quad f \mapsto xf,$$

where $(xf)(y) := \rho(x_\ell) f(yx)$. Note that $uf = f$ for all $u \in U$.

Assume that $U \subset \mathcal{G}(\mathbb{Z}_\ell) \times G(\mathbb{A}_F^{\infty,\ell})$ and let $\mathcal{H}_U$ be the $\mathcal{O}$-algebra of all compactly supported U-bi-invariant functions

$$T : U\backslash G(\mathbb{A}_F^\infty)/U \to \mathcal{O}$$

such that $\mathrm{supp}(T) \subset \mathcal{G}(\mathbb{Z}_\ell) \times G(\mathbb{A}_F^{\infty,\ell})$. This is a free $\mathcal{O}$-module with basis

$$\{[UxU] : x \in \mathcal{G}(\mathbb{Z}_\ell) \times G(\mathbb{A}_F^{\infty,\ell})\},$$

where $[UxU]$ denotes the indicator function of the double coset UxU. Multiplication is given by convolution

$$(T * T')(y) = \sum_{x \in G(\mathbb{A}_F^\infty)/U} T(x)\, T'(x^{-1}y)$$

and the multiplicative identity is the indicator function of U. $\mathcal{H}_U$ acts on $S_M(U, A)$ by

$$(Tf)(y) = \sum_{x \in G(\mathbb{A}_F^\infty)/U} T(x)\,(xf)(y).$$

Fix a field isomorphism $\iota : \overline{K} \xrightarrow{\sim} \mathbb{C}$. We have a representation

$$\rho_{\overline{K}} : \mathcal{G}(\overline{K}) = G(F \otimes_{\mathbb{Q}} \overline{K}) \to \mathrm{GL}(M \otimes_{\mathcal{O}} \overline{K}),$$

which we can identify (under ι) with a representation

$$\rho_{\mathbb{C},\iota} : G(F \otimes_{\mathbb{Q}} \mathbb{C}) \to \mathrm{GL}(M \otimes_{\mathcal{O},\iota} \mathbb{C}).$$

We write W_∞ for the finite-dimensional complex vector space $M \otimes_{\mathcal{O},\iota} \mathbb{C}$ endowed with the continuous $G(F \otimes_{\mathbb{Q}} \mathbb{R})$-action given by (the restriction of) $\rho_{\mathbb{C},\iota}$ and trivial $G(\mathbb{A}_F^\infty)$-action.

PROPOSITION 2.11. *Fix $\iota : \overline{K} \cong \mathbb{C}$, as above. We have an isomorphism of $\mathcal{H}_U$-modules*

$$S_M(U, \mathbb{C}) \cong \mathrm{Hom}_{U \times G(F \otimes_{\mathbb{Q}} \mathbb{R})}(W_\infty^\vee, L^2(G(\mathbb{Q})\backslash G(\mathbb{A}_F))).$$

PROOF. The isomorphism $f \mapsto \varphi := \varphi(f)$ is defined by

$$\varphi(\theta) : x \mapsto \theta(\rho_{\mathbb{C},\iota}(x_\infty)^{-1}\rho(x_\ell)f(x^\infty)),$$

for $\theta \in W_\infty^\vee$, $x \in G(\mathbb{A}_F)$. □

We henceforth assume that W_∞ is an irreducible representation of $G(F \otimes_{\mathbb{Q}} \mathbb{R})$. Note that, if $L^2(G(F)\backslash G(\mathbb{A}_F)) = \widehat{\bigoplus}_\pi m(\pi)\,\pi$, where the sum runs through $\pi \in \mathcal{A}_{\mathrm{disc}}(G)$, then

$$\mathrm{Hom}_{U\times G(F\otimes_{\mathbb{Q}}\mathbb{R})}(W_\infty^\vee, L^2(G(\mathbb{Q})\backslash G(\mathbb{A}_F)) = \bigoplus_{\pi_\infty = W_\infty^\vee} m(\pi)\,(\pi^\infty)^U.$$

3.2. Galois representations. Let R be a set containing all but finitely many of the places v of F which split in E. Assume that every $v \in R$ is such that

(i) v splits in E and is prime to ℓ, and
(ii) G_{F_v} is quasi-split (i.e. isomorphic to $\mathrm{GL}(N, F_v)$).

Given a place $v \in R$, let w be a place of E above v. The choice of the place w induces a canonical isomorphism $i_w : U_{E/F}(N)_{F_v} \to \mathrm{GL}(N, F_v)$ (we have $i_{w^c}(x) = J_N i_w(x)^{-t} J_N^{-1}$). For each $v \in R$, we choose $a \in G(\overline{F_v})$ such that $\mathrm{Ad}(a) \circ \xi_{\overline{F_v}}$ is defined over F_v; let ξ_v be the F_v-isomorphism $U_{E/F}(N)_{F_v} \to G_{F_v}$ such that $(\xi_v)_{\overline{F_v}} = \mathrm{Ad}(a) \circ \xi_{\overline{F_v}}$. We define

$$\theta_w := i_w \circ \xi_v^{-1} : G_{F_v} \xrightarrow{\sim} \mathrm{GL}(N, F_v).$$

Let $U = \prod_v U_v$ be a sufficiently small open compact subgroup of $\mathcal{G}(\mathbb{Z}_\ell) \times G(\mathbb{A}_F^{\infty,\ell})$, such that $U_v := \theta_w^{-1}(\mathrm{GL}_N(\mathcal{O}_{F_v}))$ for all $v \in R$ (U_v is independent of the choice of $w \mid v$). For each $v \in R$ and each choice of a place w of E above v, we have Hecke operators

$$T_{w,j} := \left[U\, \theta_w^{-1} \left(\begin{pmatrix} \varpi_v 1_j & 0 \\ 0 & 1_{N-j} \end{pmatrix} \right) U \right] \quad \text{for } j = 0, 1, \cdots, N,$$

acting on $S_M(U, A)$. We have $T_{w^c,j} = T_{w,N-j}\, T_{w,N}^{-1}$.

Let $\mathbf{T} := \mathbf{T}_M(U)$ be the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(S_M(U, \mathcal{O}))$ generated by the Hecke operators $T_{w,j}$, $T_{w,N}^{-1}$ for $0 \le j \le N$, $w \mid R$. The algebra $\mathbf{T}$ has the following properties:

(a) $\mathbf{T}$ is commutative.
(b) $\mathbf{T}$ is finite and flat over $\mathcal{O}$.
(c) $\mathbf{T}$ is noetherian, of Krull dimension 1, and every minimal prime ideal of $\mathbf{T}$ lies below a unique maximal ideal.
(d) $\mathbf{T}$ is a finite product of local rings, i.e. the natural map

$$\mathbf{T} \to \prod_{\mathfrak{m}} \mathbf{T}_{\mathfrak{m}},$$

is an isomorphism (the product is over all maximal ideals $\mathfrak{m}$ of $\mathbf{T}$).

(e) Let $e_{\mathfrak{m}}$ be the element of $\mathbf{T}$ such that $e_{\mathfrak{m}} = 1$ in $\mathbf{T}_{\mathfrak{m}}$ and $e_{\mathfrak{m}} = 0$ in $\mathbf{T}_{\mathfrak{n}}$ for all maximal ideals $\mathfrak{n} \neq \mathfrak{m}$. For any $\mathbf{T}$-module M, we have

$$M = \bigoplus_{\mathfrak{m}} e_{\mathfrak{m}} M$$

and each $e_{\mathfrak{m}} M$ is canonically isomorphic to $M_{\mathfrak{m}}$.

Proposition 2.12. *Let $\phi : \mathbf{T} \to \overline{k}$ be a homomorphism of $\mathcal{O}$-algebras. There exists a continuous semisimple* mod ℓ *Galois representation*

$$\overline{r_\phi} : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_N(\overline{k}),$$

which is unramified at all places $w \mid R$ and $\overline{r_\phi}(\mathrm{Frob}_w)$ has the characteristic polynomial

$$\sum_{j=0}^{N} (-1)^{N-j} q_w^{\frac{(N-j)(1-j-N)}{2}} \phi(T_{w,N}^{-1} T_{w,j}) X^j.$$

Moreover, we have $\overline{r_\phi}^{\vee,c} \cong \overline{r_\phi} \otimes \varepsilon_\ell^{N-1}$.

Remark. In the proposition above (and later in the text), Frob_w denotes the arithmetic Frobenius at the place w. The characteristic polynomial of $\overline{r_\phi}$ evaluated at a geometric Frobenius at w is

$$\sum_{j=0}^{N} (-1)^{N-j} q_w^{\frac{(N-j)(N-j-1)}{2}} \phi(T_{w,N-j}) X^j.$$

Proof. Let $\mathfrak{p}$ be a minimal prime lying below $\mathfrak{m} := \ker \phi$. There is a $\mathbf{T}$-eigenfunction $f \in S_M(U, \overline{K})$ such that $\mathfrak{p}$ is the kernel of the $\mathcal{O}$-homomorphism $\lambda : \mathbf{T} \to \overline{K}$ sending T to its f-eigenvalue (the elements of $\mathbf{T}$ are simultaneously diagonalisable endomorphisms of $S_M(U, \overline{K})$, by Proposition 2.11).

Fix $\iota : \overline{K} \cong \mathbb{C}$. There is a subrepresentation $\pi \subset L^2(G(F)\backslash G(\mathbb{A}_F))$ such that the eigenvalue of $T_{w,j}$ on π^U is $\iota \circ \lambda(T_{w,k})$. We define $\overline{r_\phi}$ to be the semisimplification of the mod ℓ reduction of $r_\iota(\pi)$ (cf. Corollary 2.3).

It remains to show that $\overline{r_\phi}(\mathrm{Frob}_w)$ has the correct characteristic polynomial for all $w \mid R$. Let $v \in R$ and let w be a place of E lying over v. Let $\Pi_w = \pi_v \circ \theta_w^{-1}$; Π_w is an unramified representation of $\mathrm{GL}_N(E_w)$. We have $\mathrm{rec}_{E_w}(\Pi_w) = \psi_1 \oplus \cdots \oplus \psi_N$, where the ψ_j are unramified characters $W_{E_w} \to \mathbb{C}^\times$. Then

$$\mathrm{rec}_{E_w}(\Pi_w |\det|_w^{\frac{1-N}{2}}) = (\psi_1 \oplus \cdots \oplus \psi_N) \otimes |\mathrm{Art}_{E_w}^{-1}(\,\cdot\,)|_w^{\frac{1-N}{2}}.$$

Let $\alpha_j := \iota^{-1}\psi_j(\mathrm{Frob}_w)$ for $1 \le j \le N$. The characteristic polynomial of $r_\iota(\pi)(\mathrm{Frob}_w)$ is

$$(X - \alpha_1 q_w^{\frac{1-N}{2}}) \cdots (X - \alpha_N q_w^{\frac{1-N}{2}}),$$

where q_w is the cardinality of the residue field of E_w. On the other hand, Π_w is isomorphic to the normalised parabolic induction of $\chi_1 \boxtimes \cdots \boxtimes \chi_N$, where $\chi_j = \psi_j \circ \mathrm{Art}_{E_w}$. Let σ_j $(0 \le j \le N)$ be the jth elementary symmetric polynomial in the $\chi_j(\varpi_w)$ (where ϖ_w is a uniformiser for E_w). Thus, $\sigma_0 = 1$, $\sigma_1 = \sum_{j=1}^N \chi_j(\varpi_j)$ and $\sigma_N = \prod_{j=1}^N \chi_j(\varpi_j)$. The Hecke operator $T_{w,j}$ acts on $\Pi_w^{\mathrm{GL}_N(\mathcal{O}_{E_w})}$ by multiplication by the scalar

$$t_{w,j} = q_w^{\frac{j(N-j)}{2}} \sigma_j$$

(cf. [**CHT08**] Lemma 3.1.1). We have $\iota(\alpha_j) = \chi_j(\varpi_w)^{-1}$ and, hence,

$$\begin{aligned}(X - \alpha_1 q_w^{\frac{1-N}{2}}) \cdots (X - \alpha_N q_w^{\frac{1-N}{2}}) &= \sum_{j=0}^N (-1)^{N-j} q_w^{\frac{(N-j)(1-N)}{2}} \iota^{-1}(\sigma_N^{-1}\sigma_j) X^j \\ &= \sum_{j=0}^N (-1)^{N-j} q_w^{\frac{(N-j)(1-j-N)}{2}} \iota^{-1}(t_{w,N}^{-1} t_{w,j}) X^j \\ &= \sum_{j=0}^N (-1)^{N-j} q_w^{\frac{(N-j)(1-j-N)}{2}} \lambda(T_{w,N}^{-1} T_{w,j}) X^j.\end{aligned}$$

We conclude that $\overline{r_\phi}(\mathrm{Frob}_w)$ has the desired characteristic polynomial. □

3.3. Duality. Let $\langle \cdot, \cdot \rangle_M : M \times M^\vee \to \mathcal{O}$ be the natural pairing $\langle m, \vartheta \rangle_M = \vartheta(m)$. Assuming that $U \subset \mathcal{G}(\mathbb{Z}_\ell) \times G(\mathbb{A}_F^{\infty,\ell})$ is sufficiently small, we have a perfect pairing

$$\langle \cdot, \cdot \rangle_U : S_M(U, A) \times S_{M^\vee}(U, A) \to A$$

defined by

$$\langle f, g \rangle_U = \sum_{x \in X_U} |\Gamma_{U,x}|^{-1} \langle f(x), g(x) \rangle_M \qquad (\text{where } X_U := G(F)\backslash G(\mathbb{A}_F^\infty)/U).$$

We record the following properties of $\langle \cdot, \cdot \rangle_U$:

(a) For all $x \in \mathcal{G}(\mathbb{Z}_\ell) \times G(\mathbb{A}_F^{\infty,\ell})$, we have $\langle xf, xg \rangle_{xUx^{-1}} = \langle f, g \rangle_U$.
(b) For all $U_1 \subset U$, $(U : U_1)\, \langle f, g \rangle_U = \langle f, g \rangle_{U_1}$.
(c) For all $x \in \mathcal{G}(\mathbb{Z}_\ell) \times G(\mathbb{A}_F^{\infty,\ell})$, the adjoint of the Hecke operator $[UxU]$ (acting on $S_M(U, A)$) with respect to $\langle \cdot, \cdot \rangle_U$ is $[UxU]^\vee = [Ux^{-1}U]$ (acting on $S_{M^\vee}(U, A)$).
(d) For all $w \mid R$, $j \in \{0, 1, \cdots, N\}$, we have $T_{w,j}^\vee = T_{w,N-j}\, T_{w,N}^{-1} = T_{w^c,j}$.

We write $\mathbf{T}$ for $\mathbf{T}_M(U) \subset \mathrm{End}_{\mathcal{O}}(S_M(U, \mathcal{O}))$ and $\mathbf{T}^\vee$ for $\mathbf{T}_{M^\vee}(U) \subset \mathrm{End}_{\mathcal{O}}(S_{M^\vee}(U, \mathcal{O}))$. We have an isomorphism $\mathbf{T} \cong \mathbf{T}^\vee$ given by $T \mapsto T^\vee$. Given a maximal ideal $\mathfrak{m}$ of $\mathbf{T}$, we write $\mathfrak{m}^\vee$ be the image of $\mathfrak{m}$ under the isomorphism $\mathbf{T} \cong \mathbf{T}^\vee$. Then we have $(e_{\mathfrak{m}})^\vee = e_{\mathfrak{m}^\vee}$ and $\langle \cdot\,, \cdot \rangle_U$ restricts to a perfect pairing

$$S_M(U, A)_{\mathfrak{m}} \times S_{M^\vee}(U, A)_{\mathfrak{m}^\vee} \to A.$$

PROPOSITION 2.13. *Let $\phi : \mathbf{T} \to \overline{k}$ be a homomorphism of $\mathcal{O}$-algebras. Write $\phi^\vee$ for the homomorphism $\mathbf{T}^\vee \to \overline{k}$ defined by $T \mapsto \phi(T^\vee)$. We then have $\overline{r_{\phi^\vee}} \cong \overline{r_\phi}^c \cong \overline{r_\phi}^\vee \otimes \varepsilon_\ell^{1-N}$.*

PROOF. Since $T^\vee_{w,j} = T_{w^c,j}$, we have $\overline{r_{\phi^\vee}}(\mathrm{Frob}_w) = \overline{r_\phi}(\mathrm{Frob}_{w^c})$, for all places $w \mid R$. $\square$

CHAPTER 3

Level raising

In this final chapter, we state and prove our level raising theorems.

Fix, once and for all, a field isomorphism $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, and let E be a CM number field with maximal totally real subfield F. Let Π_1, Π_2 be two conjugate self-dual, unitary cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A}_E)$. Our aim is to prove the existence of a level raising congruence between the representation $\Pi = \Pi_1 \boxplus \Pi_2$ of $\mathrm{GL}_N(\mathbb{A}_E)$ (where we write N for $2m$) and some unitary cuspidal automorphic representation Π' of $\mathrm{GL}_N(\mathbb{A}_{E_1})$, for some CM biquadratic extension E_1/E. More precisely, we prove the following theorem:

Theorem 3.1. *Assume that*

(a) The representation $\Pi = \Pi_1 \boxplus \Pi_2$ *is regular* C*-algebraic.*

(b) For some finite place $\mathfrak{P}$ *of* E *which does not divide* ℓ, $\Pi_{i,\mathfrak{P}}$ *is isomorphic to an unramified twist of the Steinberg representation of* $\mathrm{GL}_m(E_\mathfrak{P})$*:*

$$\Pi_{i,\mathfrak{P}} \cong \mathrm{St}_m(\xi_i) := \mathrm{St}_m \otimes (\xi_i \circ \det),$$

for some unramified character $\xi_i : E_\mathfrak{P}^\times \to \mathbb{C}^\times$ $(i = 1, 2)$*.*

(c) Π *satisfies the following local congruence condition at the place* $\mathfrak{P}$*: If* ϖ *is a uniformiser for* $E_\mathfrak{P}$*, then*

$$\iota^{-1}(\xi_1(\varpi)/\xi_2(\varpi)) \equiv (\mathbb{N}\mathfrak{P})^{\pm m} \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_\ell}.$$

(d) The residual $\bmod \ell$ *Galois representation*

$$\overline{r_\iota}(\Pi) : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_N(\overline{\mathbb{F}}_\ell)$$

of Π *is not isomorphic to a twist of*

$$1 \oplus \overline{\varepsilon}_\ell^{-1} \oplus \cdots \oplus \overline{\varepsilon}_\ell^{1-N}.$$

Then there exists a CM biquadratic extension E_1/E *and, for each choice of a place* $\mathfrak{P}_1$ *of* E_1 *above* $\mathfrak{P}$*, a regular* C*-algebraic, conjugate self-dual, unitary cuspidal automorphic representation* Π' *of* $\mathrm{GL}_N(\mathbb{A}_{E_1})$*, such that:*

(i) $\overline{r_\iota}(\Pi')$ *is isomorphic to* $\overline{r_\iota}(\Pi)|_{\mathrm{Gal}(\overline{E_1}/E_1)}$*.*

(ii) $\Pi'_{\mathfrak{P}_1}$ *is isomorphic to an unramified twist of the Steinberg representation of* $\mathrm{GL}_N(E_{\mathfrak{P}_1})$.

(iii) If Π_w *is unramified, for some finite place* $w \neq \mathfrak{P}$ *of* E*, then* Π'_{w_1} *is unramified for all places* $w_1 \mid w$ *of* E_1.

REMARK. If Π satisfies properties (a)-(c), then so does the base change of Π under any finite solvable CM extension.

The first step in the proof of this theorem is to relate Π to an automorphic representation of a totally definite unitary group G. Lemma 2.4 implies that we can replace E/F by some extension E_1/F_1, Π by its base change to E_1, and $\mathfrak{P}$ by any place $\mathfrak{P}_1 \mid \mathfrak{P}$ of E_1 (where F_1/F is a biquadratic extension of totally real number fields and $E_1 = E \cdot F_1$), such that the following conditions are satisfied:

(e) E/F is unramified at all finite places and any place w of E, which is such that Π_w is ramified, is split over F. Moreover, $\mathfrak{P}$ is split over F (of course, $\Pi_{\mathfrak{P}}$ is ramified, unless $m = 1$).

(f) There is an inner twist (G, ξ) of $U_{E/F}(N)$ and irreducible $G(\mathbb{A}_F)$-subspace π of $L^2(G(F)\backslash G(\mathbb{A}_F))$ such that
   - (i) $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact and $G_{F_{\mathfrak{p}}}$ is isomorphic to $\mathrm{GL}(2, D)$, where D is a central division $F_{\mathfrak{p}}$-algebra and $\mathfrak{p} = \mathfrak{P} \cap F$. We fix a choice of isomorphism $G_{F_{\mathfrak{p}}} \cong \mathrm{GL}(2, D)$ throughout.
   - (ii) The Galois representation $r_\iota(\pi)$ is isomorphic to $r_\iota(\Pi)$. In particular, π has $\mathrm{GL}_2(\mathcal{O}_D)$-fixed vectors.

We need to choose the biquadratic extension F_1/F so that the restriction of $\overline{r_\iota}(\Pi)$ to $\mathrm{Gal}(\overline{E_1}/E_1)$ is not isomorphic to a twist of $1 \oplus \bar{\varepsilon}_\ell^{-1} \oplus \cdots \oplus \bar{\varepsilon}^{1-N}$. Observe that a continuous representation $\bar{r} : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_N(\overline{\mathbb{F}}_\ell)$ is isomorphic to a twist of $1 \oplus \bar{\varepsilon}_\ell^{-1} \oplus \cdots \oplus \bar{\varepsilon}^{1-N}$ if and only if the image of

$$\bar{r} \oplus \bar{\varepsilon}_\ell : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_N(\overline{\mathbb{F}}_\ell) \times \overline{\mathbb{F}}_\ell^\times$$

is contained in a conjugate of the subgroup

$$\left\{ \left(\lambda \begin{pmatrix} 1 & & & \\ & \mu^{-1} & & \\ & & \ddots & \\ & & & \mu^{1-N} \end{pmatrix}, \mu \right) : \lambda, \mu \in \overline{\mathbb{F}}_\ell^\times \right\}$$

of $\mathrm{GL}_N(\overline{\mathbb{F}}_\ell) \times \overline{\mathbb{F}}_\ell^\times$. Let $K \subset \overline{E}$ be the fixed field of the subgroup $\ker(\overline{r_\iota}(\Pi) \oplus \bar{\varepsilon}_\ell)$ of $\mathrm{Gal}(\overline{E}/E)$; thus, K/E is a finite Galois extension and the representation $\overline{r_\iota}(\Pi) \oplus \bar{\varepsilon}_\ell$ factors through $\mathrm{Gal}(\overline{E}/E) \to \mathrm{Gal}(K/E)$. We may choose F_1/F so that $\mathrm{Gal}(K \cdot E_1/E_1) \to \mathrm{Gal}(K/E)$ is surjective. Then

$$\mathrm{im}((\overline{r_\iota}(\Pi) \oplus \bar{\varepsilon}_\ell)|_{\mathrm{Gal}(\overline{E_1}/E_1)}) = \mathrm{im}(\overline{r_\iota}(\Pi) \oplus \bar{\varepsilon}_\ell)$$

and hence $\overline{r_\iota}(\Pi)|_{\mathrm{Gal}(\overline{E_1}/E_1)}$ is not isomorphic to a twist of $1 \oplus \bar{\varepsilon}_\ell^{-1} \oplus \cdots \oplus \bar{\varepsilon}^{1-N}$.

For the rest of the chapter, we assume that Π satisfies the assumptions (a)-(f). We also fix an open compact subgroup $\mathcal{U}^{\mathfrak{P}} = \prod_{w \neq \mathfrak{P}} \mathcal{U}_w$ of $\mathrm{GL}_N(\mathbb{A}_E^{\infty,\mathfrak{P}})$ such that $\Pi^{\mathcal{U}^{\mathfrak{P}}} \neq 0$ and we assume that

(g) There is some place $w \neq \mathfrak{P}$ which splits over F, the image of $\mathcal{U}^{\mathfrak{P}}$ under the projection $\mathrm{GL}_N(\mathbb{A}_E^{\infty,\mathfrak{P}}) \to \mathrm{GL}_N(E_w)$ contains no non-trivial elements of finite order.

Whenever w is split over F, we have $\pi_{w^c} \cong \pi_w^\vee$ and we may assume that $\mathcal{U}_{w^c}$ is the image of $\mathcal{U}_w$ under the involution $x \mapsto J_N c(x)^{-t} J_N^{-1}$. We then have an open compact subgroup $U = \prod_{v \nmid \infty} U_v = U_{\mathfrak{p}} U^{\mathfrak{p}}$ of $G(\mathbb{A}_F)$ such that $\pi^U \neq 0$, $U_{\mathfrak{p}} = \mathrm{GL}_2(\mathcal{O}_D)$, and U is *sufficiently small*. The subgroup U is such that $U_v = \theta_w^{-1}(\mathcal{U}_w)$ for any $v \neq \mathfrak{p}$ which splits in E and is such that G_{F_v} is quasi-split (and any choice of $w \mid v$) and such that U_v is a hyperspecial subgroup for any v which does not split in E. We are reduced to proving the following:

Theorem 3.2. *Under the assumptions (a)-(g) above, there exist conjugate self-dual, unitary cuspidal automorphic representation Π' of $\mathrm{GL}_N(\mathbb{A}_E)$ such that:*

(i) Π' has the same residual Galois representation as Π:

$$\overline{r_\iota}(\Pi') \cong \overline{r_\iota}(\Pi).$$

(ii) $\Pi'_{\mathfrak{P}}$ is an unramified twist of the Steinberg representation St_N.

(iii) Π'_w has $\mathcal{U}_w$-fixed vectors, for all finite places w of E lying over a place v of F such that G_{F_v} is quasi-split.

Theorem 3.2 will be proven in Sections 3.1 and 3.2 below. The following theorem is also a corollary of Theorem 3.2:

Theorem 3.3. *Let Π be a conjugate self-dual, unitary cuspidal automorphic representation of $\mathrm{GL}_N(\mathbb{A}_E)$, where $N = 2m$. Assume that* *(a)* Π *is regular* C*-algebraic and* $m[F:\mathbb{Q}]$ *is even.*

(b) For some finite place $\mathfrak{P}$ *of* E *which does not divide* ℓ *and is split over* F*,* $\Pi_{\mathfrak{P}}$ *is isomorphic to*

$$\mathrm{St}_m(\xi_1) \boxplus \mathrm{St}_m(\xi_2)$$

for some unramified unitary characters $\xi_1, \xi_2 : E_{\mathfrak{P}}^\times \to \mathbb{C}^\times$*.*

(c) Π *satisfies the following local congruence condition at the place* $\mathfrak{P}$*: If* ϖ *is a uniformiser for* $E_{\mathfrak{P}}$*, then*

$$\iota^{-1}(\xi_1(\varpi)/\xi_2(\varpi)) \equiv (\mathbb{N}\mathfrak{P})^{\pm m} \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_\ell}.$$

(d) The residual $\bmod \ell$ *Galois representation*

$$\overline{r_\iota}(\Pi) : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_N(\overline{\mathbb{F}}_\ell)$$

of Π *is not isomorphic to a twist of*

$$1 \oplus \overline{\varepsilon}_\ell^{-1} \oplus \cdots \oplus \overline{\varepsilon}_\ell^{1-N}.$$

Then there exists a regular C*-algebraic, conjugate self-dual, unitary cuspidal automorphic representation* Π' *of* $\mathrm{GL}_N(\mathbb{A}_E)$ *with the same residual* $\bmod \ell$ *Galois representation as* Π *(i.e.* $\overline{r_\iota}(\Pi') \cong \overline{r_\iota}(\Pi)$*), such that* $\Pi'_{\mathfrak{P}}$ *is isomorphic to an unramified twist of the Steinberg representation of* $\mathrm{GL}_N(E_{\mathfrak{P}})$*.*

Proof. We explain how to reduce this theorem to a level raising theorem for automorphic representations of a unitary group. Since $m[F:\mathbb{Q}]$ is even, there exists an inner twist (G,ξ) of $U_{E/F}(N)$ such that

(i) $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact,
(ii) $G_{F_{\mathfrak{p}}}$ is isomorphic to $\mathrm{GL}(2,D)$, where D is a central division $F_{\mathfrak{p}}$-algebra and $\mathfrak{p}$ is the place of F lying below $\mathfrak{P}$, and
(iii) G_{F_v} is quasi-split for all finite places $v \neq \mathfrak{p}$ of F.

Let $\psi^N \in \Psi(\mathrm{GL}(N,E))$ be the formal global parameter corresponding to Π. ψ^N extends to a parameter $\psi = (\psi^N, \widetilde{\psi}) \in \Psi_2(U_{E/F}(N), \eta_1)$, by taking

$$\widetilde{\psi} : \mathcal{L}_{\psi^N} \times \mathrm{SL}_2(\mathbb{C}) = {}^L U_{E/F}(N) \times \mathrm{SL}_2(\mathbb{C}) \to {}^L U_{E/F}(N)$$

to be the identity map on ${}^L U_{E/F}(N)$ and the trivial map on $\mathrm{SL}_2(\mathbb{C})$. The A-packet $\Pi_\psi(G,\xi)$ is non-empty, since $\psi_v = \mathrm{rec}_{F_v}(\Pi_v)$ is bounded and G_{F_v}-relevant for all places v of F. Moreover, $\Pi_\psi(G,\xi,\epsilon_\psi) = \Pi_\psi(G,\xi)$, since $\overline{\mathcal{S}}_\psi$ is the trivial group in this case. Hence, there exists a subrepresentation π of $L^2(G(F)\backslash G(\mathbb{A}_F))$ with $r_\iota(\pi) \cong$

$r_\iota(\Pi)$. Now we can apply the arguments of Sections 3.1 and 3.2 to the representation π to deduce Theorem 3.3, in the same way as in the proof of Theorem 3.2. $\square$

1. Ihara's lemma

We adopt the notation of Chapter 2, Section 3. We choose our set R of split places so that R does not contain $\mathfrak{p}$, and fix a $\mathcal{G}_{\mathcal{O}}$-module M such that $\pi_\infty \cong W_\infty^\vee := M \otimes_{\mathcal{O},\iota} \mathbb{C}$. In particular, we can then identify $(\pi^\infty)^U$ with a Hecke-invariant subspace of $S_M(U, \mathbb{C})$.

Let ϖ_D be a uniformiser for $\mathcal{O}_D$, and let

$$\eta := \begin{pmatrix} & 1 \\ \varpi_D & \end{pmatrix} \in \mathrm{GL}_2(D) = G(F_\mathfrak{p}).$$

Let $U_{1,\mathfrak{p}} := U_\mathfrak{p} \cap (\eta U_\mathfrak{p} \eta^{-1})$, i.e.

$$U_{1,\mathfrak{p}} := \left\{ x \in \mathrm{GL}_2(\mathcal{O}_D) : x \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \bmod \varpi_D \right\}.$$

Set $U_1 := U^\mathfrak{p} U_{1,\mathfrak{p}}$. We have a map of $\mathbf{T}$-modules

$$\delta_M : S_M(U, \mathcal{O}) \oplus S_M(U, \mathcal{O}) \to S_M(U_1, \mathcal{O}), \quad (f, g) \mapsto f + \eta g.$$

As U is sufficiently small, $S_M(U, k) = S_M(U, \mathcal{O}) \otimes_{\mathcal{O}} k$ and $S_M(U_1, k) = S_M(U_1, \mathcal{O}) \otimes_{\mathcal{O}} k$. Let $\delta_{M,k} := \delta_M \otimes \mathrm{id}_k : S_M(U, k)^{\oplus 2} \to S_M(U_1, k)$.

We state the analogue of Ihara's lemma below. It asserts that the map δ_M is injective, after localisation at a suitable maximal ideal.

Lemma 3.4. *Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{T}$ in the support of $\ker \delta_{M,k}$ (i.e. such that $(\ker \delta_{M,k})_\mathfrak{m} \neq 0$). Then, for any k-embedding $\phi : \mathbf{T}/\mathfrak{m} \hookrightarrow \overline{k}$, the Galois representation $\overline{r_\phi}$ is of the form*

$$\overline{\psi} \otimes (1 \oplus \overline{\varepsilon}_\ell^{-1} \oplus \cdots \oplus \overline{\varepsilon}_\ell^{1-N}),$$

for some (conjugate self-dual) character $\overline{\psi} : \mathrm{Gal}(\overline{E}/E) \to \overline{k}^\times$, where $\overline{\varepsilon}_\ell : \mathrm{Gal}(\overline{E}/E) \to (\mathbb{Z}/\ell\mathbb{Z})^\times$ denotes the $\bmod \ell$ reduction of the ℓ-adic cyclotomic character.

Proof. Let $\delta_{M,\overline{k}} = \delta_M \otimes \mathrm{id}_{\overline{k}}$. We have $\ker \delta_{M,\overline{k}} = \ker \delta_{M,k} \otimes_k \overline{k}$ (as $k \to \overline{k}$ is flat). Thus,

$$(\ker \delta_{M,\overline{k}})_\mathfrak{m} = (\ker \delta_{M,k})_\mathfrak{m} \otimes_k \overline{k} \neq 0.$$

Let $(f, g) \in \ker \delta_{M,\overline{k}}$. We have $g = -\eta f \in S_M(U^\mathfrak{p} H, \overline{k})$, where H is the subgroup of $\mathrm{GL}_2(D)$ generated by $U_\mathfrak{p}$ and $\eta U_\mathfrak{p} \eta^{-1}$.

Claim. H contains $\mathrm{SL}_2(D)$, the subgroup of all elements of $\mathrm{GL}_2(D)$ with reduced norm equal to 1.

Proof of Claim. Let

$$U^+ := \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \in \mathrm{SL}_2(D) \right\}, \qquad U^- := \left\{ \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix} \in \mathrm{SL}_2(D) \right\}.$$

U^+ and U^- generate $\mathrm{SL}_2(D)$. Let us briefly justify this:

(a) Let $\mathcal{T}$ be the subgroup of $\mathrm{SL}_2(D)$ generated by U^+ and U^-. We have

$$\begin{pmatrix} 1 & \\ x^{-1} - x^{-1}y^{-1}x^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ x^{-1} - y & 1 \end{pmatrix} \begin{pmatrix} 1 & y^{-1} \\ & 1 \end{pmatrix} = \begin{pmatrix} xy & \\ & x^{-1}y^{-1} \end{pmatrix}$$

for all $x, y \in D^\times$. In particular,

$$\begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} [x, y] & \\ & 1 \end{pmatrix} = \begin{pmatrix} xy & \\ & x^{-1}y^{-1} \end{pmatrix} \begin{pmatrix} (yx)^{-1} & \\ & yx \end{pmatrix}$$

lie in $\mathcal{T}$ for all $x, y, z \in D^\times$, i.e. $\mathcal{T}$ contains all diagonal matrices

$$\begin{pmatrix} x & \\ & y \end{pmatrix} \qquad (x, y \in D^\times)$$

such that xy lies in the commutator subgroup of $D^\times$. The commutator subgroup of $D^\times$ is the kernel of the reduced norm map (this is a result of Nakayama and Matsushima [**NM43**]), so $\mathcal{T}$ contains all diagonal matrices in $\mathrm{SL}_2(D)$.

(b) Consider an element

$$t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $\mathrm{SL}_2(D)$ with $d \neq 0$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & \\ & d \end{pmatrix} \begin{pmatrix} 1 & \\ d^{-1}c & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} a - bd^{-1}c & \\ & d \end{pmatrix}$$

is a diagonal element of $\mathrm{SL}_2(D)$ (so it lies in $\mathcal{T}$, by (a)). Hence, t lies in $\mathcal{T}$.

(c) Finally, any element $t \in \mathrm{SL}_2(D)$ of the form

$$t = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$$

must satisfy $b, c \neq 0$, so

$$t = \begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ a+c & b \end{pmatrix} \in \mathcal{T}.$$

This shows that $\mathcal{T} = \mathrm{SL}_2(D)$.

Since

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} U^+ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = U^- \text{ and } \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_D) \subset H,$$

it suffices to show $U^+ \subset H$. Let

$$U^+(n) := \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in \mathrm{SL}_2(D) : x \in \varpi_D^{-n}\mathcal{O}_D \right\}.$$

We have $U^+(0) \subset \mathrm{GL}_2(\mathcal{O}_D) \subset H$. Moreover,

$$\begin{pmatrix} \varpi_D & \\ & \varpi_D^{-1} \end{pmatrix} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \cdot \eta \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \eta^{-1} \in H$$

and

$$\begin{pmatrix} \varpi_D & \\ & \varpi_D^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi_D & \\ & \varpi_D^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \varpi_D^{-1} x \varpi_D^{-1} \\ & 1 \end{pmatrix};$$

we conclude that if $U^+(n) \subset H$, then $U^+(n+2) \subset H$. Hence, $U^+ \subset H$, as desired. □

Recall that G_1 is the derived subgroup of G. We have $G_1(F_\mathfrak{p}) = \mathrm{SL}_2(D)$.

CLAIM. $g = -\eta f$ is invariant under (right) translation by $G_1(\mathbb{A}_F^\infty)$.

PROOF OF CLAIM. Let $\widetilde{U}_\ell \subset \prod_{v|\ell} U_v$ be an open compact subgroup of $\mathcal{G}(\mathbb{Z}_\ell)$ such that $\rho|_{\widetilde{U}_\ell}$ acts trivially on $M \otimes_\mathcal{O} \overline{k}$. Then g is invariant under right translation by $\widetilde{U}^\mathfrak{p} = U^{\ell,\mathfrak{p}}\widetilde{U}_\ell$. Let $W^\mathfrak{p} := \widetilde{U}^\mathfrak{p} \cap G_1(\mathbb{A}_F^{\infty,\mathfrak{p}})$. The function g is constant on double cosets of the form

$$G(F)xW^\mathfrak{p}\mathrm{SL}_2(D) = (G(F)xW^\mathfrak{p}x^{-1})x\mathrm{SL}_2(D) = G(F)G_1(\mathbb{A}_F^{\infty,\mathfrak{p}})x\mathrm{SL}_2(D) = G(F)xG_1(\mathbb{A}_F^\infty)$$

(the second equality follows by strong approximation: $G_1(F)$ is dense in $G_1(\mathbb{A}_F^{\infty,\mathfrak{p}})$).

$\square$

Let $\widetilde{U} = U^\ell \widetilde{U}_\ell \subset U$ be an open compact subgroup of $G(\mathbb{A}_F^\infty)$, where $\widetilde{U}_\ell \subset \mathcal{G}(\mathbb{Z}_\ell)$ is chosen such that $\rho|_{\widetilde{U}_\ell}$ acts trivially on $M \otimes_{\mathcal{O}} \overline{k}$. The second claim implies that, for any $(f, g) \in \ker \delta_{M,\overline{k}}$, we have $g = \widetilde{g} \circ \det$, where $\widetilde{g}$ is a $(M \otimes_{\mathcal{O}} \overline{k})$-valued function on the finite abelian group

$$C := \det G(F) \backslash (\mathbb{A}_E^\infty)^{\mathbb{N}_{E/F}=1} / \det \widetilde{U}.$$

Recall that $\det G(F) = U_{E/F}(1)(F) = (E^\times)^{\mathbb{N}_{E/F}=1}$, by Proposition 1.2. Hence, $G(\mathbb{A}_F^\infty)$ acts on $\ker \delta_{M,\overline{k}}$ by right translation, and the action factors through the map

$$G(\mathbb{A}_F^\infty) \xrightarrow{\det} (\mathbb{A}_E^\infty)^{\mathbb{N}_{E/F}=1} \to C.$$

The actions of C and $\mathbf{T}$ commute, so $(\ker \delta_{M,\overline{k}})_{\mathfrak{m}}$ is stable under the action of C.

Let $(f, g) \in (\ker \delta_{M,\overline{k}})_{\mathfrak{m}}$ be a simultaneous eigenvector for all $t \in C$: Thus, there exists a character $\chi : C \to \overline{k}^\times$ such that

$$g(xt) = \chi(\det t)\, g(x) \quad \text{for all } x \in G(\mathbb{A}_F^\infty), t \in C.$$

We compute the action of the Hecke operators $T_{w,j}$ on g: The operator $T_{w,j}$ (where $w \mid R$, χ unramified at w) acts on g by multiplication by the scalar:

$$\lambda_{w,j} = \sum_{x \in \mathrm{GL}_N(\mathcal{O}_{F_v})\beta_{v,j}\mathrm{GL}_N(\mathcal{O}_{F_v})/\mathrm{GL}_N(\mathcal{O}_{F_v})} \chi \circ \det(\theta_w^{-1}(\beta_{v,j})),$$

where $\beta_{v,j}$ denotes the matrix

$$\begin{pmatrix} \varpi_v 1_j & \\ & 1_{N-j} \end{pmatrix} \in \mathrm{GL}_N(F_v).$$

We have

$$\lambda_{w,j} = |\mathrm{GL}_N(\mathcal{O}_{F_v})\beta_{v,j}\mathrm{GL}_N(\mathcal{O}_{F_v})/\mathrm{GL}_N(\mathcal{O}_{F_v})|\, \chi_v(i_w^{-1}(\varpi_v))^j,$$

where $v \in R$ is the place of F below w and i_w denotes the canonical isomorphism $U_{E/F}(1)(F_v) \cong F_v^\times$ corresponding to the choice of the place w above v. Let

$$\widetilde{\chi} : E^\times \backslash (\mathbb{A}_E^\infty)^\times \to \overline{k}^\times, \quad \widetilde{\chi}(x) := \chi\left(\frac{x}{x^c}\right),$$

so that $\chi_v(i_w^{-1}(\varpi_v)) = \widetilde{\chi}_w(\varpi_w)$. We have a bijection

$$\mathrm{GL}_N(\mathcal{O}_{F_v})\beta_{v,j}\mathrm{GL}_N(\mathcal{O}_{F_v})/\mathrm{GL}_N(\mathcal{O}_{F_v}) \xrightarrow{\sim} \mathrm{GL}_N(\mathcal{O}_{F_v})/\mathrm{GL}_N(\mathcal{O}_{F_v}) \cap \beta_{v,j}\mathrm{GL}_N(\mathcal{O}_{F_v})\beta_{v,j}^{-1}$$

(given by $[x\beta_{v,j}] \mapsto [x]$, for $x \in \mathrm{GL}_N(\mathcal{O}_{F_v})$), and

$$\mathrm{GL}_N(\mathcal{O}_{F_v}) \cap \beta_{v,j}\mathrm{GL}_N(\mathcal{O}_{F_v})\beta_{v,j}^{-1} = \begin{pmatrix} \mathrm{GL}_j(\mathcal{O}_{F_v}) & \varpi_v \mathrm{Mat}_{j\times(N-j)}(\mathcal{O}_{F_v}) \\ \mathrm{Mat}_{(N-j)\times j}(\mathcal{O}_{F_v}) & \mathrm{GL}_{N-j}(\mathcal{O}_{F_v}) \end{pmatrix}.$$

Let k_v be the residue field of $\mathcal{O}_{F_v}$, and let P_j denote the parabolic subgroup

$$\begin{pmatrix} \mathrm{GL}(j) & 0 \\ * & \mathrm{GL}(N-j) \end{pmatrix}$$

of $\mathrm{GL}(N)$. Reduction mod ϖ_v gives a bijection

$$\mathrm{GL}_N(\mathcal{O}_{F_v})/\mathrm{GL}_N(\mathcal{O}_{F_v}) \cap \beta_{v,j}\mathrm{GL}_N(\mathcal{O}_{F_v})\beta_{v,j}^{-1} \xrightarrow{\sim} \mathrm{GL}_N(k_v)/P_j(k_v).$$

We have

$$\lambda_{w,j} = |\mathrm{GL}_N(k_v)/P_j(k_v)|\, \widetilde{\chi}_w(\varpi_w)^j = q^{-j(N-j)} \frac{|\mathrm{GL}_N(k_v)|}{|\mathrm{GL}_j(k_v)| \cdot |\mathrm{GL}_{N-j}(k_v)|}\, \widetilde{\chi}_w(\varpi_w)^j,$$

where $q = q_w := \mathbb{N}w = |k_v|$. We have

$$|\mathrm{GL}_m(k_v)| = \prod_{i=0}^{m-1}(q^m - q^i) = q^{\frac{m(m-1)}{2}} \prod_{i=1}^{m}(q^i - 1)$$

and, therefore,

$$\lambda_{w,j} = \begin{bmatrix} N \\ j \end{bmatrix}_q \widetilde{\chi}_w(\varpi_w)^j, \quad \text{where } \begin{bmatrix} N \\ j \end{bmatrix}_q := \frac{\prod_{i=1}^{N}(q^i - 1)}{\left(\prod_{i=1}^{j}(q^i-1)\right) \cdot \left(\prod_{i=1}^{N-j}(q^i-1)\right)}.$$

Remark. The $\begin{bmatrix} N \\ j \end{bmatrix}_q$ are the q-binomial coefficients. They satisfy the *q-binomial theorem*:

$$\prod_{j=0}^{N-1}(X + q^j) = \sum_{j=0}^{N} q^{\frac{(N-j)(N-j-1)}{2}} \begin{bmatrix} N \\ j \end{bmatrix}_q X^j.$$

We have a map $\phi : \mathbf{T} \to \overline{k}$ sending $T \in \mathbf{T}$ to its (f,g)-eigenvalue. The map ϕ factors through $\mathbf{T}_{\mathfrak{m}}$ and, hence, $\ker\phi = \mathfrak{m}$. We have $\phi(T_{w,j}) = \lambda_{w,j}$ (for all w as above, $1 \le j \le N$). In particular, the characteristic polynomial of $\overline{r_\phi}(\mathrm{Frob}_w)$ is

$$\sum_{j=0}^{N}(-1)^{N-j} q_w^{\frac{(N-j)(1-j-N)}{2}} \begin{bmatrix} N \\ j \end{bmatrix}_{q_w} \widetilde{\chi}_w(\varpi_w)^{j-N} X^j = \prod_{j=0}^{N-1}(X - \widetilde{\chi}_w(\varpi_w)^{-1} q_w^{1+j-N}).$$

We conclude that $\overline{r_\phi}$ is the form $\overline{\psi} \otimes (1 \oplus \overline{\varepsilon}_\ell^{-1} \oplus \cdots \oplus \overline{\varepsilon}_\ell^{1-N})$, where $\overline{\psi} = \widetilde{\chi} \circ \mathrm{Art}_E^{-1}$ (so that $\overline{\psi}(\mathrm{Frob}_w) = \widetilde{\chi}_w(\varpi_w)^{-1}$ for almost all $w \mid R$). $\square$

2. Proof of the main theorem

We define the following Hecke operators at the place $\mathfrak{p}$:

$$T_{\mathfrak{p},1} := \left[U_{\mathfrak{p}} \begin{pmatrix} \varpi_D & \\ & 1 \end{pmatrix} U_{\mathfrak{p}} \right] = [U_{\mathfrak{p}}\, \eta\, U_{\mathfrak{p}}], \quad T_{\mathfrak{p},2} := \left[U_{\mathfrak{p}} \begin{pmatrix} \varpi_D & \\ & \varpi_D \end{pmatrix} U_{\mathfrak{p}} \right] = [\varpi_D\, U_{\mathfrak{p}}].$$

and let a_i be the eigenvalue of $T_{\mathfrak{p},i}$ on $\pi_{\mathfrak{p}}^{U_{\mathfrak{p}}}$. We remark that $\pi_{\mathfrak{p}}$ is isomorphic to the (normalised) parabolic induction of $(\xi_1 \circ \mathrm{Nrd}) \boxtimes (\xi_2 \circ \mathrm{Nrd})$ to $\mathrm{GL}_2(D)$. In particular, the eigenvalues of the Hecke operators $T_{\mathfrak{p},1}$, $T_{\mathfrak{p},2}$ are $a_1 = q^{m/2}\big(\xi_1(\varpi_{\mathfrak{p}}) + \xi_2(\varpi_{\mathfrak{p}})\big)$ and $a_2 = \xi_1(\varpi_{\mathfrak{p}})\,\xi_2(\varpi_{\mathfrak{p}})$ (where $\varpi_{\mathfrak{p}}$ denotes a uniformiser of $F_{\mathfrak{p}}$). Hence, our local congruence condition satisfied by $\Pi_{\mathfrak{P}}$, can be written as

$$\iota^{-1}(a_1^2 a_2^{-1}) \equiv (q^m+1)^2 \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_\ell}.$$

Recall that we also assume that the residual Galois representation of π is not a twist of

$$1 \oplus \overline{\varepsilon}_\ell^{-1} \oplus \cdots \oplus \overline{\varepsilon}_\ell^{1-N}.$$

Let $\mathfrak{m}$ be the maximal ideal containing the kernel of $\mathbf{T} \to \mathrm{End}_{\overline{K}}(\iota^{-1}\pi^U)$. Then $\delta_{M,k}$ is injective after localisation at $\mathfrak{m}$ (by Ihara's lemma). In particular, $(\delta_M)_{\mathfrak{m}}$ has torsion-free cokernel. Let $S_M(U_1, \mathcal{O})^{\mathrm{new}}$ be the cokernel of δ_M. We have a short exact sequence:

$$0 \to S_M(U,\mathcal{O})_{\mathfrak{m}}^{\oplus 2} \xrightarrow{(\delta_M)_{\mathfrak{m}}} S_M(U_1,\mathcal{O})_{\mathfrak{m}} \to S_M(U_1,\mathcal{O})_{\mathfrak{m}}^{\mathrm{new}} \to 0.$$

PROPOSITION 3.5. *Under the assumptions above, we have* $S_M(U_1,\mathcal{O})_{\mathfrak{m}}^{\mathrm{new}} \neq 0$.

PROOF. It suffices to show that

$$(\delta_{M,k})_{\mathfrak{m}} : S_M(U,k)_{\mathfrak{m}}^{\oplus 2} \to S_M(U_1,k)_{\mathfrak{m}}$$

is not surjective. First, we compute the adjoint map of the map

$$(\delta_{M^\vee,k})_{\mathfrak{m}^\vee} : S_{M^\vee}(U,k)_{\mathfrak{m}^\vee}^{\oplus 2} \to S_{M^\vee}(U_1,k)_{\mathfrak{m}^\vee}$$

with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_U$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{U_1}$:

(a) The adjoint of the map $S_{M^\vee}(U,k) \hookrightarrow S_{M^\vee}(U_1,k)$, $f \mapsto f$ is

$$S_M(U_1,k) \to S_M(U,k), \quad f \mapsto \mathrm{tr}_{U/U_1} f = \sum_{x \in U/U_1} xf.$$

(b) The adjoint of the map $S_{M^\vee}(U,k) \hookrightarrow S_{M^\vee}(U_1,k)$, $f \mapsto \eta f$ is

$$S_M(U_1,k) \to S_M(U,k), \quad f \mapsto \mathrm{tr}_{U/U_1}\eta^{-1}f = \sum_{x\in U/U_1} x\eta^{-1}f.$$

(c) The idempotent $e_{\mathfrak{m}} \in \mathbf{T}$ commutes with the maps above, so $(\delta_{M^\vee,k})^\vee_{\mathfrak{m}^\vee}$ is given by

$$S_M(U_1,k)_{\mathfrak{m}} \to S_M(U,k)^{\oplus 2}_{\mathfrak{m}}, \quad f \mapsto (\mathrm{tr}_{U/U_1} f, \mathrm{tr}_{U/U_1}\eta^{-1}f).$$

Fix k-embedding $\phi : \mathbf{T}/\mathfrak{m} \hookrightarrow \overline{k}$. We write $\overline{r_{\mathfrak{m}}}$ for $\overline{r_\phi}$; we also write $\overline{r_{\mathfrak{m}^\vee}}$ for $\overline{r_{\phi^\vee}}$, where $\phi^\vee : \mathbf{T}^\vee/\mathfrak{m}^\vee \hookrightarrow \overline{k}$ is defined by $\phi^\vee(T) = \phi(T^\vee)$. We have $\overline{r_{\mathfrak{m}^\vee}} \cong \overline{r_{\mathfrak{m}}}^c \cong \overline{r_{\mathfrak{m}}}^\vee \otimes \overline{\varepsilon}_\ell^{1-N}$. It follows, by Lemma 3.4, that $(\delta_{M^\vee,k})_{\mathfrak{m}^\vee}$ is injective and, thus, $(\delta_{M^\vee,k})^\vee_{\mathfrak{m}^\vee}$ is surjective.

It suffices to show that the map

$$\Delta_{\mathfrak{m}} = (\delta_{M^\vee,k})^\vee_{\mathfrak{m}^\vee} \circ (\delta_{M,k})_{\mathfrak{m}} : S_M(U,k)^{\oplus 2}_{\mathfrak{m}} \to S_M(U,k)^{\oplus 2}_{\mathfrak{m}}$$

is not surjective. The map above is given by

$$\begin{pmatrix} q^m+1 & T_{\mathfrak{p},1} \\ T_{\mathfrak{p},1}T_{\mathfrak{p},2}^{-1} & q^m+1 \end{pmatrix}.$$

(Note that $(U:U_1) = |\mathcal{O}_D/\varpi_D| + 1 = q^m+1$, where $m = N/2$).

Let f be a Hecke eigenfunction in $S_M(U,\overline{K})$ corresponding to an element of $(\pi^\infty)^U$ (under the isomorphism of Proposition 9). Enlarging K, we may assume that $f \in S_M(U,K)$. Rescaling f, we may assume that $f \in S_M(U,\mathcal{O})$, $f \not\equiv 0 \bmod \varpi$. Then the image $\overline{f}$ of f in $S_M(U,k)_{\mathfrak{m}}$ is non-zero. The Hecke operator $T_{\mathfrak{p},i}$ acts on $\overline{f}$ by multiplication by the scalar $\overline{a}_i := \iota^{-1}(a_i) \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_\ell}$ $(i=1,2)$. Since

$$\det\begin{pmatrix} q^m+1 & \overline{a}_1 \\ \overline{a}_1\overline{a}_2^{-1} & q^m+1 \end{pmatrix} = 0,$$

there is a non-zero linear combination of $(\overline{f},0)$ and $(0,\overline{f})$ in the kernel of $\Delta_{\mathfrak{m}}$. In particular, $\Delta_{\mathfrak{m}}$ is not invertible and, thus, not surjective. ☐

We finally turn to the proof of the main theorem of this chapter:

Proof of Theorem 3.2. We have a short exact sequence:

$$0 \to S_M(U,\overline{K})^{\oplus 2}_{\mathfrak{m}} \xrightarrow{(\delta_M)_{\mathfrak{m}}\otimes \mathrm{id}_{\overline{K}}} S_M(U_1,\overline{K})_{\mathfrak{m}} \to S_M(U_1,\overline{K})^{\mathrm{new}}_{\mathfrak{m}} \to 0,$$

where $S_M(U_1,\overline{K})^{\mathrm{new}}_{\mathfrak{m}} = S_M(U_1,\mathcal{O})^{\mathrm{new}}_{\mathfrak{m}} \otimes_{\mathcal{O}} \overline{K} \neq 0$. Under the isomorphism of Proposition 2.11, the cokernel of the map $(\delta_M)_{\mathfrak{m}} \otimes \mathrm{id}_{\overline{K}}$ is given by

$$\bigoplus_{\sigma} m(\sigma)\ \dim(\sigma^{\mathfrak{p},\infty})^{U^{\mathfrak{p}}} \left[\sigma_{\mathfrak{p}}^{U_{1,\mathfrak{p}}}/\langle \sigma_{\mathfrak{p}}^{U_{\mathfrak{p}}}, \sigma_{\mathfrak{p}}^{\eta U_{\mathfrak{p}}\eta^{-1}}\rangle\right],$$

where the sum runs through all automorphic representations $\sigma \subset L^2(G(F)\backslash G(\mathbb{A}_F))$ with $\sigma^{U_1} \neq 0$, $\sigma_\infty = W_\infty^\vee$, and the same residual Galois representation as π. Moreover, $\sigma_{\mathfrak{p}}^{U_{1,\mathfrak{p}}}/\langle \sigma_{\mathfrak{p}}^{U_{\mathfrak{p}}}, \sigma_{\mathfrak{p}}^{\eta U_{\mathfrak{p}}\eta^{-1}}\rangle$ is non-trivial if and only if $\sigma_{\mathfrak{p}}$ is an unramified twist of the Steinberg representation St_2 of $\mathrm{GL}_2(D)$. Indeed, if τ is an irreducible admissible representations of $\mathrm{GL}_2(D)$ with $U_{1,\mathfrak{p}}$-fixed vectors, then τ is isomorphic to one of the following representations:

(1) $\tau \cong \chi \circ \det$, where $\det : \mathrm{GL}_2(D) \to F_{\mathfrak{p}}^\times$ denotes the reduced norm and χ is an unramified character of $F_{\mathfrak{p}}^\times$.
(2) τ is (isomorphic to) the (normalised) parabolic induction of $\chi_1 \boxtimes \chi_2$ to $\mathrm{GL}_2(D)$, where χ_1, χ_2 are unramified characters of $D^\times$ and $\chi_1 \neq \chi_2 \otimes |\mathrm{Nrd}|^{\pm 1}$. Then τ satisfies $\dim \tau^{U_{\mathfrak{p}}} = 1$ and $\dim \tau^{U_{1,\mathfrak{p}}} = 2$ (and, thus, we also have $\dim \tau^{\eta U_{\mathfrak{p}}\eta^{-1}} = 1$).
(3) $\tau \cong \mathrm{St}_2(\chi) := \mathrm{St}_2 \otimes (\chi \circ \det)$, where χ is an unramified character of $F_{\mathfrak{p}}^\times$ and St_2 is the unique irreducible quotient of the parabolic induction $|\mathrm{Nrd}|^{-\frac{1}{2}} \boxtimes |\mathrm{Nrd}|^{\frac{1}{2}}$. In this case, τ satisfies $\dim \tau^{U_{\mathfrak{p}}} = 0$ and $\dim \tau^{U_{1,\mathfrak{p}}} = 1$.

We remark that St_2 corresponds to the representation St_N of $\mathrm{GL}_N(F)$, under the Jacquet-Langlands correspondence.

Remark. The Hecke algebra of compactly supported $U_{1,\mathfrak{p}}$-biinvariant functions

$$f : \mathrm{GL}_2(D) \to \mathbb{C}$$

is isomorphic to the (non-commutative) $\mathbb{C}$-algebra $\mathscr{H}$ generated by the generators ρ, ξ, ξ' with the relations

$$(\rho+1)(\rho - q^m) = 0, \quad \xi \cdot \xi' = \xi' \cdot \xi = 1, \quad \rho \cdot \xi^2 = \xi^2 \cdot \rho.$$

This is proved in [**GSZ01**], Section 4. Irreducible representations of $\mathrm{GL}_2(D)$ with $U_{1\mathfrak{p}}$-vectors are in bijection with simple $\mathscr{H}$-modules; classifying all simple modules over $\mathscr{H}$ leads to the above classification.

We conclude, under the assumptions above, that there is an automorphic representation $\sigma \subset L^2(G(F)\backslash G(\mathbb{A}_F))$ with $\sigma^{U^{\mathfrak{p}}} \neq 0$, $\sigma_\infty = \pi_\infty$, the same residual Galois representation as π, and $\sigma_{\mathfrak{p}} \cong \mathrm{St}_2(\chi)$ for some unramified character $\chi : F_{\mathfrak{p}}^\times \to \mathbb{C}^\times$.

We take Π' to be the base change of σ to $\mathrm{GL}_N(\mathbb{A}_E)$ (cf. Theorem 2.2); Π' is necessarily cuspidal because $\Pi'_{\mathfrak{P}}$ is an unramified twist of the Steinberg representation. Finally, Π' satisfies $(\Pi')^{\mathcal{U}_w} \neq 0$ for all places $w \neq \mathfrak{P}$, since

(i) $\Pi'_w \cong \sigma_v \circ \theta_w$, for all places w of E which are split over a place v of F and are such that G_{F_v} is quasi-split.

(ii) Π'_w is unramified, for all places w of E which are inert over F.

This completes the proof. □

3. An inductive level raising theorem

Recall that we have fixed an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. As before, E denotes a CM number field with maximal totally real subfield F. The following theorem follows, by induction on k, from our previous theorems:

THEOREM 3.6. *Let Π_i $(i = 1, 2, \cdots, 2^k)$ be conjugate self-dual, unitary cupsidal automorphic representations of $\mathrm{GL}_m(\mathbb{A}_E)$. We write Π for*

$$\Pi_1 \boxplus \Pi_2 \boxplus \cdots \boxplus \Pi_{2^k},$$

a conjugate self-dual, unitary representation of $\mathrm{GL}_N(\mathbb{A}_E)$, where $N = 2^k m$.

We assume that

(a) Π is regular C-algebraic (in particular, $\Pi_i \| \det \|^{\frac{1}{2}}$ is regular L-algebraic for all $i = 1, 2, \cdots, 2^k$).

(b) For some finite place $\mathfrak{P}$ of E which does not divide ℓ, $\Pi_{i,\mathfrak{P}}$ is isomorphic to an unramified twist of the Steinberg representation of $\mathrm{GL}_m(E_{\mathfrak{P}})$:

$$\Pi_{i,\mathfrak{P}} \cong \mathrm{St}_m(\xi_i) := \mathrm{St}_m \otimes (\xi_i \circ \det),$$

for some unramified character $\xi_i : E_{\mathfrak{P}}^\times \to \mathbb{C}^\times$ $(i = 1, 2, \cdots, 2^k)$.

(c) The following local congruence conditions at the place $\mathfrak{P}$ are satisfied: If ϖ is a uniformiser for $E_{\mathfrak{P}}$, then

$$\iota^{-1}(\xi_i(\varpi)/\xi_{i+1}(\varpi)) \equiv (\mathbb{N}\mathfrak{P})^{-m} \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_\ell}$$

for all $1 \leq i < 2^k$.

(d) For any $1 \leq r \leq k$ and any $0 \leq s < 2^{k-r}$, the residual $\bmod\ \ell$ Galois representation

$$\overline{r_\iota}\big(\boxplus_{i=2^r s+1}^{2^r(s+1)} \Pi_i \big) : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_{2^r m}(\overline{\mathbb{F}}_\ell)$$

of $\boxplus_{i=2^r s+1}^{2^r(s+1)}\Pi_i$ *is not isomorphic to a twist of*

$$1 \oplus \overline{\varepsilon}_\ell^{-1} \oplus \cdots \oplus \overline{\varepsilon}_\ell^{1-2^r m}.$$

Then there exists a CM biquadratic extension E_1/E *and, for each choice of a place* $\mathfrak{P}_1$ *of* E_1 *above* $\mathfrak{P}$*, a regular* C*-algebraic, conjugate self-dual, unitary cuspidal automorphic representation* Π' *of* $\mathrm{GL}_N(\mathbb{A}_{E_1})$*, such that:*

(i) $\overline{r_\iota}(\Pi')$ *is isomorphic to* $\overline{r_\iota}(\Pi)|_{\mathrm{Gal}(\overline{E_1}/E_1)}$.
(ii) $\Pi'_{\mathfrak{P}_1}$ *is isomorphic to an unramified twist of the Steinberg representation of* $\mathrm{GL}_N(E_{\mathfrak{P}_1})$.
(iii) If Π_w *is unramified, for some finite place* $w \neq \mathfrak{P}$ *of* E*, then* Π'_{w_1} *is unramified for all places* $w_1 \mid w$ *of* E_1.

PROOF. Let S be a finite set of places of F containing $\mathfrak{p} := \mathfrak{P} \cap F$, all places that ramify in E, and all places v such that Π_w is ramified for some place $w \mid v$. We choose a totally real quadratic extension F_0/F, as in Lemma 2.4, such that any place v_0 of F_0 lying over a place in S splits in $E_0 = E \cdot F_0$ and $\mathrm{Gal}(K \cdot E_0/E_0) \to \mathrm{Gal}(K/E)$ is an isomorphism, where K/E is a finite Galois extension such that

$$\overline{r_\iota}\big(\boxplus_{i=2^r s+1}^{2^r(s+1)}\Pi_i\big) : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_{2^r m}(\overline{\mathbb{F}}_\ell)$$

factors through $\mathrm{Gal}(K/E)$ for all $1 \le r \le k$ and $0 \le s < 2^{k-r}$.

For each $1 \le r \le k$, we choose an extended pure inner twist (G_r, ξ_r, z_r) of $U_{E_0/F_0}(2^r m)$, such that

(i) $G_r(F_0 \otimes_{\mathbb{Q}} \mathbb{R})$ is compact,
(ii) $G_{r,F_{\mathfrak{p}_0}}$ is isomorphic to $\mathrm{GL}(2, D_{r,\mathfrak{p}_0})$ for any place $\mathfrak{p}_0 \mid \mathfrak{p}$ of F_0, where $D_{r,\mathfrak{p}_0}$ is a central division $F_{\mathfrak{p}_0}$-algebra, and
(iii) $G_{r,F_{v_0}}$ is quasi-split for all finite places $v_0 \nmid \mathfrak{p}$ of F_0.

Following the proof of Lemma 2.4, we choose $F_1 = F_0(\sqrt{d})$ (where $d > 1$ is a square-free integer) such that:

(i) If for some place v_0 of F_0 and some $1 \le r \le k$, the cocycle z_r has non-trivial image under the localisation map $B(F_0, G_r)_{\mathrm{bsc}} \to B(F_{0,v_0} G_r)_{\mathrm{bsc}}$, then v_0 splits in F_1.
(ii) $\mathrm{Gal}(K \cdot E_1/E_1) \to \mathrm{Gal}(K \cdot E_0/E_0)$ is an isomorphism, where $E_1 = E \cdot F_1 = E_0(\sqrt{d})$.

Suppose that τ is a regular C-algebraic representation of $\mathrm{GL}_{2^r m}(\mathbb{A}_E)$ (for some $1 \leq r \leq k$) of the form $\tau = \tau_1 \boxplus \tau_2$, where τ_i are conjugate self-dual, unitary cuspidal automorphic representations of $\mathrm{GL}_{2^{r-1}m}(\mathbb{A}_E)$ and $\tau_{i,\mathfrak{P}}$ are isomorphic to unramified twists of the Steinberg representation of $\mathrm{GL}_{2^{r-1}m}(E_{\mathfrak{P}})$. Then there exists a subrepresentation π of $L^2(G_r(F_1)\backslash G_r(\mathbb{A}_{F_1}))$ such that $r_\iota(\pi) \cong r_\iota(\tau)|_{\mathrm{Gal}(\overline{E_1}/E_1)}$.

Fix a place $\mathfrak{P}_1 \mid \mathfrak{P}$ of E_1. Theorem 3.2 implies that there exist conjugate self-dual, unitary cuspidal automorphic representations Π'_i of $\mathrm{GL}_{2m}(\mathbb{A}_{E_1})$ ($i = 1, 2, \cdots, 2^{k-1}$) such that

(i) $\overline{r_\iota}(\Pi'_i) \cong \overline{r_\iota}(\Pi_{2i-1} \boxplus \Pi_{2i})|_{\mathrm{Gal}(\overline{E_1}/E_1)}$ for all $1 \leq i \leq 2^{k-1}$.

(ii) $\Pi'_{i,\mathfrak{P}_1} \cong \mathrm{St}_{2m}(\xi'_i)$, where ξ'_i is an unramified character of $E^\times_{\mathfrak{P}_1}$, for all $1 \leq i \leq 2^{k-1}$. We remark that condition (i) implies that

$$\iota^{-1}\xi'_i \equiv \iota^{-1}(\xi_{2i-1} \circ \mathbb{N}_{E_{1,\mathfrak{P}_1}/E_{\mathfrak{P}}}) \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_\ell}.$$

(iii) For any archimedean place w_1 of E_1 and any $1 \leq i \leq 2^{k-1}$, Π'_{i,w_1} is isomorphic to $\Pi_{2i-1,w} \boxplus \Pi_{2i,w}$ (where we write w for the restriction of w_1 to E and we identify E_{1,w_1} and E_w). In particular, $\Pi'_1 \boxplus \cdots \boxplus \Pi'_{2^{k-1}}$ is regular C-algebraic.

We can now apply Theorem 3.2, in the same way as above, to Π'_i ($i = 1, 2, \cdots, 2^{k-1}$). The theorem follows by induction on k. □

4. An application of the theorem

The following result is a corollary of Theorem 3.6.

Corollary 3.7. *Let $K/\mathbb{Q}$ be an imaginary quadratic field, let $\ell > 3$ be a rational prime and let $\overline{\psi} : \mathrm{Gal}(\overline{K}/K) \to \overline{\mathbb{F}}_\ell^\times$ be a continuous character. Let*

$$\overline{r} := \overline{\psi} \oplus \overline{\psi}^c : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell).$$

Then, for every integer $k \geq 1$, there exists a CM biquadratic extension E/K, and a regular C-algebraic, essentially conjugate self-dual, unitary cuspidal automorphic representation Π of $\mathrm{GL}_{2^k}(\mathbb{A}_E)$, such that

(i) $\overline{r_\iota}(\Pi) \cong (\mathrm{Sym}^{2^k-1}\,\overline{r})|_{\mathrm{Gal}(\overline{E}/E)}$, and

(ii) There exists a finite place w of E such that Π_w is isomorphic to an unramified twist of the Steinberg representation of $\mathrm{GL}_{2^k}(E_w)$.

Proof. Choose a continuous character $\overline{\omega} : \mathrm{Gal}(\overline{K}/K) \to \overline{\mathbb{F}}_\ell^\times$ such that

$$\overline{\omega} \cdot \overline{\omega}^c = (\overline{\psi} \cdot \overline{\psi}^c \cdot \overline{\varepsilon}_\ell)^{-1}$$

(by Lemma 4.1.5 of [**CHT08**], there exists a continuous algebraic character

$$\omega_0 : \mathrm{Gal}(\overline{K}/K) \to \overline{\mathbb{Q}}_\ell^\times$$

such that $\omega_0 \cdot \omega_0^c = \varepsilon_\ell^{-1}$; we can take $\overline{\omega} = \overline{\omega}_0 \cdot \overline{\psi}^{-1}$, where $\overline{\omega}_0$ is the mod ℓ reduction of ω_0). We write $\overline{\rho}$ for $\overline{r} \otimes \overline{\omega}$. Then $\overline{\rho}$ satisfies $\overline{\rho}^{c,\vee} \cong \overline{\rho} \otimes \overline{\varepsilon}_\ell$ and

$$\mathrm{Sym}^d \overline{\rho} \cong (\mathrm{Sym}^d \overline{r}) \otimes \overline{\omega}^d = \left(\bigoplus_{i=0}^{d} \overline{\psi}^{d-i} \cdot (\overline{\psi}^c)^i \right) \otimes \overline{\omega}^d$$

for all integers $d \geq 1$. Using Lemma 4.1.6 of [**CHT08**], we choose Hecke characters

$$\chi, \phi : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$$

such that

(i) $\chi^c = \chi^{-1}$ and $\phi^c = \phi^{-1}$,
(ii) χ, $\phi\| \cdot \|^{-\frac{d}{2}}$ are algebraic and χ_∞ is non-trivial,
(iii) $\overline{r_\iota}(\chi) = \overline{\psi}^{-1} \cdot \overline{\psi}^c$ and $\overline{r_\iota}(\phi\| \cdot \|^{-\frac{d}{2}}) = (\overline{\psi} \cdot \omega)^d$.

Then $\sigma := \boxplus_{i=0}^{d} (\phi \cdot \chi^i)$ is an isobaric sum of unitary, conjugate self-dual (cuspidal) automorphic representations of $\mathrm{GL}_1(\mathbb{A}_K) = \mathbb{A}_K^\times$.

Let $d = 2^k - 1$. We apply Theorem 3.6 to σ, taking $\Pi_i := \phi \cdot \chi^{i-1}$ $(1 \leq i \leq 2^k)$. We note that σ is a regular C-algebraic representation with $\overline{r_\iota}(\sigma) \cong \mathrm{Sym}^d \overline{\rho}$.

We claim that there is a finite place $\mathfrak{P}$ of K such that conditions (b) and (c) are satisfied at $\mathfrak{P}$. Indeed, by Chebotarev density theorem, $(\overline{r_\iota}(\chi) \oplus \overline{\varepsilon}_\ell)(\mathrm{Frob}_{\mathfrak{P}})$ is trivial for infinitely many primes $\mathfrak{P}$. Let $\mathfrak{P}$ be such a prime, not dividing ℓ, and such that ϕ, χ and $\overline{\omega}$ are unramified at $\mathfrak{P}$. Then $\iota^{-1}(\chi(\varpi)) \equiv 1 \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_\ell}$, where ϖ is a uniformiser for $K_{\mathfrak{P}}$, and $\mathbb{N}\mathfrak{P} \equiv 1 \bmod \ell$, so condition (c) is satisfied.

Suppose that condition (d) fails. In particular, $\overline{r_\iota}(\sigma)$ contains two one-dimensional irreducible constituents $\overline{\theta}_1$ and $\overline{\theta}_2$, such that $\overline{\theta}_1 = \overline{\theta}_2 \otimes \overline{\varepsilon}_\ell$. Hence, $\overline{\varepsilon}_\ell$ must be equal to

$$(\overline{\psi}^{-1} \cdot \overline{\psi}^c)^i$$

for some integer i; this implies that $\overline{\varepsilon}_\ell^c = \overline{\varepsilon}_\ell^{-1}$. But on the other hand, $\overline{\varepsilon}_\ell^c = \overline{\varepsilon}_\ell$, so we must have $\overline{\varepsilon}_\ell^2 = 1$. This is not possible, as $\ell > 3$ and K is an *imaginary* quadratic field.[5]

[5]Note that the mod 5 cyclotomic character has order 2 when restricted to the absolute Galois group of $K = \mathbb{Q}(\sqrt{5})$.

We conclude that there exists a CM biquadratic extension E/K and a regular C-algebraic conjugate self-dual, unitary cuspidal automorphic representation σ of $\mathrm{GL}_{2^k}(\mathbb{A}_E)$ such that

(i) $\overline{r_\iota}(\sigma) \cong (\mathrm{Sym}^{2^k-1}\,\overline{\rho})|_{\mathrm{Gal}(\overline{E}/E)}$, and
(ii) σ_w is an unramified twist of the Steinberg representation of $\mathrm{GL}_{2^k}(E_w)$, for some finite place w of E.

Finally, we take $\Pi = \sigma \otimes (\omega \circ \mathrm{Art}_E)^{-d}$, where ω is the Teichmuller lift of $\overline{\omega}$. □

As mentioned in the introduction, one can try to combine this result with a suitable automorphy lifting theorem (for residually reducible Galois representations) to prove the existence of the $(2^k - 1)$-th symmetric power of an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with residual mod ℓ Galois representation isomorphic to $\mathrm{Ind}_K^\mathbb{Q}\overline{\psi}$.

Clozel and Thorne, in their work on symmetric power functoriality ([**CT14**], [**CT15**], [**CT17**]), make use of the main theorem of [**Tho15**], which is an automorphy lifting theorem for residually reducible ℓ-adic Galois representations ρ, such that $\overline{\rho}$ (the mod ℓ reduction of ρ) has two Jordan-Hölder factors. For the application described above, one needs a generalisation of this theorem (in which $\overline{\rho}$ can have 2^k factors).

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