Automorphy lifting for residually reducible l-adic Galois representations

Jack A. Thorne*

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Abstract

We prove automorphy lifting theorems for residually reducible Galois representations in the setting of unitary groups over CM fields. Our methods are inspired by those of Skinner–Wiles in the setting of GL_2.

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Introduction

In this paper we prove new automorphy lifting theorems for l-adic Galois representations over CM fields satisfying a self-duality hypothesis. The main novelty is that we can prove lifting results for Galois representations which are *residually reducible*. This paper can therefore be viewed as a sequel to [Tho12], where we treated the residually irreducible case. However, there are a number of serious new obstacles.

Let F be an imaginary CM field with totally real subfield F^+, and let $c \in \mathrm{Gal}(F/F^+)$ denote the non-trivial element. Let $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ be a continuous irreducible representation. We suppose that ρ is conjugate self-dual, in the sense that $\rho^c \cong \rho^\vee \epsilon^{1-n}$ (ϵ denoting the l-adic cyclotomic character), and de Rham with distinct Hodge–Tate weights, in the sense of l-adic Hodge theory. The central problem for us is to show that ρ is *automorphic*, i.e. that ρ arises from automorphic forms as in Theorem 2.2 below. Choosing a suitable finite extension K of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ with ring of integers $\mathcal{O}_K$ and an invariant $\mathcal{O}_K$-lattice inside K^n, we can view ρ as a representation $G_F \to \mathrm{GL}_n(\mathcal{O}_K)$. Then (writing $\lambda \subset \mathcal{O}_K$ for the unique maximal ideal) the reduced representation $\overline{\rho} = \rho \bmod \lambda$ makes sense, and its semisimplification $\overline{\rho}^{\mathrm{ss}}$ is independent of the choice of invariant lattice. Previous efforts have centered around the case where $\overline{\rho}$ is absolutely irreducible. In this case, assuming the existence of a lift of $\overline{\rho}$ which arises from automorphic forms, we obtain a map $\varphi : R \to \mathbb{T}$, where R is the universal deformation ring of $\overline{\rho}$ classifying deformations of a certain well-chosen type (for example, conjugate self-dual, de Rham with fixed Hodge–Tate weights), and $\mathbb{T}$ is the quotient classifying deformations which arise from a suitable space of automorphic forms (for example, automorphic forms on a unitary group which have cohomology for a fixed system of coefficients). We can then proceed by trying to show that φ is close to being an isomorphism.

If $\overline{\rho}$ is not absolutely irreducible, then problems arise. First, the universal deformation ring need not exist in general. For some choices of invariant lattice, one can arrive at a $\overline{\rho}$ with scalar centralizer. (This is the approach adopted in the work of Skinner–Wiles [SW99], where the authors choose $\overline{\rho}$ to take values in the group $\mathrm{B}_2 \subset \mathrm{GL}_2$ of upper-triangular matrices.) In this case, the universal deformation ring exists, but there need not exist a map $R \to \mathbb{T}$. In general one can expect a map $R^{\mathrm{tr}} \to \mathbb{T}$, where R^{tr} denotes the universal pseudodeformation ring of the pseudocharacter $\mathrm{tr}\,\overline{\rho}$, but the ring R^{tr} is difficult to control using Galois cohomology, a tool which is essential in other arguments.

In this paper we circumvent these problems in some cases by permitting $\overline{\rho}$ which are 'reducible, but not too reducible'. In fact, we allow residual representations which are Schur, in the sense of Definition 3.2. (This property was first defined in [CHT08].) If $\overline{\rho}$ is Schur then the universal deformation ring R exists, and is related to the universal pseudodeformation ring R^{tr} in a simple way. This behavior is related to the existence of elliptic endoscopic groups of $U(n)$.

Having restricted our attention to this class of representations, we try to apply the Taylor–Wiles–Kisin argument. The relevant arguments in Galois cohomology require the residual representation to be absolutely irreducible (at the very least; current technology asks for it further to be adequate, in the sense of [Tho12]). In order to circumvent this difficulty, we follow the strategy of Skinner–Wiles [SW99], who have proved automorphy lifting theorems for GL_2, working with residually reducible representations over totally real fields. The basic idea is that by working with Hida families we can move from the residual representation $\overline{\rho}$ to an irreducible representation with coefficients in a one-dimensional quotient of the Iwasawa algebra. One can then try to apply the usual arguments to a localization of R at the dimension one prime corresponding to such a representation. (This means we must restrict to representations ρ which are not only de Rham but even ordinary at primes dividing l.)

There is one final hiccup. At a key point in the argument, we must show that the locus inside $\mathrm{Spec}\, R$ of reducible Galois representations has large codimension. In contrast to the case of GL_2, when working with unitary Galois representations, there is no *a priori* reason for this to be the case; the endoscopic parameters can contribute irreducible components of $\operatorname{Spec} R$ which have full dimension. (In the context in which we work, one expects a universal deformation ring allowing representations ordinary at l, of variable Hodge–Tate weights, to be equidimensional of dimension $1 + nd$, where $d = [F^+ : \mathbb{Q}]$.) For this reason, we must impose an additional hypothesis. For example, we can ask for ρ to admit a place v at which the associated Weil–Deligne representation of $\rho|_{G_{F_v}}$ corresponds under the local Langlands correspondence to a twist of the Steinberg representation. Since this is incompatible with ρ being a direct sum of two representations of strictly smaller dimension, we can force the locus of reducible deformations to be small. Our main theorem, Theorem 7.1, is an automorphy lifting theorem which makes use of this assumption.

Let us briefly describe one possible application of our work. Part of the interest of automorphy lifting theorems in the residually reducible case is that it is often easier to verify the residual automorphy hypothesis. For example, Skinner–Wiles take the approach of showing that the constant term of a GL_2 Eisenstein series vanishes mod l, so one can apply the Deligne–Serre lemma to obtain a congruence with cusp form. A different approach can be taken with automorphic forms on unitary groups. For example, one can take an endoscopic lift from a product of smaller unitary groups and then apply a level-raising result (as in, for example, the work of Bellaiche–Graftieaux [BG06] or our paper [Tho]) to obtain a congruence with an automorphic representation which is stable. Since the automorphic representations we eventually consider are for other reasons assumed to be square-integrable at a finite place, this approach seems particularly effective here.

In the final section below we discuss a theorem (Theorem 8.1) which combines this idea with Serre's conjecture for GL_2 over $\mathbb{Q}$ (now a theorem of Khare–Wintenberger and Kisin) to prove an automorphy result for irreducible three-dimensional Galois representations over a quadratic imaginary field, with no hypothesis of residual automorphy. In joint work with L. Clozel [CTa], [CTb], we will apply similar ideas to the problem of symmetric power functoriality for GL_2, proving for example the following theorem[1], for which this paper represents an essential input:

Theorem. *Let E be an elliptic curve over $\mathbb{Q}$, without complex multiplication. Then the 5^{th} symmetric power L-function of E is automorphic, and thus has an analytic continuation to the entire complex plane.*

We now describe the organization of this paper. In §1, we collect some useful results in commutative algebra. In §2, we recall the definition of a RAECSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. In §3, we describe the basic objects in deformation theory with which we work. In particular, we make our first important observation, about the relation between the rings R and R^{tr} (denoted $R_{\mathcal{S}}^{\mathrm{univ}}$ and $P_{\mathcal{S}}$ in the body of the paper). Namely, we show that when $\overline{\rho}$ is Schur, the natural map $R^{\mathrm{tr}} \to R$ is a finite ring homomorphism. This generalizes the well-known fact (due to Carayol for GL_n) that if $\overline{\rho}$ is absolutely irreducible, then $R^{\mathrm{tr}} \to R$ is surjective.

In §4, we define the relevant spaces of automorphic forms and recall some basic facts from Hida theory. We prove an '$R_{\mathfrak{p}} = \mathbb{T}_{\mathfrak{p}}$' theorem under some stringent hypotheses. Here $\mathfrak{p}$ denotes a dimension one prime of R, and $(\cdot)_{\mathfrak{p}}$ denotes localization and completion at that prime. Then §5 is devoted to giving some situations when these hypotheses can be expected to hold.

In §6, we show how to upgrade an '$R_{\mathfrak{p}} = \mathbb{T}_{\mathfrak{p}}$' theorem into information about the relation between R and $\mathbb{T}$. Since $R_{\mathfrak{p}}$ only knows about the irreducible components of $\operatorname{Spec} R$ which contain $\mathfrak{p}$, we need a way to move between different components of $\operatorname{Spec} R$; to do this we use some input from commutative algebra, in the form of the notion of connectedness dimension of local rings. In §7 we give our main result, an automorphy lifting theorem using all of the ideas discussed in this introduction. Finally, in §8 we describe an application of our work to the Fontaine-Mazur conjecture for $U(3)$.

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[1] In fact, at the time of writing, the theorem as stated is conditional on the stabilization of the twisted trace formula; see [CTb] for further discussion.

Notation

If F is a field of characteristic zero with a fixed algebraic closure $\overline{F}$, then we write $G_F = \mathrm{Gal}(\overline{F}/F)$ for its absolute Galois group. If F/F^+ is a quadratic extension of such fields, we write δ_{F/F^+} for the nontrivial character of $\mathrm{Gal}(F/F^+)$. We write $\epsilon_l : G_F \to \mathbb{Z}_l^\times$ for the l-adic cyclotomic character. If the prime l is understood, we will write $\epsilon_l = \epsilon$. We use the term 'CM field' to mean a totally imaginary quadratic extension of a totally real number field. If F is a CM field, then it is endowed with a canonical involution c with the property that for all $x \in F$ and for all embeddings $\tau : F \hookrightarrow \mathbb{C}$, we have $\tau \circ c(x) = \overline{\tau(x)}$. Then the subfield $F^+ = F^{c=1}$ is the maximal totally real subfield of F.

We fix throughout this paper an algebraic closure $\overline{\mathbb{Q}}_l$ of $\mathbb{Q}_l$ for each prime l, and write $\mathrm{val}_l : \overline{\mathbb{Q}}_l^\times \to \mathbb{Q}$ for the l-adic valuation, normalized so that $\mathrm{val}_l(l) = 1$. If F is a number field, then we will fix choices of algebraic closure $\overline{F}$ of F, algebraic closures $\overline{F}_v$ of the completion F_v for each place v of F, and embeddings $\overline{F} \hookrightarrow \overline{F}_v$ extending the canonical embeddings $F \hookrightarrow F_v$. These choices determine maps $G_{F_v} \hookrightarrow G_F$. If χ is a character $\mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ of type A_0 (i.e. the restriction of χ to $(F \otimes_{\mathbb{Q}} \mathbb{R})_0^\times$ is given by $\prod_{\tau : F \hookrightarrow \mathbb{C}} x_\tau^{a_\tau}$ for some integers a_τ), and ι is an isomorphism $\overline{\mathbb{Q}}_l \to \mathbb{C}$, then we write $r_\iota(\chi) : G_F \to \overline{\mathbb{Q}}_l^{\times}$ for the associated l-adic character, given by the formula

$$\iota\left((r_\iota(\chi) \circ \mathrm{Art}_F)(x) \prod_{\tau \in \mathrm{Hom}(F,\mathbb{C})} x_{\iota^{-1}\tau}^{-a_\tau}\right) = \chi(x) \prod_{\tau \in \mathrm{Hom}(F,\mathbb{C})} x_\tau^{-a_\tau},$$

where Art_F is the global Artin map

$$\mathrm{Art}_F = \prod_v \mathrm{Art}_{F_v} : \mathbb{A}_F^\times \to G_F^{\mathrm{ab}}.$$

We normalize the local Artin maps Art_{F_v} to take uniformizers to geometric Frobenius elements. If v is a finite place of F, we will write $\mathcal{O}_{F_v}$ for the ring of integers of F_v, $k(v)$ for its residue field, and q_v for the cardinality of $k(v)$. We write $I_{F_v} \subset G_{F_v}$ for the inertia group, and $\mathrm{Frob}_v \in G_{F_v}/I_{F_v}$ for the geometric Frobenius element. If $v|l$ and $\rho : G_{F_v} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is a continuous representation, and $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_l$ is a continuous embedding, then we write $\mathrm{HT}_\tau(\rho)$ for the multiset of integers whose elements are the integers i such that $\mathrm{gr}^i(\rho \otimes_{\tau, F_v} B_{dR})^{G_{F_v}} \neq 0$, with multiplicity $\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i(\rho \otimes_{\tau, F_v} B_{dR})^{G_{F_v}}$. Here B_{dR} denotes Fontaine's ring of p-adic (or l-adic) periods, cf. [Ber04]. If ρ is de Rham, then this set has n distinct elements. If $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is a continuous representation, and $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ is an embedding, then we write $\mathrm{HT}_\tau(\rho)$ to mean $\mathrm{HT}_\tau(\rho|_{G_{F_v}})$, where v is the place of F induced by the embedding τ. Thus for the character $r_\iota(\chi)$ defined above, we have

$$\mathrm{HT}_\tau(r_\iota(\chi)) = \{-a_{\iota\tau}\},$$

and $\mathrm{HT}_\tau(\epsilon) = \{-1\}$ for any τ.

If v is a finite place of F, then we write rec_{F_v} for the local Langlands correspondence for F_v, as defined in [HT01]. By definition, it is a bijection between the set of isomorphism classes of irreducible admissible representations of $\mathrm{GL}_n(F_v)$ over $\mathbb{C}$, and the set of Frobenius–semisimple Weil–Deligne representations (r, N) of W_{F_v} over $\mathbb{C}$. It is characterized uniquely by the equality of certain ϵ– and L–factors. If π is an irreducible admissible representation of $\mathrm{GL}_n(F_v)$, then we define $\mathrm{rec}_{F_v}^T(\pi) = \mathrm{rec}_{F_v}(\pi \otimes |\cdot|^{(1-n)/2})$. Then $\mathrm{rec}_{F_v}^T$ commutes with the action of $\mathrm{Aut}(\mathbb{C})$, and therefore makes sense over any field Ω which is abstractly isomorphic to $\mathbb{C}$ (e.g. $\overline{\mathbb{Q}}_l$). If $\rho : G_{F_v} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is a continuous representation (which is de Rham if $v|l$), then we write $\mathrm{WD}(\rho) = (r, N)$ for the associated Weil–Deligne representation. If (r, N) is a Weil–Deligne representation, then we write $(r, N)^{\text{F-ss}}$ for its Frobenius–semisimplification.

We write $\mathbb{Z}_+^n \subset \mathbb{Z}^n$ for the set of tuples $\lambda = (\lambda_1, \ldots, \lambda_n)$ of integers with $\lambda_1 \geq \cdots \geq \lambda_n$.

If R is a ring and $\mathfrak{p} \subset R$ is a prime ideal, then we write $R_{(\mathfrak{p})}$ for the localization of R at $\mathfrak{p}$ (i.e. the ring $(R - \mathfrak{p})^{-1}R$), $R_\mathfrak{p}$ for the completion of $R_{(\mathfrak{p})}$ at the ideal $\mathfrak{p}_{(\mathfrak{p})}$, and $\kappa(\mathfrak{p}) = \mathrm{Frac}\, R/\mathfrak{p}$ for the residue field of $\mathfrak{p}$. If R is a local ring, then we write $\mathfrak{m}_R$ for its unique maximal ideal.

1 Preliminaries in commutative algebra

Let l be a prime and let K be a finite extension of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$. We write $\mathcal{O}$ for the ring of integers of K, $\lambda \subset \mathcal{O}$ for the maximal ideal, and $k = \mathcal{O}/\lambda$ for the residue field. If R is a complete Noetherian local $\mathcal{O}$-algebra with residue field k, then we write $\mathcal{C}_R$ for the category of complete Noetherian local R-algebras with residue field k.

Definition 1.1. *A ring $A \in \mathcal{C}_\mathcal{O}$ is said to be* geometrically integral *(resp.* geometrically irreducible*) if for every finite extension E/K in $\overline{\mathbb{Q}}_l$, the algebra $A \otimes_\mathcal{O} \mathcal{O}_E$ is a domain (resp.* $\operatorname{Spec} A \otimes_\mathcal{O} \mathcal{O}_E$ *is irreducible). A ring $A \in \mathcal{C}_k$ is said to be* geometrically integral *(resp.* geometrically irreducible*) if for every finite extension k'/k, the algebra $A \otimes_k k'$ is a domain (resp.* $\operatorname{Spec} A \otimes_k k'$ *is irreducible).*

Definition 1.2. *Let R, S be complete Noetherian local $\mathcal{O}$-algebras (not necessarily with residue field k). We view R and S as being endowed with their $\mathfrak{m}_R$-adic and $\mathfrak{m}_S$-adic topologies, respectively. We define the completed tensor product $R\widehat{\otimes}_\mathcal{O}S$ to be the completion of the algebra $R \otimes_\mathcal{O} S$ for the I-adic topology generated by the ideal $I = (\mathfrak{m}_R, \mathfrak{m}_S) \subset R\widehat{\otimes}_\mathcal{O}S$.*

The completed tensor product $R\widehat{\otimes}_\mathcal{O}S$ is equipped with canonical continuous maps $\rho : R \to R\widehat{\otimes}_\mathcal{O}S$, $\sigma : S \to R\widehat{\otimes}_\mathcal{O}S$. This operation has natural commutativity and associativity properties; see [Gro60, Ch. 0, §7.7].

Lemma 1.3. *Let R, S be complete Noetherian local $\mathcal{O}$-algebras. Suppose that $R/\mathfrak{m}_R$ is a finite extension of k. Then:*

1. *If S is $\mathcal{O}$-flat, then $R\widehat{\otimes}_\mathcal{O}S$ is R-flat.*
2. *$R\widehat{\otimes}_\mathcal{O}S$ is a semi-local Noetherian ring, and its maximal ideals are in bijection with the maximal ideals of $R/\mathfrak{m}_R \otimes_k S/\mathfrak{m}_S$. In particular, if $R/\mathfrak{m}_R = k$ then $R\widehat{\otimes}_\mathcal{O}S$ is a complete Noetherian local $\mathcal{O}$-algebra, and if further $S/\mathfrak{m}_S = k$, then $R\widehat{\otimes}_\mathcal{O}S$ is a complete Noetherian local $\mathcal{O}$-algebra with residue field k.*
3. *Suppose that $R/\mathfrak{m}_R = k$. Then $R\widehat{\otimes}_\mathcal{O}S$ has the following universal property: let T be a complete Noetherian local $\mathcal{O}$-algebra, and let $u : R \to T$, $v : S \to T$ be continuous morphisms of $\mathcal{O}$-algebras. Then there is a unique continuous $\mathcal{O}$-algebra morphism $w : R\widehat{\otimes}_\mathcal{O}S \to T$ such that $w \circ \rho = u$ and $w \circ \sigma = v$.*

Proof. The first 2 parts follow from [Gro64, Ch. 0, (19.7.1.2)], the third from [Gro60, Ch. 0, (7.7.6)]. ☐

Lemma 1.4.

1. *Let A, $B \in \mathcal{C}_\mathcal{O}$ be $\mathcal{O}$-flat domains. Then $\dim A\widehat{\otimes}_\mathcal{O}B = \dim A + \dim B - 1$.*
2. *Let A, $B \in \mathcal{C}_\mathcal{O}$ be $\mathcal{O}$-flat and geometrically integral (resp. geometrically irreducible). Then $A\widehat{\otimes}_\mathcal{O}B$ is $\mathcal{O}$-flat and geometrically integral (resp. geometrically irreducible).*
3. *Let A, $B \in \mathcal{C}_k$ be geometrically integral (resp. geometrically irreducible). Then $A\widehat{\otimes}_kB$ is geometrically integral (resp. geometrically irreducible).*
4. *Suppose $A \in \mathcal{C}_\mathcal{O}$ (resp. $B \in \mathcal{C}_\mathcal{O}$) has distinct minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ (resp. $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$). Suppose that the rings $A/\mathfrak{p}_i$ and $B/\mathfrak{q}_j$ are all $\mathcal{O}$-flat and geometrically integral. Then the distinct minimal primes of $A\widehat{\otimes}_\mathcal{O}B$ are $(\mathfrak{p}_i, \mathfrak{q}_j)$, $i = 1, \ldots, r$, $j = 1, \ldots, s$.*
5. *Suppose $A \in \mathcal{C}_k$ (resp. $B \in \mathcal{C}_k$) has distinct minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ (resp. $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$). Suppose that the rings $A/\mathfrak{p}_i$ and $B/\mathfrak{q}_j$ are all geometrically integral. Then the distinct minimal primes of $A\widehat{\otimes}_kB$ are $(\mathfrak{p}_i, \mathfrak{q}_j)$, $i = 1, \ldots, r$, $j = 1, \ldots, s$.*
6. *Let A, $B \in \mathcal{C}_\mathcal{O}$, and suppose that for each minimal prime $\mathfrak{p} \subset A$ (resp. $\mathfrak{q} \subset B$), the quotient $A/\mathfrak{p}$ (resp. $B/\mathfrak{q}$) is $\mathcal{O}$-flat and geometrically integral. Suppose moreover that for each minimal prime $\mathfrak{p} \subset A/(\lambda)$ (resp. $\mathfrak{q} \subset B/(\lambda)$), $\mathfrak{p}$ contains a unique minimal prime of A (resp. $\mathfrak{q}$ contains a unique minimal prime of B). Then each minimal prime of $A\widehat{\otimes}_\mathcal{O}B/(\lambda)$ contains a unique minimal prime of $A\widehat{\otimes}_\mathcal{O}B$.*

7. *Let A, $B \in \mathcal{C}_k$, and suppose that for each minimal prime $\mathfrak{p} \subset A$ (resp. $\mathfrak{q} \subset B$), $A/\mathfrak{p}$ is geometrically integral over k (resp. $B/\mathfrak{q}$ is geometrically integral over k). Suppose moreover that A and B are generically reduced. Then $A\widehat{\otimes}_k B$ is generically reduced.*
8. *Let A, $B \in \mathcal{C}_{\mathcal{O}}$, and suppose that for each minimal prime $\mathfrak{p} \subset A/(\lambda)$ (resp. $\mathfrak{q} \subset B/(\lambda)$), $A/\mathfrak{p}$ is geometrically integral over k (resp. $B/\mathfrak{q}$ is geometrically integral over k). Suppose moreover that both $A/(\lambda)$ and $B/(\lambda)$ are generically reduced. Then $A\widehat{\otimes}_{\mathcal{O}} B/(\lambda)$ is generically reduced.*

Proof. Parts 1–5 are contained in [BLGHT11, Lemma 3.3]. Part 6 is contained in [BLGHT11, Lemma 3.3], except the authors assume in addition that for each minimal prime $\mathfrak{p} \subset A/(\lambda)$, $\mathfrak{q} \subset B/(\lambda)$, the quotients $A/\mathfrak{p}$ and $B/\mathfrak{q}$ are geometrically integral over k. We thank the referee for pointing out that the conclusion holds without making this assumption, and for providing the following proof. Let $\mathfrak{q} \subset A\widehat{\otimes}_{\mathcal{O}} B/(\lambda)$ be a minimal prime. We must show that there is a unique minimal prime of $A\widehat{\otimes}_{\mathcal{O}} B$ contained inside it. Let $\mathfrak{q}_A$ (resp. $\mathfrak{q}_B$) denote the pullback of $\mathfrak{q}$ to $A/(\lambda)$ (resp. $B/(\lambda)$). The maps

$$A/(\lambda) \to A\widehat{\otimes}_{\mathcal{O}} B/(\lambda) \cong A/(\lambda)\widehat{\otimes}_k B/(\lambda)$$

and

$$B/(\lambda) \to A\widehat{\otimes}_{\mathcal{O}} B/(\lambda) \cong A/(\lambda)\widehat{\otimes}_k B/(\lambda)$$

are flat, and so $\mathfrak{q}_A$ and $\mathfrak{q}_B$ are minimal primes. By assumption, there is a unique minimal prime $\mathfrak{p}_A$ of A contained in $\mathfrak{q}_A$ (resp. a unique minimal prime $\mathfrak{p}_B$ of B contained in $\mathfrak{q}_B$). By part 4 of the lemma, $\mathfrak{p} = (\mathfrak{p}_A, \mathfrak{p}_B)$ is a minimal prime of $A\widehat{\otimes}_{\mathcal{O}} B$, which is contained in $\mathfrak{q}$.

Suppose that $\mathfrak{p}'$ is another minimal prime of $A\widehat{\otimes}_{\mathcal{O}} B$ which is contained in $\mathfrak{q}$. We can find minimal primes $\mathfrak{p}'_A \subset A$, $\mathfrak{p}'_B \subset B$ such that $\mathfrak{p}' = (\mathfrak{p}'_A, \mathfrak{p}'_B)$. We have $\mathfrak{p}'_A \subset \mathfrak{q}_A$ and $\mathfrak{p}'_B \subset \mathfrak{q}_B$; by assumption, this implies $\mathfrak{p}'_A = \mathfrak{p}_A$ and $\mathfrak{p}'_B = \mathfrak{p}_B$, hence $\mathfrak{p}' = \mathfrak{p}$. This establishes the desired uniqueness, and completes the proof of part 6 of the lemma.

Part 8 follows on applying part 7 to the ring $(A\widehat{\otimes}_{\mathcal{O}} B)/(\lambda) = A/(\lambda)\widehat{\otimes}_k B/(\lambda)$. We prove part 7. Let $I \subset A$, $J \subset B$ be the respective nilpotent ideals. The assertion that A is generically reduced is equivalent to the assertion that every element $x \in I$ is annihilated by an element $f \in A$ which is not contained in any minimal prime of A. (More precisely, it means that for every $x \in I$ and for every minimal prime $\mathfrak{p}$ of A, there is an element $f \in A - \mathfrak{p}$ such that $fx = 0$. Since $\mathrm{Ann}_A(x) \subset A$ is an ideal, prime avoidance implies that we can find $f \in A$ such that $fx = 0$ and f is not contained in any minimal prime of A.) Similar remarks apply to B. On the other hand, if $f \in A$ and $g \in B$ are not contained in any minimal prime, it follows from part 5 of the lemma that $f \otimes g \in A\widehat{\otimes}_k B$ is not contained in any minimal prime.

The nilpotent ideal of $A\widehat{\otimes}_k B$ is equal to $(I, J) = I \cdot A\widehat{\otimes}_k B + J \cdot A\widehat{\otimes}_k B$ (as $A/I\widehat{\otimes}_k B/J$ is reduced, by [Gro65, Ch. IV, (7.5.7)]). Let $x \in I$, $y \in J$, and choose elements $f \in A$, $g \in B$ not contained in any minimal primes and with $fx = 0$ and $gy = 0$. Then $f \otimes g \cdot (xA\widehat{\otimes}_k B + yA\widehat{\otimes}_k B) = 0$. Since x, y were arbitrary, this shows that $A\widehat{\otimes}_k B$ is generically reduced. ☐

Lemma 1.5. *Let R, $S \in \mathcal{C}_{\mathcal{O}}$. Let $P \subset R/(\lambda)$ be a prime, and let $P' = (P, \mathfrak{m}_S) \subset R\widehat{\otimes}_{\mathcal{O}} S$. Then there is a canonical isomorphism $R_P\widehat{\otimes}_{\mathcal{O}} S \cong (R\widehat{\otimes}_{\mathcal{O}} S)_{P'}$.*

Proof. We construct the maps in either direction. There are canonical isomorphisms

$$R\widehat{\otimes}_{\mathcal{O}} S \cong \varprojlim_i R \otimes_{\mathcal{O}} S/\mathfrak{m}_S^i,\ R_P\widehat{\otimes}_{\mathcal{O}} S \cong \varprojlim_i R_P \otimes_{\mathcal{O}} S/\mathfrak{m}_S^i,$$

so passing to the limit with respect to the natural maps $R \otimes_{\mathcal{O}} S/\mathfrak{m}_S^i \to R_P \otimes_{\mathcal{O}} S/\mathfrak{m}_S^i$, we obtain a homomorphism $R\widehat{\otimes}_{\mathcal{O}} S \to R_P\widehat{\otimes}_{\mathcal{O}} S$. The pre-image of the maximal ideal $(P, \mathfrak{m}_S) \subset R_P\widehat{\otimes}_{\mathcal{O}} S$ is P', so after localization and completion we obtain a continuous map $f : (R\widehat{\otimes}_{\mathcal{O}} S)_{P'} \to R_P\widehat{\otimes}_{\mathcal{O}} S$. To go in the other direction, we observe that there are natural maps

$$R \to R\widehat{\otimes}_{\mathcal{O}} S \to (R\widehat{\otimes}_{\mathcal{O}} S)_{P'},\ S \to R\widehat{\otimes}_{\mathcal{O}} S \to (R\widehat{\otimes}_{\mathcal{O}} S)_{P'},$$

and the first extends by continuity to a map $R_P \to (R\widehat{\otimes}_{\mathcal{O}} S)_{P'}$. The universal property of $R_P\widehat{\otimes}_{\mathcal{O}} S$ then gives rise to a map $g : R_P\widehat{\otimes}_{\mathcal{O}} S \to (R\widehat{\otimes}_{\mathcal{O}} S)_{P'}$. To finish the proof of the lemma, we must show fg and gf

equal the identity. It is clear from the construction that the map $fg : R_P\widehat{\otimes}_{\mathcal{O}}S \to R_P\widehat{\otimes}_{\mathcal{O}}S$ agrees with the identity on the image of $R \otimes_{\mathcal{O}} S$, hence on the image of $R_{(P)} \otimes_{\mathcal{O}} S$. Since this image is dense, fg equals the identity. A similar argument shows that gf is the identity, and completes the proof. $\square$

A useful special case of the lemma arises when $S = \mathcal{O}[\![X_1, \ldots, X_n]\!]$, for some $n \geq 0$. We then obtain a canonical isomorphism

$$R_P\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[\![X_1, \ldots, X_n]\!] \cong R[\![X_1, \ldots, X_n]\!]_{P'}.$$

The next result generalizes [Tay08, Lemma 2.7].

Proposition 1.6. *Let A be an excellent local $\mathcal{O}$-algebra, and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the distinct minimal primes of A. Suppose that A satisfies the following conditions.*

1. *For each $i = 1, \ldots, r$, $A/\mathfrak{p}_i$ is $\mathcal{O}$-flat of dimension $d + 1$, and $A/(\lambda)$ is generically reduced.*
2. *Each minimal prime of $A/(\lambda)$ contains a unique minimal prime of A.*

Let $B = \widehat{A}$. Then:

1. *For each minimal prime $\mathfrak{q} \subset B$, $B/\mathfrak{q}$ is $\mathcal{O}$-flat of dimension $d + 1$, and $B/(\lambda)$ is generically reduced.*
2. *Each minimal prime of $B/(\lambda)$ contains a unique minimal prime of B.*

Proof. We may suppose without loss of generality that A is reduced. The ring A is equidimensional of dimension $d + 1$, so B is equidimensional of dimension $d + 1$ by [Mat89, Corollary 31.5]. It is also $\mathcal{O}$-flat, so $A/(\lambda)$ is equidimensional of dimension d (by [Mat89, Theorem 31.5]) and B is $\mathcal{O}$-flat (since $A \to B$ is faithfully flat). Moreover, $B/(\lambda)$ is generically reduced since $A/(\lambda)$ is generically reduced and A is excellent (cf. [Mat89, Theorem 23.9]). This proves the first point in the statement of the proposition.

Let $\widetilde{A}$ denote the normalization of A. Thus $\widetilde{A} = \prod \widetilde{A}_i$, where $\widetilde{A}_i$ is the normalization of $A/\mathfrak{p}_i$. Let $\mathfrak{q}_{i,j}$ be the distinct minimal primes of $A/(\mathfrak{p}_i, \lambda)$. The rings $A/\mathfrak{p}_i$ and $\widetilde{A}_i$ are $\mathcal{O}$-flat domains, and the maps $A/\mathfrak{p}_i \to \widetilde{A}_i$ are finite (since A is excellent). Consequently, the rings $A/(\mathfrak{p}_i, \lambda)$ and $\widetilde{A}_i/(\lambda)$ are equidimensional of dimension d, by [Mat89, Theorem 31.5], and the maps $A/(\mathfrak{p}_i, \lambda) \to \widetilde{A}_i/(\lambda)$ are finite.

In fact, the $\mathfrak{q}_{i,j}$ are the distinct minimal primes of $A/(\lambda)$. Indeed, they are distinct, since if $\mathfrak{q}_{i,j} = \mathfrak{q}_{i',j'}$ then $\mathfrak{p}_i \subset \mathfrak{q}_{i,j} = \mathfrak{q}_{i',j'}$, so $i = i'$ (since each minimal prime of $A/(\lambda)$ contains a unique minimal prime of A, by hypothesis), and then $j = j'$ (by construction). They exhaust the minimal primes of $A/(\lambda)$, since if $\wp \subset A/(\lambda)$ is any minimal prime, then $\dim A/\wp = d = \dim A - 1$, so $\wp$ contains $\mathfrak{p}_i$ for some i.

We now observe that $A/(\mathfrak{p}_i)_{(\mathfrak{q}_{i,j})} = A_{(\mathfrak{q}_{i,j})}$ is a DVR. Indeed, the ideal (λ) is principal and $A_{(\mathfrak{q}_{i,j})}/(\lambda) = A/(\lambda)_{(\mathfrak{q}_{i,j})}$ is a field, since $A/(\lambda)$ is generically reduced. Since localization commutes with normalization, it follows that $A_{(\mathfrak{q}_{i,j})} = \widetilde{A}_{i,(\mathfrak{q}_{i,j})}$. Let $Q = \widetilde{A}/A$, a finite A-module. Then $Q_{(\mathfrak{q}_{i,j})} = 0$ for all i, j, and we have exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & \widetilde{A} \arrow[r] & Q \arrow[r] & 0 \end{tikzcd} \tag{1.1}$$

and (after applying $- \otimes_A B$ to (1.1))

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] & \prod_{i,k} \widehat{\widetilde{A}_{i,\mathfrak{m}_{i,k}}} \arrow[r] & \widehat{Q} \arrow[r] & 0, \end{tikzcd} \tag{1.2}$$

where the product is taken over the finitely many maximal ideals $\mathfrak{m}_{i,k}$ of $\widetilde{A}_i$. Let $\mathfrak{r}_{i,k} = \ker(B \to \widehat{\widetilde{A}_{i,\mathfrak{m}_{i,k}}})$. If $\wp \subset B$ is a minimal prime, then $\wp \cap A$ is minimal in A (by the going down theorem, cf. [Mat89, Theorem 15.1]), so equals $\mathfrak{p}_i$, for some i. Then $Q_{(\mathfrak{p}_i)} = \widehat{Q}_{(\wp)} = 0$. Since $B_{(\wp)}$ is a local ring, it follows that $B_{(\wp)} = (\widehat{\widetilde{A}_{i,\mathfrak{m}_{i,k}}})_{(\wp)}$ for some pair (i, k) which is uniquely determined by $\wp$, and then $\wp = \mathfrak{r}_{i,k}$. Since each $\mathfrak{r}_{i,k}$ contains a minimal prime of B, it follows that the $\mathfrak{r}_{i,k}$ are the distinct minimal primes of B.

Now suppose that $\wp \subset B/(\lambda)$ is a minimal prime. Then $\wp \cap A/(\lambda)$ is minimal, hence equals $\mathfrak{q}_{i,j}$ for some i, j, and $Q_{(\mathfrak{q}_{i,j})} = \widehat{Q}_{(\wp)} = 0$. Again using the fact that $B_{(\wp)}$ is a local ring, we see that exactly one of the localizations $(\widehat{\widetilde{A}_{i,\mathfrak{m}_{i,k}}})_{(\wp)}$ is non-zero, and it follows that $\wp$ contains exactly one of the ideals $\mathfrak{r}_{i,k}$. Combining this with the reasoning of the previous paragraph, we see that $\wp$ contains a unique minimal prime of B. This completes the proof. $\square$

The following definition plays an important role in §6.

Definition 1.7. *Let $R \in \mathcal{C}_{\mathcal{O}}$. The* connectedness dimension *of R is*

$$c(R) = \inf_{\mathcal{C}_1, \mathcal{C}_2} \left\{ \dim \cup_{C \in \mathcal{C}_1, D \in \mathcal{C}_2} C \cap D \right\},$$

where the infimum is taken over the set of partitions of the set of irreducible components of $\operatorname{Spec} R$ *into two disjoint non-empty subsets $\mathcal{C}_1, \mathcal{C}_2$.*

If $I \subset R$ is an ideal, the arithmetic rank $r(I)$ *of I is the minimal integer r such that there exist elements $f_1, \dots, f_r$ with $\sqrt{(f_1, \dots, f_r)} = \sqrt{I}$.*

Proposition 1.8. *With R, I as above, let $S = R/I$. Then we have $c(S) \geq c(R) - r(I) - 1$.*

Proof. This follows immediately from [BR86, Theorem 2.4]. □

We finish this section with some miscellaneous lemmas.

Lemma 1.9. *Let R be an object of $\mathcal{C}_k$ of dimension $d \geq 1$, and suppose given countably many ideals $I_1, I_2, \dots$ such that for all i, we have $\dim R/I_i \leq d - 1$. Then there exists a dimension one prime $\mathfrak{p} \subset R$ such that $\mathfrak{p}$ does not contain I_i for any i.*

Proof. If $d = 1$ then the result is clear. Otherwise, by the Noether normalization theorem for complete local rings, we can find an injective finite map $k[\![x_1, \dots, x_d]\!] \hookrightarrow R$. We may therefore assume $R = k[\![x_1, \dots, x_d]\!]$ and that each $I_i = (f_i)$ is principal. Then R is a UFD and there exist uncountably many pairwise non-associate prime elements $g \in \mathfrak{m}_R - \mathfrak{m}_R^2$, as follows easily from the Weierstrass preparation theorem. Choosing g coprime to each f_i and passing to $R/(g)$, we can reduce by induction to the case $d = 1$. □

Lemma 1.10.

1. *Let A be a Noetherian local ring, and let M be a finite A-module. Suppose that* $\operatorname{depth}_A M \geq \dim A$. *Then equality holds, and* $\operatorname{Supp}_A M \subset \operatorname{Spec} A$ *is a union of irreducible components of* $\operatorname{Spec} A$ *of dimension* $\dim A$.
2. *Let A be a Noetherian local ring, and let M be a finite A-module. Suppose that M is nearly faithful (i.e.* $\operatorname{Ann}_A(M) \subset A$ *is a nilpotent ideal), and let $I \subset A$ be a proper ideal. Then $M/(I)$ is a nearly faithful A/I-module.*
3. *Let A be a Noetherian local $\mathcal{O}$-algebra, and let M be a finite A-module which is flat over $\mathcal{O}$. Let $\wp \subset A/(\lambda)$ be a prime minimal in* $\operatorname{Supp}_{A/(\lambda)} M/(\lambda)$. *Then $\wp$ is not minimal in* $\operatorname{Supp}_A M$.

Proof. The first part follows from the proof of [Tay08, Lemma 2.3] (if not quite its statement). The second part is contained in [Tay08, Lemma 2.2]. We now prove the third part. Suppose that $\wp$ is minimal in $\operatorname{Supp}_A M$. Then $\wp$ is an associated prime of M (by [Mat89, Theorem 6.5]), hence consists of zero-divisors on M (by [Mat89, Theorem 6.1]). Since $\lambda \in \wp$, this contradicts the hypothesis that M is $\mathcal{O}$-flat. □

2 Automorphic forms on $\mathrm{GL}_n(\mathbb{A}_F)$

In this section we define the class of automorphic representations whose attached Galois representations we wish to study. Let F be a CM field with maximal totally real subfield F^+.

Definition 2.1. *We say that a pair (π, χ) of an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ and a continuous character $\chi : (F^+)^\times \backslash \mathbb{A}_{F^+}^\times \to \mathbb{C}^\times$ is* RAECSDC (regular, algebraic, essentially conjugate self-dual, cuspidal) *if it satisfies the following properties:*

1. *π is cuspidal.*
2. *$\pi^c \cong \pi^\vee \otimes \chi \circ \mathbb{N}_{F/F^+}$.*
3. *$\chi_v(-1) = (-1)^n$ for each place $v | \infty$ of F^+.*

4. *The infinitesimal character of π_∞ agrees with the infinitesimal character of an algebraic representation of the group* $\mathrm{Res}^F_{\mathbb{Q}} \mathrm{GL}_n$.

We say that an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ *is* RACSDC *if it satisfies these conditions with* $\chi = \delta^n_{F/F^+}$.

If $\lambda = (\lambda_\tau)_{\tau:F\hookrightarrow\mathbb{C}} \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,\mathbb{C})}$, let Ξ_λ denote the irreducible representation of $\mathrm{GL}_n^{\mathrm{Hom}(F,\mathbb{C})}$ which is the tensor product over $\tau \in \mathrm{Hom}(F,\mathbb{C})$ of the irreducible representation of GL_n with highest weight λ_τ. If π_∞ has the same infinitesimal character as $\Xi^\vee_\lambda$, we say that π has weight λ.

Theorem 2.2. *Let (π, χ) be a RAECSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight λ. Fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$. Then there exists a continuous semisimple representation*

$$r_\iota(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

satisfying the following conditions.

1. $r_\iota(\pi)^c \cong r_\iota(\pi)^\vee \epsilon^{1-n} r_\iota(\chi)|_{G_F}$.
2. *For each place $v|l$ of F, $r_\iota(\pi)|_{G_{F_v}}$ is de Rham, and for each embedding $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ we have*
$$\mathrm{HT}_\tau(r_\iota(\pi)) = \{\lambda_{\iota\tau,n}, \lambda_{\iota\tau,n-1} + 1, \ldots, \lambda_{\iota\tau,1} + n - 1\}.$$
3. *For each finite place v of F, we have* $\mathrm{WD}(r_\iota(\pi)|_{G_{F_v}})^{F\text{-}ss} \cong \mathrm{rec}^T_{F_v}(\iota^{-1}\pi_v)$.

These conditions characterize $r_\iota(\pi)$ uniquely up to isomorphism.

Proof. This theorem represents the culmination of the work of many people. We refer to [CH13, Theorem 3.2.3] for the existence of $r_\iota(\pi)$, and [Car12] (resp. [Car]) for the completion of the proof of local–global compatibility in the case $v \nmid l$ (resp. $v|l$). The uniqueness of $r_\iota(\pi)$ is an easy consequence of the Chebotarev density theorem. □

Let $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_l)}$. Given an isomorphism $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$, we define $\iota\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,\mathbb{C})}$ by the formula $(\iota\lambda)_\tau = \lambda_{\iota^{-1}\tau}$. If $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is a continuous representation and there exists a RAECSDC automorphic representation π of weight $\iota\lambda$ such that $\rho \cong r_\iota(\pi)$, we shall say that ρ is automorphic of weight λ. This paper is dedicated to proving that certain l-adic Galois representations arise from automorphic forms in this sense.

As discussed in the introduction, we must restrict to automorphic forms which are ι-ordinary, in the sense of [Ger, Definition 5.1.2]. Here we give a different, but equivalent definition:

Lemma 2.3. *Let $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$ be an isomorphism, let $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_l)}$, and let (π, χ) be a RAECSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\iota\lambda$. Then π is ι-ordinary if and only if for each place $v|l$ of F, there exist smooth characters $\chi_{v,1}, \ldots, \chi_{v,n} : F_v^\times \to \overline{\mathbb{Q}}_l^\times$ satisfying the following conditions:*

1. *For each place $v|l$ of F, for each uniformizer ϖ_v of F_v, and for each $i = 1, \ldots, n$, we have*
$$\mathrm{val}_l(\chi_{v,i}(\varpi_v)) = \frac{1}{e_v} \sum_{\tau:F_v\hookrightarrow\overline{\mathbb{Q}}_l} \left(\lambda_{\tau,n+1-i} - \frac{(n-1)}{2} + i - 1\right).$$
(*We write e_v for the absolute ramification index of F_v, and $\varpi_v \in F_v^\times$ for a choice of uniformizer.*) *In particular, we have* $\mathrm{val}_l(\chi_{v,1}(\varpi_v)) < \mathrm{val}_l(\chi_{v,2}(\varpi_v)) < \cdots < \mathrm{val}_l(\chi_{v,n}(\varpi_v))$.
2. *For each place $v|l$ of F, π_v is a subquotient of the normalized induction* $\text{n-Ind}_B^{\mathrm{GL}_n(F_v)} \iota\chi_{v,1} \otimes \cdots \otimes \iota\chi_{v,n}$.

We observe that if π is ι-ordinary and $v|l$, then the tuple $(\chi_{v,1}, \ldots, \chi_{v,n})$ in the statement of Lemma 2.3 is uniquely determined by ι and π (or even by $\iota^{-1}\pi_v$).

Proof. This follows easily from [Ger, Lemma 5.1.1]. □

Theorem 2.4. *Let $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$ be an isomorphism, let $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F, \overline{\mathbb{Q}}_l)}$, and let (π, χ) be a RAECSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ which is ι-ordinary of weight $\iota\lambda$. Let v be a place of F dividing l, and let $(\chi_{v,1}, \dots, \chi_{v,n})$ be the tuple of characters associated to $\iota^{-1}\pi_v$ by Lemma 2.3. Then there is an isomorphism*

$$r_\iota(\pi)|_{G_{F_v}} \sim \begin{pmatrix} \psi_1 & * & * & * \\ 0 & \psi_2 & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & \psi_n \end{pmatrix},$$

where for each $i = 1, \dots, n$, $\psi_i : G_{F_v} \to \overline{\mathbb{Q}}_l^\times$ is a continuous character satisfying the identity

$$\psi_i(\mathrm{Art}_{F_v}(\sigma)) = \chi_{v,i}(\sigma) \prod_{\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_l} \tau(\sigma)^{-(\lambda_{\tau, n-i+1} + i - 1)} \tag{2.1}$$

for all $\sigma \in \mathcal{O}_{F_v}^\times$.

Proof. This follows from local–global compatibility at $l = p$ (cf. the proof of [Ger, Corollary 2.7.8], which treats the case where π_v is unramified). We sketch the proof, which uses Fontaine's theory of weakly admissible modules, cf. [Fon94], [BM02, §2]. Let $\rho = r_\iota(\pi)|_{G_{F_v}}$, and let $K \subset \overline{\mathbb{Q}}_l$ be a finite extension of $\mathbb{Q}_l$ such that ρ takes values in $\mathrm{GL}_n(K)$. Let $L \subset \overline{F}_v$ be a finite Galois extension of F_v such that $\rho|_{G_L}$ is semi-stable. After possibly enlarging K, we can suppose in addition that K contains the images of all embeddings $L \hookrightarrow \overline{\mathbb{Q}}_l$. We write L_0 for the maximal absolutely unramified subfield of L, and σ for the absolute (arithmetic) Frobenius of L_0.

Fontaine's functor $D_{\mathrm{st},L}$ associates (cf. [Fon94, §5.6.3]) to ρ a $(\varphi, N, L/F_v, K)$-module D. By definition, D is a free $L_0 \otimes_{\mathbb{Q}_l} K$-module equipped with the following data:

- A $\sigma \otimes 1$–semilinear endomorphism φ of D.
- An $L_0 \otimes_{\mathbb{Q}_l} K$–linear endomorphism N of D satisfying the relation $N\varphi = l\varphi N$.
- An L–semilinear, K–linear action of the group $\mathrm{Gal}(L/F_v)$ on D that commutes with the action of φ and N.

Moreover, D is filtered: it is endowed with a decreasing, separated, exhaustive filtration $\mathrm{Fil}_\bullet D_L$ of $D_L = D \otimes_{L_0} L$ by $L \otimes_{\mathbb{Q}_l} K$-submodules. With this additional data, D is weakly admissible. By definition, this means that $t_N(D) = t_H(D)$, and that for all sub-$(\varphi, N, L/F_v, K)$-modules $D' \subset D$, we have $t_N(D') \geq t_H(D')$, where t_N, t_H are as defined in [Fon94, §4.4] and D'_L is endowed with the induced filtration.

We have a factorization $D_L = \prod_{\tau : L \hookrightarrow K} D_\tau$, and a corresponding factorization

$$\mathrm{Fil}_\bullet D_L = \prod_{\tau : L \hookrightarrow K} \mathrm{Fil}_\bullet D_\tau.$$

The assertion that $\mathrm{HT}_\tau(r_\iota(\pi)) = \{\lambda_{\tau,n}, \lambda_{\tau,n-1} + 1, \dots, \lambda_{\tau,1} + n - 1\}$ for each embedding $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ implies that for each embedding $\tau : L \hookrightarrow K$, we have

$$\dim_K \mathrm{gr}^i \mathrm{Fil}_\bullet D_\tau = \begin{cases} 1 & i = \lambda_{\tau|_{F_v}, j} + n - j, \text{ some } j = 1, \dots, n \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

On the other hand, we can define a Weil–Deligne representation associated to ρ as follows. Given $g \in W_{F_v}$, let g act on D by $(g \bmod W_L) \circ \varphi^{-\alpha(g)}$, where the action of g on the residue field of $\overline{F}_v$ is given by the $\alpha(g)^{\text{th}}$ power of the absolute (arithmetic) Frobenius. This action of W_{F_v} is $L_0 \otimes_{\mathbb{Q}_l} K$–linear. We can therefore decompose $D = \prod_{t : L_0 \hookrightarrow K} D_t$, where $D_t = D \otimes_{L_0, t} K$, and each factor D_t is then invariant under the action of W_{F_v} and N, giving a Weil–Deligne representation $\mathrm{WD}(D) = \mathrm{WD}(\rho)$ of W_{F_v} over K. As the notation suggests, the Weil–Deligne representation $\mathrm{WD}(D)$ is independent of the choice of t, up to isomorphism. The assertion of local-global compatibility is that there is an isomorphism

$$\mathrm{WD}(D)^{\text{F-ss}} \otimes_K \overline{\mathbb{Q}}_l \cong \mathrm{rec}^T_{F_v}(\iota^{-1}\pi_v).$$

So far, we have not used the fact that π is ι-ordinary. We now use this, in the form of the assertion of Lemma 2.3 that π_v is a subquotient of a representation $\text{n-Ind}_B^G\, \iota\chi_{v,1} \otimes \cdots \otimes \iota\chi_{v,n}$, for some smooth characters $\chi_{v,i} : F_v^\times \to \overline{\mathbb{Q}}_l^\times$ with

$$\mathrm{val}_l(\chi_{v,i}(\varpi_v)) = \frac{1}{e_v} \sum_{\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_l} \left(\lambda_{\tau, n+1-i} - \frac{(n-1)}{2} + i - 1 \right). \tag{2.3}$$

Writing $\mathrm{rec}^T_{F_v}(\iota^{-1}\pi_v) = (r, N)$, we have $r \sim \chi_{v,1}\iota^{-1}|\cdot|^{(1-n)/2} \oplus \cdots \oplus \chi_{v,n}\iota^{-1}|\cdot|^{(1-n)/2}$. Using this fact, we now construct an increasing filtration

$$G^\bullet = \left(0 = G^0 \subset G^1 \subset \cdots \subset G^n = D\right)$$

of D by $(\varphi, N, L/F_v, K)$-submodules, such that each G^i, $i = 1, \ldots, n$, is free over $L_0 \otimes_{\mathbb{Q}_l} K$ of rank i. We will show that each G^i (equipped with the induced filtration) is weakly admissible, which implies (cf. [Fon94, 5.6.7, Théorème]) that ρ can be conjugated to take image in the upper-triangular subgroup of $\mathrm{GL}_n(K)$.

After possibly enlarging K, we can assume that each character $\chi_{v,1}\iota^{-1}|\cdot|^{(1-n)/2}, \ldots, \chi_{v,n}\iota^{-1}|\cdot|^{(1-n)/2}$ takes values in $K^\times$. For each $i = 1, \ldots, n$, we define $G^i \subset D$ to be the free $L_0 \otimes_{\mathbb{Q}_l} K$-submodule such that for each embedding $t : L_0 \hookrightarrow K$, $G^i \otimes_{L_0, t} K \subset D_t$ is the subspace where W_{F_v} acts by the characters $\chi_{v,1}\iota^{-1}|\cdot|^{(1-n)/2}, \ldots, \chi_{v,i}\iota^{-1}|\cdot|^{(1-n)/2}$. It is clear that G^i is stable under the action of φ, N, and $\mathrm{Gal}(L/F_v)$, so defines a $(\varphi, N, L/F_v, K)$-submodule of D. Since D is weakly admissible, we have (the second equality by (2.3), the second inequality by (2.2)):

$$\begin{aligned}
t_N(G^i) = \sum_{j=1}^i t_N(\mathrm{gr}^j G^\bullet) &= \frac{[K : \mathbb{Q}_l]}{[F_v : \mathbb{Q}_l]} \sum_{\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_l} \sum_{j=1}^i (\lambda_{\tau, n+1-j} + j - 1) \\
&\geq t_H(G^i) = \frac{[K : \mathbb{Q}_l]}{[L : \mathbb{Q}_l]} \sum_{\tau : L \hookrightarrow \overline{\mathbb{Q}}_l} \sum_j j \cdot \dim_K \mathrm{gr}^j \mathrm{Fil}_\bullet G^i_\tau \\
&\geq \frac{[K : \mathbb{Q}_l]}{[L : \mathbb{Q}_l]} \sum_{\tau : L \hookrightarrow \overline{\mathbb{Q}}_l} \sum_{j=1}^i (\lambda_{\tau, n+1-j} + j - 1) \\
&= \frac{[K : \mathbb{Q}_l]}{[F_v : \mathbb{Q}_l]} \sum_{\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_l} \sum_{j=1}^i (\lambda_{\tau, n+1-j} + j - 1).
\end{aligned}$$

It follows that G^i is weakly admissible. Moreover, an easy calculation shows that $\mathrm{gr}^i G^\bullet$ is the image under Fontaine's functor of an L–semi-stable character $\psi_i : G_{F_v} \to K^\times$ satisfying (2.1). This completes the proof. □

Definition 2.5. *Let $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ be a continuous representation, and let $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F, \overline{\mathbb{Q}}_l)}$. We say that ρ is* ordinary of weight λ *if for each place $v|l$ of F, there is an isomorphism*

$$\rho|_{G_{F_v}} \sim \begin{pmatrix} \psi_1 & * & * & * \\ 0 & \psi_2 & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & \psi_n \end{pmatrix},$$

where for each $i = 1, \ldots, n$, $\psi_i : G_{F_v} \to \overline{\mathbb{Q}}_l^\times$ is a continuous character satisfying the relation

$$\psi_i(\mathrm{Art}_{F_v}(\sigma)) = \prod_{\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_l} \tau(\sigma)^{-(\lambda_{\tau, n-i+1} + i - 1)}$$

for all σ in some open subgroup of $\mathcal{O}_{F_v}^\times$.

Corollary 2.6. *Let $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$ be an isomorphism, let $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_l)}$, and let (π, χ) be a RAECSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\iota\lambda$. Suppose that π is ι-ordinary. Then $r_\iota(\pi)$ is ordinary of weight λ.*

We conclude this section with a result about soluble base change and descent.

Lemma 2.7. *Let L/F be a soluble CM extension, and let $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$ be an isomorphism.*

1. *Let (π, χ) be a RAECSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and suppose that $r_\iota(\pi)|_{G_L}$ is irreducible. Then there exists a RAECSDC automorphic representation (π_L, χ_L) of $\mathrm{GL}_n(\mathbb{A}_L)$ such that $r_\iota(\pi_L) \cong r_\iota(\pi)|_{G_L}$.*

2. *Let $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ be a continuous representation and let $\psi : G_{F^+} \to \overline{\mathbb{Q}}_l^\times$ be a continuous character such that the value $\psi(c)$ is independent of the choice of complex conjugation $c \in G_{F^+}$, and $\rho^c \cong \rho^\vee \otimes \psi|_{G_F}$. Suppose that $\rho|_{G_L}$ is irreducible, and that there exists a RAECSDC automorphic representation (π', χ') of $\mathrm{GL}_n(\mathbb{A}_L)$ such that $\rho|_{G_L} \cong r_\iota(\pi')$. Then there exists a RAECSDC automorphic representation (π, χ) of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\rho \cong r_\iota(\pi)$.*

We will often combine Lemma 2.7 with the following observation: if (π, χ) is a RAECSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and there exists a place w of F such that π_w is square-integrable, then $r_\iota(\pi)$ is irreducible (because $r_\iota(\pi)|_{G_{F_w}}$ is indecomposable, by local-global compatibility; cf. [TY07, Theorem B]).

Proof. This follows, by reduction to the case L/F cyclic, from [AC89, Ch. 3, Theorem 4.2] and [AC89, Ch. 3, Theorem 5.1]; see [BLGHT11, Lemma 1.4]. □

3 Deformation theory

In this paper we will make use of the framework for the deformation theory of conjugate self-dual Galois representations established in [CHT08], and its modification by Geraghty [Ger] to the context of ordinary Galois representations. We begin by recalling the definition of the group $\mathcal{G}_n$ of [CHT08].

3.1 The group $\mathcal{G}_n$

We recall that $\mathcal{G}_n$ is the group over $\mathbb{Z}$ defined as the semi-direct product

$$\mathcal{G}_n = (\mathrm{GL}_n \times \mathrm{GL}_1) \rtimes \{1, j\} = \mathcal{G}_n^0 \rtimes \{1, j\},$$

where j acts on $\mathrm{GL}_n \times \mathrm{GL}_1$ by $j(g, \mu)j^{-1} = (\mu {}^t g^{-1}, \mu)$. It has a representation ad on $\mathrm{Lie}\,\mathrm{GL}_n = \mathfrak{gl}_n$, given by the formulae

$$\mathrm{ad}(g, \mu)(X) = gXg^{-1}, \ \mathrm{ad}(j)(X) = -{}^tX,$$

and a character $\nu : \mathcal{G}_n \to \mathrm{GL}_1$ given by the formulae

$$\nu(g, \mu) = \mu, \ \nu(j) = -1.$$

If Γ is a group, R is a ring and $r : \Gamma \to \mathcal{G}_n(R)$ is a homomorphism, then we write $\mathrm{ad}\, r$ for the representation of Γ on $\mathfrak{gl}_n(R)$, and $\nu \circ r$ for the induced character $\Gamma \to \mathrm{GL}_1(R) = R^\times$. If $\Delta \subset \Gamma$ is a subgroup such that $r(\Delta) \subset \mathcal{G}_n^0(R)$, then we write $r|_\Delta$ for the composite homomorphism $\Delta \to \mathcal{G}_n^0(R) \to \mathrm{GL}_n(R)$. More generally, if Δ' is another group equipped with a homomorphism $f : \Delta' \to \Delta$, then we define $r|_{\Delta'} = f \circ r|_\Delta$.

Now suppose that $\Gamma = \Delta \rtimes \{1, c\}$ is a group. The following result is then an immediate consequence of [CHT08, Lemma 2.1.1].

Lemma 3.1. *Let R be a ring. There is a natural bijection between the following two sets:*

- *The set of homomorphisms $r : \Gamma \to \mathcal{G}_n(R)$ such that $r^{-1}(\mathcal{G}_n^0(R)) = \Delta$.*
- *The set of triples $(\rho, \mu, \langle \cdot, \cdot \rangle)$, where $\rho : \Delta \to \mathrm{GL}_n(R)$, $\mu : \Gamma \to R^\times$ are homomorphisms and $\langle \cdot, \cdot \rangle : R^n \times R^n \to R$ is a perfect R-linear pairing such that for all $x, y \in R^n$ and $\delta \in \Delta$, we have*

$$\langle x, y \rangle = -\mu(c)\langle y, x \rangle \text{ and } \langle \rho(\delta)x, \rho(\delta^c)y \rangle = \mu(\delta)\langle x, y \rangle.$$

Under this correspondence we have $\mu = \nu \circ r$ and $\langle x, y \rangle = {}^t x A^{-1} y$, where $r(c) = (A, -\mu(c))j$.

The following definition is [CHT08, Definition 2.1.6].

Definition 3.2. *Let k be a field, and let $r : \Gamma \to \mathcal{G}_n(k)$ be a homomorphism such that $r^{-1}(\mathcal{G}_n^0(k)) = \Delta$. We say that r is* Schur *if all irreducible Δ-subquotients of k^n are absolutely irreducible and if for all Δ-invariant subspaces $k^n \supset W_1 \supset W_2$ with k^n/W_1 and W_2 irreducible, we have*

$$(k^n/W_1)^c \not\cong W_2^\vee \otimes (\nu \circ r).$$

The following lemma follows immediately from [CHT08, Lemma 2.1.7] (and its proof).

Lemma 3.3. *Suppose that $r : \Gamma \to \mathcal{G}_n(k)$ is Schur. Then:*

1. *$r|_\Delta$ is semisimple and multiplicity free, and each irreducible constituent ρ satisfies $\rho^c \cong \rho^\vee \otimes (\nu \circ r)$.*
2. *Suppose that $r' : \Gamma \to \mathcal{G}_n(k)$ is another Schur homomorphism such that $\operatorname{tr} r'|_\Delta = \operatorname{tr} r|_\Delta$, and that k is algebraically closed. Then r and r' are $\mathrm{GL}_n(k)$-conjugate.*
3. *Suppose that the characteristic of k is not 2. Then $H^0(\Gamma, \operatorname{ad} r) = 0$.*

Lemma 3.4. *Let k be a field and $\rho = \oplus_{i=1}^s \rho_i : \Delta \to \mathrm{GL}_n(k)$ a semisimple representation, with each ρ_i absolutely irreducible. Suppose that there is a character $\mu : \Gamma \to k^\times$ such that:*

1. *For each $i = 1, \dots, s$, there is a perfect pairing $\langle \cdot, \cdot \rangle_i$ such that $\langle x, y \rangle_i = -\mu(c)\langle y, x \rangle_i$ and $\langle \rho(\delta)x, \rho(\delta^c)y \rangle_i = \mu(\delta)\langle x, y \rangle_i$ for all $x, y \in \rho_i$, $\delta \in \Delta$.*
2. *For each $i \neq j$ we have $\rho_i \not\cong \rho_j$ and $\rho_j^c \not\cong \rho_i^\vee \otimes \mu$.*

Then ρ admits an extension to a homomorphism $r : \Gamma \to \mathcal{G}_n(k)$ such that $r^{-1}(\mathcal{G}_n^0(k)) = \Delta$. The set of $\mathrm{GL}_n(k)$-conjugacy classes of such extensions is a principal homogeneous space for the group $\prod_{i=1}^s k^\times/(k^\times)^2$, with action given as follows. Write $r(c) = (A, -\mu(c))j$. Then $A = \oplus_{i=1}^s A_i$ is a block diagonal matrix, and $(\alpha_i) \in \prod_{i=1}^s k^\times$ acts by $A_i \mapsto \alpha_i A_i$. Moreover, every choice of extension is Schur.

Proof. The proof is an easy generalization of the proof of [CHT08, Lemma 2.1.4]. ☐

3.2 Deformation of Galois representations

Let F be a CM field with maximal totally real subfield F^+, and let l be an odd prime. We fix a finite set of places S of F^+ which split in F and write $F(S)$ for the maximal extension of F unramified outside S. We suppose that S contains the set S_l of places of F^+ dividing l. We write $G_{F^+,S} = \mathrm{Gal}(F(S)/F^+)$ and $G_{F,S} \subset G_{F^+,S}$ for the subset of elements fixing F. For each $v \in S$ we choose a place $\widetilde{v}$ of F above it, and write $\widetilde{S}$ for the set of these places. We choose a complex conjugation $c \in G_{F^+,S}$.

We fix also a finite field k of characteristic l and a representation $\overline{r} : G_{F^+,S} \to \mathcal{G}_n(k)$ such that $G_{F,S} = \overline{r}^{-1}(\mathrm{GL}_n \times \mathrm{GL}_1(k))$. Let K be a finite extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ with ring of integers $\mathcal{O}$, maximal ideal λ, and residue field k. Choose a character $\chi : G_{F^+,S} \to \mathcal{O}^\times$ such that $\nu \circ \overline{r} = \overline{\chi}$.

If $v \in S_l$, then we write Λ_v for the completed group algebra $\mathcal{O}[\![(I_{F_{\widetilde{v}}}^{\mathrm{ab}}(l))^n]\!]$, where $I_{F_{\widetilde{v}}}^{\mathrm{ab}}(l)$ denotes the maximal pro-l quotient of the inertia group of the maximal abelian extension of $F_{\widetilde{v}}$. By class field theory, this group is isomorphic to $\mathcal{O}_{F_{\widetilde{v}}}^\times(l)$, the maximal pro-$l$ quotient of $\mathcal{O}_{F_{\widetilde{v}}}^\times$. Let $\Lambda = \widehat{\otimes}_{v \in S_l} \Lambda_v$, the completed tensor product being over $\mathcal{O}$. We will consider deformations of $\overline{r}$ to objects of $\mathcal{C}_\Lambda$. If $v \in S$, we write $\overline{r}|_{G_{F_{\widetilde{v}}}}$ for the composite

$$G_{F_{\widetilde{v}}} \to G_{F,S} \to \mathcal{G}_n^0(k) \to \mathrm{GL}_n(k).$$

Definition 3.5. *A lifting of* $\overline{r}$ *(resp.* $\overline{r}|_{G_{F_{\widetilde{v}}}}$*) to an object* R *of* $\mathcal{C}_{\mathcal{O}}$ *is a continuous homomorphism* $r : G_{F^+,S} \to \mathcal{G}_n(R)$ *(resp.* $r : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(R)$*) with* $r \bmod \mathfrak{m}_R = \overline{r}$ *(resp.* $= \overline{r}|_{G_{F_{\widetilde{v}}}}$*) and* $\nu \circ r = \chi$ *(resp. no further condition). Two liftings are said to be* equivalent *if they are conjugate by an element of* $1+M_n(\mathfrak{m}_R) \subset \mathrm{GL}_n(R)$*. An equivalence class of liftings is called a* deformation.

Let $T \subset S$*. By a* T-framed lifting *of* $\overline{r}$ *to* R *we mean a tuple* $(r; \alpha_v)_{v\in T}$ *where* r *is a lifting of* $\overline{r}$ *and* $\alpha_v \in 1 + M_n(\mathfrak{m}_R)$*. We call two framed liftings* $(r; \alpha_v)$ *and* $(r'; \alpha'_v)$ *equivalent if there is an element* $\beta \in 1 + M_n(\mathfrak{m}_R)$ *with* $r' = \beta r \beta^{-1}$ *and* $\alpha'_v = \beta\alpha_v$*. By a* T-framed deformation *of* $\overline{r}$ *we mean an equivalence class of* T*-framed liftings.*

Definition 3.6. *If* $v \in S - S_l$ *then we define a* local deformation problem *at* v *to be a subfunctor* $\mathcal{D}_v$ *of the functor of all liftings of* $\overline{r}|_{G_{F_{\widetilde{v}}}}$ *to objects of* $\mathcal{C}_{\mathcal{O}}$ *satisfying the following conditions:*

1. $(k, \overline{r}) \in \mathcal{D}_v$*.*
2. *Suppose that* (R_1, r_1) *and* $(R_2, r_2) \in \mathcal{D}_v$*, that* I_1 *(resp.* (I_2)*) is a closed ideal of* R_1 *(resp.* R_2*) and that* $f : R_1/I_1 \to R_2/I_2$ *is an isomorphism in* $\mathcal{C}_{\mathcal{O}}$ *such that* $f(r_1 \ mod\ I_1) = r_2 \ mod\ I_2$*. Let* R_3 *denote the subring of* $R_1 \times R_2$ *consisting of pairs with the same image in* $R_1/I_1 \cong R_2/I_2$*. Then* $(R_3, r_1 \times r_2) \in \mathcal{D}_v$*.*
3. *If* (R_j, r_j) *is an inverse system of elements of* $\mathcal{D}_v$ *then*
$$(\lim R_j, \lim r_j) \in \mathcal{D}_v.$$
4. $\mathcal{D}_v$ *is closed under equivalence.*
5. *If* $R \subset S$ *is an inclusion in* $\mathcal{C}_{\mathcal{O}}$ *and if* $r : G_F \to \mathrm{GL}_n(R)$ *is a lifting of* $\overline{r}$ *such that* $(S, r) \in \mathcal{D}_v$ *then* $(R, r) \in \mathcal{D}_v$*.*

On the other hand, if $v \in S_l$ *we define a* local deformation problem *at* v *to be a subfunctor* $\mathcal{D}_v$ *of the functor of all liftings of* $\overline{r}|_{G_{F_{\widetilde{v}}}}$ *to objects of* $\mathcal{C}_{\Lambda_v}$ *satisfying the same conditions with the category* $\mathcal{C}_{\mathcal{O}}$ *replaced by* $\mathcal{C}_{\Lambda_v}$*.*

Given a collection $\{\mathcal{D}_v\}_{v\in S}$ of local deformation problems, we have a (global) deformation problem consisting of the following data:

$$\mathcal{S} = \left(F/F^+, S, \widetilde{S}, \Lambda, \overline{r}, \chi, \{\mathcal{D}_v\}_{v\in S}\right).$$

Definition 3.7. *Let* $T \subset S$*, and let* $R \in \mathcal{C}_\Lambda$*. The ring* R *then has canonical structures of* $\mathcal{O}$*-algebra and* Λ_v*-algebra for each* $v \in S_l$*. We say that a* T*-framed lifting* $(r; \alpha_v)_{v\in T}$ *of* $\overline{r}$ *to* R *is of* type $\mathcal{S}$ *if for all* $v \in S$*, the restriction* $r|_{G_{F_{\widetilde{v}}}}$ *lies in* $\mathcal{D}_v$*. We say that a* T*-framed deformation is of* type $\mathcal{S}$ *if some (equivalently any) element of the equivalence class is of type* $\mathcal{S}$*.*

We let $\mathrm{Def}^{\square_T}_{\mathcal{S}}$ *denote the functor which associates to an object* R *of* $\mathcal{C}_\Lambda$ *the set of all* T*-framed deformations of* $\overline{r}$ *to* R *of type* $\mathcal{S}$*. If* $T = S$ *then we refer to framed deformations and write* $\mathrm{Def}^{\square}_{\mathcal{S}}$*. If* $T = \emptyset$ *we refer to deformations and write* $\mathrm{Def}_{\mathcal{S}}$*.*

If R_v *denotes the ring representing the local deformation problem* $\mathcal{D}_v$*, then we write*

$$R^{loc}_{\mathcal{S},T} = \widehat{\otimes}_{v\in T} R_v,$$

the completed tensor product being over $\mathcal{O}$*. Note that* $R^{loc}_{\mathcal{S},T}$ *is naturally a* Λ*-algebra whenever* T *contains* S_l*.*

Proposition 3.8. *Suppose that* $\overline{r}$ *is Schur. Then the functors* $\mathrm{Def}^{\square_T}_{\mathcal{S}}, \mathrm{Def}^{\square}_{\mathcal{S}}, \mathrm{Def}_{\mathcal{S}}$ *are represented by objects of* $\mathcal{C}_\Lambda$*. We write respectively* $R^{\square_T}_{\mathcal{S}}, R^{\square}_{\mathcal{S}}$ *and* $R^{univ}_{\mathcal{S}}$ *for the representing objects.*

Proof. We prove this for $\mathrm{Def}^{\square}_{\mathcal{S}}$, the other cases being similar. Consider the deformation problem in the sense of [CHT08, §2.3]

$$\mathcal{S}' = \left(F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{r}, \chi, \{\mathcal{D}'_v\}_{v\in S, v|l} \cup \{\mathcal{D}_v\}_{v\in S, v\nmid l}\right),$$

where for $v|l$ we take $\mathcal{D}'_v$ to be the unrestricted functor of liftings of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ to objects of $\mathcal{C}_{\mathcal{O}}$. Then the functor $\mathcal{D}'_v$ is represented by an object $R_v^{\square} \in \mathcal{C}_{\mathcal{O}}$. Moreover, the deformation problem $\mathcal{S}'$ defines a functor $\mathrm{Def}^{\square}_{\mathcal{S}'} : \mathcal{C}_{\mathcal{O}} \to$ Sets which is represented by an object $R^{\square}_{\mathcal{S}'} \in \mathcal{C}_{\mathcal{O}}$ (by [CHT08, Proposition 2.2.9]). On the other hand, for $v|l$ $\mathcal{D}_v$ is represented by an object R_v of $\mathcal{C}_{\Lambda_v}$, and we have a canonical homomorphism $R_v^{\square} \to R_v$. The functor $\mathrm{Def}^{\square}_{\mathcal{S}}$ is represented by$R^{\square}_{\mathcal{S}'} \widehat{\otimes}_{\widehat{\otimes}_{v|l} R_v^{\square}} \left(\widehat{\bigotimes}_{v|l} R_v \right)$ with its induced Λ-algebra structure. □

Proposition 3.9. *Suppose that $\overline{r}$ is Schur, and let $T = S$. Then the ring $R^{\square}_{\mathcal{S}}$ can be presented as a quotient of a power series ring over $R^{loc}_{\mathcal{S},T}$ in g variables by r relations, where*

$$g - r = -\dim_k H^0(G_{F^+,S}, \operatorname{ad} \overline{r}(1)) - \sum_{v|\infty} n(n + \chi(c_v))/2.$$

(If $v|\infty$ is a place of F^+, then we write $c_v \in G_{F^+}$ for a choice of complex conjugation at the place v.)

Proof. This follows from [CHT08, Corollary 2.3.5] by the same argument as in the proof of Proposition 3.8. □

3.3 Local deformation problems

In this section we define some useful local deformation problems. We shall always use the notation $R_v^{\square}$ for the ring representing the local deformation problem consisting of all liftings of $\overline{r}|_{G_{F_{\widetilde{v}}}}$. Thus $R_v^{\square}$ is an $\mathcal{O}$-algebra (resp. a Λ_v-algebra) when $v \nmid l$ (resp. when $v|l$). We recall (cf. [BLGHT11, Lemma 3.2]) that to give a local deformation problem, it suffices to give a quotient R of the unrestricted universal lifting ring $R_v^{\square}$ of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ which has the following two properties:

- R is reduced;
- and the ideal $I = \ker(R_v^{\square} \to R)$ is $(1 + M_n(\mathfrak{m}_{R_v^{\square}}))$-invariant and satisfies $I \neq \mathfrak{m}_{R_v^{\square}}$.

3.3.1 Unrestricted deformations

Proposition 3.10. *Suppose that $v \nmid l$ and that $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is unramified, and that $H^0(G_{F_{\widetilde{v}}}, \operatorname{ad} \overline{r}(1)) = 0$. Then $R_v^{\square}$ is formally smooth over $\mathcal{O}$ of dimension $1 + n^2$.*

We omit the proof, which is a standard argument in obstruction theory.

3.3.2 Ordinary deformations

Suppose that $v \in S_l$, that $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is trivial, and that K contains the images of all embeddings $F_{\widetilde{v}} \hookrightarrow \overline{\mathbb{Q}}_l$. Recall that we have defined $\Lambda_v = \mathcal{O}[\![(I^{\mathrm{ab}}_{F_{\widetilde{v}}}(l))^n]\!]$, a completed group algebra. This algebra comes equipped with the universal characters $\psi_i^v : I_{F_{\widetilde{v}}} \to \Lambda_v^{\times}$, $i = 1, \ldots, n$. For each minimal prime $Q_v \subset \Lambda_v$, Λ_v / Q_v is geometrically integral over $\mathcal{O}$.

We recall that Geraghty has defined a quotient $R_v^{\triangle}$ of the universal lifting ring $R_v^{\square}$ satisfying the following condition (cf. [Ger, Lemma 3.1.3]):

- Let E/K be a finite extension with ring of integers $\mathcal{O}_E$, and fix a map $\Lambda_v \to \mathcal{O}_E$ of $\mathcal{O}$-algebras. Let $\rho : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\mathcal{O}_E)$ be a continuous lifting of $\overline{r}|_{G_{F_{\widetilde{v}}}}$. Then the map $R_v^{\square} \to \mathcal{O}_E$ classifying ρ factors through $R_v^{\triangle}$ if and only if ρ is $\mathrm{GL}_n(\mathcal{O}_E)$-conjugate to an upper-triangular representation satisfying the following condition: if $(\chi_1, \ldots, \chi_n)$ are the characters appearing on the diagonal, then the tuple of characters $(\chi_1|_{I_{F_{\widetilde{v}}}}, \ldots, \chi_n|_{I_{F_{\widetilde{v}}}})$ is equal to the pushforward of the universal tuple $(\psi_1^v, \ldots, \psi_n^v)$ along the map $\Lambda_v \to \mathcal{O}_E$.

We briefly recall the construction. Let $\mathcal{F}$ denote the $\mathcal{O}$-scheme of full flags in $\mathcal{O}^n$, and let $\mathcal{G}_v$ denote the closed subscheme of $\mathcal{F} \otimes_{\mathcal{O}} R_v^{\square}$ whose A-points for an $\mathcal{O}$-algebra A are pairs $(\mathrm{Fil}_v^\bullet, \varphi)$, where $\varphi : R_v^{\square} \to A$ is an $\mathcal{O}$-algebra homomorphism and $\mathrm{Fil}_v^\bullet$ is an increasing filtration of A^n by A-direct summands which are preserved by the pushforward of the universal lifting under φ, and such that the action of $\rho|_{I_{F_{\widetilde{v}}}}$ on $\mathrm{gr}^i \mathrm{Fil}_v^\bullet = \mathrm{Fil}_v^i / \mathrm{Fil}_v^{i-1}$ is given by the pushforward of the universal character ψ_i^v under the homomorphism $\Lambda_v \to R_v^{\square} \to A$. We thus have a projective morphism $\pi : \mathcal{G}_v \to R_v^{\square}$, and $R_v^{\triangle}$ is defined as the maximal reduced, $\mathcal{O}$-flat quotient of the scheme-theoretic image of π. The quotient $R_v^{\square} \to R_v^{\triangle}$ defines a local deformation problem $\mathcal{D}_v^{\triangle}$.

Lemma 3.11. *Suppose that $[F_{\widetilde{v}} : \mathbb{Q}_l] > n(n-1)/2 + 1$. Then $\mathcal{G}_v$ is $\mathcal{O}$-flat and reduced. For each minimal prime $Q_v \subset \Lambda_v$, $\mathcal{G}_v \otimes_{\Lambda_v} \Lambda_v/Q_v$ is $\mathcal{O}$-flat and integral of dimension $1 + [F_{\widetilde{v}} : \mathbb{Q}_l]n(n+1)/2 + n^2$, and $\mathcal{G}_v \otimes_{\Lambda_v} \Lambda_v/(Q_v, \lambda)$ is integral.*

Corollary 3.12. *Suppose that $[F_{\widetilde{v}} : \mathbb{Q}_l] > n(n-1)/2 + 1$, and let $R \in \mathcal{C}_{\mathcal{O}}$ be integral. Let $E = \mathrm{Frac}(R)$ and choose an algebraic closure $\overline{E}$ of E. Then a homomorphism $R_v^{\square} \to R$ factors through the quotient $R_v^{\triangle}$ if and only if the following condition is satisfied:*

- *Let $\rho : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(R)$ denote the induced lifting of $\overline{r}|_{G_{F_{\widetilde{v}}}}$. There exists an increasing filtration $0 = \mathrm{Fil}_v^0 \subset \mathrm{Fil}_v^1 \subset \cdots \subset \mathrm{Fil}_v^n = \overline{E}^n$ of $\rho \otimes_R \overline{E}$ by $G_{F_{\widetilde{v}}}$-invariant subspaces, such that each graded piece $\mathrm{gr}^i \mathrm{Fil}_v^\bullet = \mathrm{Fil}_v^i / \mathrm{Fil}_v^{i-1}$ is one-dimensional, and the action of $I_{F_{\widetilde{v}}}$ on this graded piece is given by the specialization of the universal character $\psi_i^v : I_{F_{\widetilde{v}}} \to \Lambda_v^\times$ via the homomorphism $\Lambda_v \to R_v^{\square} \to \overline{E}$.*

Proof. The lemma shows that the scheme-theoretic image of π is already $\mathcal{O}$-flat and reduced, so that a $\mathrm{Spec}\, R$-point of the scheme-theoretic image of π necessarily factors through $\mathrm{Spec}\, R_v^{\triangle}$. ☐

Proof of Lemma 3.11. We study $\mathcal{G}_v$ by means of a finite type model. Let $d_v = [F_{\widetilde{v}} : \mathbb{Q}_l] > n(n-1)/2+1$. We treat the case where $F_{\widetilde{v}}$ contains an l^{th} root of unity, the other case being similar (and simpler). Then the pro-l group $G_{F_{\widetilde{v}}}(l)$ admits a presentation as a quotient of the free pro-l group on generators $x_1, \dots, x_{d_v+2}$ by the single relation

$$x_1^{l^s}[x_1, x_2][x_3, x_4] \dots [x_{d_v+1}, x_{d_v+2}] = 1, \tag{3.1}$$

where s is the largest integer such that $F_{\widetilde{v}}$ contains a root of unity of order exactly l^s (see [NSW00, Theorem 7.5.8]). (We note that in the above situation, d_v is necessarily even.) During the course of this proof, we write GL_n for the $\mathcal{O}$-scheme of invertible $n \times n$ matrices, and $\mathrm{U}_n \subset \mathrm{B}_n \subset \mathrm{GL}_n$ for its closed subschemes of unipotent upper-triangular and upper-triangular matrices, respectively. We write $\mathcal{N}$ for the $\mathcal{O}$-scheme of tuples $(g_1, \dots, g_{d_v+2}) \in \mathrm{B}_n^{d_v+2}$ satisfying the relation (3.1).

Specifying a minimal prime $Q_v \subset \Lambda_v$ is the same as choosing roots of unity $\zeta_1, \dots, \zeta_n \in \mu_{l^s}(\mathcal{O})$, the prime being given by

$$(\psi_1^v(x_1) - \zeta_1, \dots, \psi_n^v(x_1) - \zeta_n).$$

We suppose such a choice has been fixed, and write $\mathcal{N}_{Q_v}$ for the closed subscheme of $\mathcal{N}$ where the diagonal entries of g_1 are given by $\zeta_1, \dots, \zeta_n$. We then have a canonical identification (since each ζ_i has trivial image in k):

$$\mathcal{N}_{Q_v} \otimes_{\mathcal{O}} k = \{(g_1, \dots, g_{d_v+2}) \in (\mathrm{U}_n \times \mathrm{B}_n^{d_v+1}) \otimes_{\mathcal{O}} k \mid g_1^{l^s}[g_1, g_2] \dots [g_{d_v+1}, g_{d_v+2}] = 1\}.$$

For each $i = 3, \dots, d_v+2$, we write $Z_i \subset \mathcal{N}_{Q_v} \otimes_{\mathcal{O}} k$ for the open subscheme where g_i has distinct eigenvalues, and $f_i : Z_i \to (\mathrm{U}_n \times \mathrm{B}_n^{d_v}) \otimes_{\mathcal{O}} k$ for the projection which forgets g_{i+1} (if i is odd) or g_{i-1} (if i is even). We also write $V_i \subset (\mathrm{U}_n \times \mathrm{B}_n^{d_v}) \otimes_{\mathcal{O}} k$ for the open subscheme where the i-entry (if i is odd) or the $(i-1)$-entry (if i is even) has distinct eigenvalues. In either case, there is a factorization

$$f_i : Z_i \to V_i \hookrightarrow (\mathrm{U}_n \times \mathrm{B}_n^{d_v}) \otimes_{\mathcal{O}} k.$$

We write $Z = \cup_{i=3}^{d_v+2} Z_i$, an open subscheme of $\mathcal{N}_{Q_v} \otimes_{\mathcal{O}} k$.

The fibers of f_i are sets of solutions $h \in \mathrm{B}_n \otimes_{\mathcal{O}} k$ to equations $hgh^{-1} = gu$, where $g \in \mathrm{B}_n \otimes_{\mathcal{O}} k$ and $u \in \mathrm{U}_n \otimes_{\mathcal{O}} k$ and g has distinct eigenvalues. In particular, the non-empty fibers of f are smooth of dimension n, being torsors for the torus $Z_{\mathrm{B}_n}(g)$. On the other hand, f_i is a relative global complete intersection over

V_i, since Z_i is obtained by imposing $\dim \mathrm{U}_n \otimes_{\mathcal{O}} k = n(n-1)/2$ relations on $V_i \times (\mathrm{B}_n \otimes_{\mathcal{O}} k)$. It follows that f_i is smooth, and hence Z is smooth over k and irreducible of dimension

$$\dim Z = \dim(\mathrm{U}_n \times \mathrm{B}_n^{d_v}) \otimes_{\mathcal{O}} k + n = \dim \mathrm{B}_n^{d_v+1} \otimes_{\mathcal{O}} k.$$

Since $\mathcal{N}_{Q_v} \otimes_{\mathcal{O}} k$ is globally cut out inside $(\mathrm{U}_n \times \mathrm{B}_n^{d_v+1}) \otimes_{\mathcal{O}} k$ by $\dim \mathrm{U}_n \otimes_{\mathcal{O}} k$ equations, every irreducible component has dimension at least $\dim \mathrm{B}_n^{d_v+1} \otimes_{\mathcal{O}} k$. The complement of Z is contained in the closed subscheme of $(\mathrm{U}_n \times \mathrm{B}_n^{d_v+1}) \otimes_{\mathcal{O}} k$ where none of $g_3, \dots, g_{d_v+2}$ has distinct eigenvalues, which has dimension

$$\dim(\mathrm{U}_n \times \mathrm{B}_n^{d_v+1}) \otimes_{\mathcal{O}} k - d_v.$$

Provided, therefore, that

$$\dim(\mathrm{U}_n \times \dim \mathrm{B}_n^{d_v+1}) \otimes_{\mathcal{O}} k - d_v < \dim \mathrm{B}_n^{d_v+1} \otimes_{\mathcal{O}} k - 1,$$

or equivalently that $d_v > n(n-1)/2 + 1$, we see that Z is dense in $\mathcal{N}_{Q_v} \otimes_{\mathcal{O}} k$, with complement of codimension at least 2. Under our hypotheses, it thus follows that $\mathcal{N}_{Q_v} \otimes_{\mathcal{O}} k$ is a normal complete intersection. A similar argument shows that $\mathcal{N}_{Q_v}$ itself is an $\mathcal{O}$-flat normal complete intersection, and that $\mathcal{N}$ is a complete intersection. Since it is also generically reduced, $\mathcal{N}$ is reduced. It is $\mathcal{O}$-flat since each $\mathcal{N}_{Q_v}$ is $\mathcal{O}$-flat, and these are the distinct irreducible components of $\mathcal{N}$.

Let $x \in \mathcal{G}_v$ be a closed point, which therefore maps to the closed point of $R_v^{\triangle}$. The completed local ring of $\mathcal{G}_v$ at x is formally smooth over the completed local ring of $\mathcal{N}$ at the point in its special fiber where all the matrices are equal to the identity (cf. the proof of [Ger, Lemma 3.2.1]). It follows that $\mathcal{G}_v$ is itself an $\mathcal{O}$-flat reduced complete intersection, that $\mathcal{G}_v \otimes_{\Lambda_v} \Lambda_v/Q_v$ is $\mathcal{O}$-flat and normal, and that $\mathcal{G}_v \otimes_{\Lambda_v} \Lambda_v/(Q_v, \lambda)$ is integral. ☐

If $1 \leq i < j \leq n$, we write $I(i,j,v) \subset \Lambda_v$ for the ideal generated by the relations $\psi_i^v(\sigma) - \psi_j^v(\sigma)$, $\sigma \in I_{F_{\widetilde{v}}}(l)$. Let $I_v = \prod_{i<j} I(i,j,v)$. Thus $\dim \Lambda_v/I_v = 1 + (n-1)[F_{\widetilde{v}} : \mathbb{Q}_l]$, and $V(I_v) \subset \operatorname{Spec} \Lambda_v$ is the closed subset where the characters $\psi_{i,j}^v$ are not pairwise distinct.

Lemma 3.13. *Let $U \subset \operatorname{Spec} \Lambda_v$ denote the complement of $V(I_v)$. Then the map $\pi_U : \mathcal{G}_{v,U} \to \operatorname{Spec} R_{v,U}^{\triangle}$ is an isomorphism. In particular, for each minimal prime $Q_v \subset \Lambda_v$, $R_{v,U}/(Q_v, \lambda)$ is integral.*

Proof. The map π_U is projective, by construction. It is also quasi-finite; in fact, the fibers are singletons, as follows immediately using the definition of $\mathcal{G}_v$ in terms of its functor of points. It follows that π_U is a finite morphism. To show that it is an isomorphism, it therefore suffices to check that π_U induces surjections on completed local rings, and this can be proved following e.g. the proof of [CHT08, Lemma 2.4.6]. ☐

Proposition 3.14. *Suppose that $[F_{\widetilde{v}} : \mathbb{Q}_l] > n(n-1)/2 + 1$. Let $x \in \operatorname{Spec} R_v^{\triangle}[1/l]$ be a closed point, and let y denote its image in $\operatorname{Spec} \Lambda_v[1/l]$.*

1. *Suppose that for each $1 \leq i < j \leq n$, we have $\psi_i^v \bmod y \neq \psi_j^v \bmod y$, as characters $I_{F_{\widetilde{v}}}^{ab}(l) \to \kappa(y)^\times$. Then $\operatorname{Spec} R_v^{\triangle}[1/l] \otimes_{\Lambda_v} \kappa(y)$ is geometrically connected of dimension at most $[F_{\widetilde{v}} : \mathbb{Q}_l]n(n-1)/2 + n^2 + n(n-1)/2$.*
2. *Suppose that for each $1 \leq i < j \leq n$, we have $\psi_i^v \bmod y \neq \psi_j^v \bmod y$ and $\psi_i^v \bmod y \neq \epsilon\psi_j^v \bmod y$. Then $\operatorname{Spec} R_v^{\triangle}[1/l] \otimes_{\Lambda_v} \kappa(y)$ is regular of dimension $[F_{\widetilde{v}} : \mathbb{Q}_l]n(n-1)/2 + n^2$, and $\operatorname{Spec} R_v^{\triangle}[1/l]_{(x)}$ is regular of dimension $[F_{\widetilde{v}} : \mathbb{Q}_l]n(n+1)/2 + n^2$.*
3. *For each minimal prime $Q_v \subset \Lambda_v$, $R_v^{\triangle}/(Q_v)$ is geometrically irreducible of dimension $1 + [F_{\widetilde{v}} : \mathbb{Q}_l]n(n+1)/2 + n^2$, and $R_v^{\triangle}/(Q_v, \lambda)$ is generically reduced.*

Proof. The first part follows easily from Lemma 3.13, [Ger, Lemma 3.4.2], and the argument of [Ger, Lemma 3.2.3] (which is essentially to calculate the Zariski tangent space of $\mathcal{G}_v[1/l]$ at the unique point above x). The second part of the proposition follows in a similar manner. We now prove the third part. It follows from the second part of the proposition that for each minimal prime $Q_v \subset \Lambda_v$, $\pi(\mathcal{G}_v \otimes_{\Lambda_v} \Lambda_v/Q_v)$ is a geometrically irreducible closed subset of $\operatorname{Spec} R_v^{\triangle}$ of dimension $1 + [F_{\widetilde{v}} : \mathbb{Q}_l]n(n+1)/2 + n^2$. Thus the $V(Q_v) \subset \operatorname{Spec} R_v^{\triangle}$ are the distinct irreducible components of $\operatorname{Spec} R_v^{\triangle}$, and they are each geometrically irreducible of this dimension. The claim that $R_v^{\triangle}/(Q_v, \lambda)$ is generically reduced follows from Lemma 3.13 and the fact (Lemma 3.11) that $\mathcal{G}_v \otimes_{\Lambda_v} \Lambda_v/(Q_v, \lambda)$ is reduced. ☐

3.3.3 Level raising deformations

Now suppose that $q_v \equiv 1 \bmod l$, and that $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is trivial. We recall a local deformation problem from [Tay08]. Choose characters

$$\chi_{v,1}, \dots, \chi_{v,n} : \mathcal{O}_{F_{\widetilde{v}}}^\times \to \mathcal{O}^\times,$$

necessarily of finite order, which become trivial on reduction modulo λ. We write $\mathcal{D}_v^\chi$ for the functor of liftings ρ of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ to objects of $\mathcal{C}_\mathcal{O}$ such that for all $\sigma \in I_{F_{\widetilde{v}}}$, we have

$$\mathrm{char}_{\rho(\sigma)}(X) = \prod_{i=1}^n \left(X - \chi_{v,i}(\mathrm{Art}_{F_{\widetilde{v}}}^{-1}(\sigma)\right)^{-1}).$$

This defines a local deformation problem, and we write $R_v^{\chi_v}$ for the corresponding local lifting ring.

Proposition 3.15. *1. Suppose that $\chi_{v,j} = 1$ for each j. Then each minimal prime of $R_v^1/(\lambda)$ contains a unique minimal prime of R_v^1, and each for each minimal prime $\mathfrak{p} \subset R_v^1$, $R_v^1/\mathfrak{p}$ is $\mathcal{O}$-flat of dimension n^2+1. Moreover, $R_v^1/(\lambda)$ is generically reduced. If K is sufficiently large, then for every minimal prime $\mathfrak{p} \subset R_v^1$, $R_v^1/\mathfrak{p}$ is geometrically integral over $\mathcal{O}$; and for every minimal prime $\mathfrak{p} \subset R_v^1/(\lambda)$, $R_v^1/\mathfrak{p}$ is geometrically integral over k.*

2. Suppose that the $\chi_{v,j}$ are pairwise distinct. Then $\operatorname{Spec} R_v^{\chi_v}$ is geometrically irreducible of dimension n^2+1, and its generic point is of characteristic zero. Moreover, $\operatorname{Spec} R_v^{\chi_v}[1/l]$ is formally smooth over K.

Proof. The assertion that $R_v^1/(\lambda)$ is generically reduced follows from [Tho12, Lemma 3.15] and [Mat89, Theorem 23.9]. The formal smoothness of $\operatorname{Spec} R_v^{\chi_v}[1/l]$ when the $\chi_{v,j}$ are pairwise distinct follows from [Tay08, Lemma 3.3]. (It is assumed in *loc. cit.* that $l > n$. However, the proof of this lemma goes through without change in the case $l \leq n$.) The fact that the irreducible components of $\operatorname{Spec} R_v^1$ and $\operatorname{Spec} R_v^1/(\lambda)$ are geometrically irreducible when K is sufficiently large follows from the last two parts of [BLGHT11, Lemma 3.3]. The rest of the proposition is contained in [Tho12, Proposition 3.16]. $\square$

Proposition 3.16. *Suppose that the $\chi_{v,j}$ are pairwise distinct. Let A be a complete Noetherian local $\mathcal{O}$-algebra.* (*We do not assume that the residue field of A is k.*) *Suppose that $\operatorname{Spec} A[1/l]$ is* (*non-trivial and*) *connected. Then $\operatorname{Spec} A\widehat{\otimes}_\mathcal{O} R_v^{\chi_v}[1/l]$ is connected.*

Proof. In the case that $A = \mathcal{O}$, this is [Tay08, Lemma 3.4]. Let $\phi \in G_{F_{\widetilde{v}}}$ be a lift of arithmetic Frobenius, and let $t \in I_{F_{\widetilde{v}}}(l)$ be a generator of the l-part of tame inertia. Via the assignment $\rho \mapsto (\rho(\phi), \rho(t)) = (\Phi, \Sigma)$, we see (cf. the discussion after [Tay08, Proposition 3.1]) that $R_v^{\chi_v}$ represents the functor $\mathcal{C}_\mathcal{O} \to$ Sets which assigns to $R \in \mathcal{C}_\mathcal{O}$ the set of pairs (Φ, Σ), where $\Phi \in 1 + M_n(\mathfrak{m}_R)$ and $\Sigma \in 1 + M_n(\mathfrak{m}_R)$, the characteristic polynomial of Σ is $\prod_{i=1}^n (X - \chi_{v,i}(\mathrm{Art}_{F_{\widetilde{v}}}^{-1}(t)^{-1})$, and $\Phi\Sigma\Phi^{-1} = \Sigma^{q_v}$.

Let $\rho_0 \in \mathcal{D}_v^{\chi_v}(\mathcal{O})$ be the representation corresponding to the pair

$$(1_n, \mathrm{diag}(\chi_{v,1}(\mathrm{Art}_{F_{\widetilde{v}}}^{-1}(t)^{-1}), \dots, \chi_{v,n}(\mathrm{Art}_{F_{\widetilde{v}}}^{-1}(t)^{-1}))).$$

Then ρ_0 determines a homomorphism $R_v^{\chi_v} \to \mathcal{O}$, hence a homomorphism $A\widehat{\otimes}_\mathcal{O} R_v^{\chi_v}[1/l] \to A[1/l]$. We will imitate the proof of [Tay08, Lemma 3.4] to show that every closed point of $\operatorname{Spec} A\widehat{\otimes}_\mathcal{O} R_v^{\chi_v}[1/l]$ lies in a connected component which intersects the closed subscheme $\operatorname{Spec} A[1/l] \subset \operatorname{Spec} A\widehat{\otimes}_\mathcal{O} R_v^{\chi_v}[1/l]$ determined by ρ_0. Since $A\widehat{\otimes}_\mathcal{O} R_v^{\chi_v}[1/l]$ is a Jacobson ring, this will show that $\operatorname{Spec} A\widehat{\otimes}_\mathcal{O} R_v^{\chi_v}[1/l]$ is itself connected.

Let $P \subset A\widehat{\otimes}_\mathcal{O} R_v^{\chi_v}[1/l]$ be a maximal ideal, and let P^c denote the pullback of P to $A\widehat{\otimes}_\mathcal{O} R_v^{\chi_v}$. Let B denote the normalization of $A\widehat{\otimes}_\mathcal{O} R_v^{\chi_v}/P^c$, and let $(\Phi, \Sigma) \in \mathrm{GL}_n(B)^2$ denote the specialization of the universal pair. Then B is a DVR, finite over A, and $\lambda \in \mathfrak{m}_B$. We write C for the complete topological ring $B\langle \{X_{i,j}, Y\}_{1 \leq i,j \leq n}\rangle$, where the brackets $\langle \cdot \rangle$ denote power series whose coefficients tend to 0 in the $\mathfrak{m}_B$-adic topology. Consider the pair of $\mathrm{GL}_n(B)$-valued matrices

$$((X_{i,j})\Phi(X_{i,j})^{-1}, (X_{i,j})\Sigma(X_{i,j})^{-1})$$

This pair defines a map (notation as in the proof of [Tay08, Lemma 3.4])

$$\operatorname{Spec} C \to \mathcal{M}(\prod_{i=1}^{n}(X - \chi_{v,i}(\operatorname{Art}_{F_{\tilde{v}}}^{-1}(t))^{-1}), q_v)$$

such that $\operatorname{Spec} C/(\mathfrak{m}_B)$ maps to the closed point

$$(1_n, 1_n) \in \mathcal{M}(\prod_{i=1}^{n}(X - \chi_{v,i}(\operatorname{Art}_{F_{\tilde{v}}}^{-1}(t))^{-1}), q_v)(k).$$

In particular, we get a continuous homomorphism from the completed local ring of $\mathcal{M}(\prod_{i=1}^{n}(X-\chi_{v,i}(\operatorname{Art}_{F_{\tilde{v}}}^{-1}(t))^{-1}), q_v)$ at this point:

$$R_v^{\chi_v} \cong \widehat{\mathcal{O}}_{\mathcal{M}(\prod_{i=1}^{n}(X-\chi_{v,i}(\operatorname{Art}_{F_{\tilde{v}}}^{-1}(t))^{-1}),q_v),(1_n,1_n)} \to C$$

Since C is also tautologically a topological A-algebra, we get, by Lemma 1.3, a continuous map:

$$A\widehat{\otimes}_{\mathcal{O}} R_v^{\chi_v} \to C.$$

For any choice of $E \in \mathrm{GL}_n(B)$, we get a continuous composite homomorphism

$$f_E : A\widehat{\otimes}_{\mathcal{O}} R_v^{\chi_v} \to C \to B,$$

under which the pair (Φ, Σ) gets mapped to $(E\Phi E^{-1}, E\Sigma E^{-1})$. Since C is a domain, this shows that the point $P \in \operatorname{Spec} A\widehat{\otimes}_{\mathcal{O}} R_v^{\chi_v}[1/l]$ is contained in the same connected component as $\ker f_E[1/l]$, for any E. We can thus assume that Φ and Σ are upper-triangular, and that the diagonal entries of Σ are equal to $\chi_{v,1}(\operatorname{Art}_{F_{\tilde{v}}}^{-1}(t))^{-1}, \ldots, \chi_{v,n}(\operatorname{Art}_{F_{\tilde{v}}}^{-1}(t))^{-1}$.

Now let $D = B\langle X\rangle$, $\Lambda = \operatorname{diag}(1, X, \ldots, X^{n-1}) \in M_n(D)$, and consider the pair $(\Lambda^{-1}\Phi\Lambda, \Lambda^{-1}\Sigma\Lambda) \in \mathrm{GL}_n(D)^2$. This again determines a map

$$\operatorname{Spec} D \to \mathcal{M}(\prod_{i=1}^{n}(X - \chi_{v,i}(\operatorname{Art}_{F_{\tilde{v}}}^{-1}(t))^{-1}), q_v)$$

with the property that $\operatorname{Spec} D/(\mathfrak{m}_B)$ maps to the closed point $(1_n, 1_n)$ in the special fiber. We then get a continuous homomorphism $R_v^{\chi_v} \to D$, hence $A\widehat{\otimes}_{\mathcal{O}} R_v^{\chi_v} \to D$, hence for any choice of $x \in \mathfrak{m}_B$, a composite map

$$g_x : A\widehat{\otimes}_{\mathcal{O}} R_v^{\chi_v} \to D \to B.$$

We have $\ker g_1[1/l] = P$, while $g_0(\Phi) = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$ and $g_0(\Sigma) = \operatorname{diag}(\chi_{v,1}(\operatorname{Art}_{F_{\tilde{v}}}^{-1}(t))^{-1}, \ldots, \chi_{v,n}(\operatorname{Art}_{F_{\tilde{v}}}^{-1}(t))^{-1})$. Replacing P by $\ker g_0[1/l]$, we can thus assume that Φ and Σ are in fact diagonal.

Now let $E = B[\![Y_1, \ldots, Y_n]\!]$, and consider the pair

$$(\operatorname{diag}(1 + Y_1, \ldots, 1 + Y_n)^{-1}\Phi, \Sigma) \in \mathrm{GL}_n(E)^2.$$

As in the previous two stages of the argument, this map determines a continuous homomorphism $R_v^{\chi_v} \to E$, and hence for every choice of $y_1, \ldots, y_n \in \mathfrak{m}_B$, a continuous homomorphism

$$h_y : A\widehat{\otimes}_{\mathcal{O}} R_v^{\chi_v} \to E \to B.$$

We have $\ker h_0[1/l] = P$. On the other hand, choosing $y_i = \alpha_i - 1$ for each $i = 1, \ldots, n$, we have $h_y(\Phi) = 1$, $h_y(\Sigma) = \operatorname{diag}(\chi_{v,1}(\operatorname{Art}_{F_{\tilde{v}}}^{-1}(t))^{-1}, \ldots, \chi_{v,n}(\operatorname{Art}_{F_{\tilde{v}}}^{-1}(t))^{-1}))$. Then $\ker h_y[1/l]$ lies in the closed subscheme $\operatorname{Spec} A[1/l] \subset \operatorname{Spec} A\widehat{\otimes}_{\mathcal{O}} R_v^{\chi_v}[1/l]$ determined by ρ_0. Since we have shown that $\ker h_0[1/l]$ and $\ker h_y[1/l]$ lie in the same connected component of $\operatorname{Spec} A\widehat{\otimes}_{\mathcal{O}} R_v^{\chi_v}[1/l]$, this completes the proof. □

3.3.4 Steinberg deformations

Now suppose that $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is trivial and that $q_v \equiv 1 \bmod l$. Then in [Tay08, §3] is defined a quotient $R_v^{\mathrm{St}} = R_v^{\square}/\mathcal{I}_v^{\mathrm{Stein}}$ of $R_v^{\square}$. (In fact the assumption $l > n$ is made in this reference, but this is not needed for the definition or the results that follow.) We recall that by definition, $R_v^{\square}/\mathcal{I}_v^{\mathrm{Stein},1}$ is the quotient of R_v^1 defined by the condition that the characteristic polynomial of a Frobenius lift has the form $\prod_{i=1}^n (X - \alpha q_v^{n-i})$ for some α. Then $R_v^{\square}/\mathcal{I}_v^{\mathrm{Stein}}$ is the maximal $\mathcal{O}$-flat quotient of this ring, and defines a local deformation problem.

Proposition 3.17. *The ring R_v^{St} is $\mathcal{O}$-flat and geometrically integral of dimension n^2+1, and $R_v^{St}/(\lambda)$ is generically reduced. Moreover,* $\operatorname{Spec} R_v^{St}[1/l]$ *is formally smooth over K. If K is sufficiently large, then every irreducible component of* $\operatorname{Spec} R_v^{St}/(\lambda)$ *is geometrically irreducible.*

Proof. Except for the final sentence and the assertion that $R_v^{\mathrm{St}}/(\lambda)$ is generically reduced, this is contained in [Tay08, Proposition 3.1] and [Tay08, Lemma 3.3]. To see that $R_v^{\mathrm{St}}/(\lambda)$ is generically reduced, we observe that R_v^{St} is quotient of the ring R_v^1 already defined in §3.3.3, hence $R_v^{\mathrm{St}}/(\lambda)$ is a quotient of $R_v^1/(\lambda)$. The rings $R_v^1/(\lambda)$ and $R_v^{\mathrm{St}}/(\lambda)$ are each equidimensional of dimension n^2. It follows that if $\mathfrak{p} \subset R_v^{\mathrm{St}}/(\lambda)$ is a minimal prime, then its pullback to $R_v^1/(\lambda)$ is also minimal, and $R_v^1/(\lambda)_{(\mathfrak{p})}$ is a field (since $R_v^1/(\lambda)$ is generically reduced, by Proposition 3.15). Since the map $R_v^1/(\lambda)_{(\mathfrak{p})} \to R_v^{\mathrm{St}}/(\lambda)_{(\mathfrak{p})}$ is surjective, we deduce that $R_v^{\mathrm{St}}/(\lambda)_{(\mathfrak{p})}$ is a field. Since $\mathfrak{p}$ was arbitrary, this shows that $R_v^{\mathrm{St}}/(\lambda)$ is generically reduced.

To complete the proof, we must show that if K is sufficiently large, then every irreducible component of $\operatorname{Spec} R_v^{\mathrm{St}}/(\lambda)$ is geometrically irreducible. If K is sufficiently large, in the sense of Proposition 3.15, then every irreducible component of $\operatorname{Spec} R_v^1/(\lambda)$ is geometrically irreducible. The argument above shows that $\operatorname{Spec} R_v^{\mathrm{St}}/(\lambda)$ is a union of components of $\operatorname{Spec} R_v^1/(\lambda)$, so the result follows. ☐

Proposition 3.18. *Let A be a complete Noetherian local $\mathcal{O}$-algebra.* (*We do not assume that the residue field of A is k.*) *Suppose that* $\operatorname{Spec} A[1/l]$ *is* (*non-trivial and*) *connected. Then* $\operatorname{Spec} A \widehat{\otimes}_{\mathcal{O}} R_v^{St}[1/l]$ *is connected.*

Proof. The proof is the essentially the same as that of Proposition 3.16 above. ☐

3.3.5 Taylor–Wiles deformations

Finally suppose that $q_v \equiv 1 \bmod l$ and that $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is unramified. Choose an eigenvalue $\overline{\alpha}_v \in k$ of $\overline{r}|_{G_{F_{\widetilde{v}}}}(\mathrm{Frob}_{\widetilde{v}})$ of multiplicity n_v, $1 \leq n_v \leq n$, such that $\overline{r}|_{G_{F_{\widetilde{v}}}}(\mathrm{Frob}_{\widetilde{v}})$ acts semisimply on its generalized $\overline{\alpha}_v$-eigenspace. Then we can write $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{s}_v \oplus \overline{\psi}_v$, where $\overline{\psi}_v(\mathrm{Frob}_{\widetilde{v}})$ is equal to $\overline{\alpha}_v \cdot 1_{n_v}$. We define $\mathcal{D}_v^{\mathrm{TW}}(\overline{\alpha}_v)$ to be the functor of lifts $\rho = s \oplus \psi$, where this decomposition lifts the previous one and s is unramified and ψ may be ramified, but the restriction to inertia is scalar. Then $\mathcal{D}_v^{\mathrm{TW}}(\overline{\alpha}_v)$ is a local deformation problem (cf. [Tho12, Lemma 4.2]).

Fix a deformation problem

$$\mathcal{S} = \left(F/F^+, S, \widetilde{S}, \Lambda, \overline{r}, \chi, \{\mathcal{D}_v\}_{v \in S}\right),$$

a positive integer N, and a finite set Q_N of primes v of F^+ split in F, disjoint from S. Choose for each $v \in Q_N$ a prime $\widetilde{v}$ of F above it, and let $\widetilde{Q}_N = \{\widetilde{v} \mid v \in Q_N\}$. We suppose that Q_N has q elements and that for each $v \in Q_N$, we have $q_v \equiv 1 \bmod l^N$. Choose for each $v \in Q_N$ an eigenvalue $\overline{\alpha}_v$ of $\overline{r}(\mathrm{Frob}_{\widetilde{v}})$, such that $\overline{r}(\mathrm{Frob}_{\widetilde{v}})$ acts semisimply on its generalized $\overline{\alpha}_v$-eigenspace; then the local deformation problem $\mathcal{D}_v^{\mathrm{TW}}(\overline{\alpha}_v)$ is defined. We refer to the tuple $(Q_N, \widetilde{Q}_N, \{\overline{\alpha}_v\}_{v \in Q_N})$ as a choice of Taylor–Wiles data of order q and level N.

In this case we define an auxiliary deformation problem

$$\mathcal{S}_N = \left(F/F^+, S \cup Q_N, \widetilde{S} \cup \widetilde{Q}_N, \Lambda, \overline{r}, \chi, \{\mathcal{D}_v\}_{v \in S} \cup \{\mathcal{D}_v^{\mathrm{TW}}(\overline{\alpha}_v)\}_{v \in Q_N}\right).$$

This is an abuse of notation since $\mathcal{S}_N$ depends on the choice of Taylor–Wiles data of level N, and not just on N. This abuse is not serious, since in practice we will use only a single choice of Taylor–Wiles data for each integer N.

Lemma 3.19. *With the above choices, let Δ_N denote the maximal l-power order quotient of $\prod_{v \in Q_N} k(v)^\times$. Suppose that $\overline{r}$ is Schur, so that $R^{univ}_{\mathcal{S}}$ is defined. Then $R^{univ}_{\mathcal{S}_N}$ has a canonical $\mathcal{O}[\Delta_N]$-algebra structure, and the natural surjection $R^{univ}_{\mathcal{S}_N} \to R^{univ}_{\mathcal{S}}$ induces an isomorphism*

$$R^{univ}_{\mathcal{S}_N}/(\mathfrak{a}_N) \cong R^{univ}_{\mathcal{S}},$$

where $\mathfrak{a}_N \subset \mathcal{O}[\Delta_N]$ is the augmentation ideal.

Proof. Let $r^{\mathrm{univ}}_{\mathcal{S}_N} : G_{F^+, S_N} \to \mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{S}_N})$ be a representative of the universal deformation. For each $v \in Q_N$, we can find a decomposition $r^{\mathrm{univ}}_{\mathcal{S}_N}|_{G_{F_{\widetilde{v}}}} = s_v \oplus \psi_v$ lifting the decomposition of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ in the definition of the local deformation problem at v. There is a character $\phi_v : I^{\mathrm{ab}}_{F_{\widetilde{v}}} \to (R^{\mathrm{univ}}_{\mathcal{S}_N})^\times$ such that $\psi_v|_{I_{F_{\widetilde{v}}}} = \phi_v \oplus \cdots \oplus \phi_v$ (n_v times). Composing with the Artin map, we get a homomorphism $\mathcal{O}^\times_{F_{\widetilde{v}}} \to (R^{\mathrm{univ}}_{\mathcal{S}_N})^\times$ which factors through the maximal pro-l quotient of $\mathcal{O}^\times_{F_{\widetilde{v}}}$, hence the maximal l-power order quotient of $k(v)^\times$ (since $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is unramified). Taking the product of these characters gives the desired homomorphism $\Delta_N \to (R^{\mathrm{univ}}_{\mathcal{S}_N})^\times$. □

3.3.6 The ring $R^{\mathrm{loc}}_{\mathcal{S},T}$

We now apply Lemma 1.4 to understand the ring $R^{\mathrm{loc}}_{\mathcal{S},T}$ for the global deformation problem

$$\mathcal{S} = \left(F/F^+, S, \widetilde{S}, \Lambda, \overline{r}, \chi, \{\mathcal{D}_v\}_{v \in S}\right), \tag{3.2}$$

where S is a disjoint union $S = S_l \cup S(B) \cup R \cup S_a$, and the deformation problems $\mathcal{D}_v$ are as follows:

- If $v \in S_l$, then $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is trivial, $[F_{\widetilde{v}} : \mathbb{Q}_l] > 1 + n(n-1)/2$, and $\mathcal{D}_v = R^{\triangle}_v$.
- If $v \in S(B)$, then $q_v \equiv 1 \bmod l$, $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is trivial, and $\mathcal{D}_v = R^{\mathrm{St}}_v$.
- If $v \in R$, then $q_v \equiv 1 \bmod l$, $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is trivial, and $\mathcal{D}_v = R^{\chi_v}_v$ for some tuple $\chi_v = \chi_{v,1} \times \cdots \times \chi_{v,n}$ of characters $\chi_{v,i} : k(v)^\times \to \mathcal{O}^\times$ which are trivial mod λ.
- If $v \in S_a$, then $q_v \not\equiv 1 \bmod l$, $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is unramified and $\overline{r}|_{G_{F_{\widetilde{v}}}}(\mathrm{Frob}_{\widetilde{v}})$ is a scalar matrix, and $\mathcal{D}_v = R^{\square}_v$.

We set $T = S$. For this choice of $\mathcal{S}$ and T, we have

$$R^{\mathrm{loc}}_{\mathcal{S},T} = \left(\widehat{\otimes}_{v \in S_l} R^{\triangle}_v\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\otimes}_{v \in S(B)} R^{\mathrm{St}}_v\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\otimes}_{v \in R} R^{\chi_v}_v\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\otimes}_{v \in S_a} R^{\square}_v\right).$$

Lemma 3.20. *Let $\mathcal{S}$ be the global deformation problem (3.2).*

1. *Suppose that for each $v \in R$, the characters $\chi_{v,1}, \ldots, \chi_{v,n}$ are pairwise distinct. Let $Q \subset \Lambda$ be a minimal prime. Then $\operatorname{Spec} R^{loc}_{\mathcal{S},T}/(Q)$ is geometrically irreducible, and its generic point is of characteristic 0. We have $\dim R^{loc}_{\mathcal{S},T}/(Q) = 1 + n(n+1)[F^+ : \mathbb{Q}]/2 + n^2|T|$.*
2. *Suppose that for each $v \in R$, the characters $\chi_{v,1}, \ldots, \chi_{v,n}$ are trivial, and that K is sufficiently large. Let $Q \subset \Lambda$ be a minimal prime. Then for each minimal prime $\mathfrak{p} \subset R^{loc}_{\mathcal{S},T}/(Q)$, $R^{loc}_{\mathcal{S},T}/\mathfrak{p}$ is $\mathcal{O}$-flat and geometrically integral of dimension $1 + n(n+1)[F^+ : \mathbb{Q}]/2 + n^2|T|$, and for each minimal prime $\mathfrak{q} \subset R^{loc}_{\mathcal{S},T}/(Q, \lambda)$, $R^{loc}_{\mathcal{S},T}/\mathfrak{q}$ is geometrically integral. Moreover, each minimal prime of $R^{loc}_{\mathcal{S},T}/(Q, \lambda)$ is contained in a unique minimal prime of $R^{loc}_{\mathcal{S},T}/(Q)$, and $R^{loc}_{\mathcal{S},T}/(Q, \lambda)$ is generically reduced.*

Proof. Let $Q \subset \Lambda = \widehat{\otimes}_{v \in S_l} \Lambda_v$ be a minimal prime. Then there are minimal primes $Q_v \subset \Lambda_v$ such that $Q = (\{Q_v\}_{v \in S_l})$, and we can write

$$R^{\mathrm{loc}}_{\mathcal{S},T}/(Q) = \left(\widehat{\otimes}_{v \in S_l} R^{\triangle}_v/(Q_v)\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\otimes}_{v \in S(B)} R^{\mathrm{St}}_v\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\otimes}_{v \in R} R^{\chi_v}_v\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\otimes}_{v \in S_a} R^{\square}_v\right). \tag{3.3}$$

Suppose that for each $v \in R$, the characters $\chi_{v,1}, \ldots, \chi_{v,n}$ are pairwise distinct. Then each of the rings appearing in the completed tensor product (3.3) is geometrically irreducible over $\mathcal{O}$ with generic point of characteristic 0 (by Proposition 3.14, if $v \in S_l$; by Proposition 3.17, if $v \in S(B)$; by Proposition 3.15, if $v \in R$; and by Proposition 3.10, if $v \in S_a$). The first part of the lemma then follows from Lemma 1.4.

Now suppose instead that for each $v \in R$, the characters $\chi_{v,1}, \ldots, \chi_{v,n}$ are trivial, and that K is sufficiently large, in the sense of Proposition 3.15 and Proposition 3.17. If $v \in S_l \cup S(B) \cup S_a$, then the corresponding factor R_v in (3.3) is geometrically irreducible with generic point of characteristic 0 (by the same results as in the first part of the lemma), and each irreducible component of $R_v/(\lambda)$ is geometrically irreducible. If $v \in R$, then each irreducible component of $R_v = R_v^{\chi_v}$ (resp. $R_v^{\chi_v}/(\lambda)$) is geometrically irreducible, and each minimal prime of $R_v^{\chi_v}/(\lambda)$ contains a unique minimal prime of $R_v^{\chi_v}$. The second part of the lemma then also follows from Lemma 1.4. $\square$

Lemma 3.21. *Let $\mathcal{S}$ be the global deformation problem (3.2), and suppose that for each $v \in R$, the characters $\chi_{v,1}, \ldots, \chi_{v,n}$ are trivial. Suppose moreover that $\overline{r}|_{G_{F^+(\zeta_l)}}$ is Schur and for each complex conjugation $c \in G_{F^+}$, we have $\chi(c) = -1$. Let $r = |R|$. Then the connectededness dimension of $R_{\mathcal{S}}^{univ}$ satisfies $c(R_{\mathcal{S}}^{univ}) \geq n[F^+ : \mathbb{Q}] - rn - 2$.*

Proof. We first give a lower bound for $c(R_{\mathcal{S},T}^{\mathrm{loc}})$. By Lemma 1.4, the minimal primes of the ring

$$R_0 = \left(\widehat{\bigotimes}_{v \in S_l} R_v^{\triangle}\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\bigotimes}_{v \in S(B)} R_v^{\mathrm{St}}\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\bigotimes}_{v \in S_a} R_v^{\square}\right)$$

are in bijection with the minimal primes $Q \subset \Lambda$, each minimal prime of R_0 being of the form $\sqrt{QR_0}$. In particular, if $\mathfrak{p}_1, \mathfrak{p}_2 \subset R_0$ are any distinct minimal primes, then we can find distinct minimal primes $Q_1, Q_2 \subset \Lambda$ such that $\dim R_0/(\mathfrak{p}_1 + \mathfrak{p}_2) = \dim R_0/(Q_1 + Q_2) \geq \dim R_0/(\lambda) = \dim R_{\mathcal{S},T}^{\mathrm{loc}} - 1 - rn^2$. (As ideals of Λ, we have $\sqrt{Q_1 + Q_2} = \lambda\Lambda$.) By definition of the connectedness dimension, $c(R_0) \geq \dim R_{\mathcal{S},T}^{\mathrm{loc}} - 1 - rn^2$. It follows from the description of R_v^1 given in [Tay08, §3] that this ring admits a presentation as a quotient of a power series ring over $\mathcal{O}$ in $2n^2$ variables by $n^2 + n$ relations. Similarly, $R_{\mathcal{S},T}^{\mathrm{loc}}$ admits a presentation as a quotient of a power series ring over R_0 in $2rn^2$ variables by $rn(n+1)$ relations. Using Proposition 1.8 we see that

$$c(R_{\mathcal{S},T}^{\mathrm{loc}}) \geq \dim R_{\mathcal{S},T}^{\mathrm{loc}} - 1 - rn^2 + 2rn^2 - rn(n+1) - 1 = [F^+ : \mathbb{Q}]n(n+1)/2 + n^2|T| - rn - 1.$$

Applying Proposition 1.8 once more to the presentation of $R_{\mathcal{S}}^{\square}$ given in Proposition 3.9, we find

$$c(R_{\mathcal{S}}^{\square}) \geq c(R_{\mathcal{S},T}^{\mathrm{loc}}) - n(n-1)/2[F^+ : \mathbb{Q}] - 1 \geq n[F^+ : \mathbb{Q}] + n^2|T| - rn - 2.$$

(In applying Proposition 3.9, we use our assumptions that $\overline{r}|_{G_{F^+(\zeta_l)}}$ is Schur and $\chi(c) = -1$ for each complex conjugation $c \in G_{F^+}$. The vanishing of the term $\dim_k H^0(G_{F^+,S}, \operatorname{ad} \overline{r}(1))$ follows from Lemma 3.3.) Finally, it is clear from the definition of connectedness dimension that $c(R_{\mathcal{S}}^{\square_T}) = c(R_{\mathcal{S}}^{\mathrm{univ}}) + n^2|T|$. $\square$

3.4 Pseudodeformations of Galois representations

In this section we consider pseudodeformations of $\overline{r}$. Since we do not wish to exclude the case $l < n$, we need to use group determinants rather than pseudocharacters. We follow here the exposition of [Che]. We begin by recalling the relevant definitions.

Definition 3.22. *Let A be a ring, and M, N be A-modules. Then M (resp. N) defines a functor $\underline{M}$ (resp. $\underline{N}$) : A-alg $\to$ Sets by the rule $B \mapsto M \otimes_A B$ (resp. $B \mapsto N \otimes_A B$). An A-polynomial law from M to N is a natural transformation $P : \underline{M} \to \underline{N}$. Such an A-polynomial law is said to be homogeneous of degree n if for all $B \in A$-alg, $b \in B$, and $m \in M \otimes_A B$, we have $P(bm) = b^n P(m)$.*

If G is a group then an A-valued determinant on G of dimension n is an A-polynomial law $D : A[G] \to A$, homogeneous of degree n, which is multiplicative, in the sense that $D(1) = 1$ and for all $B \in A$-alg, $r, s \in B[G] = A[G] \otimes_A B$, we have $D(rs) = D(r) \cdot D(s)$.

Definition 3.23. *Given an A-valued determinant $D : A[G] \to A$ of dimension n, we define the characteristic polynomial of $g \in G$ by the formula*

$$D_{A[t]}(t-g) = \sum_{i=0}^{n} (-1)^i \Lambda_i(g) t^{n-i}.$$

If G and A are endowed with topologies, we say that D is continuous if the maps $\Lambda_i : G \to A$ are continuous for each $i = 0, \dots, n$.

The following lemma collects some basic facts in the theory of group determinants.

Lemma 3.24. *Let G be a group.*

1. *Let A be a ring, and $\rho : G \to \mathrm{GL}_n(A)$ a representation. Extend ρ to an algebra homomorphism $A[G] \to M_{n\times n}(A)$. Then the formula $D = \det \circ \rho$ defines an A-valued determinant on G of dimension n.*
2. *Let k be an algebraically closed field, and let $D : G \to k$ be a determinant of dimension n. Then there exists a semisimple representation $\rho : G \to \mathrm{GL}_n(k)$, unique up to isomorphism, with $D = \det \circ \rho$.*

Proof. The first part is elementary. The second part is [Che, Theorem 2.12]. □

We set $\overline{D} = \det \circ \overline{r}|_{G_{F,S}}$.

Definition 3.25. *A* pseudodeformation *of $\overline{D}$ to an object R of $\mathcal{C}_{\mathcal{O}}$ is a continuous determinant $D : G_{F,S} \to R$ of dimension n such that $D \otimes_R k = \overline{D}$. (Here R is given its natural profinite topology.) We write* PDef_S *for the set-valued functor which associates to an object R of $\mathcal{C}_{\mathcal{O}}$ the set of all pseudodeformations of $\overline{D}$ to R.*

Proposition 3.26.

1. *The functor* PDef_S *is represented by an object Q_S of $\mathcal{C}_{\mathcal{O}}$.*
2. *Write $\Lambda_i^{univ} : G \to Q_S$ for the coefficients of the universal characteristic polynomial, and let $\mathcal{L}$ be a set of finite places of F, disjoint from S, of Dirichlet density 1. Then Q_S is topologically generated as an $\mathcal{O}$-algebra by the elements $\Lambda_i^{univ}(\mathrm{Frob}_w)$ for $w \in \mathcal{L}, i = 0, \dots, n$.*

Proof. The first part follows easily from [Che, Proposition 3.3] and [Che, Proposition 3.7]. For the second part, we must show that for every $r \geq 0, Q_S/\mathfrak{m}_{Q_S}^r$ is generated as an $\mathcal{O}$-algebra by the elements $\Lambda_i^{\mathrm{univ}}(\mathrm{Frob}_w)$. This follows from the Chebotarev density theorem and [Che, Corollary 1.14]. □

Definition 3.27. *Fix a global deformation problem $\mathcal{S}$, and suppose that $\overline{r}$ is Schur. Then there is a natural map $Q_S \widehat{\otimes}_{\mathcal{O}} \Lambda \to R_{\mathcal{S}}^{univ}$ classifying the determinant of the universal deformation. We write $P_{\mathcal{S}}$ for the image of this map.*

By Proposition 3.26, we could have defined $P_{\mathcal{S}}$ as the closed Λ-subalgebra of $R_{\mathcal{S}}^{\mathrm{univ}}$ topologically generated by the coefficients of the characteristic polynomials of elements of $G_{F,S}$. We have followed a slightly circuitous route to its definition in order to have access to the following lemma.

Lemma 3.28. *Suppose that $\overline{r}$ is Schur. Fix an integer $q \geq 0$. Then there exists an integer $C > 0$ depending only on $q, \overline{r}$ and S such that for any set $S' \supset S$ of finite primes of F^+ split in F such that $|S' - S| \leq q$ and any deformation problem $\mathcal{S}'$ unramified outside S', $P_{\mathcal{S}'}$ can be written as a quotient of a power ring over $\mathcal{O}$ in C variables.*

Proof. It suffices to prove the result for $Q_{S'}$, and hence it is enough to show that $\dim_k \mathrm{PDef}_{S'}(k[\epsilon])$ can be bounded independently of S'. Let $F_0 \subset \overline{F}$ denote the extension of F^+ cut out by $\overline{r}$, and let $F_1 \subset \overline{F}$ denote the maximal extension of F_0 which is pro-l and unramified outside S'. [Che, Lemma 3.8] shows that any pseudodeformation to $k[\epsilon]$ factors through $\mathrm{Gal}(F_1/F)$, and [Che, Proposition 2.38] then implies that to prove the lemma it suffices to bound the number of topological generators of $\mathrm{Gal}(F_1/F)$ solely in terms of $|S' - S|$. It is clearly enough to bound the number of topological generators of $\mathrm{Gal}(F_1/F_0)$ in terms of $|S' - S|$. By Frattini's argument, this is equivalent to giving a bound for the degree of the maximal elementary abelian l-extension of F_0, unramified outside S'. This is an exercise in class field theory. □

We now attempt to clarify the relation between $P_{\mathcal{S}}$ and $R_{\mathcal{S}}^{\text{univ}}$. Suppose that the residual representation $\overline{r}$ is Schur and that $\overline{r}|_{G_{F,S}} = \overline{\rho}_1 \oplus \overline{\rho}_2$ is a direct sum of two absolutely irreducible representations. Then the centralizer $Z_{\mathrm{GL}_n(k)}(\overline{r})$ is equal to $\mu_2 \times \mu_2 \subset k^\times \times k^\times$. Let $\mathcal{D}_v$ be one of the local deformation problems defining $\mathcal{S}$, and let $R \in \mathcal{C}_{\mathcal{O}}$ (if $v \notin S_l$) or $R \in \mathcal{C}_{\Lambda_v}$ (if $v \in S_l$). Then the pre-image of the group $Z_{\mathrm{GL}_n(k)}(\overline{r})$ in $\mathrm{GL}_n(R)$ acts on the set $\mathcal{D}_v^\square(R)$ of all liftings of $\overline{r}|_{G_{F_{\widetilde{v}}}}$, but a priori it need not leave the subset $\mathcal{D}_v(R) \subset \mathcal{D}_v^\square(R)$ invariant.

Suppose that for each local deformation problem $\mathcal{D}_v$ defining $\mathcal{S}$, and for each R as above, the pre-image of the group $\mu_2 \times \mu_2 = Z_{\mathrm{GL}_n(k)}(\overline{r})$ in $\mathrm{GL}_n(R)$ leaves $\mathcal{D}_v(R) \subset \mathcal{D}_v^\square(R)$ invariant. (This is the case for each of the local deformation problems defined in §3.3 above.) Then the group $\mu_2 \times \mu_2$ acts on the ring $R_{\mathcal{S}}^{\text{univ}}$ by conjugation of the universal deformation, as follows. If $[r]$ is a deformation represented by a choice of lifting $r : G_{F^+,S} \to \mathcal{G}_n(R)$, and $\gamma \in \mu_2 \times \mu_2$, then let $\widetilde{\gamma} \in \mathrm{GL}_n(R)$ be an arbitrary pre-image of γ, and let ${}^\gamma[r] = [{}^{\widetilde{\gamma}}r]$. The equivalence class of this deformation is independent of the choice of lift $\widetilde{\gamma}$. Moreover, this deformation is of type $\mathcal{S}$, because of our assumption on the local deformation problems $\mathcal{D}_v$. By universality, this operation determines a map $\gamma^* : R_{\mathcal{S}}^{\text{univ}} \to R_{\mathcal{S}}^{\text{univ}}$, which defines the action of the element $\gamma \in \mu_2 \times \mu_2$.

Proposition 3.29. *1. Suppose that $\overline{r}|_{G_{F,S}}$ is absolutely irreducible. Then the inclusion $P_{\mathcal{S}} \subset R_{\mathcal{S}}^{univ}$ is an equality.*

2. Suppose that $\overline{r}$ is Schur. Then the inclusion $P_{\mathcal{S}} \subset R_{\mathcal{S}}^{univ}$ is finite.

3. Suppose that $\overline{r}$ is Schur, that $\overline{r}|_{G_{F,S}} = \overline{\rho}_1 \oplus \overline{\rho}_2$, and that the local deformation problems defining $\mathcal{S}$ are among those defined in §3.3, as in the above discussion. Let $\mathfrak{p} \subset R_{\mathcal{S}}^{univ}$ be a dimension one prime, let E be the fraction field of $R_{\mathcal{S}}^{univ}/\mathfrak{p}$, and suppose that the induced representation $r : G_{F^+,S} \to \mathcal{G}_n(E)$ is such that $r|_{G_{F,S}}$ is absolutely irreducible. Let $\mathfrak{q} = P_{\mathcal{S}} \cap \mathfrak{p}$. Then the group $\mu_2 \times \mu_2$ permutes transitively the primes of $R_{\mathcal{S}}^{univ}$ above $\mathfrak{q}$.

Proof. For the first part, it suffices to show that the map $P_{\mathcal{S}} \to R_{\mathcal{S}}^{\text{univ}}$ induces a surjection on Zariski tangent spaces, or equivalently that if r_1, r_2 are liftings of $\overline{r}$ to $k[\epsilon]$ such that for every $x \in k[\epsilon][G_{F,S}]$, $r_1(x)$ and $r_2(x)$ have the same characteristic polynomial, then r_1 and r_2 are $1 + \epsilon M_n(k)$-conjugate. By the corresponding result for deformations valued in GL_n, we may assume that $r_1|_{G_{F,S}} = r_2|_{G_{F,S}}$.

By Lemma 3.1, the data of r_i is equivalent to the data of $r_i|_{G_{F,S}}$ and a matrix A_i realizing the conjugate self-duality of r_i, in particular satisfying the relation

$$\chi(\delta)A_i^{-1} = {}^t r_i(\delta) A_i^{-1} r_i(\delta^c)$$

for all $\delta \in G_{F,S}$. If $B \in \mathrm{GL}_n(k[\epsilon])$, then conjugating by B sends the matrix A_i to $BA_i{}^tB$. By Schur's lemma (in the guise of [CHT08, Lemma 2.1.8]), A_1 and A_2 differ by a scalar. Since the characteristic is not 2, we can choose a scalar matrix in $1 + \epsilon M_n(k)$ which takes A_1 to A_2.

For the second part, it suffices to show that $R_{\mathcal{S}}^{\text{univ}}/(\mathfrak{m}_{P_{\mathcal{S}}})$ is Artinian. Suppose not; then there exists a dimension one prime $\mathfrak{p} \subset R_{\mathcal{S}}^{\text{univ}}/(\mathfrak{m}_{P_{\mathcal{S}}})$. Let A denote the normalization of $R_{\mathcal{S}}^{\text{univ}}/\mathfrak{p}$ in its fraction field E. Fix an isomorphism $A \cong k'[\![T]\!]$, where k'/k is a finite field extension. Let $r : G_{F^+,S} \to \mathcal{G}_n(A)$ denote a representative of the corresponding deformation. By construction, we have $\det \circ r|_{G_{F,S}} = \det \circ \overline{r}|_{G_{F,S}}$. Moreover, the representations $r|_{G_{F,S}} \otimes_A E$ and $\overline{r}|_{G_{F,S}} \otimes_k E$ are semisimple, by Lemma 3.3 and because $\overline{r}|_{G_{F,S}}$ is Schur. It follows that we can find $\gamma \in \mathrm{GL}_n(E)$ with $\gamma r|_{G_{F,S}}\gamma^{-1} = \overline{r}|_{G_{F,S}}$.

Let $E^n = \oplus_i V_i$ denote the isotypic decomposition under $r|_{G_{F,S}} \otimes_A E$, and let $L_i = A^n \cap V_i$. The map $L_i/TL_i \to A^n/TA^n$ is injective, so $\oplus_i L_i \to A^n$ is an isomorphism ($\overline{r}|_{G_{F,S}}$ is multiplicity-free). Similarly, let $E^n = \oplus_i V_i'$ denote the isotypic decomposition with respect to $\overline{r}|_{G_{F,S}} \otimes_k E$, and set $L_i' = A^n \cap V_i'$. Using once more the fact that $\overline{r}|_{G_{F,S}}$ is semisimple, we see that we can write $\gamma = (\gamma_i)_i \in \oplus_i \mathrm{Hom}_{G_{F,S}}(V_i, V_i')$, and after scaling each γ_i we can assume that $\gamma(L_i) = L_i'$, and hence $\gamma \in \mathrm{GL}_n(A)$. Since $\overline{r} \equiv r \bmod T$, we can even assume that $\gamma \in 1 + M_{n\times n}(\mathfrak{m}_A)$. Indeed, the reduction of γ_i mod $\mathfrak{m}_A$ centralizes $\overline{r}_i$, so is scalar under the identification $L_i/TL_i = L_i'/TL_i'$. After multiplying by an element of $A^\times$ we can therefore assume that $\gamma_i \equiv 1 \bmod \mathfrak{m}_A$. Arguing as above, we find that we can choose $\gamma \in 1 + M_{n\times n}(\mathfrak{m}_A)$ and $\gamma r \gamma^{-1} = \overline{r}$. This contradicts the universal property of $R_{\mathcal{S}}^{\text{univ}}$.

For the third part, we first prove the following statement (we treat the case of $\mathfrak{p}$ of characteristic l here, the mixed characteristic case being similar). Let $A = k'[\![T]\!]$, where k'/k is a finite extension, and let E denote its fraction field. Let $r_1, r_2 : G_{F^+,S} \to \mathcal{G}_n(A)$ be lifts of $\overline{r}$ such that $r_1|_{G_{F,S}} \otimes_A E$ and $r_2|_{G_{F,S}} \otimes_A E$ are absolutely irreducible and $\det \circ r_1|_{G_{F,S}} = \det \circ r_2|_{G_{F,S}}$. Then there exists $\gamma \in \mathrm{GL}_n(A)$ with $\gamma r_1|_{G_{F,S}} \gamma^{-1} = r_2|_{G_{F,S}}$.

Write A_1, A_2 for the matrices realizing the conjugate self-duality of these representations. Since the characteristic polynomials are equal, there exists $\gamma \in \mathrm{GL}_n(E)$ with $\gamma r_1|_{G_{F,S}} \gamma^{-1} = r_2|_{G_{F,S}}$. By the Cartan decomposition, we can write $\gamma = k_2^{-1} \gamma' k_1$ with $k_i \in \mathrm{GL}_n(A)$ and $\gamma' = \mathrm{diag}(T^{a_1}, \dots, T^{a_n})$ and $a_1 \geq a_2 \geq \cdots \geq a_n$. Replacing r_i with $k_i r_i k_i^{-1}$, we can suppose that $\gamma = \gamma'$. Then Schur's lemma shows that $\gamma A_1 {}^t\gamma = \lambda A_2$, for some scalar λ.

The proof will be complete if we can show that $a_1 = \cdots = a_n$. Suppose instead that $a_m > a_{m+1} = \cdots = a_n$, say. After adjoining a suitable root of T to A we can suppose that λ is a unit and $a_1 + \cdots + a_n = 0$. For $\sigma \in G_{F,S}$, we write

$$r_1(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}, \ A_1 = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where the diagonal block matrices have size m and $n - m$, respectively. If $j \leq m < i$ then $a_i - a_j < 0$, so the equation

$$r_2|_{G_{F,S}}(\sigma) = \mathrm{diag}(T^{a_1}, \dots, T^{a_n}) \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \mathrm{diag}(T^{-a_1}, \dots, T^{-a_n}) \in \mathrm{GL}_n(A)$$

implies that $c(\sigma)$ is divisible by T. Similarly, the equation

$$A_2 = \lambda^{-1} \, \mathrm{diag}(T^{a_1}, \dots, T^{a_n}) \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \mathrm{diag}(T^{a_1}, \dots, T^{a_n}) \in \mathrm{GL}_n(A)$$

implies that W is divisible by T. We thus have

$$\overline{r}_1(\sigma) = \begin{pmatrix} \overline{a}(\sigma) & \overline{b}(\sigma) \\ 0 & \overline{d}(\sigma) \end{pmatrix}, \ \overline{A}_1 = \begin{pmatrix} \overline{X} & \overline{Y} \\ \overline{Z} & 0 \end{pmatrix}.$$

Since $\overline{r}$ is Schur (and hence $\overline{r}|_{G_{F,S}}$ is semisimple) we can find a matrix $u \in M_{m \times (n-m)}(k)$ with

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{a}(\sigma) & \overline{b}(\sigma) \\ 0 & \overline{d}(\sigma) \end{pmatrix} \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \overline{a}(\sigma) & 0 \\ 0 & \overline{d}(\sigma) \end{pmatrix}.$$

Again using the fact that $\overline{r}$ is Schur, we see that the matrix

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{X} & \overline{Y} \\ \overline{Z} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u^t & 1 \end{pmatrix} = \begin{pmatrix} \overline{X} + u\overline{Z} + \overline{Y}u^t & \overline{Y} \\ \overline{Z} & 0 \end{pmatrix}$$

must be block diagonal. Since it is also non-singular, this contradiction concludes the proof of the statement.

Now suppose that $\mathfrak{p}_1, \mathfrak{p}_2$ are primes of $R_{\mathcal{S}}^{\mathrm{univ}}$ above $\mathfrak{q}$, as in the statement of the proposition. We can find a finite extension E' of the fraction field of $P_{\mathcal{S}}/\mathfrak{q}$ with ring of integers A as above and representations r_1, $r_2 : G_{F^+,S} \to \mathcal{G}_n(A^0)$, where $A^0 \subset A$ is the subring of elements with image in the residue field contained inside k, such that the induced homomorphisms $R_{\mathcal{S}} \to A^0$ have kernels $\mathfrak{p}_1$, $\mathfrak{p}_2$. By what we have just proved, there exists $\gamma \in \mathrm{GL}_n(A)$ with $\gamma r_1|_{G_{F,S}} \gamma^{-1} = r_2|_{G_{F,S}}$, and hence $\gamma A_1 \gamma^t = \mu A_2$ for some $\mu \in A^\times$. After possibly modifying γ by a scalar, we see that γ realizes the action of an element of the group $\mu_2 \times \mu_2$. This completes the proof.

□

3.5 Reducible deformations

In this section we suppose that $\overline{r}$ is Schur and that its restriction to $G_{F,S}$ has the form $\overline{r}|_{G_{F,S}} = \overline{\rho}_1 \oplus \overline{\rho}_2$, where the $\overline{\rho}_i$ are absolutely irreducible. Let $n_i = \dim \overline{\rho}_i$. Let $\Lambda_i = \widehat{\otimes}_{v \in S_l} \mathcal{O}[\![I_{F_{\widetilde{v}}}^{\mathrm{ab}}(l)^{n_i}]\!]$, the completed tensor product being over $\mathcal{O}$. Then the representations $\overline{\rho}_i$ admit unique extensions to homomorphisms $\overline{r}_i : G_{F^+,S} \to \mathcal{G}_n(k)$

with the property that $\nu \circ \overline{r}_i = \nu \circ \overline{r}$ and such that, writing $\overline{r}(c) = (A, -\chi(c))j$, $\overline{r}_i(c) = (A_i, -\chi(c))j$, the matrix A is block diagonal with the diagonal blocks given by A_1, A_2.

Suppose that $r_i : G_{F^+,S} \to \mathcal{G}_{n_i}(R)$, $i = 1, 2$, are liftings of $\overline{r}_i$ to R, where $R \in \mathcal{C}_\mathcal{O}$ is endowed with structures of Λ_1– and Λ_2–algebra. Suppose further that for each i we have $\nu \circ r_i = \chi$. Then we can define the direct sum

$$r_1 \oplus r_2 : G_{F^+,S} \to \mathcal{G}_n(R)$$

in an obvious manner, and $\nu \circ (r_1 \oplus r_2) = \chi$.

Lemma 3.30. *1. Let $A \in \mathcal{C}_\Lambda$ be a discrete valuation ring with fraction field E and residue field k, and let $r : G_{F^+,S} \to \mathcal{G}_n(A)$ be a lifting of $\overline{r}$ such that $r|_{G_{F,S}} \otimes_A E$ is reducible. Then, after possibly conjugating by an element $\gamma \in 1 + M_n(\mathfrak{m}_A)$, there is a direct sum decomposition $r = r_1 \oplus r_2$ lifting $\overline{r} = \overline{r}_1 \oplus \overline{r}_2$. In particular, $r|_{G_{F,S}} \otimes_A E$ is semisimple and multiplicity-free.*

2. Let A be as above and let $r : G_{F^+,S} \to \mathcal{G}_n(A)$ be a lifting of $\overline{r}$ such that $r|_{G_{F,S}} \otimes_A E$ is irreducible. Then $r|_{G_{F,S}} \otimes_A E$ is absolutely irreducible.

3. Let R be an Artinian $\mathcal{O}$-algebra with residue field k, and let $r : G_{F^+,S} \to \mathcal{G}_n(R)$ (resp. r') be a lifting of $\overline{r}$ of the form $r_1 \oplus r_2$ (resp. $r'_1 \oplus r'_2$), where r_i (resp. r'_i) lifts $\overline{r}_i$. Suppose there exists $\gamma \in \mathrm{GL}_n(R)$ with $\gamma \equiv 1 \bmod \mathfrak{m}_R$, $\gamma r \gamma^{-1} = r'$. Then γ is a block diagonal matrix, with blocks of size n_1, n_2.

Proof. For the first part, taking the reduction of a Jordan-Hölder filtration of $r|_{G_{F,S}} \otimes_A E$ shows that this representation has at most two Jordan-Hölder factors and is Schur, hence semisimple. Let $E^n = V_1 \oplus V_2$ be a decomposition into simple submodules, and let $L_i = V_i \cap A^n$. Let T be a uniformizer of A. Then the natural map $L_i/TL_i \to A^n/TA^n$ is injective and $G_{F,S}$-equivariant, hence $A^n = L_1 \oplus L_2$. After renumbering, we can assume that the image of L_i/TL_i in A^n/TA^n is $\overline{r}_i|_{G_{F,S}}$. Let $e_1, \dots, e_{n_1}$ be a basis of L_1 lifting the standard one, and let $f_1, \dots, f_{n_2}$ be a basis of L_2 lifting the standard one. Then in the basis $e_1, \dots, e_{n_1}, f_1, \dots, f_{n_2}$, the representation r takes the desired form.

For the second part, consider the extension of $r|_{G_{F,S}}$ to an algebra homomorphism $A[G_{F,S}] \to M_{n \times n}(A)$, and let $\mathcal{A}$ denote the image. Write $\mathcal{E} = \mathcal{A} \otimes_A E$. Then $\mathcal{E}$ is a simple algebra by Schur's lemma. On the other hand, by [Che, Theorem 2.22] we can construct a family of orthogonal idempotents $e_1 + \cdots + e_n = 1$ in $\mathcal{E}$. We must therefore have $\mathcal{E} = M_{n \times n}(E)$, and the representation is absolutely irreducible.

Finally, for the third part we may suppose by induction on the length of R that $r \equiv r' \bmod I$, where $I \subset R$ is an ideal with $\mathfrak{m}_R \cdot I = 0$, and that $\gamma \equiv 1 \bmod I$. Then we can write $r' = r(1 + \phi), \gamma = 1 + m$, where $\phi \in Z^1(G_{F^+,S}, \operatorname{ad} \overline{r}) \otimes_k I$ and $m \in 1 + M_{n \times n}(I)$. A calculation shows that, writing m in block form as

$$m = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

we have $Y \in \mathrm{Hom}_{G_{F,S}}(\overline{r}_2, \overline{r}_1) \otimes_k I = 0$, $Z \in \mathrm{Hom}_{G_{F,S}}(\overline{r}_1, \overline{r}_2) \otimes_k I = 0$. The result follows. □

Fix a global deformation problem

$$\mathcal{S} = \left(F/F^+, S, \widetilde{S}, \Lambda, \overline{r}, \chi, \{\mathcal{D}_v\}_{v \in S} \right).$$

Definition 3.31. *Let R be an object of $\mathcal{C}_\Lambda$. By a reducible deformation of $\overline{r}$ to R, we mean a deformation whose equivalence class contains a lifting of the form $r = r_1 \oplus r_2$, where for $i = 1, 2$ $r_i : G_{F^+,S} \to \mathcal{G}_{n_i}(R)$ is a lifting of $\overline{r}_i$. We write $\mathrm{Def}^{red}_{\mathcal{S}}$ for the subfunctor of $\mathrm{Def}_{\mathcal{S}}$ of reducible deformations.*

Proposition 3.32. $\mathrm{Def}^{red}_{\mathcal{S}}$ *is a closed subfunctor of* $\mathrm{Def}_{\mathcal{S}}$*, hence is represented by a quotient $R^{red}_{\mathcal{S}}$ of $R^{univ}_{\mathcal{S}}$.*

Proof. We must show that $\mathrm{Def}^{\mathrm{red}}_{\mathcal{S}} \subset \mathrm{Def}_{\mathcal{S}}$ is relatively representable. This means that for every diagram of Artinian $\mathcal{C}_\Lambda$-algebras

$$\begin{tikzcd} A \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & C & \end{tikzcd}$$

the diagram of sets

$$\begin{tikzcd}
\mathrm{Def}^{\mathrm{red}}_{\mathcal{S}}(A \times_C B) \arrow[r] \arrow[d] & \mathrm{Def}^{\mathrm{red}}_{\mathcal{S}}(A) \times_{\mathrm{Def}^{\mathrm{red}}_{\mathcal{S}}(C)} \mathrm{Def}^{\mathrm{red}}_{\mathcal{S}}(B) \arrow[d] \\
\mathrm{Def}_{\mathcal{S}}(A \times_C B) \arrow[r] & \mathrm{Def}_{\mathcal{S}}(A) \times_{\mathrm{Def}_{\mathcal{S}}(C)} \mathrm{Def}_{\mathcal{S}}(B)
\end{tikzcd}$$

is Cartesian. We are immediately reduced to showing that the top horizontal arrow is surjective. This is an easy exercise using the third part of the lemma above. ☐

Lemma 3.33. *Let $\mathfrak{p}$ be a dimension one prime of $R^{univ}_{\mathcal{S}}$ not containing the kernel of the map $R^{univ}_{\mathcal{S}} \to R^{red}_{\mathcal{S}}$. Let Let A denote the normalization of $R^{univ}_{\mathcal{S}}/\mathfrak{p}$ in its fraction field E, and let $r : G_{F^+,S} \to \mathcal{G}_n(A)$ denote the induced representation. Then $r|_{G_{F,S}} \otimes_A E$ is absolutely irreducible.*

Proof. Let $A^0 \subset A$ denote the subring of elements whose image modulo the maximal ideal of A lies in k. Suppose for contradiction that $r|_{G_{F,S}} \otimes_A E$ is reducible. By the first part of Lemma 3.30 above, there is $\gamma \in 1 + M_n(\mathfrak{m}_{A^0})$ such that $\gamma r \gamma^{-1}$ admits a direct sum decomposition $r = r_1 \oplus r_2$, where r_i lifts $\overline{r}_i$, and this contradicts the hypothesis on $\mathfrak{p}$. This shows that $r|_{G_{F,S}} \otimes_A E$ is irreducible. The absolute irreducibility now follows from the second part of Lemma 3.30. ☐

3.6 Twisting

We suppose in this section that $\overline{r}$ is Schur and l does not divide n. We fix a global deformation problem

$$\mathcal{S} = \left(F/F^+, S, \widetilde{S}, \Lambda, \overline{r}, \chi, \{\mathcal{D}_v\}_{v \in S} \right),$$

where the local deformation problems $\mathcal{D}_v$ are among those defined in §3.3.

Lemma 3.34. *Suppose that $R \in \mathcal{C}_\Lambda$, and that $r : G_{F^+,S} \to \mathcal{G}_n(R)$ is a lifting of type $\mathcal{S}$. Let $c \in G_{F^+,S}$ be a choice of complex conjugation. Suppose that $\psi : G_{F,S} \to R^\times$ is a character with trivial reduction modulo $\mathfrak{m}_R$ such that $\psi\psi^c = 1$, and ψ is unramified away from the places of F dividing l. Then there exists a unique lifting $r \otimes \psi : G_{F^+,S} \to \mathcal{G}_n(R)$ such that $(r \otimes \psi)|_{G_{F,S}} = r|_{G_{F,S}} \otimes \psi$ and $(r \otimes \psi)(c) = r(c)$. Moreover, $r \otimes \psi$ is of type $\mathcal{S}$.*

Proof. This is an immediate consequence of Lemma 3.1 and the invariance of our chosen local deformation problems under twisting by characters (which are unramified, if $v \in S - S_l$). ☐

Let Δ denote the Galois group of the maximal abelian pro-l extension of F unramified outside l. Then Δ is a finitely generated $\mathbb{Z}_l$-module, and receives an action of the group $\mathrm{Gal}(F/F^+) = \{1, c\}$. We write $\Delta/(c+1)$ for the maximal quotient of Δ on which c acts by -1. The completed group algebra $\mathcal{O}[\![\Delta/(c+1)]\!]$ represents the functor of lifts of the trivial character to characters $\psi : G_{F,S} \to R^\times$, $R \in \mathcal{C}_{\mathcal{O}}$ which are unramified away from the primes dividing l and satisfy the relation $\psi\psi^c = 1$.

Lemma 3.35. *There is a commutative diagram*

$$\begin{tikzcd}
P_{\mathcal{S}} \arrow[r] \arrow[d] & R^{univ}_{\mathcal{S}} \arrow[d] \\
P_{\mathcal{S}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Delta/(c+1)]\!] \arrow[r] & R^{univ}_{\mathcal{S}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Delta/(c+1)]\!].
\end{tikzcd}$$

The map $R^{univ}_{\mathcal{S}} \to R^{univ}_{\mathcal{S}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Delta/(c+1)]\!]$ is the one classifying the deformation $[r_{\mathcal{S}} \otimes \Psi]$, where $r_{\mathcal{S}} : G_{F^+,S} \to \mathcal{G}_n(R^{univ}_{\mathcal{S}})$ is a representative of the universal deformation, and $\Psi : G_{F,S} \to \mathcal{O}[\![\Delta/(c+1)]\!]^\times$ is the universal character.

Proof. It suffices to note that the map $P_{\mathcal{S}} \to R_{\mathcal{S}} \to R^{\mathrm{univ}}_{\mathcal{S}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Delta/(c+1)]\!]$ has image in the subring $P_{\mathcal{S}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Delta/(c+1)]\!] \subset R^{\mathrm{univ}}_{\mathcal{S}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Delta/(c+1)]\!]$. ☐

Let $\psi_0 : G_{F,S} \to \mathcal{O}^\times$ denote the Teichmüller lift of $\det \overline{r}|_{G_{F,S}}$. We define a quotient $R^{\mathrm{univ}}_{\mathcal{S},\psi_0}$ of $R^{\mathrm{univ}}_{\mathcal{S}}$ by the condition that $\det r_{\mathcal{S}}|_{G_{F,S}} = \psi_0$, where $r_{\mathcal{S}}$ is a representative of the universal deformation.

Lemma 3.36. *There is a canonical isomorphism* $R^{univ}_{\mathcal{S}} \cong R^{univ}_{\mathcal{S},\psi_0} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}\llbracket \Delta/(c+1) \rrbracket$.

Proof. Let $r_{\mathcal{S},\psi_0} : G_{F^+,S} \to \mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{S},\psi_0})$ denote a representative of the universal deformation, and let $\Psi : G_{F,S} \to \mathcal{O}\llbracket \Delta/(c+1) \rrbracket^\times$ denote the universal character. The map

$$R^{\mathrm{univ}}_{\mathcal{S}} \to R^{\mathrm{univ}}_{\mathcal{S},\psi_0} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}\llbracket \Delta/(c+1) \rrbracket$$

is induced by the lifting $r_{\mathcal{S},\psi_0} \otimes \Psi$, which is of type $\mathcal{S}$. We construct its inverse. Indeed, let $\psi_{\mathcal{S}} = \det r_{\mathcal{S}}|_{G_{F,S}}$. Since l does not divide n, there is a unique character $\psi : \Delta/(c+1) \to (R^{\mathrm{univ}}_{\mathcal{S}})^\times$ such that $\psi^n = \psi_{\mathcal{S}} \psi_0^{-1}$. The lifting $r_{\mathcal{S}} \otimes \psi^{-1}$ has determinant ψ_0, and the pair $(r_{\mathcal{S}} \otimes \psi^{-1}, \psi)$ induces a homomorphism

$$R^{\mathrm{univ}}_{\mathcal{S},\psi_0} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}\llbracket \Delta/(c+1) \rrbracket \to R^{\mathrm{univ}}_{\mathcal{S}}$$

which is the inverse of the isomorphism of the lemma. $\square$

3.7 Localizing and completing at a dimension one prime

We suppose in this section that $\overline{r}$ is Schur, that $\overline{r}|_{G_{F,S}} = \overline{\rho}_1 \oplus \overline{\rho}_2$ is a direct sum of two absolutely irreducible representations, and that l does not divide n. Let

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \Lambda, \overline{r}, \chi, \{\mathcal{D}_v\}_{v \in S})$$

be a global deformation problem, and let $\mathfrak{p} \subset R^{\mathrm{univ}}_{\mathcal{S}}$ be a dimension one prime of characteristic l. Let $\mathfrak{q} = \mathfrak{p} \cap P_{\mathcal{S}}$. Write A for the normalization of $R^{\mathrm{univ}}_{\mathcal{S}}/\mathfrak{p}$ in its residue field $E = \operatorname{Frac} R^{\mathrm{univ}}_{\mathcal{S}}/\mathfrak{p}$, and $E' = \operatorname{Frac} P_{\mathcal{S}}/\mathfrak{q}$. Thus $A \cong k'\llbracket T \rrbracket$. After possibly enlarging $\mathcal{O}$, we can suppose that $k = k'$, and we will assume this in what follows. We will also assume that $[F^+ : \mathbb{Q}] > 1$.

We suppose that $\Lambda \to A$ is finite; in this case, we can choose a finite faithfully flat extension $\Lambda \to \widetilde{\Lambda}$ inducing a bijection on minimal primes, together with a surjective map $\widetilde{\Lambda} \to A$ with kernel $\widetilde{P}$ making the diagram

$$\begin{tikzcd}
\Lambda \arrow[r] \arrow[rr, bend right] & \widetilde{\Lambda} \arrow[r] & A
\end{tikzcd}$$

commute. We can further suppose that for each minimal prime $Q \subset \Lambda$, $\widetilde{\Lambda}/(Q)$ is isomorphic to a power series ring over $\mathcal{O}$, and that the map $\Lambda/(Q,\lambda) \to \widetilde{\Lambda}/(Q,\lambda)$ induces a separable extension of fraction fields. Indeed, we can find an isomorphism $\Lambda \cong \mathcal{O}\llbracket X_1, \ldots, X_s \rrbracket \otimes_{\mathcal{O}} \mathcal{O}[N]$, where N is a finite abelian l-group. The homomorphism $\Lambda \to A \cong k\llbracket T \rrbracket$ is given on the first factor by $X_i \mapsto T^{n_i} u_i$, where each $u_i \in A$ is either zero or a unit, and on the second factor by the augmentation homomorphism $\mathcal{O}[N] \to k$. We may suppose that u_1 is a unit. We set $\widetilde{\Lambda} = \mathcal{O}\llbracket W_1, \ldots, W_n \rrbracket \otimes_{\mathcal{O}} \mathcal{O}[N]$. The homomorphism $\widetilde{\Lambda} \to A$ is given by $W_1 \mapsto T, W_i \mapsto 0, i = 2, \ldots, s$. For $i = 1, \ldots, s$, let $\widetilde{u}_i$ denote an arbitrary lift of u_i to $\mathcal{O}\llbracket W_1 \rrbracket \subset \widetilde{\Lambda}$. Then the homomorphism $\Lambda \to \widetilde{\Lambda}$ is given by the formulae $X_1 \mapsto W_1^{n_1} \widetilde{u}_1 + W_1 W_2$, $X_i \mapsto -W_i + W_1^{n_i} \widetilde{u}_i$, $i = 2, \ldots, s$.

Let $r_{\mathcal{S}}$ be a representative of the universal deformation. Then the homomorphism $R^{\mathrm{univ}}_{\mathcal{S}} \to A$ induces a representation

$$r_{\mathcal{S}} \bmod \mathfrak{p} : G_{F^+,S} \to \mathcal{G}_n(A),$$

which we will denote by $r_{\mathfrak{p}}$. Suppose that $r_{\mathfrak{p}}|_{G_{F,S}} \otimes_A E$ is absolutely irreducible. In this case, the group $\mu_2 \times \mu_2$, acting on $R^{\mathrm{univ}}_{\mathcal{S}}$, permutes transitively the set of primes of $R^{\mathrm{univ}}_{\mathcal{S}}$ above $\mathfrak{q}$, by Proposition 3.29. We define primes $\widetilde{\mathfrak{p}} = \ker(R^{\mathrm{univ}}_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda} \to A)$ and $\widetilde{\mathfrak{q}} = \ker(P_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda} \to A)$. The group $\mu_2 \times \mu_2$ acts on $R^{\mathrm{univ}}_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda}$, acting trivially on the second factor.

Proposition 3.37. *Let assumptions be as above.*

1. *Let* $(Q_N, \widetilde{Q}_N, \{\overline{\alpha}_v\}_{v\in Q_N})$ *be a choice of Taylor–Wiles data of order* q *and level* N*, and let* $\mathcal{S}_N$ *be the associated augmented deformation problem (cf. §3.3.5). Let* $\mathfrak{p}_N$ *(resp.* $\mathfrak{q}_N$*) denote the pullback of* $\mathfrak{p}$ *(resp.* $\mathfrak{q}$*) to* $R^{univ}_{\mathcal{S}_N}$ *(resp.* $P_{\mathcal{S}_N}$*), and let* $\widetilde{\mathfrak{p}}_N$ *(resp.* $\widetilde{\mathfrak{q}}_N$*) denote the pullback of* $\widetilde{\mathfrak{p}}$ *(resp.* $\widetilde{\mathfrak{q}}$*) to* $R^{univ}_{\mathcal{S}_N} \otimes_\Lambda \widetilde{\Lambda}$ *(resp.* $P_{\mathcal{S}_N} \otimes_\Lambda \widetilde{\Lambda}$*). Then there exists an integer* $C > 0$ *depending only on* $\mathcal{S}$*,* q *and* $\mathfrak{p}$*, and not on* N *or the choice of Taylor–Wiles data, such that* A*-module* $\widetilde{\mathfrak{p}}_N/(\widetilde{\mathfrak{q}}_N + \widetilde{\mathfrak{p}}_N^2)$ *is finite, of cardinality at most* C*.*

2. *Suppose that* $E = E'$*. Then the primes of* $R^{univ}_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda}$ *above* $\widetilde{\mathfrak{q}}$ *are in bijection with the primes of* $R^{univ}_{\mathcal{S}}$ *above* $\mathfrak{q}$*. The natural map on localizations and completions* $(P_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda})_{\widetilde{\mathfrak{q}}} \to (R^{univ}_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda})_{\widetilde{\mathfrak{p}}}$ *is an isomorphism.*

(We remind the reader of our convention that if R is a ring and $\mathfrak{p} \subset R$ is a prime, then $R_{(\mathfrak{p})}$ denotes the localization of R at $\mathfrak{p}$, and $R_{\mathfrak{p}}$ the completion of this ring at its unique maximal ideal.)

Proof. Elements in the finite A-module $\operatorname{Hom}_A(\widetilde{\mathfrak{p}}/(\widetilde{\mathfrak{q}} + \widetilde{\mathfrak{p}}^2), E/A)$ correspond to equivalence classes of lifts $\widetilde{r}$ of $r_{\mathfrak{p}}$ to $A \oplus \epsilon E/A$ with $\det \circ \widetilde{r}|_{G_{F,S}} = \det \circ r_{\mathfrak{p}}|_{G_{F,S}}$, subject to certain ramification conditions at places in S. (This correspondence, and the fact that the relative tangent space $\widetilde{\mathfrak{p}}/(\widetilde{\mathfrak{q}} + \widetilde{\mathfrak{p}}^2)$ is a module over the ring A, which is a discrete valuation ring, is one of the main reasons for introducing $\widetilde{\Lambda}$.) We can write $\widetilde{r} = (1 + \epsilon\phi) r_{\mathfrak{p}}$, where $\phi \in Z^1(G_{F^+,S}, \operatorname{ad} r_{\mathfrak{p}} \otimes_A E/A)$. Similar remarks apply to the finite A-modules $\widetilde{\mathfrak{p}}_N/(\widetilde{\mathfrak{q}}_N + \widetilde{\mathfrak{p}}_N^2)$. To prove the first part of the proposition, it therefore suffices to show the following two claims: first, that for any finite set S' of places of F^+ split in F which contains S, the minimal number of generators of the finitely generated A-module $\operatorname{Hom}_A(H^1(G_{F^+,S'}, \operatorname{ad} r_{\mathfrak{p}} \otimes_A E/A), E/A)$ can be bounded solely in terms of $|S' - S|$; second, that there exists an N_0, depending only on $r_{\mathfrak{p}}$, such that for any $\phi \in Z^1(G_{F^+,S'}, \operatorname{ad} r_{\mathfrak{p}} \otimes_A E/A)$, $T^{N_0}\phi$ is a coboundary.

For the first claim, it suffices since $\overline{r}$ is Schur to bound $\dim_k H^1(G_{F,S'}, \operatorname{ad} \overline{r})$ solely in terms of $|S' - S|$, and this is immediate. For the second, it suffices (since l is odd) to show that we can find N_0 such that the image of $T^{N_0}\phi$ in $Z^1(G_{F,S}, \operatorname{ad} r_{\mathfrak{p}} \otimes E/A)$ is a coboundary. We extend the representation $r_{\mathfrak{p}}|_{G_{F,S}}$ to an algebra map $r_{\mathfrak{p}}|_{G_{F,S}} : A[G_{F,S}] \to M_{n\times n}(A)$. Since $r_{\mathfrak{p}}|_{G_{F,S}} \otimes_A E$ is absolutely irreducible, this algebra homomorphism is surjective after extending scalars to E, and so $\mathcal{A} = r_{\mathfrak{p}}|_{G_{F,S}}(A[G_{F,S}])$ is an order in $M_{n\times n}(A)$. Choose $N \geq 0$ such that $T^N M_{n\times n}(A) \subset \mathcal{A}$.

Writing $\widetilde{r}(a) = r_{\mathfrak{p}}(a) + \epsilon\delta(a)$, we have the formula $\delta(ab) = r_{\mathfrak{p}}(a)\delta(b) + \delta(a) r_{\mathfrak{p}}(b)$, and $\operatorname{tr} \delta(a) \in E/A$ is zero for all $a \in A[G_{F,S}]$. Suppose that a belongs to the ideal $\ker r_{\mathfrak{p}}|_{G_{F,S}} \subset A[G_{F,S}]$. Then $\delta(ab) = \delta(a) r_{\mathfrak{p}}(b)$. We deduce that $\operatorname{tr} T^N \delta(a) M_{n\times n}(A) = 0$, hence $T^N\delta(a) = 0$. Replacing δ with $T^N\delta$, we may assume that δ is pulled back from an A-linear derivation $\mathcal{A} \to M_{n\times n}(E/A)$.

Let $B = \sum_{j=1}^n \delta(T^N e_{j,1}) T^N e_{1,j} \in M_{n\times n}(E/A)$, where $e_{j,i}$ is the matrix with a 1 in the (j,i) spot and 0 everywhere else. For any $\gamma \in \mathcal{A}$, we have

$$T^{2N}\delta(\gamma) = \delta(T^{2N}\gamma) = \sum_{j=1}^n \delta(T^N e_{j,1} T^N e_{1,j}\gamma) = \sum_{j=1}^n \delta(T^N e_{j,1}) T^N e_{1,j}\gamma + T^N e_{j,1}\delta(T^N e_{1,j}\gamma),$$

and this last term equals $B\gamma + \sum_{j=1}^n T^N e_{j,1}\delta(T^N e_{1,j}\gamma)$. On the other hand, we have

$$\sum_{j=1}^n T^N e_{j,1}\delta\left(T^N e_{1,j}\gamma\right) = \sum_{j=1}^n T^N e_{j,1}\delta\left(\sum_{i=1}^n \gamma_{j,i} T^N e_{1,i}\right) = \sum_{j=1}^n \sum_{i=1}^n \gamma_{j,i} T^N e_{j,1}\delta(T^N e_{1,i}) = \sum_{i=1}^n \gamma T^N e_{i,1}\delta(T^N e_{1,i}),$$

and this equals $-\gamma B$. It follows that after multiplying δ by T^{2N}, it becomes a coboundary. Thus we can take $N_0 = 3N$. This concludes the proof of the first part of the proposition.

For the second part, note that the fiber of $\operatorname{Spec} R^{\text{univ}}_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda} \to \operatorname{Spec} P_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda}$ above $\widetilde{\mathfrak{q}}$ is identified with $\operatorname{Spec} R^{\text{univ}}_{\mathcal{S}} \otimes_{P_{\mathcal{S}}} E$. The map

$$(P_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda})_{\widetilde{\mathfrak{q}}} \to (R^{\text{univ}}_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda})_{\widetilde{\mathfrak{p}}}$$

is surjective; indeed, it follows from the proof of the first part of this proposition that the relative tangent space is $\widetilde{\mathfrak{p}}/(\widetilde{\mathfrak{q}} + \widetilde{\mathfrak{p}}^2) \otimes_A E = 0$. For the injectivity, note that for any prime $\widetilde{\mathfrak{r}}$ of $R^{\text{univ}}_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda}$ above $\mathfrak{q}$, the completions $(R_{\mathcal{S}}^{\mathrm{univ}} \otimes_\Lambda \widetilde{\Lambda})_{\widetilde{\mathfrak{p}}}$ and $(R_{\mathcal{S}}^{\mathrm{univ}} \otimes_\Lambda \widetilde{\Lambda})_{\widetilde{\mathfrak{r}}}$ are isomorphic as $(P_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda})_{\widetilde{\mathfrak{q}}}$-algebras, via the action of the group $\mu_2 \times \mu_2$ (by Proposition 3.29). The map

$$(P_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda})_{\widetilde{\mathfrak{q}}} \to R_{\mathcal{S}}^{\mathrm{univ}} \otimes_{P_{\mathcal{S}}} (P_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda})_{\widetilde{\mathfrak{q}}} \cong \prod_{\widetilde{\mathfrak{r}}} (R_{\mathcal{S}}^{\mathrm{univ}} \otimes_\Lambda \widetilde{\Lambda})_{\widetilde{\mathfrak{r}}}$$

is injective, since $(P_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda})_{\widetilde{\mathfrak{q}}}$ is a flat $P_{\mathcal{S}}$-algebra. It follows that the induced map $(P_{\mathcal{S}} \otimes_\Lambda \widetilde{\Lambda})_{\widetilde{\mathfrak{q}}} \to (R_{\mathcal{S}}^{\mathrm{univ}} \otimes_\Lambda \widetilde{\Lambda})_{\widetilde{\mathfrak{p}}}$ must also be injective. □

It need not always be the case that $\mathrm{Frac}\, P_{\mathcal{S}}/\mathfrak{q} = \mathrm{Frac}\, R_{\mathcal{S}}^{\mathrm{univ}}/\mathfrak{p}$. However, let $\mathcal{S}$ denote the deformation problem

$$\mathcal{S} = \left(F/F^+, S, \widetilde{S}, \Lambda, \overline{r}, \chi, \{\mathcal{D}_v\}_{v \in S} \right),$$

where the local deformation problems $\mathcal{D}_v$ are among those defined in §3.3, and let Δ denote the Galois group of the maximal pro-l abelian extension of L, unramified outside the places dividing l. Let $\psi : \Delta/(c+1) \to A^\times$ denote a continuous character which has trivial reduction modulo T. Then we have (cf. Lemma 3.34) a lifting $r_{\mathfrak{p}} \otimes \psi : G_{F^+,S} \to \mathcal{G}_n(A)$, and we write $\mathfrak{p}_\psi$ and $\mathfrak{q}_\psi$ for the induced ideals of $R_{\mathcal{S}}^{\mathrm{univ}}$ and $P_{\mathcal{S}}$, respectively.

Lemma 3.38. *1. Let Q be a minimal prime of $R_{\mathcal{S}}^{univ}$. Then $Q \subset \mathfrak{p}$ if and only if $Q \subset \mathfrak{p}_\psi$.*

2. There exists a choice of ψ with the property that $\mathrm{Frac}\, P_{\mathcal{S}}/\mathfrak{q}_\psi = \mathrm{Frac}\, R_{\mathcal{S}}^{univ}/\mathfrak{p}_\psi = E$.

Proof. For the first part it is enough, by symmetry, to show that if $Q \subset \mathfrak{p}$ then $Q \subset \mathfrak{p}_\psi$. We use the isomorphism $R_{\mathcal{S}}^{\mathrm{univ}} \cong R_{\mathcal{S},\psi_0}^{\mathrm{univ}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Delta/(c+1)]\!]$ of Lemma 3.36. Let Q_0 denote the pullback of Q under the homomorphism $R_{\mathcal{S},\psi_0}^{\mathrm{univ}} \to R_{\mathcal{S}}^{\mathrm{univ}}$. Then there is a surjection

$$R_{\mathcal{S},\psi_0}^{\mathrm{univ}}/Q_0 \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Delta/(c+1)]\!] \to R_{\mathcal{S}}^{\mathrm{univ}}/Q,$$

and for any minimal prime $P \subset \mathcal{O}[\![\Delta/(c+1)]\!]$, the ring

$$R_{\mathcal{S},\psi_0}^{\mathrm{univ}}/Q_0 \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Delta/(c+1)]\!]/P$$

is isomorphic to a power series ring over the domain $R_{\mathcal{S},\psi_0}^{\mathrm{univ}}/Q_0$. It follows that there exists a minimal prime $P_0 \subset \mathcal{O}[\![\Delta/(c+1)]\!]$ such that $Q = (Q_0, P_0)$. The composite $R_{\mathcal{S},\psi_0}^{\mathrm{univ}} \to R_{\mathcal{S}}^{\mathrm{univ}} \to A$ has kernel containing Q_0 (since $Q \subset \mathfrak{p}$). It follows that there is a homomorphism

$$R_{\mathcal{S}}^{\mathrm{univ}}/Q = R_{\mathcal{S},\psi_0}^{\mathrm{univ}}/Q_0 \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Delta/(c+1)]\!]/P_0 \to A \widehat{\otimes}_k k[\![\Delta/(c+1)]\!]/P_0, \tag{3.4}$$

arising from the universal property of the completed tensor product $R_{\mathcal{S},\psi_0}^{\mathrm{univ}}/Q_0 \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Delta/(c+1)]\!]/P_0$ via the map

$$R_{\mathcal{S},\psi_0}^{\mathrm{univ}}/Q_0 \to A \to A \widehat{\otimes}_k k[\![\Delta/(c+1)]\!]$$

and the diagonal map

$$\mathcal{O}[\![\Delta/(c+1)]\!]/P_0 \to A \widehat{\otimes}_k k[\![\Delta/(c+1)]\!]/P_0.$$

The map (3.4) cuts out an irreducible Zariski closed subset of $\mathrm{Spec}\, R_{\mathcal{S}}^{\mathrm{univ}}/Q$ which contains both $\mathfrak{p}$ and $\mathfrak{p}_\psi$. In particular, we have $Q \subset \mathfrak{p}_\psi$, as desired.

For the second part of the lemma, choose a surjection $\Delta/(c+1) \to \mathbb{Z}_l$, and $\sigma \in G_{F,S}$ that is mapped to a topological generator of $\mathbb{Z}_l$. Write $\det r_{\mathfrak{p}}|_{G_{F,S}}(\sigma) = x\alpha$, where $x \in k^\times$ and $\alpha \in A^\times$ satisfies $\alpha \equiv 1 \bmod T$. Define a character $\psi : \Delta/(c+1) \to A^\times$ by pulling back the character $\mathbb{Z}_l \to A^\times$ which sends the image of σ to the unique n^{th} root of $(1+T)/\alpha$ in $1 + TA$. Then $\det(r \otimes \psi)|_{G_{F,S}}(\sigma) = x(1+T)$, and $\mathrm{Frac}\, P_{\mathcal{S}}/\mathfrak{q}_\psi$ is identified with a closed subfield of $E = k((T))$ containing T. The only possibility is $\mathrm{Frac}\, P_{\mathcal{S}}/\mathfrak{q}_\psi = E$, and this implies the lemma. □

We now suppose that $\mathcal{S}$ is the global deformation problem (3.2). For the convenience of the reader, we recall its definition here:

$$\mathcal{S} = \left(F/F^+, S, \widetilde{S}, \Lambda, \overline{r}, \chi, \{\mathcal{D}_v\}_{v \in S} \right), \tag{3.5}$$

where S is a disjoint union $S = S_l \cup S(B) \cup R \cup S_a$, and the deformation problems $\mathcal{D}_v$ are as follows:

- If $v \in S_l$, then $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is trivial, $[F_{\widetilde{v}} : \mathbb{Q}_l] > n(n-1)/2 + 1$, and $\mathcal{D}_v = R_v^{\triangle}$.
- If $v \in S(B)$, then $q_v \equiv 1 \bmod l$, $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is trivial, and $\mathcal{D}_v = R_v^{\mathrm{St}}$.
- If $v \in R$, then $q_v \equiv 1 \bmod l$, $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is trivial, and $\mathcal{D}_v = R_v^{\chi_v}$ for some tuple $\chi_v = \chi_{v,1} \times \cdots \times \chi_{v,n}$ of characters $\chi_{v,i} : k(v)^\times \to \mathcal{O}^\times$ which are trivial mod λ.
- If $v \in S_a$, then $q_v \not\equiv 1 \bmod l$, $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is unramified and $\overline{r}|_{G_{F_{\widetilde{v}}}}(\mathrm{Frob}_{\widetilde{v}})$ is a scalar matrix, and $\mathcal{D}_v = R_v^{\square}$.

Let $T = S$. We write $P^{\mathrm{loc}} \subset R^{\mathrm{loc}}_{\mathcal{S},T}$ for the kernel of the composite $R^{\mathrm{loc}}_{\mathcal{S},T} \to R_{\mathcal{S}} \to A$, and $\widetilde{P}^{\mathrm{loc}} \subset \widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T} = R^{\mathrm{loc}}_{\mathcal{S},T} \otimes_\Lambda \widetilde{\Lambda}$ for the kernel of the natural surjection $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T} \to A$. Finally, we fix an integer $q' \geq 0$ and define $R^\infty = \widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}[\![x_1, \ldots, x_{q'}]\!]$, and P^∞ to be the kernel of the map $R^\infty \to A$ sending each of $x_1, \ldots, x_{q'}$ to 0.

Lemma 3.39. *Let $Q \subset \Lambda$ be a minimal prime and let $V \subset \operatorname{Spec} \Lambda/Q[1/l]$ denote the open subset where for each $v \in S_l$ and for each $1 \leq i < j \leq n$, we have $\psi_i^v \neq \psi_j^v$ and $\psi_i^v \neq \epsilon\psi_j^v$. Let $\widetilde{V}$ denote the pre-image of V in $\operatorname{Spec} \widetilde{\Lambda}/(Q)[1/l]$. Suppose finally that for each $v \in S_l$, we have $[F_{\widetilde{v}} : \mathbb{Q}_l] > n(n-1)/2 + 1$. Then:*

1. *The pre-image of $\widetilde{V}$ in $\operatorname{Spec} \widehat{\otimes}_{v \in S_l} R_v^{\triangle} \otimes_\Lambda \widetilde{\Lambda}/(Q)[1/l]$ is regular.*
2. *The ring $\widehat{\otimes}_{v \in S_l} R_v^{\triangle} \otimes_\Lambda \widetilde{\Lambda}/(Q)$ is geometrically irreducible over $\mathcal{O}$.*

Proof. Consider the two projections

$$f : \operatorname{Spec} \widehat{\otimes}_{v \in S_l} R_v^{\triangle} \to \operatorname{Spec} \Lambda,$$

$$\widetilde{f} : \operatorname{Spec} \widehat{\otimes}_{v \in S_l} R_v^{\triangle} \otimes_\Lambda \widetilde{\Lambda} \to \operatorname{Spec} \widetilde{\Lambda}.$$

We make the following observations.

- $\widetilde{f}$ admits a section. Indeed, it is easy to construct a section of f, and we get a section of $\widetilde{f}$ by base extension to $\widetilde{\Lambda}$.
- $\widetilde{f}^{-1}(\widetilde{V})$ is connected. The set $\widetilde{V}$ is irreducible, the fibers of $\widetilde{f}$ above closed points of $\widetilde{V}$ are connected (by Proposition 3.14), and $\widetilde{f}$ admits a section.
- $\widetilde{f}^{-1}(\widetilde{V})$ is regular. Because $\widehat{\otimes}_{v \in S_l} R_v^{\triangle} \otimes_\Lambda \widetilde{\Lambda}$ is excellent, and $\widehat{\otimes}_{v \in S_l} R_v^{\triangle} \otimes_\Lambda \widetilde{\Lambda}[1/l]$ is a Jacobson ring, it suffices to check regularity at closed points of $\widetilde{f}^{-1}(\widetilde{V})$. If $x \in \widetilde{f}^{-1}(\widetilde{V})$ is such a point, then regularity at x follows from the regularity of the fiber $\widetilde{f}^{-1}\widetilde{f}(x)$, regularity of $\widetilde{\Lambda}[1/l]$, and the equality

  $$\dim \widetilde{f}^{-1}(\widetilde{V}) = \dim \widetilde{V} + \dim \widetilde{f}^{-1}\widetilde{f}(x)$$

  (see Proposition 3.14 again).

This already proves the first part of the lemma, and shows that $\widetilde{f}^{-1}(\widetilde{V})$ is irreducible (even geometrically irreducible, since we reach the same conclusion if we first enlarge the field of coefficients). We claim that $\widetilde{f}^{-1}(\widetilde{V})$ is Zariski dense in $\operatorname{Spec} \widehat{\otimes}_{v \in S_l} R_v^{\triangle} \otimes_\Lambda \widetilde{\Lambda}/(Q)[1/l]$. To show this, it is enough to show that each generic point $\widetilde{\eta} \in \operatorname{Spec} \widehat{\otimes}_{v \in S_l} R_v^{\triangle} \otimes_\Lambda \widetilde{\Lambda}/(Q)[1/l]$ is contained in $\widetilde{f}^{-1}(\widetilde{V})$. Let $\widetilde{\eta}$ be such a point, and let $\eta \in \operatorname{Spec} \widehat{\otimes}_{v \in S_l} R_v^{\triangle} \otimes_\Lambda \Lambda/Q[1/l]$ denote its image under the finite, faithfully flat morphism

$$\operatorname{Spec} \widehat{\otimes}_{v \in S_l} R_v^{\triangle} \otimes_\Lambda \widetilde{\Lambda}/(Q)[1/l] \to \operatorname{Spec} \widehat{\otimes}_{v \in S_l} R_v^{\triangle} \otimes_\Lambda \Lambda/Q[1/l].$$

Then η is a generic point. Since $\operatorname{Spec} \widehat{\otimes}_{v \in S_l} R_v^{\triangle} \otimes_\Lambda \Lambda/Q[1/l]$ is irreducible, we have $\eta \in f^{-1}(V)$, hence $\widetilde{\eta} \in \widetilde{f}^{-1}(\widetilde{V})$. This completes the proof of the claim. The lemma itself now follows from the observation that the ring $\widehat{\otimes}_{v \in S_l} R_v^{\triangle} \otimes_\Lambda \widetilde{\Lambda}$ is $\mathcal{O}$-flat (by Lemma 1.4). ☐

Lemma 3.40. *With hypothesis as above, suppose in addition that:*

1. *For each $v \in S_l$, the characters ψ_i^v mod $\mathfrak{p} : I^{ab}_{F_{\widetilde{v}}}(l) \to \Lambda_v^\times \to A^\times$, $i = 1, \dots, n$, are pairwise distinct.*
2. *For each $v \in S(B)$, $r_{\mathfrak{p}}|_{G_{F_{\widetilde{v}}}}$ is unramified, and $r_{\mathfrak{p}}|_{G_{F_{\widetilde{v}}}}(\mathrm{Frob}_{\widetilde{v}}) \in \mathrm{GL}_n(A)$ is a scalar matrix $\alpha_v \cdot 1_n$, $\alpha_v \in 1 + \mathfrak{m}_A$.*
3. *For each $v \in R$, $r_{\mathfrak{p}}|_{G_{F_{\widetilde{v}}}}$ is trivial.*

Then:

1. *Suppose further that for each $v \in R$, the characters $\chi_{v,1}, \dots, \chi_{v,n}$ are pairwise distinct. Then for each minimal prime $Q \subset \Lambda$, $\mathrm{Spec}\, R^\infty_{P^\infty}/(Q)$ is irreducible of dimension $n(n+1)[F^+ : \mathbb{Q}]/2 + n^2|T| + q'$, and its generic point is of characteristic 0.*
2. *Suppose instead that for each $v \in R$, the characters $\chi_{v,1}, \dots, \chi_{v,n}$ are trivial, and that K is sufficiently large, in the sense of Proposition 3.15. Then for each minimal prime $Q \subset \Lambda$, $\mathrm{Spec}\, R^\infty_{P^\infty}/(Q)$ is equidimensional of dimension $n(n+1)[F^+ : \mathbb{Q}]/2 + n^2|T| + q'$, and all of its generic points are of characteristic 0. Moreover, each minimal prime of $R^\infty_{P^\infty}/(Q, \lambda)$ contains a unique minimal prime of $R^\infty_{P^\infty}/(Q)$.*

(The hypotheses of Lemma 3.40 will hold when we apply it later during the proof of Theorem 4.19.)

Proof. We observe that we have already shown in Lemma 3.20 that the ring $R^{\mathrm{loc}}_{\mathcal{S},T}$ satisfies the analogue of the above properties. The thrust of this lemma is thus to show that these properties are preserved under the operations of extension of scalars to $\widetilde{\Lambda}$ and localization and completion at P^∞. We begin with some preliminary reductions. First, we can assume by Lemma 1.5 that $q' = 0$, and hence $R^\infty = \widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}$ and $P^\infty = \widetilde{P}^{\mathrm{loc}}$. We can write

$$\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T} = \left(\widehat{\bigotimes}_{v \in S_a} R_v^\square\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\bigotimes}_{v \in S(B)} R_v^{\mathrm{St}}\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\bigotimes}_{v \in S_l} R_v^\triangle \otimes_\Lambda \widetilde{\Lambda}\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\bigotimes}_{v \in R} R_v^{\chi_v}\right).$$

Suppose that for each $v \in R$, the characters $\chi_{v,1}, \dots, \chi_{v,n}$ are pairwise distinct, and let $Q \subset \Lambda$ be a minimal prime. By Lemma 3.20, the ring $R^{\mathrm{loc}}_{\mathcal{S},T}/(Q)$ contains a unique minimal prime $\mathfrak{Q}$, and $R^{\mathrm{loc}}_{\mathcal{S},T}/\mathfrak{Q}$ is $\mathcal{O}$-flat of dimension $1 + n(n+1)[F^+ : \mathbb{Q}]/2 + n^2|T|$. It follows that $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}/(\mathfrak{Q}) = R^{\mathrm{loc}}_{\mathcal{S},T}/\mathfrak{Q} \otimes_\Lambda \widetilde{\Lambda}$ is $\mathcal{O}$-flat of the same dimension, hence $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T,\widetilde{P}^{\mathrm{loc}}}/(\mathfrak{Q})$ is $\mathcal{O}$-flat of dimension $n(n+1)[F^+ : \mathbb{Q}]/2 + n^2|T|$. (The ring $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}/(\mathfrak{Q})$ is excellent, hence catenary, and $\widetilde{P}^{\mathrm{loc}}$ has dimension 1.) We will show that the ring $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T,\widetilde{P}^{\mathrm{loc}}}[1/l]/(Q)$ is a domain. This will then imply the first part of the lemma.

Choose for each $v \in S(B)$ a lift $\widetilde{\alpha}_v$ of α_v to $\widetilde{\Lambda}$. There is an induced homomorphism $f_{\widetilde{\alpha}} : \widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T} \to \widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}$ which classifies the universal lifting for $v \in T - S(B)$, and the unramified twist of the universal lifting by $\widetilde{\alpha}_v$ for $v \in S(B)$. Then $f_{\widetilde{\alpha}}(P^{\mathrm{loc}})$ is the kernel of the surjective homomorphism

$$\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T} \to \left(\widehat{\bigotimes}_{v \in S_a} R_v^\square\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\bigotimes}_{v \in S(B)} k\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\bigotimes}_{v \in S_l} R_v^\triangle \widehat{\otimes}_\Lambda \widetilde{\Lambda}\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\bigotimes}_{v \in R} k\right) \to A,$$

which classifies $r_{\mathfrak{p}}|_{G_{F_{\widetilde{v}}}}$ for $v \in S_a \cup S_l$, and the trivial lifting $\overline{r}|_{G_{F_{\widetilde{v}}}}$ for $v \in S(B) \cup R$. Writing $\mathfrak{P}'$ for the kernel of this homomorphism, we get an isomorphism $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T,\widetilde{P}^{\mathrm{loc}}} \cong \widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T,\mathfrak{P}'}$. We define

$$C = \left(\widehat{\bigotimes}_{v \in S_a} R_v^\square\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\bigotimes}_{v \in S_l} R_v^\triangle \widehat{\otimes}_\Lambda \widetilde{\Lambda}\right)$$

and write $\mathfrak{P}$ for the kernel of the map $C \to A$. We define

$$D = \left(\widehat{\bigotimes}_{v \in S(B)} R_v^{\mathrm{St}}\right) \widehat{\otimes}_{\mathcal{O}} \left(\widehat{\bigotimes}_{v \in R} R_v^\chi\right).$$

Thus $\mathfrak{P}' = (\mathfrak{P}, \mathfrak{m}_D) \subset C \widehat{\otimes}_{\mathcal{O}} D$, and by Lemma 1.5 there is a canonical isomorphism $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T,\mathfrak{P}'} \cong C_{\mathfrak{P}} \widehat{\otimes}_{\mathcal{O}} D$. For each minimal prime $Q \subset \Lambda$, the ring $(C/(Q))_{\mathfrak{P}}$ is a formally smooth $(\widetilde{\Lambda}/(Q))_{\widetilde{P}}$-algebra. To see this, we compute the relative tangent space $\mathfrak{P}/(\mathfrak{P}^2, \widetilde{P})$ of the morphism $(\widetilde{\Lambda}/(Q))_{\widetilde{P}} \to (C/(Q))_{\mathfrak{P}}$. By Lemma 3.13, the E-dual of this tangent space is isomorphic to the space of data of the following type: a choice for each $v|l$ of a pair $(\mathrm{Fil}_v^\bullet, r)$ of a lifting of $r_{\mathfrak{p}}|_{G_{L_{\widetilde{v}}}}$ to $E[\epsilon]$ and a filtration preserved by it, such that the diagonal characters are the pushfowards of $(\psi_1^v, \ldots, \psi_n^v)$ via the homomorphism $\Lambda/P \hookrightarrow E$, and for each $v \in S_a$, an unramified lifting of $r_{\mathfrak{p}}|_{G_{L_{\widetilde{v}}}}$ to $E[\epsilon]$. Arguing as in the proof of [Ger, Lemma 3.2.3] shows that this tangent space has dimension equal to $n(n-1)/2[F^+ : \mathbb{Q}] + n^2|S_l| + n^2|S_a| = \dim(C/Q)_{\mathfrak{P}} - \dim(\widetilde{\Lambda}/(Q))_{\widetilde{P}}$. It follows that $(C/(Q))_{\mathfrak{P}}$ is a formally smooth $(\widetilde{\Lambda}/(Q))_{\widetilde{P}}$-algebra, and hence that this ring can be presented as a power series ring over $(\widetilde{\Lambda}/(Q))_{\widetilde{P}}$. In particular, $(C/(Q))_{\mathfrak{P}}$ is a regular local domain. (We use here that for each $1 \leq i < j \leq n$, we have $\psi_i^v \not\equiv \psi_j^v \bmod \mathfrak{p}$ and $\psi_i^v \not\equiv \epsilon\psi_j^v \bmod \mathfrak{p}$. The first condition holds by hypothesis. For the second, we observe that both the characters ψ_i^v and ψ_j^v are trivial modulo $\mathfrak{m}_{\Lambda_v}$. If $\psi_i^v \equiv \epsilon\psi_j^v \bmod \mathfrak{p}$ then ϵ is trivial modulo λ, which implies that $\psi_i^v \equiv \psi_j^v \bmod \mathfrak{p}$, a contradiction.)

Successively applying Proposition 3.16 and Proposition 3.18, we see that $\operatorname{Spec} C_{\mathfrak{P}}/(Q)\widehat{\otimes}_{\mathcal{O}} D[1/l]$ is connected, hence $\operatorname{Spec} \widetilde{R}^{\mathrm{loc}}_{\mathcal{S}_\chi, T, \mathfrak{P}'}/(Q)[1/l]$ is connected. We now make the following observations.

- The morphism $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S}_\chi, T, (\mathfrak{P}')}/(Q)[1/l] \to \widetilde{R}^{\mathrm{loc}}_{\mathcal{S}_\chi, T, \mathfrak{P}'}/(Q)[1/l]$ is regular, being a localization of a regular morphism (see [Mat89, §32] for the definition and basic properties of this notion), and faithfully flat.
- The ring $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S}_\chi, T, (\mathfrak{P}')}/(Q)[1/l]$ is regular. Indeed, $\mathfrak{P}'$ lies in the set $\widetilde{f}^{-1}(\widetilde{V})$ of Lemma 3.39.

It follows (by [Mat89, Theorem 32.2]) that $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S}_\chi, T, \mathfrak{P}'}/(Q)[1/l]$ is regular. Since we have shown this ring to have connected spectrum, it is a domain. This completes the proof of the first part of the lemma.

We now come to the second part of the lemma. We therefore now assume that for each $v \in R$, the characters $\chi_{v,1}, \ldots, \chi_{v,n}$ are trivial, and fix a prime $Q \subset \Lambda$. We are going to apply Proposition 1.6. First, we observe that the ring $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}/(Q, \lambda)$ is generically reduced. We have seen (Lemma 3.20) that $R^{\mathrm{loc}}_{\mathcal{S},T}/(Q, \lambda)$ is generically reduced (equivalently, satisfies the condition (R_0) of being reduced in codimension 0). Moreover, under the map

$$\operatorname{Spec} R^{\mathrm{loc}}_{\mathcal{S},T}/(Q, \lambda) \to \operatorname{Spec} \Lambda/(Q, \lambda),$$

every generic point of $\operatorname{Spec} R^{\mathrm{loc}}_{\mathcal{S},T}/(Q, \lambda)$ maps to the generic point of $\operatorname{Spec} \Lambda/(Q, \lambda)$. Since the map

$$\operatorname{Spec} \widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}/(Q, \lambda) \to \operatorname{Spec} R^{\mathrm{loc}}_{\mathcal{S},T}/(Q, \lambda)$$

is faithfully flat, it follows that every generic point of $\operatorname{Spec} \widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}/(Q, \lambda)$ maps to the generic point of $\operatorname{Spec} \widetilde{\Lambda}/(Q, \lambda)$. To show that the ring $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}/(Q, \lambda)$ satisfies condition (R_0), it is therefore enough to show that the ring

$$\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}/(Q, \lambda) \otimes_{\widetilde{\Lambda}} \operatorname{Frac} \widetilde{\Lambda}/(Q, \lambda) \tag{3.6}$$

satisfies (R_0). Let us write η_Q for the generic point of $\operatorname{Spec} \Lambda/(Q, \lambda)$, and $\widetilde{\eta}_Q$ for the generic point of $\operatorname{Spec} \widetilde{\Lambda}/(Q, \lambda)$. The finite extension $\kappa(\widetilde{\eta}_Q)/\kappa(\eta_Q)$ is separable, by construction, and the ring (3.6) is canonically isomorphic to

$$(R^{\mathrm{loc}}_{\mathcal{S},T}/(Q, \lambda) \otimes_\Lambda \kappa(\eta_Q)) \otimes_{\kappa(\eta_Q)} \kappa(\widetilde{\eta}_Q).$$

Since the ring $R^{\mathrm{loc}}_{\mathcal{S},T}/(Q, \lambda) \otimes_\Lambda \kappa(\eta_Q)$ satisfies (R_0), we deduce the corresponding condition for the ring (3.6) and hence for $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}/(Q, \lambda)$.

Second, we observe that for each minimal prime $\mathfrak{Q} \subset \widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}/(Q)$, $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}/(\mathfrak{Q})$ is $\mathcal{O}$-flat and geometrically integral; and that each minimal prime of $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}/(Q, \lambda)$ contains a unique minimal prime of $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S},T}/(Q)$. By Lemma 1.4, to show this it is enough to show that for each of the rings

$$B = R_v^{\square} \ (v \in S_a), \ R_v^{\mathrm{St}} \ (v \in S(B)), \ R_v^1 \ (v \in R), \text{ and } \left(\widehat{\otimes}_{v \in S_l} R_v^{\triangle}\right) \otimes_\Lambda \widetilde{\Lambda}/(Q),$$

the following hold:

- for each minimal prime $\mathfrak{Q} \subset B$, $B/\mathfrak{Q}$ is $\mathcal{O}$-flat and geometrically integral;

- and each minimal prime of $B/(\lambda)$ contains a unique minimal prime of B.

If $v \in S_a$, then this follows from Proposition 3.10. If $v \in S(B)$, then it follows from Proposition 3.17. If $v \in R$, then it follows from Proposition 3.15, provided that the coefficient field K is sufficiently large. For the remaining ring $B = \widehat{\otimes}_{v \in S_l} R_v^{\triangle} \otimes_\Lambda \widetilde{\Lambda}/(Q)$, it follows from Lemma 3.39.

We can now complete the proof of the lemma. We have shown that the ring $\widetilde{R}^{\text{loc}}_{\mathcal{S},T,(\widetilde{P}^{\text{loc}})}/(Q)$ has the following properties:

- the quotient $\widetilde{R}^{\text{loc}}_{\mathcal{S},T,(\widetilde{P}^{\text{loc}})}/(Q,\lambda)$ is generically reduced;
- for each minimal prime $\mathfrak{Q} \subset \widetilde{R}^{\text{loc}}_{\mathcal{S},T,(\widetilde{P}^{\text{loc}})}/(Q)$, the quotient $\widetilde{R}^{\text{loc}}_{\mathcal{S},T,(\widetilde{P}^{\text{loc}})}/\mathfrak{Q}$ is $\mathcal{O}$-flat;
- and each minimal prime of $\widetilde{R}^{\text{loc}}_{\mathcal{S},T,(\widetilde{P}^{\text{loc}})}/(Q,\lambda)$ contains a unique minimal prime of $\widetilde{R}^{\text{loc}}_{\mathcal{S},T,(\widetilde{P}^{\text{loc}})}/(Q)$.

Indeed, we have shown that the ring $\widetilde{R}^{\text{loc}}_{\mathcal{S},T}/(Q)$ has these properties, and they are preserved under localization at $\widetilde{P}^{\text{loc}}$. The hypotheses of Proposition 1.6 are now fulfilled, and this implies that the above properties are preserved under completion at $\widetilde{P}^{\text{loc}}$. This completes the proof of the lemma. □

4 Automorphic forms

In this section we define spaces of algebraic modular forms on definite unitary groups. We work with ordinary forms; by Hida theory, it suffices therefore to work throughout with forms of weight zero (or equivalently, with trivial coefficients). A development of this theory in the context of Galois representations was given by Geraghty [Ger]. We will refer mostly to this work for structural results. We remark that Geraghty works with 'true' unitary groups (associated to a Hermitian vector space), whereas we prefer to work with a division algebra endowed with involution at the second kind, which may therefore be ramified at a finite set of places (the set $S(B)$ below). This does not, however, require any modifications to the Hida-theoretic arguments, for which we continue to use Geraghty's work as a reference.

4.1 Definitions

Fix an integer $n \geq 1$ and an odd prime l. Let L be a CM field with maximal totally real subfield L^+. We suppose that every prime above l splits in L/L^+ and that L/L^+ is everywhere unramified. Let S_l denote the set of places of L^+ dividing l. For each place in $v \in S_l$ we choose a place $\widetilde{v}$ of L above it and denote the set of these by $\widetilde{S}_l$. We write I_l for the set of embeddings $L \hookrightarrow \overline{\mathbb{Q}}_l$, and $\widetilde{I}_l \subset I_l$ for the those embeddings inducing an element of $\widetilde{S}_l$. We fix also a finite non-empty set of places $S(B)$ of L^+ such that

- Every element of $S(B)$ splits in L.
- $S(B)$ contains no place above l.
- If n is even then $n[L^+ : \mathbb{Q}]/2 + |S(B)| \equiv 0 \bmod 2$.

Under these hypotheses we can find a central division algebra B over L equipped with an involution † satisfying the following properties (see [CHT08, §3.3]):

- $\dim_L B = n^2$.
- $B^{\text{op}} \cong B \otimes_{L,c} L$.
- B splits outside $S(B)$.
- For each prime w of L above $S(B)$, $B_w = B \otimes_L L_w$ is a division algebra.
- $\dagger|_L = c$.

- Defining a group G over L^+ by the formula

$$G(R) = \{g \in B \otimes_{L^+} R \mid g^\dagger g = 1\}$$

  for any L^+-algebra R, we have that $G(L^+ \otimes_{\mathbb{Q}} \mathbb{R})$ is compact and G is quasi-split at every finite place $v \notin S(B)$ of L^+.

We can find a maximal order $\mathcal{O}_B \subset B$ such that $\mathcal{O}_B^\dagger = \mathcal{O}_B$ and such that $\mathcal{O}_{B,w} = \mathcal{O}_B \otimes_{\mathcal{O}_L} \mathcal{O}_{L_w}$ is a maximal order in B_w for every place $w \in L$ split over L^+. This defines an integral model for G over $\mathcal{O}_{L^+}$, which we continue to denote by G.

Let $v \notin S(B)$ be a finite place of L^+ which splits as $v = ww^c$ in L. Then we can find an isomorphism

$$\iota_v : \mathcal{O}_B \otimes_{\mathcal{O}_{L^+}} \mathcal{O}_{L_v^+} \to M_n(\mathcal{O}_{L_w}) \times M_n(\mathcal{O}_{L_{w^c}})$$

such that $\iota_v(g^\dagger) = {}^t\iota_v(g)^c$. Projection to the first factor then gives rise to an isomorphism

$$\iota_w : G(L_v^+) \to \mathrm{GL}_n(L_w),$$

with the property that $\iota_w(G(\mathcal{O}_{L_v^+})) = \mathrm{GL}_n(\mathcal{O}_{L_w})$. If $v \in S(B)$ then we get an isomorphism $\iota_w : G(L_v^+) \to B_w^\times$, with the property that $\iota_w(G(\mathcal{O}_{L_v^+})) = \mathcal{O}_{B_w}^\times$.

Let K be a finite extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$, with ring of integers $\mathcal{O}$ and residue field k. We write λ for the maximal ideal of $\mathcal{O}$. We will suppose K large enough to contain the image of every embedding of L in $\overline{\mathbb{Q}}_l$.

Let R be a finite set of finite places of L^+, disjoint from $S_l \cup S(B)$ and containing only places which split in L. Let $T \supset S_l \cup S(B) \cup R$ be a finite set of places of L^+ which split in L. For each $v \in T$ we choose a place $\widetilde{v}$ of L above it, extending our previous choice for $v \in S_l$.

We suppose that $U = \prod_v U_v$ is an open compact subgroup of $G(\mathbb{A}_{L^+}^\infty)$ such that $U_v \subset \iota_{\widetilde{v}}^{-1}\,\mathrm{Iw}(\widetilde{v})$ for $v \in R$. Here given a place $\widetilde{v}$ of L, $\mathrm{Iw}(\widetilde{v})$ is the subgroup of $\mathrm{GL}_n(\mathcal{O}_{L_{\widetilde{v}}})$ consisting of matrices whose image in $\mathrm{GL}_n(k(\widetilde{v}))$ is upper triangular. We will also write $\mathrm{Iw}_1(\widetilde{v}) \subset \mathrm{Iw}(\widetilde{v})$ for the subgroup of matrices whose image in $\mathrm{GL}_n(k(\widetilde{v}))$ is upper-triangular and unipotent. We say that U is sufficiently small if there exists a place $v \notin S_l$ of L^+ such that U_v contains no non-trivial elements of finite order. We shall often assume this in the applications below.

For each $v \in R$, we choose a character $\chi_v : \mathrm{Iw}(\widetilde{v})/\mathrm{Iw}_1(\widetilde{v}) \to \mathcal{O}^\times$. This is equivalent to the data of an n-tuple of characters $\chi_{v,1}, \ldots, \chi_{v,n} : k(\widetilde{v})^\times \to \mathcal{O}^\times$. We write $M = \otimes_{v\in R}\mathcal{O}(\chi_v)$. Then M is an $\mathcal{O}[\prod_{v\in R}\mathrm{Iw}(\widetilde{v})/\mathrm{Iw}_1(\widetilde{v})]$-module, free of rank 1 over $\mathcal{O}$.

Definition 4.1. *Let $U, \chi = \{\chi_v\}$ be as above. If A is an $\mathcal{O}$-module, we write $S_\chi(U, A)$ for the set of functions*

$$f : G(L^+)\backslash G(\mathbb{A}_{L^+}^\infty) \to M \otimes_{\mathcal{O}} A$$

such that for every $u \in U$, we have $f(gu) = u_R^{-1} f(g)$, where u_R denotes the image of u under the projection $U \to \prod_{v\in R}\mathrm{Iw}(\widetilde{v})$. If V is an arbitrary subgroup of $G(\mathbb{A}_{L^+}^\infty)$ whose projection to the places $v \in R$ is contained in $\iota_{\widetilde{v}}^{-1}\,\mathrm{Iw}(\widetilde{v})$, then we will write

$$S_\chi(V, \overline{\mathbb{Q}}_l) = \varinjlim_{U \supset V} S_\chi(U, \overline{\mathbb{Q}}_l),$$

the limit being taken over the directed system of all open compact subgroups $U = \prod_v U_v \subset G(\mathbb{A}_{L^+}^\infty)$ such that for each $v \in R$, we have $U_v \subset \iota_{\widetilde{v}}^{-1}\,\mathrm{Iw}(\widetilde{v})$. The group $G(\mathbb{A}_{L^+}^{\infty,R}) \times \prod_{v\in R}\iota_{\widetilde{v}}^{-1}\,\mathrm{Iw}(\widetilde{v})$ acts on $S_\chi(\{1\}, \overline{\mathbb{Q}}_l)$ by right translation.

For integers $0 \le b \le c$, and $v \in S_l$, we consider the subgroup $\mathrm{Iw}(\widetilde{v}^{b,c}) \subset \mathrm{GL}_n(\mathcal{O}_{L_{\widetilde{v}}})$ defined as those matrices which are congruent to an upper-triangular matrix modulo $\widetilde{v}^c$ and congruent to a unipotent upper-triangular matrix modulo $\widetilde{v}^b$. We set $U(\mathfrak{l}^{b,c}) = U^l \times \prod_{v\in S_l}\iota_{\widetilde{v}}^{-1}\,\mathrm{Iw}(\widetilde{v}^{b,c})$.

We now define certain Hecke operators as double coset operators (cf. [Ger, §2.3]). If w is a place of L split over L^+ and not in T, let ϖ_w be a uniformizer of L_w. Then we define for $j = 1, \ldots, n$ the Hecke operator

$$T_w^j = \iota_w^{-1}\left(\left[\mathrm{GL}_n(\mathcal{O}_{L_w})\begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{n-j}\end{pmatrix}\mathrm{GL}_n(\mathcal{O}_{L_w})\right]\right).$$

This is independent of the choice of uniformizer ϖ_w.

If v is a place of L^+ dividing l and if $u \in T_n(\mathcal{O}_{L_{\widetilde{v}}})$, where $T_n \subset \mathrm{GL}_n$ is the diagonal torus, then we write

$$\langle u \rangle = \iota_{\widetilde{v}}^{-1} \left(\left[\mathrm{Iw}(\widetilde{v}^{b,c}) u \, \mathrm{Iw}(\widetilde{v}^{b,c}) \right] \right).$$

We fix a choice of uniformizer $\varpi_{\widetilde{v}} \in L_{\widetilde{v}}$ and define

$$U_{\widetilde{v}}^j = \iota_{\widetilde{v}}^{-1} \left(\left[\mathrm{Iw}(\widetilde{v}^{b,c}) \begin{pmatrix} \varpi_{\widetilde{v}} 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix} \mathrm{Iw}(\widetilde{v}^{b,c}) \right] \right).$$

This operator depends on the choice of uniformizer. Now [Ger, Lemma 2.3.3] shows that these operators act on the spaces $S_\chi(U(\mathfrak{l}^{b,c}), A)$ for any $\mathcal{O}$-module A and commute with the inclusions

$$S_\chi(U(\mathfrak{l}^{b,c}), A) \subset S_\chi(U(\mathfrak{l}^{b',c'}), A),$$

when $b \leq b'$ and $c \leq c'$.

For any $\mathcal{O}$-module A there is defined a subspace of ordinary forms

$$S_\chi^{\mathrm{ord}}(U(\mathfrak{l}^{b,c}), A) = e S_\chi(U(\mathfrak{l}^{b,c}), A) \subset S_\chi(U(\mathfrak{l}^{b,c}), A)$$

which is preserved by all of the above defined Hecke operators, defined as the image of the ordinary projector e, see [Ger, Definition 2.4.2].

Definition 4.2. *We write* $\mathbb{T}_\chi^T(U(\mathfrak{l}^{b,c}), A)$ *for the* $\mathcal{O}$*-subalgebra of*

$$\mathrm{End}_{\mathcal{O}}(S_\chi^{ord}(U(\mathfrak{l}^{b,c}), A))$$

generated by the operators $T_w^j, j = 1, \ldots, n$ *and* $(T_w^n)^{-1}$ *as above and* $\langle u \rangle$ *for*

$$u \in T_n(\mathcal{O}_{L^+,l}) = \prod_{v \in S_l} T_n(\mathcal{O}_{L_{\widetilde{v}}}).$$

We write $\mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O}) = \varprojlim_c \mathbb{T}_\chi^T(U(\mathfrak{l}^{c,c}), \mathcal{O})$*. This algebra acts naturally on the spaces*

$$S_\chi(U(\mathfrak{l}^\infty), K/\mathcal{O}) = \varinjlim_c S_\chi^{ord}(U(\mathfrak{l}^{c,c}), K/\mathcal{O})$$

and

$$S_\chi(U(\mathfrak{l}^\infty), \mathcal{O}) = S_\chi(U(\mathfrak{l}^\infty), K/\mathcal{O})^\vee = \mathrm{Hom}_{\mathcal{O}}(S_\chi(U(\mathfrak{l}^\infty), K/\mathcal{O}), K/\mathcal{O}).$$

Let Λ' denote the completed group ring of the group $T_n(\mathcal{O}_{L^+,l})$. Via the Artin map, there is a canonical quotient map $\Lambda' \to \Lambda = \widehat{\otimes}_{v \in S_l} \mathcal{O}[\![I_{L_{\widetilde{v}}}^{\mathrm{ab}}(l)]\!]$. On the other hand, there is a canonical splitting $T_n(\mathcal{O}_{L^+,l}) \cong \prod_{v \in S_l} T_n(\mathcal{O}_{L_{\widetilde{v}}})(l) \times \prod_{v \in S_l} T_n(k(\widetilde{v}))$, which makes Λ' into an augmented Λ-algebra. Define a homomorphism $T_n(\mathcal{O}_{L^+,l}) \to \mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})^\times$ by the formula

$$(u_{\widetilde{v}})_{v \in S_l} \mapsto \prod_{v \in S_l} \langle u_{\widetilde{v}} \rangle \cdot \prod_{\tau \in \widetilde{I}_l} \prod_{i=1}^n \tau(u_{\widetilde{v}(\tau),i})^{1-i},$$

where we write $u = (u_1, \ldots, u_n)$ for a typical element of T_n, and $\widetilde{v}(\tau) \in \widetilde{S}_l$ denotes the place of L induced by an embedding $\tau \in \widetilde{I}_l$. This choice defines a homomorphism $\Lambda' \to \mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})$. We will always view the Hecke algebra $\mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})$ as endowed with this structure of Λ'-algebra (hence Λ-algebra).

Proposition 4.3. *Suppose that* U *is sufficiently small. Then* $\mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})$ *is a finite faithful* Λ*-algebra, and* $S_\chi(U(\mathfrak{l}^\infty), \mathcal{O})$ *is a faithful* $\mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})$*-module which is finite free over* Λ *with the induced* Λ*-module structure.*

Proof. This is proved exactly as in [Ger, Proposition 2.5.3] and [Ger, Corollary 2.5.4]. ☐

Proposition 4.4. *Let $\pi = \otimes'_w \pi_w$ be a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$ of weight $\lambda = 0$. Suppose that for each finite place w of L which is inert over L^+, π_w is unramified, and for each place w of L lying above a place of $S(B)$, π_w is an unramified twist of the Steinberg representation. Then:*

1. *There exists an automorphic representation $\sigma = \otimes'_v \sigma_v$ of $G(\mathbb{A}_{L^+})$ satisfying the following conditions:*
   - *(a) σ_∞ is the trivial representation.*
   - *(b) For each finite place v of L^+ which is inert in L, σ_v has a fixed vector under a hyperspecial maximal compact subgroup of L^+.*
   - *(c) For each finite place $v \notin S(B)$ of L^+ which is split as $v = ww^c$ in L, $\sigma_w \cong \pi_w \circ \iota_w^{-1}$.*
2. *Let σ be as in part 1, and let $U = \prod_v U_v \subset G(\mathbb{A}^\infty_{L^+})$ be an open compact subgroup such that for all finite places v of L^+ inert in L, U_v is a hyperspecial maximal compact subgroup. Then there exists an automorphic representation σ' of $G(\mathbb{A}_{L^+})$ satisfying the following conditions:*
   - *(a) σ'_∞ is the trivial representation.*
   - *(b) For each finite place v of L^+ which is inert in L, σ'_v has a fixed vector under U_v.*
   - *(c) For each finite place $v \notin S(B)$ of L^+ which is split as $v = ww^c$ in L, $\sigma'_w \cong \pi_w \circ \iota_w^{-1} (\cong \sigma_w)$.*

Proof. The first part is contained in [CHT08, Proposition 3.3.2]. In fact, essentially the same argument also gives the second part. Let G^* denote the quasi-split inner form of G, and let R denote the right regular representation of $G(\mathbb{A}_{L^+})$ on $L^2(G(L^+)\backslash G(\mathbb{A}_{L^+}))$. Let $f^\infty \in C_c^\infty(G(\mathbb{A}^\infty_{L^+}))$ be a locally constant function of compact support, and let $f_\infty \in C_c^\infty(G(L^+ \otimes_{\mathbb{Q}} \mathbb{R}))$ be the constant function 1. Let $f = f^\infty \otimes f_\infty \in C_c^\infty(G(\mathbb{A}_{L^+}))$. According to the proof of [Lab99, Theorem A.3.1], there is an identity (for an appropriate choice of measures)

$$\operatorname{tr} R(f) = \mathrm{ST}_e^{G^*}(f^{G^*}),$$

where $f^{G^*} \in C_c^\infty(G^*(\mathbb{A}_{L^+}))$ is a transfer of f. (We refer to *loc. cit.* for the terms which are not defined here.) In particular, the distribution $f^\infty \mapsto \operatorname{tr} R(f)$ is stable, and the existence of σ' follows from that of σ. ☐

4.2 Galois representations

Theorem 4.5. *Let $\pi = \otimes'_v \pi_v$ be an irreducible $G(\mathbb{A}^{\infty, R}_{L^+}) \times \prod_{v \in R} \iota_{\widetilde{v}}^{-1} \mathrm{Iw}(\widetilde{v})$-submodule of the space of automorphic forms $S_\chi(\{1\}, \overline{\mathbb{Q}}_l)$. Then there exists a continuous semisimple representation $r_l(\pi) : G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ with the following properties:*

1. *$r_l(\pi)^c \cong r_l(\pi)^\vee \epsilon^{1-n}$.*
2. *If $v \in S_l$ then $r_l(\pi)|_{G_{L_{\widetilde{v}}}}$ is de Rham and for each $\tau \in \widetilde{I}_l$, we have $\mathrm{HT}_\tau(r_l(\pi)) = \{0, 1, \ldots, n-1\}$.*
3. *If $v \notin S(B) \cup R$ is a place of L^+ which splits as $v = ww^c$ in L, then*
$$\mathrm{WD}(r_l(\pi)|_{G_{L_w}})^{F\text{-}ss} \cong \mathrm{rec}^T_{L_w}(\pi_v \circ \iota_w^{-1}).$$
4. *If v is an inert place and π_v has a fixed vector for a hyperspecial maximal compact in $G(L_v^+)$ then $r_l(\pi)$ is unramified above v.*
5. *If $v \in R$ and $\pi_v^{\iota_{\widetilde{v}}^{-1} \mathrm{Iw}(\widetilde{v})} \neq 0$, then for every $\sigma \in I_{L_{\widetilde{v}}}$, we have*
$$\mathrm{char}_{r_l(\pi)(\sigma)}(X) = \prod_{j=1}^n \left(X - \chi_{v,j}^{-1}(\mathrm{Art}^{-1}_{L_{\widetilde{v}}}(\sigma)) \right).$$

Proof. This follows from [CHT08, Proposition 3.3.4], [CHT08, Lemma 3.1.6] and Theorem 2.2. $\square$

We record for later use the following consequence of the proof of [CHT08, Proposition 3.3.4] and [TY07, Theorem B].

Lemma 4.6. *Let π be an irreducible $G(\mathbb{A}_{L^+}^{\infty,R}) \times \prod_{v\in R} \iota_{\widetilde{v}}^{-1} \operatorname{Iw}(\widetilde{v})$-submodule of the space of automorphic forms $S_\chi(\{1\}, \overline{\mathbb{Q}}_l)$. Then exactly one of the following holds.*

1. *There exists a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_L)$, an isomorphism $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$, and an isomorphism $r_l(\pi) \cong r_\iota(\Pi)$. Moreover, Π satisfies the following properties:*
    - *(a) If v is a finite place of L^+ inert in L, and π_v has a fixed vector for a hyperspecial maximal compact subgroup of $G(L_v^+)$, then Π_v is unramified.*
    - *(b) If $v \notin S(B) \cup R$ is a finite place of L^+ split in L as $v = ww^c$, then $\Pi_w \cong \pi_v \circ \iota_w^{-1}$.*
    - *(c) If $v \in S(B)$ and $\pi_v^{G(\mathcal{O}_{L_v^+})} \neq 0$, then $\Pi_{\widetilde{v}}$ is an unramified twist of the Steinberg representation.*

    Finally, $r_l(\pi)$ is irreducible.

2. *There exists an integer $m > 1$ with $m|n$, a continuous representation $\rho : G_L \to \mathrm{GL}_{n/m}(\overline{\mathbb{Q}}_l)$ with $\rho^c \cong \rho^\vee \epsilon^{m-n}$, and an isomorphism $r_l(\pi) \cong \rho \oplus \rho \otimes \epsilon^{-1} \oplus \cdots \oplus \rho \otimes \epsilon^{1-m}$.*

Proposition 4.7. *Suppose that U is sufficiently small. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})$. Then, after possibly enlarging K, we can identify $\mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})/\mathfrak{m} = k$ and there is a continuous semisimple representation*

$$\overline{r}_\mathfrak{m} : G_L \to \mathrm{GL}_n(k),$$

unique up to isomorphism, such that:

1. *$\overline{r}_\mathfrak{m}^c \cong \overline{r}_\mathfrak{m}^\vee \epsilon^{1-n}$.*
2. *$\overline{r}_\mathfrak{m}$ is unramified outside T. For all $v \notin T$ splitting in L as $v = ww^c$, the characteristic polynomial of $\overline{r}_\mathfrak{m}(\mathrm{Frob}_w)$ is*

$$X^n + \cdots + (-1)^j (q_w)^{j(j-1)/2} T_w^j X^{n-j} + \cdots + (-1)^n (q_w)^{n(n-1)/2} T_w^n.$$

Proof. This is proved exactly as in [Ger, Proposition 2.7.3]. $\square$

We remark that the proof of [Ger, Proposition 2.7.3] shows that any maximal ideal $\mathfrak{m} \subset \mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})$ is the pullback of a maximal ideal $\mathfrak{m}_1 \subset \mathbb{T}_\chi^T(U(\mathfrak{l}^{1,1}), \mathcal{O})$. If $c \geq 1$ is an integer, then we define

$$\mathfrak{m}_c = \ker(\mathbb{T}_\chi^T(U(\mathfrak{l}^{c,c}), \mathcal{O}) \to \mathbb{T}_\chi^T(U(\mathfrak{l}^{1,1}), \mathcal{O}) \to \mathbb{T}_\chi^T(U(\mathfrak{l}^{1,1}), \mathcal{O})/\mathfrak{m}_1.$$

Definition 4.8. *We say that a maximal ideal $\mathfrak{m}$ as above is* residually Schur *if writing $\overline{r}_\mathfrak{m} = \oplus_{i=1}^s \overline{\rho}_i$ as a sum of irreducible subrepresentations, each $\overline{\rho}_i$ is absolutely irreducible and for each $1 \leq i, j \leq s$, $i \neq j$, we have $\overline{\rho}_i \ncong \overline{\rho}_j$ and $\overline{\rho}_i^c \ncong \overline{\rho}_j^\vee \otimes \epsilon^{1-n}$.*

Suppose that π is an irreducible $G(\mathbb{A}_{L^+}^{\infty,R}) \times \prod_{v\in R} \iota_{\widetilde{v}}^{-1} \operatorname{Iw}(\widetilde{v})$-submodule of $S_\chi(\{1\}, \overline{\mathbb{Q}}_l)$ such that $\pi \cap S_\chi^{\mathrm{ord}}(U(\mathfrak{l}^{c,c}), \overline{\mathbb{Q}}_l) \neq 0$ for some $c \geq 1$. We can then associate to π a maximal ideal $\mathfrak{m}$ of the algebra $\mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})$. Indeed, the Hecke eigenvalues of π give rise to a homomorphism $\mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O}) \to \overline{\mathbb{Z}}_l \to \overline{\mathbb{F}}_l$, and we take $\mathfrak{m}$ to be the kernel of this homomorphism. If this maximal ideal $\mathfrak{m}$ is residually Schur, then the first alternative of Lemma 4.6 holds for π.

Proposition 4.9. *Let $\mathfrak{m} \subset \mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})$ be a residually Schur maximal ideal. Then, after possibly enlarging K, $\overline{r}_\mathfrak{m}$ admits an extension to a continuous homomorphism*

$$\overline{r}_\mathfrak{m} : G_{L^+} \to \mathcal{G}_n(k),$$

with the property that $\overline{r}_\mathfrak{m}^{-1}(\mathrm{GL}_n \times \mathrm{GL}_1(k)) = G_L$ and $\nu \circ \overline{r}_\mathfrak{m} = \epsilon^{1-n} \delta_{L/L^+}^n$. Moreover, this extension is Schur.

Proof. Choose an irreducible submodule π of $S_\chi(\{1\}, \overline{\mathbb{Q}}_l)$ such that $r_l(\pi)$ has reduction equal to $\overline{r}_{\mathfrak{m}}$. By [CHT08, Lemma 2.1.5], after possibly extending K we can assume that $r_l(\pi)$ is valued in $\mathrm{GL}_n(\mathcal{O})$ and find an extension to a representation $r_l(\pi) : G_{F^+,S} \to \mathcal{G}_n(\mathcal{O})$. By the main result of [BC11], we have $\nu \circ r_l(\pi) = \epsilon^{1-n}\delta^n_{L/L^+}$. It now suffices to take the reduction mod λ of this extension. The representation obtained in this way is Schur, by Lemma 3.4. $\square$

Proposition 4.10. *Suppose that U is sufficiently small, and let $\mathfrak{m} \subset \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})$ be a residually Schur maximal ideal. Fix $c \geq 1$, and let $\mathfrak{p} \subset \mathfrak{m}_c$ be a minimal prime. Then there exists a finite field extension $E_\mathfrak{p}$ of the fraction field of $\mathbb{T}^T_\chi(U(\mathfrak{l}^{c,c}), \mathcal{O})/\mathfrak{p}$ with ring of integers $\mathcal{O}_\mathfrak{p}$ and a continuous representation $r_\mathfrak{p} : G_{L^+,T} \to \mathcal{G}_n(\mathcal{O}_\mathfrak{p})$ such that $r_\mathfrak{p}$ mod $\mathfrak{m}_{\mathcal{O}_\mathfrak{p}} = \overline{r}_\mathfrak{m}$ and satisfying the following: for any embedding $E_\mathfrak{p} \hookrightarrow \overline{\mathbb{Q}}_l$, there is an irreducible $G(\mathbb{A}^{\infty,R}_{L^+}) \times \prod_{v\in R} \iota^{-1}_{\widetilde{v}} \mathrm{Iw}(\widetilde{v})$-submodule of the space $S_\chi(\{1\}, \overline{\mathbb{Q}}_l)$ inducing the homomorphism $\mathbb{T}^T_\chi(U(\mathfrak{l}^{c,c}), \mathcal{O}) \to \overline{\mathbb{Q}}_l$, and such that $r_\mathfrak{p}|_{G_{L,T}} \otimes_{\mathcal{O}_\mathfrak{p}} \overline{\mathbb{Q}}_l \cong r_l(\pi)$.*

Proof. This follows the existence of $r_l(\pi)$ and [CHT08, Lemma 2.1.5]. $\square$

In contrast to the residually irreducible case, $\overline{r}_\mathfrak{m}$ need not admit a lifting to $\mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m}$. The best we can hope for is the following.

Proposition 4.11. *Suppose that U is sufficiently small, and let $\mathfrak{m} \subset \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})$ be a residually Schur maximal ideal. Define a group determinant over k by $\overline{D}_\mathfrak{m} = \det \circ \overline{r}_\mathfrak{m}|_{G_{L,S}}$. Then $\overline{D}_\mathfrak{m}$ admits a unique lifting to a pseudodeformation*

$$D_\mathfrak{m} : G_{L,T} \to \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m}$$

satisfying the following property: for all $v \notin T$ splitting in L as $v = ww^c$, the characteristic polynomial of $D_\mathfrak{m}(\mathrm{Frob}_w)$ is

$$X^n + \cdots + (-1)^j (q_w)^{j(j-1)/2} T^j_w X^{n-j} + \cdots + (-1)^n (q_w)^{n(n-1)/2} T^n_w.$$

Proof. It suffices to construct such a lifting to $D_{\mathfrak{m}_c} : G_L \to \mathbb{T}^T_\chi(U(\mathfrak{l}^{c,c}), \mathcal{O})_{\mathfrak{m}_c}$, as the uniqueness ensures that we can then pass to the limit. Let $A \subset k \oplus \bigoplus_{\mathfrak{p}\supset\mathfrak{m}_c} \mathcal{O}_\mathfrak{p}$ denote the subring of elements $(x, (y_\mathfrak{p}))$ such that $x \equiv y_\mathfrak{p}$ mod $\mathfrak{m}_c$ for each $\mathfrak{p}$. By the previous proposition, there exists a continuous representation $r_c = \oplus_\mathfrak{p} r_\mathfrak{p} : G_{L^+,T} \to \mathcal{G}_n(A)$, and we now take $D_{\mathfrak{m}_c} = \det \circ r_c|_{G_{L,T}}$. By the second part of Proposition 3.26, $D_{\mathfrak{m}_c}$ is actually valued in $\mathbb{T}^T_\chi(U(\mathfrak{l}^{c,c}), \mathcal{O})_{\mathfrak{m}_c} \subset A$. It is now easy to see that $D_{\mathfrak{m}_c}$ satisfies the required properties. $\square$

4.3 Deformation rings and Hecke algebras

We now specialize to the following situation. Let $S = T = S_l \cup R \cup S(B) \cup S_a$, and that $U = \prod_v U_v$ has the following form:

- For v inert in L, $U_v \subset G(L^+_v)$ is a hyperspecial maximal compact subgroup.
- For $v \notin T$ split in L, $U_v = G(\mathcal{O}_{L^+_v})$.
- For $v \in S(B), U_v$ is a maximal compact subgroup.
- For $v \in S_l, U_v = G(\mathcal{O}_{L^+_v})$.
- For $v \in R, U_v = \iota^{-1}_{\widetilde{v}} \mathrm{Iw}(\widetilde{v})$.
- For $v \in S_a, U_v = \iota^{-1}_{\widetilde{v}} \ker(\mathrm{GL}_n(\mathcal{O}_{L_{\widetilde{v}}}) \to \mathrm{GL}_n(k(\widetilde{v})))$.

We suppose that S_a is non-empty and that for every $v \in S_a$, v is absolutely unramified, $\overline{r}_\mathfrak{m}$ is unramified above v, $\mathrm{ad}\, \overline{r}(\mathrm{Frob}_v) = 1$ and v does not split in $L(\zeta_l)$. Then $H^0(G_{L_{\widetilde{v}}}, \mathrm{ad}\, \overline{r}(1)) = 0$, and U is sufficiently small. We leave the characters χ_v for $v \in R$ unspecified, but suppose that they are trivial mod λ. We fix a residually Schur maximal ideal $\mathfrak{m} \subset \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})$.

Suppose also that for each $v \in S_l \cup R \cup S(B)$, $\overline{r}_{\mathfrak{m}}|_{G_{L_{\widetilde{v}}}}$ is trivial and that for each $v \in R \cup S(B)$, $q_v \equiv 1 \bmod l$. Under these assumptions we can define a global deformation problem (using the local deformation problems defined in §3):

$$\mathcal{S}_\chi = \left(L/L^+, S, \widetilde{S}, \Lambda, \overline{r}_{\mathfrak{m}}, \epsilon^{1-n}\delta^n_{L/L^+}, \{R_v^{\triangle}\}_{v \in S_l} \cup \{R_v^{\chi_v}\}_{v \in R} \cup \{R_v^{\mathrm{St}}\}_{v \in S(B)} \cup \{R_v^{\square}\}_{v \in S_a} \right).$$

Because $\overline{r}_{\mathfrak{m}}|_{G_{L_{\widetilde{v}}}}$ is trivial for each $v \in S_l$, it follows from Theorem 4.5 and Theorem 2.4 that the natural homomorphism $\Lambda' \to \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}$ factors through its quotient Λ. Moreover, the ring $\mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}$ is topologically generated as a Λ-algebra by the unramified Hecke operators T_w^j at the places $w \notin L$ which are split over L^+.

Proposition 4.12. *The natural map $Q_S \widehat{\otimes}_{\mathcal{O}} \Lambda \to \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}$ is surjective and factors through the quotient $Q_S \widehat{\otimes}_{\mathcal{O}} \Lambda \to P_{\mathcal{S}_\chi}$.*

Proof. The map is surjective since since $\mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}$ is topologically generated as a Λ-algebra by the coefficients of the characteristic polynomial of the group determinant $D_{\mathfrak{m}}$, evaluated at the Frobenius elements at places of L split over L^+. Let $I = \ker(Q_S \widehat{\otimes}_{\mathcal{O}} \Lambda \to R^{\mathrm{univ}}_{\mathcal{S}_\chi})$, and let $J = \ker(Q_S \widehat{\otimes}_{\mathcal{O}} \Lambda \to \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}})$. To complete the proof of the proposition, we must show that $I \subset J$. We claim that there exists a Zariski dense set Y of maximal ideals $\mathfrak{p} \subset \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}[1/l]$ with the following property:

- For each $\mathfrak{p} \in Y$, there is a finite extension $E_{\mathfrak{p}}$ of $\mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}[1/l]/\mathfrak{p}$ with ring of integers $\mathcal{O}_{\mathfrak{p}}$ and a lifting $r_{\mathfrak{p}} : G_{L^+,S} \to \mathcal{G}_n(\mathcal{O}^0_{\mathfrak{p}})$, where $\mathcal{O}^0_{\mathfrak{p}} \subset \mathcal{O}_{\mathfrak{p}}$ is the subring of elements whose image in the residue field lies in k, such that $r_{\mathfrak{p}}$ is of type $\mathcal{S}_\chi$ and the following diagram commutes:

$$\begin{tikzcd}
Q_S \widehat{\otimes}_{\mathcal{O}} \Lambda \arrow[r] \arrow[d] & \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}} \arrow[d] \\
R^{\mathrm{univ}}_{\mathcal{S}_\chi} \arrow[r] & \mathcal{O}^0_{\mathfrak{p}}.
\end{tikzcd}$$

The claim implies the proposition. Indeed, if $\mathfrak{p} \in Y$ then after identifying $\mathfrak{p}$ with a maximal ideal of $Q_S \widehat{\otimes}_{\mathcal{O}} \Lambda[1/l]$, we have $I[1/l] \subset \mathfrak{p}$. Since $\mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}[1/l]$ is reduced, it follows from this and the fact that Y is Zariski dense that $I[1/l] \subset J[1/l]$. Since $\mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}$ is $\mathcal{O}$-flat, it follows that $I \subset J$.

We now prove the claim. Let $X \subset \operatorname{Spec} \Lambda[1/l]$ be the set of maximal ideals corresponding to tuples of characters

$$\varphi_1^v, \varphi_2^v \epsilon^{-1}, \ldots, \varphi_n^v \epsilon^{1-n} : I^{\mathrm{ab}}_{L_{\widetilde{v}}}(l) \to \overline{\mathbb{Q}}_l^\times, \ v \in S_l,$$

where each $\varphi_i^v : I^{\mathrm{ab}}_{L_{\widetilde{v}}} \to \overline{\mathbb{Q}}_l^\times$ is of finite order. Let Y denote the pre-image of X in $\operatorname{Spec} \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}[1/l]$. Then X is Zariski dense in $\operatorname{Spec} \Lambda[1/l]$, so Y is Zariski dense in $\operatorname{Spec} \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}[1/l]$, by Proposition 4.3. Furthermore, if $\mathfrak{p} \in Y$ lies above a point $\mathfrak{q} \in X$ corresponding to a tuple of characters $(\varphi_v^i)_{v \in S_l, i=1,\ldots,n}$ which are all of conductor at most c, for some $c \geq 1$, then there exists an irreducible $G(\mathbb{A}^{\infty,R}_{L^+}) \times \prod_{v \in R} \iota_{\widetilde{v}}^{-1} \operatorname{Iw}(\widetilde{v})$-submodule π of $S_\chi(\{1\}, \overline{\mathbb{Q}}_l)$ such that $\pi \cap S_\chi^{\mathrm{ord}}(U(\mathfrak{l}^{c,c}), \mathcal{O})_{\mathfrak{m}} \neq 0$, and $\mathfrak{p}$ is the kernel of the homomorphism

$$\mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}[1/l] \to \mathbb{T}^T_\chi(U(\mathfrak{l}^{c,c}), \mathcal{O})_{\mathfrak{m}}[1/l] \to \overline{\mathbb{Q}}_l$$

associated to the Hecke eigenvalues of π. The existence of the Galois representation $r_{\mathfrak{p}}$ follows from Proposition 4.10. The fact that it is of type $\mathcal{S}_\chi$ follows from local-global compatibility (cf. Theorem 4.5 and Theorem 2.4). $\square$

Let us write Δ for the Galois group of the maximal abelian pro-l extension of L, unramified outside l. If $t \geq 1$ is an integer we write Δ_t for its quotient, the Galois group of the maximal abelian pro-l extension of L, unramified outside l and of conductor t at each place of L above l.

Proposition 4.13. *There is a commutative diagram of* Λ*-algebras*

$$\begin{tikzcd}
R^{univ}_{\mathcal{S}_\chi} \arrow[d] & P_{\mathcal{S}_\chi} \arrow[l] \arrow[r] \arrow[d] & \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m} \arrow[d] \\
R^{univ}_{\mathcal{S}_\chi} \widehat{\otimes}_\mathcal{O} \mathcal{O}[\![\Delta/(c+1)]\!] & P_{\mathcal{S}_\chi} \widehat{\otimes}_\mathcal{O} \mathcal{O}[\![\Delta/(c+1)]\!] \arrow[l] \arrow[r] & \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m} \widehat{\otimes}_\mathcal{O} \mathcal{O}[\![\Delta/(c+1)]\!],
\end{tikzcd}$$

which extends the diagram of Lemma 3.35.

Proof. Let $t \geq 1$ be an integer. We construct maps $\mathbb{T}^T_\chi(U(\mathfrak{l}^{b,b}), \mathcal{O})_{\mathfrak{m}_b} \to \mathbb{T}^T_\chi(U(\mathfrak{l}^{b,b}), \mathcal{O})_{\mathfrak{m}_b} \otimes_\mathcal{O} \mathcal{O}[\Delta_t/(c+1)]$ for each $b \geq t$. It will then be clear from the construction that these maps fit into an inverse system, and passing to the limit with respect to b and t gives the diagram of the proposition. We therefore fix a choice of integer $b \geq t$ for the remainder of the proof.

We will give another construction of $\mathbb{T}^T_\chi(U(\mathfrak{l}^{b,b}), \mathcal{O})_{\mathfrak{m}_b}$. Fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$. Let P_b denote the finite set of RACSDC automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_L)$ satisfying the following conditions:

- π is ι-ordinary of weight $\lambda = 0$, and there is an isomorphism of residual representations $\overline{r_\iota(\pi)}^{\mathrm{ss}} \cong \overline{r}_\mathfrak{m}$.
- If v is a finite place of L^+ inert in L, then π_v is unramified.
- If $v = ww^c$ is a finite place of L^+ which splits in L and not lying in $S(B) \cup S_l \cup R$ then $\pi_w^{\iota_w U_v} \neq 0$.
- If $v \in S(B)$, then $\pi_{\widetilde{v}}$ is an unramified twist of the Steinberg representation
- If $v \in S_l$, then the subspace of $\iota^{-1}\pi_{\widetilde{v}}^{\mathrm{Iw}_{\widetilde{v}}(b,b)}$ where each operator $\iota_{\widetilde{v}} U^j_{\widetilde{v}}$, $j = 1, \ldots, n$ acts with eigenvalues which are l-adic units is non-zero.
- If $v \in R$, then $\pi_{\widetilde{v}}$ is a subquotient of a normalized induction $\text{n-Ind}_B^{\mathrm{GL}_n(L_{\widetilde{v}})} \widetilde{\chi}_{v,1} \otimes \ldots \widetilde{\chi}_{v,n}$, where each $\widetilde{\chi}_{v,i} : L_{\widetilde{v}}^\times \to \mathbb{C}^\times$ is a smooth character satisfying $\widetilde{\chi}_{v,i}|_{\mathcal{O}^\times_{L_{\widetilde{v}}}} = \iota\chi^{-1}_{v,i}$.

We make the following observations, which follow from [CHT08, Proposition 3.3.2] and Proposition 4.4:

- If $\pi \in P_b$, then there is an irreducible $G(\mathbb{A}^{\infty,R}_{L^+}) \times \prod_{v \in R} \iota_{\widetilde{v}}^{-1} \mathrm{Iw}(\widetilde{v})$-submodule σ of the space of automorphic forms $S_\chi(\{1\}, \overline{\mathbb{Q}}_l)$ such that $r_\iota(\pi) \cong r_l(\sigma)$, and $\sigma \cap S^{\mathrm{ord}}_\chi(U(\mathfrak{l}^{b,b}), \mathcal{O})_\mathfrak{m} \neq 0$.
- Conversely, if σ is an irreducible $G(\mathbb{A}^{\infty,R}_{L^+}) \times \prod_{v \in R} \iota_{\widetilde{v}}^{-1} \mathrm{Iw}(\widetilde{v})$-submodule of the space of automorphic forms $S_\chi(\{1\}, \overline{\mathbb{Q}}_l)$ and $\sigma \cap S^{\mathrm{ord}}_\chi(U(\mathfrak{l}^{b,b}), \mathcal{O})_\mathfrak{m} \neq 0$, then there exists a representation $\pi \in P_b$ such that $r_\iota(\pi) \cong r_l(\sigma)$, and π is then unique.

Let S_b denote the set of representations σ of $G(\mathbb{A}^{\infty,R}_{L^+}) \times \prod_{v \in R} \iota_{\widetilde{v}}^{-1} \mathrm{Iw}(\widetilde{v})$ satisfying the second point. Then there is a surjective map $S_b \to P_b$, and if $\sigma \mapsto \pi$ and $v = ww^c$ is a place of L^+ split in L, not lying in $S(B)$, then there is an isomorphism $\iota^{-1}\pi_w \cong \sigma_v \circ \iota_w^{-1}$.

It follows that $\mathbb{T}^T_\chi(U(\mathfrak{l}^{b,b}), \mathcal{O})_{\mathfrak{m}_b}$ can be identified with the $\mathcal{O}$-subalgebra of $\prod_{\pi \in P_b} \overline{\mathbb{Q}}_l$ generated by the unramified Hecke operators $\iota_w T^j_w$, $j = 1, \ldots, n$ and w a place of F split over F^+ and not dividing T, and the diamond operators $\iota_{\widetilde{v}}\langle u \rangle$ for $v \in S_l$, $u \in T_n(\mathcal{O}_{L_{\widetilde{v}}})(l)$.

Now suppose given a character $\psi : \Delta_t/(c+1) \to \overline{\mathbb{Q}}_l^\times$. If $\pi \in P_b$ then $\pi \otimes \iota\psi \in P_b$, and we write $f_\psi : \prod_{\pi \in P_b} \overline{\mathbb{Q}}_l \to \prod_{\pi \in P_b} \overline{\mathbb{Q}}_l$ for the automorphism induced by the permutation $\pi \mapsto \pi \otimes \iota\psi$ of the finite set P_b. An easy calculation shows that if ψ takes values in $\mathcal{O}^\times$ then f_ψ takes the generators of $\mathbb{T}^T_\chi(U(\mathfrak{l}^{b,b}), \mathcal{O})_{\mathfrak{m}_b}$ to $\mathcal{O}^\times$-multiples of themselves, and so induces an automorphism of this Hecke algebra. In any case, the product of these homomorphisms over all characters $\psi : \Delta_t/(c+1) \to \overline{\mathbb{Q}}_l^\times$ gives a homomorphism

$$\mathbb{T}^T_\chi(U(\mathfrak{l}^{b,b}), \mathcal{O})_{\mathfrak{m}_b} \to \prod_{\psi : \Delta_t/(c+1) \to \overline{\mathbb{Q}}_l^\times} \mathbb{T}^T_\chi(U(\mathfrak{l}^{b,b}), \mathcal{O})_{\mathfrak{m}_b} \otimes_\mathcal{O} \overline{\mathbb{Q}}_l \cong \mathbb{T}^T_\chi(U(\mathfrak{l}^{b,b}), \mathcal{O})_{\mathfrak{m}_b} \otimes_\mathcal{O} \overline{\mathbb{Q}}_l[\Delta_t/(c+1)]$$

which actually takes values in the subring $\mathbb{T}^T_\chi(U(\mathfrak{l}^{b,b}), \mathcal{O})_{\mathfrak{m}_b} \otimes_\mathcal{O} \mathcal{O}[\Delta_t/(c+1)]$. Under this homomorphism, the Hecke operator T^j_w is mapped to $T^j_w \otimes \mathrm{Frob}^j_w$. This concludes the proof. □

Corollary 4.14. *Let $\mathfrak{p} \subset R^{univ}_{\mathcal{S}_\chi}$ be a dimension one prime of characteristic l, and let* $J_{\mathcal{S}_\chi} = \ker(P_{\mathcal{S}_\chi} \to \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m}$*. Let A denote the normalization of $R^{univ}_{\mathcal{S}_\chi}/\mathfrak{p}$ in its fraction field E, and let $\psi : \Delta \to 1 + \mathfrak{m}_A \subset A^\times$ be a continuous character such that $\psi\psi^c = 1$. Let $\mathfrak{p}_\psi$ denote the ideal obtained by twisting $\mathfrak{p}$ by ψ* (*see Lemma 3.38*)*. Suppose that $J_{\mathcal{S}_\chi} R^{univ}_{\mathcal{S}_\chi} \subset \mathfrak{p}$. Then $J_{\mathcal{S}_\chi} R^{univ}_{\mathcal{S}_\chi} \subset \mathfrak{p}_\psi$.*

Proof. The ideal $\mathfrak{p}_\psi$ is the kernel of the composite homomorphism

$$R^{\text{univ}}_{\mathcal{S}} \to R^{\text{univ}}_{\mathcal{S}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Delta/(c+1)]\!] \to A,$$

the homomorphism $\mathcal{O}[\![\Delta/(c+1)]\!] \to A$ being induced by the character ψ. The previous proposition now implies that $J_{\mathcal{S}_\chi}$ is mapped to zero by this homomorphism. $\square$

4.4 Auxiliary levels

We continue with the assumptions of the previous section. Let q, N be positive integers and fix a choice of Taylor–Wiles data

$$(Q_N, \widetilde{Q}_N, \{\overline{\alpha}_v\}_{v \in Q_N})$$

of order q and level N, where $\overline{\alpha}_v$ is an eigenvalue of $\overline{r}_\mathfrak{m}(\text{Frob}_{\widetilde{v}})$ of multiplicity n_v. We assume that for each $v \in Q_N$, l does not divide n_v. A choice of Taylor–Wiles data having been fixed, we have defined an auxiliary deformation problem (cf. the discussion preceding Lemma 3.19)

$$\mathcal{S}_{\chi,N} = \Big(L/L^+, S_N, \widetilde{S}_N, \Lambda, \overline{r}_\mathfrak{m}, \epsilon^{1-n}\delta^n_{L/L^+},$$
$$\{R^\triangle_v\}_{v \in S_l} \cup \{R^{\chi_v}_v\}_{v \in R} \cup \{R^{\text{St}}_v\}_{v \in S(B)} \cup \{R^\square_v\}_{v \in S_a} \cup \{\mathcal{D}^{\text{TW}}_v(\overline{\alpha}_v)\}_{v \in Q_N}\Big).$$

Let Δ_N denote the maximal l-power order quotient of $\prod_{v \in Q_N} k(\widetilde{v})^\times$. For $v \in Q_N$ we let $\mathfrak{p}^{\widetilde{v}}_N \subset \text{GL}_n(\mathcal{O}_{L_{\widetilde{v}}})$ denote the standard parahoric subgroup corresponding to the partition $n = (n - n_v) + n_v$, and $\mathfrak{p}^{\widetilde{v}}_{N,1}$ denote the kernel of the homomorphism

$$\mathfrak{p}^{\widetilde{v}}_N \longrightarrow \text{GL}_{n_v}(\mathcal{O}_{L_{\widetilde{v}}}) \xrightarrow{\det} k(\widetilde{v})^\times \longrightarrow k(\widetilde{v})^\times(l).$$

Thus $\prod_{v \in Q_N} \mathfrak{p}^{\widetilde{v}}_N / \mathfrak{p}^{\widetilde{v}}_{N,1} \cong \Delta_N$. Finally we set $U_0(Q_N) = \prod_v U_0(Q_N)_v$, where $U_0(Q_N)_v = U_v$ if $v \notin Q_N$ and $U_0(Q_N)_v = \iota^{-1}_{\widetilde{v}} \mathfrak{p}^{\widetilde{v}}_N$ if $v \in Q_N$. We set $U_1(Q_N) = \prod_v U_1(Q_N)_v$, where $U_1(Q_N)_v = U_v$ if $v \notin Q_N$ and $U_1(Q_N)_v = \iota^{-1}_{\widetilde{v}} \mathfrak{p}^{\widetilde{v}}_{N,1}$ if $v \in Q_N$.

We have natural maps of Λ-algebras

$$\mathbb{T}^{T \cup Q_N}_\chi(U_1(Q_N)(\mathfrak{l}^\infty), \mathcal{O}) \to \mathbb{T}^{T \cup Q_N}_\chi(U_0(Q_N)(\mathfrak{l}^\infty), \mathcal{O}) \to \mathbb{T}^{T \cup Q_N}_\chi(U(\mathfrak{l}^\infty), \mathcal{O}) \to \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O}).$$

The first two maps are surjective, the third is injective. In an abuse of notation, we write $\mathfrak{m}$ for the pullback of the maximal ideal $\mathfrak{m}$ to each of these algebras. After localizing at $\mathfrak{m}$ the third map becomes an isomorphism, since each localized Hecke algebra can be viewed as the image of the universal pseudo-deformation ring $P_{\mathcal{S}_{\chi,N}}$.

Fix for each $v \in Q_N$ a choice of uniformizer $\varpi_{\widetilde{v}}$ of $\mathcal{O}_{L_{\widetilde{v}}}$. In [Tho12, §5] we have constructed operators $\text{pr} = \prod_{v \in Q_N} \text{pr}_{\varpi_{\widetilde{v}}}$ which act on the spaces $S_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m}$, $S_\chi(U_0(Q_N)(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m}$, and $S_\chi(U_1(Q_N)(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m}$ compatibly with all maps and in a manner commuting with the action of the Hecke operators at primes not in Q_N. See in particular the proof of [Tho12, Theorem 8.6]. We set

$$H_\chi = S_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m} = (S_\chi(U(\mathfrak{l}^\infty), K/\mathcal{O})_\mathfrak{m})^\vee,$$

$$H_{\chi,N,0} = (\text{pr}\, S_\chi(U_0(Q_N)(\mathfrak{l}^\infty), K/\mathcal{O})_\mathfrak{m})^\vee,$$

$$H_{\chi,N,1} = (\text{pr}\, S_\chi(U_1(Q_N)(\mathfrak{l}^\infty), K/\mathcal{O})_\mathfrak{m})^\vee.$$

For $v \in Q_N$, $\alpha \in k(\widetilde{v})^\times$, we define a Hecke operator V_α on $S_\chi(U_1(Q_N)(\mathfrak{l}^\infty), \mathcal{O})^\vee$ as follows. Let $\widetilde{\alpha} \in \mathcal{O}_{L_{\widetilde{v}}}$ be any lift of α. Then we define

$$V_\alpha = \iota_{\widetilde{v}}^{-1}\left(\left[\mathfrak{p}_{N,1}^{\widetilde{v}} \begin{pmatrix} 1_{n-n_v} & 0 \\ 0 & \mathrm{diag}(\widetilde{\alpha}, 1, \dots, 1) \end{pmatrix} \mathfrak{p}_{N,1}^{\widetilde{v}}\right]\right).$$

This definition is independent of the choice of $\widetilde{\alpha}$. This operator preserves the spaces $H_{\chi,N,1}$. For $\alpha = (\alpha_i) \in \Delta_N$, we set $V_\alpha = \prod_i V_{\alpha_i}$. The assignment $\alpha \mapsto V_\alpha$ defines an action of the group Δ_N on the spaces $H_{\chi,N,1}$.

Theorem 4.15. *Let $\mathbb{T}_\chi = \mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m}$ and let $\mathbb{T}_{\chi,N,i}$ denote the image of the map*

$$\mathbb{T}_\chi^{T \cup Q_N}(U_i(Q_N)(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m} \to \mathrm{End}_\Lambda(H_{\chi,N,i}).$$

1. *The natural homomorphism $\Lambda[\Delta_N] \to R^{univ}_{\mathcal{S}_{\chi,N}}$ factors through $P_{\mathcal{S}_{\chi,N}} \subset R^{univ}_{\mathcal{S}_{\chi,N}}$, and the pseudodeformation $D_\mathfrak{m} : G_{L,S \cup Q_N} \to \mathbb{T}_{\chi,N,1}$ constructed above induces a surjective map $P_{\mathcal{S}_{\chi,N}} \to \mathbb{T}_{\chi,N,1}$.*
2. *The operator* pr *induces an isomorphism $H_{\chi,N,0} \cong H_\chi$ of $\mathbb{T}_\chi^{T \cup Q_N}(U_0(Q_N)(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m}$-modules.*
3. *$H_{\chi,N,1}$ is free over $\Lambda[\Delta_N]$ and we have a canonical isomorphism*

   $$(H_{\chi,N,1})_{\Delta_N} \to H_{\chi,N,0}$$

   induced by restriction.
4. *The natural map $\mathbb{T}_{\chi,N,0} \to \mathbb{T}_\chi$ is an isomorphism.*
5. *The two structures on $H_{\chi,N,1}$ of $\Lambda[\Delta_N]$-module, induced by the Hecke operators V_α, $\alpha \in \Delta_N$, and the homomorphism $\Lambda[\Delta_N] \to P_{\mathcal{S}_{\chi,N}} \to \mathbb{T}_{\chi,N,1}$, are the same. In particular, for each $\alpha \in \Delta_N$, $V_\alpha \in \mathrm{End}_\Lambda(H_{\chi,N,1})$ actually lies in $\mathbb{T}_{\chi,N,1}$.*

Proof. For $v \in Q_N$, $\sigma \in I_{L_{\widetilde{v}}}$, the universal trace in $R^{\mathrm{univ}}_{\mathcal{S}_{\chi,N}}$ has the form $(n - n_v) + n_v \phi(\sigma)$, where $\phi : I_{L_{\widetilde{v}}} \to (R^{\mathrm{univ}}_{\mathcal{S}_{\chi,N}})^\times$ is the character through which, by construction, inertia at $\widetilde{v}$ acts on the universal deformation of type $\mathcal{S}_{\chi,N}$; see the proof of Lemma 3.19. Since we have assumed that l does not divide n_v, it follows that $\phi(\sigma) \in P_{\mathcal{S}_{\chi,N}}$. The surjectivity of the map $P_{\mathcal{S}_{\chi,N}} \to \mathbb{T}_{\chi,N,1}$ is proved in the same way as the first part of Proposition 4.12. This proves the first part of the theorem. The remainder of the theorem can be deduced from the finite level case just as in the proof of [Tho12, Theorem 8.6]. ☐

4.5 Soluble base change

We put ourselves in the setting of §4.3. In this section we suppose that M/L is a soluble CM extension, linearly disjoint over L from the extension of $L(\zeta_l)$ cut out by $\overline{r}_\mathfrak{m}|_{G_{L(\zeta_l)}}$, in which every prime above $S_l \cup S_a \cup R$ splits. (Thus M/L is a good extension, in the language of the proof of Theorem 6.1 below.) We write $T_M, S_{l,M}, R_M, S(B)_M, S_{a,M}$ for the sets of primes of M^+ above the sets $T, S_l, R, S(B)$, and S_a, respectively. We write $\widetilde{T}_M$ for the set of places of M above $\widetilde{T}$. If $v \in T_M$, then we write $\widetilde{v}$ for the unique place of $\widetilde{T}_M$ above v.

Let $\Lambda_M = \widehat{\otimes}_{v \in S_{l,M}} \mathcal{O}[\![I^{\mathrm{ab}}_{M_{\widetilde{v}}}(l)^n]\!]$ denote the Iwasawa algebra of M. There is a natural homomorphism $\Lambda_M \to \Lambda$, induced by the norm homomorphism $(\mathcal{O}_{M^+} \otimes \mathbb{Z}_l)^\times \to (\mathcal{O}_{L^+} \otimes \mathbb{Z}_l)^\times$. We will frequently use the following simple lemma.

Lemma 4.16. *Suppose that $A \in \mathcal{C}_\mathcal{O}$, and for each $v \in S_l$, suppose given continuous characters $\varphi_1^v, \dots, \varphi_n^v : I^{ab}_{L_{\widetilde{v}}}(l) \to A^\times$. Then the homomorphism $\Lambda_M \to A$ induced by the restrictions $\varphi_i^v|_{G_{M_{\widetilde{u}}}}$ for $u \in S_{l,M}$ dividing $v \in S_l$ factors through the homomorphism $\Lambda_M \to \Lambda$ above.*

Proof. We reduce immediately to the universal case $A = \Lambda$. In this case the lemma follows from the commutativity of the diagram

$$\begin{tikzcd}
M_{\widetilde{u}}^{\times} \arrow[r, "\mathrm{Art}_{M_{\widetilde{u}}}"] \arrow[d, "\mathbb{N}_{M_{\widetilde{u}}/L_{\widetilde{v}}}"'] & G_{M_{\widetilde{u}}}^{\mathrm{ab}} \arrow[d] \\
L_{\widetilde{v}}^{\times} \arrow[r, "\mathrm{Art}_{L_{\widetilde{v}}}"] & G_{L_{\widetilde{v}}}^{\mathrm{ab}}.
\end{tikzcd}$$

□

The characters $\chi_{v,i} : k(\widetilde{v})^{\times} \to \mathcal{O}^{\times}$ for $v \in R$ induce characters of the groups $k(\widetilde{v})^{\times}$ for $v \in R_M$. By abuse of notation, we denote this collection of characters for $v \in R_M$ by χ. We can then define a new deformation problem

$$\mathcal{S}_{\chi,M} = \Big(M/M^+, T_M, \widetilde{T}_M, \Lambda_M, \overline{r}_{\mathfrak{m}}|_{G_{M^+}}, \epsilon^{1-n}\delta^n_{M/M^+}, \{R_v^{\triangle}\}_{v \in S_{l,M}} \cup \{R_v^{\chi_v}\}_{v \in R_M}$$

$$\cup \{R_v^{\mathrm{St}}\}_{v \in S(B)_M} \cup \{R_v^{\square}\}_{v \in S_{a,M}}\Big).$$

Thus the universal deformation ring $R^{\mathrm{univ}}_{\mathcal{S}_{\chi,M}}$ and its Λ_M-subalgebra $P_{\mathcal{S}_{\chi,M}}$ are defined. We write $U_M = \prod U_{v,M}$ for the (sufficiently small) open compact subgroup of $G(\mathbb{A}^{\infty}_{M^+})$ defined as follows:

- For v inert in M, $U_v \subset G(M_v^+)$ is a hyperspecial maximal compact subgroup.
- For $v \notin T_M$ split in M, $U_v = G(\mathcal{O}_{M_v^+})$.
- For $v \in S(B)_M, U_v$ is a maximal compact subgroup.
- For $v \in S_{l,M}, U_v = G(\mathcal{O}_{M_v^+})$.
- For $v \in R_M, U_v = \iota_{\widetilde{v}}^{-1}\,\mathrm{Iw}(\widetilde{v})$.
- For $v \in S_{a,M}, U_v = \iota_{\widetilde{v}}^{-1} \ker(\mathrm{GL}_n(\mathcal{O}_{M_{\widetilde{v}}}) \to \mathrm{GL}_n(k(\widetilde{v})))$.

Here for $v \in T_M - S(B)_M$, $\iota_{\widetilde{v}}$ is an isomorphism $G(M_v^+) \cong \mathrm{GL}_n(M_{\widetilde{v}})$, defined in the same way as the isomorphism $\iota_{\widetilde{v}}$ for $v \in T - S(B)$. The Hecke algebra $\mathbb{T}^{T_M}_{\chi}(U_M(\mathfrak{l}^{\infty}), \mathcal{O})$ is defined in the same way as above. By [CHT08, Proposition 3.3.2], Lemma 2.7, and Proposition 4.4, there exists a homomorphism $\mathbb{T}^{T_M}_{\chi}(U_M(\mathfrak{l}^{\infty}), \mathcal{O}) \to \overline{\mathbb{F}}_l$, whose kernel we denote by $\mathfrak{m}_M$, such that $\overline{r}_{\mathfrak{m}_M}|_{G_M} \cong \overline{r}_{\mathfrak{m}}|_{G_M}$. The maximal ideal $\mathfrak{m}_M$ is residually Schur and we can make our choices so that $\overline{r}_{\mathfrak{m}_M} = \overline{r}_{\mathfrak{m}}|_{G_{M^+}}$.

We now have a diagram of Λ_M-algebras

$$\begin{tikzcd}
R^{\mathrm{univ}}_{\mathcal{S}_{\chi,M}} & P_{\mathcal{S}_{\chi,M}} \arrow[l] \arrow[r] & \mathbb{T}^{T_M}_{\chi}(U_M(\mathfrak{l}^{\infty}), \mathcal{O})_{\mathfrak{m}_M}.
\end{tikzcd}$$

Proposition 4.17. *Restriction of deformations $r \mapsto r|_{G_{M^+}}$ induces a map $R^{univ}_{\mathcal{S}_{\chi,M}} \to R^{univ}_{\mathcal{S}_{\chi}}$. This is a finite homomorphism of Λ_M-algebras, where $R^{univ}_{\mathcal{S}_{\chi}}$ is given a Λ_M-algebra structure by the homomorphism $\Lambda_M \to \Lambda$.*

Proof. Let $r^{\mathrm{univ}}_{\mathcal{S}_{\chi}}$ denote a representative of the universal deformation. The existence of the map $R^{\mathrm{univ}}_{\mathcal{S}_{\chi,M}} \to R^{\mathrm{univ}}_{\mathcal{S}_{\chi}}$ is equivalent to the assertion that $r^{\mathrm{univ}}_{\mathcal{S}_{\chi}}|_{G_{M^+}}$ is of type $\mathcal{S}_{\chi,M}$. This is a local problem. Since we chose M to be split above the primes in $S_l \cup S_a \cup R$, we only need to show that if $w \in S(B)_M$ is a place dividing $v \in S(B)$, then the restriction of the universal lifting induces a natural map $R^{\mathrm{St}}_w \to R^{\mathrm{St}}_v$. However, this is clear from the definitions. We now obtain a commutative diagram of Λ_M-algebras

$$\begin{tikzcd}
P_{\mathcal{S}_{\chi,M}} \arrow[r] \arrow[d] & P_{\mathcal{S}_{\chi}} \arrow[d] \\
R^{\mathrm{univ}}_{\mathcal{S}_{\chi,M}} \arrow[r] & R^{\mathrm{univ}}_{\mathcal{S}_{\chi}}.
\end{tikzcd}$$

To finish the proof of the proposition, it suffices to prove that the ring $R^{\mathrm{univ}}_{\mathcal{S}_\chi}/(\mathfrak{m}_{R^{\mathrm{univ}}_{\mathcal{S}_{\chi,M}}})$ is an Artinian k-algebra. This ring classifies deformations of $\overline{r}_\mathfrak{m}$ containing a lifting whose restriction to G_{M^+} equals $\overline{r}_\mathfrak{m}|_{G^+_M}$. Let M_0 denote the extension of M^+ cut out by $\overline{r}_\mathfrak{m}|_{G_{M^+}}$. Then any such lifting is trivial on the finite index subgroup G_{M_0} of G_{L^+}.

If $\mathfrak{p} \subset R^{\mathrm{univ}}_{\mathcal{S}_\chi}/(\mathfrak{m}_{R^{\mathrm{univ}}_{\mathcal{S}_{\chi,M}}})$ is a prime ideal, then the coefficients of the characteristic polynomial of $r^{\mathrm{univ}}_{\mathcal{S}_\chi}(\sigma)$ mod $\mathfrak{p}$ for $\sigma \in G_L$ are amongst the sums of roots of unity of bounded order, so are finite in number. Arguing as in the proof of [BLGGT, Lemma 1.2.2], we see that the subring of $R^{\mathrm{univ}}_{\mathcal{S}_\chi}/\mathfrak{p}$ topologically generated by these elements is a finite k-algebra, and hence $R^{\mathrm{univ}}_{\mathcal{S}_\chi}/\mathfrak{p}$ is itself a finite k-algebra (being finite over the subring topologically generated by the coefficients of these characteristic polynomials, by Proposition 3.29). It follows that $R^{\mathrm{univ}}_{\mathcal{S}_\chi}/(\mathfrak{m}_{R^{\mathrm{univ}}_{\mathcal{S}_{\chi,M}}})$ is a k-algebra of dimension 0, hence an Artinian k-algebra. ☐

Proposition 4.18. *There is a commutative diagram of Λ_M-algebras*

$$\begin{tikzcd}
R^{univ}_{\mathcal{S}_\chi} & P_{\mathcal{S}_\chi} \arrow[l] \arrow[r] & \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m} \\
R^{univ}_{\mathcal{S}_{\chi,M}} \arrow[u] & P_{\mathcal{S}_{\chi,M}} \arrow[l] \arrow[r] \arrow[u] & \mathbb{T}^{T_M}_\chi(U_M(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}_M} \arrow[u]
\end{tikzcd}$$

In particular, if we write $J_{\mathcal{S}_\chi} = \ker(P_{\mathcal{S}_\chi} \to \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m})$ and similarly for $J_{\mathcal{S}_{\chi,M}}$, then we have $J_{\mathcal{S}_{\chi,M}} P_{\mathcal{S}_\chi} \subset J_{\mathcal{S}_\chi}$.

Proof. It remains to construct a map $\mathbb{T}^{T_M}_\chi(U_M(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}_M} \to \mathbb{T}^T_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m}$ and show that the right-hand square commutes. For each integer $c > 0$, let $\mathfrak{m}_{M,c}$ and $\mathfrak{m}_c$ denote the images of the maximal ideals $\mathfrak{m}_M$ and $\mathfrak{m}$ in the Hecke algebras $\mathbb{T}^{T_M}_\chi(U_M(\mathfrak{l}^{c,c}), \mathcal{O})$ and $\mathbb{T}^T_\chi(U(\mathfrak{l}^{c,c}), \mathcal{O})$. It suffices to construct maps $\mathbb{T}^{T_M}_\chi(U_M(\mathfrak{l}^{c,c}), \mathcal{O})_{\mathfrak{m}_{M,c}} \to \mathbb{T}^T_\chi(U(\mathfrak{l}^{c,c}), \mathcal{O})_{\mathfrak{m}_c}$ and show that the resulting square

$$\begin{tikzcd}
P_{\mathcal{S}_\chi} \arrow[r] & \mathbb{T}^T_\chi(U(\mathfrak{l}^{c,c}), \mathcal{O})_{\mathfrak{m}_c} \\
P_{\mathcal{S}_{\chi,M}} \arrow[r] \arrow[u] & \mathbb{T}^{T_M}_\chi(U_M(\mathfrak{l}^{c,c}), \mathcal{O})_{\mathfrak{m}_{M,c}} \arrow[u]
\end{tikzcd}$$

commutes. We fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$, and write P_c for the set of RACSDC automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_L)$ which satisfy the following conditions:

- π is ι-ordinary of weight $\lambda = 0$, and there is an isomorphism of residual representations $\overline{r_\iota(\pi)}^{\mathrm{ss}} \cong \overline{r}_\mathfrak{m}$.
- If v is a finite place of L^+ inert in L, then π_v is unramified.
- If $v = ww^c$ is a finite place of L^+ which splits in L and not lying in $S(B) \cup S_l \cup R$ then $\pi_w^{\iota_w U_v} \neq 0$.
- If $v \in S(B)$, then $\pi_{\widetilde{v}}$ is an unramified twist of the Steinberg representation
- If $v \in S_l$, then the subspace of $\iota^{-1}\pi_{\widetilde{v}}^{\mathrm{Iw}_{\widetilde{v}}(c,c)}$ where each operator $\iota_{\widetilde{v}} U^j_{\widetilde{v}}$, $j = 1, \ldots, n$ acts with eigenvalues which are l-adic units is non-zero.
- If $v \in R$, then $\pi_{\widetilde{v}}$ is a subquotient of a normalized induction $\text{n-Ind}_B^{\mathrm{GL}_n(L_{\widetilde{v}})} \widetilde{\chi}_{v,1} \otimes \ldots \widetilde{\chi}_{v,n}$, where each $\widetilde{\chi}_{v,i} : L^\times_{\widetilde{v}} \to \mathbb{C}^\times$ is a smooth character satisfying $\widetilde{\chi}_{v,i}|_{\mathcal{O}^\times_{L_{\widetilde{v}}}} = \iota \chi^{-1}_{v,i}$.

We write S_c for the set of irreducible $G(\mathbb{A}^{\infty,R}_{L^+}) \times \prod_{v \in R} \iota^{-1}_{\widetilde{v}} \mathrm{Iw}(\widetilde{v})$-submodules σ of the space $S_\chi(\{1\}, \overline{\mathbb{Q}}_l)$ such that $\sigma \cap S^{\mathrm{ord}}_\chi(U(\mathfrak{l}^{c,c}), \mathcal{O})_{\mathfrak{m}_c} \neq 0$. Similarly, we write $P_{c,M}$ for the set of RACSDC automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_M)$ which satisfy the following conditions:

- π is ι-ordinary of weight $\lambda = 0$, and there is an isomorphism of residual representations $\overline{r_\iota(\pi)}^{\mathrm{ss}} \cong \overline{r}_{\mathfrak{m}}|_{G_M}$.
- If v is a finite place of M^+ inert in M, then π_v is unramified.
- If $v = ww^c$ is a finite place of M^+ which splits in M and not lying in $S(B)_M \cup S_{l,M} \cup R_M$ then $\pi_w^{\iota_w U_v} \neq 0$.
- If $v \in S(B)_M$, then $\pi_{\widetilde{v}}$ is an unramified twist of the Steinberg representation
- If $v \in S_{l,M}$, then the subspace of $\iota^{-1}\pi_{\widetilde{v}}^{\mathrm{Iw}_{\widetilde{v}}(c,c)}$ where each operator $\iota_{\widetilde{v}} U_{\widetilde{v}}^j$, $j = 1, \dots, n$ acts with eigenvalues which are l-adic units is non-zero.
- If $v \in R_M$, then $\pi_{\widetilde{v}}$ is a subquotient of a normalized induction $\text{n-Ind}_B^{\mathrm{GL}_n(M_{\widetilde{v}})} \widetilde{\chi}_{v,1} \otimes \dots \widetilde{\chi}_{v,n}$, where each $\widetilde{\chi}_{v,i} : M_{\widetilde{v}}^\times \to \mathbb{C}^\times$ is a smooth character satisfying $\widetilde{\chi}_{v,i}|_{\mathcal{O}_{M_{\widetilde{v}}}^\times} = \iota\chi_{v,i}^{-1}$.

We write $S_{M,c}$ for the set of irreducible $G(\mathbb{A}_{M^+}^{\infty,R_M}) \times \prod_{v\in R_M} \iota_{\widetilde{v}}^{-1} \mathrm{Iw}(\widetilde{v})$-submodules σ of the space $S_\chi(\{1\}, \overline{\mathbb{Q}}_l)$ such that $\sigma \cap S_\chi^{\mathrm{ord}}(U_M(\mathfrak{l}^{c,c}), \mathcal{O})_{\mathfrak{m}_{M,c}} \neq 0$. Just as in the proof of Proposition 4.13, there are surjective maps $S_c \to P_c$ and $S_{M,c} \to P_{M,c}$ given by base change. By Lemma 2.7, there is a map $f_c : P_c \to P_{M,c}$ given by base change $\pi \mapsto \pi_M$. The algebra $\mathbb{T}_\chi^T(U(\mathfrak{l}^{c,c}), \mathcal{O})_{\mathfrak{m}_c}$ may be identified with the $\mathcal{O}$-subalgebra of $\prod_{\pi\in P_c} \overline{\mathbb{Q}}_l$ generated by the images of the Hecke operators described in §4.1, and also with the image of the natural ring homomorphism $P_{\mathcal{S}_\chi} \to \prod_{\pi\in P_c} \overline{\mathbb{Q}}_l$. Similar remarks apply to the algebra $\mathbb{T}_\chi^{T_M}(U_M(\mathfrak{l}^{c,c}), \mathcal{O})_{\mathfrak{m}_{M,c}}$. There is a map $\prod_{\Pi\in P_{M,c}} \overline{\mathbb{Q}}_l \to \prod_{\pi\in P_c} \overline{\mathbb{Q}}_l$ given by $(x_\Pi)_{\Pi\in P_{M,c}} \mapsto (x_{\pi_M})_{\pi\in P_c}$, and a commutative diagram

$$\begin{tikzcd}
P_{\mathcal{S}_\chi} \arrow[r] & \prod_{\pi\in P_c} \overline{\mathbb{Q}}_l \\
P_{\mathcal{S}_{\chi,M}} \arrow[r] \arrow[u] & \prod_{\Pi\in P_{M,c}} \overline{\mathbb{Q}}_l. \arrow[u]
\end{tikzcd}$$

After identifying the images of the horizontal arrows in this diagram with the respective Hecke algebras, this gives the desired commutative square. $\square$

4.6 A patching argument

We put ourselves in the setting of §4.3. We leave the characters χ_v for $v \in R$ unspecified, but suppose that they are all trivial mod λ. Note that the rings $R_{\mathcal{S}_\chi}^{\mathrm{univ}}/(\lambda)$ for varying χ are canonically identified. Similar remarks apply to the rings $P_{\mathcal{S}_\chi}$, $R_{\mathcal{S}_\chi,T}^{\mathrm{loc}}$ etc. and the spaces of automorphic forms $S_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}$. In particular, the sets of prime ideals containing λ in each of these rings are in canonical bijection. In the following we will abuse notation and view such a prime ideal as belonging to any one of these rings, depending on the context. The level U will be fixed throughout this section, so we write $H_\chi = S_\chi(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}$ and $\mathbb{T}_\chi = \mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}$ for this space of automorphic forms and Hecke algebra, respectively, as in §4.4. We will also adopt the notations $H_{\chi,N,1}$ and $\mathbb{T}_{\chi,N,1}$ of that section for the corresponding objects at auxiliary levels, once we make a choice of Taylor–Wiles data of level N.

Let $J_{\mathcal{S}_\chi} = \ker(P_{\mathcal{S}_\chi} \to \mathbb{T}_\chi)$. We suppose that $\mathfrak{p} \supset J_{\mathcal{S}_\chi} R_{\mathcal{S}_\chi}^{\mathrm{univ}}$ is a prime ideal of dimension one and characteristic l. Write A for the normalization of $R_{\mathcal{S}_\chi}^{\mathrm{univ}}/\mathfrak{p}$ in its fraction field E. We let $\mathfrak{q} = P_{\mathcal{S}_\chi} \cap \mathfrak{p}$. Suppose that the map $\Lambda \to A$ is finite. Arguing as in §3.7, we can choose a finite faithfully flat extension $\Lambda \to \widetilde{\Lambda}$ inducing a bijection on minimal primes, together with a surjective map $\widetilde{\Lambda} \to A$ with kernel $\widetilde{P}$ and making the diagram

$$\begin{tikzcd}
\Lambda \arrow[r] \arrow[rr, bend right] & \widetilde{\Lambda} \arrow[r] & A
\end{tikzcd}$$

commute, and extensions $\widetilde{\mathfrak{p}}$ (resp. $\widetilde{\mathfrak{q}}$) of $\mathfrak{p}$ (resp. $\mathfrak{q}$) to the rings $R_{\mathcal{S}_\chi}^{\mathrm{univ}} \otimes_\Lambda \widetilde{\Lambda}$ (resp. $P_{\mathcal{S}_\chi} \otimes_\Lambda \widetilde{\Lambda}$).

We set $\widetilde{R}_{\mathcal{S}_\chi}^{\mathrm{univ}} = R^{\mathrm{univ}} \otimes_\Lambda \widetilde{\Lambda}$, $\widetilde{P}_{\mathcal{S}_\chi} = P_{\mathcal{S}_\chi} \otimes_\Lambda \widetilde{\Lambda}$, and $\widetilde{\mathbb{T}}_\chi = \mathbb{T}_\chi \otimes_\Lambda \widetilde{\Lambda}$. We set $\mathcal{T} = \Lambda[\![\{X_{v,i,j}\}_{v\in T, 1\leq i,j\leq n}]\!]$ and $\widetilde{\mathcal{T}} = \mathcal{T} \otimes_\Lambda \widetilde{\Lambda}$. For each χ we fix a choice of lifting $r_{\mathcal{S}_\chi}^{\mathrm{univ}} : G_{L^+,S} \to \mathcal{G}_n(R_{\mathcal{S}_\chi}^{\mathrm{univ}})$ representing the universal deformation $[r^{\mathrm{univ}}_{\mathcal{S}_\chi}]$. We suppose that the different choices are identified modulo λ. Having made this choice we get an isomorphism $R^{\square_T}_{\mathcal{S}_\chi} \cong R^{\mathrm{univ}}_{\mathcal{S}_\chi} \widehat{\otimes}_\Lambda \mathcal{T}$, which classifies the T-framed lifting $(r^{\mathrm{univ}}_{\mathcal{S}_\chi}; \{(X_{v,i,j})_{i,j}\}_{v\in T})$. In order to simplify the notation slightly, we now choose an ordering of the variables $X_{v,i,j}$ and write them as $X_1, \dots, X_{n^2t}$, where $t = |T|$. Thus we have $\mathcal{T} = \Lambda[\![X_1, \dots, X_{n^2t}]\!]$.

We define $P^{\square_T}_{\mathcal{S}_\chi} = P_{\mathcal{S}_\chi} \widehat{\otimes}_\Lambda \mathcal{T}$ so that there is a commutative diagram

$$\begin{tikzcd}
P_{\mathcal{S}_\chi} \arrow[r] \arrow[d] & P_{\mathcal{S}_\chi} \widehat{\otimes}_\Lambda \mathcal{T} \arrow[r] \arrow[d] & P^{\square_T}_{\mathcal{S}_\chi} \arrow[d] \\
R^{\mathrm{univ}}_{\mathcal{S}_\chi} \arrow[r] & R^{\mathrm{univ}}_{\mathcal{S}_\chi} \widehat{\otimes}_\Lambda \mathcal{T} \arrow[r] & R^{\square_T}_{\mathcal{S}_\chi}.
\end{tikzcd}$$

For varying χ these diagrams are identified modulo λ, by the choice of universal lifting. For any choice of Taylor–Wiles data of level q and level N, we obtain a diagram

$$\begin{tikzcd}
 & P^{\square_T}_{\mathcal{S}_\chi, N} \arrow[r] \arrow[d] & P_{\mathcal{S}_\chi} \arrow[r] \arrow[d] & A \arrow[d] \\
R^{\mathrm{loc}}_{\mathcal{S}_\chi, T} \arrow[r] & R^{\square_T}_{\mathcal{S}_\chi, N} \arrow[r] & R^{\mathrm{univ}}_{\mathcal{S}_\chi} \arrow[r] & A
\end{tikzcd}$$

We write $\mathfrak{p}_N$, $\mathfrak{q}_N$ for the kernels of the respective maps $R^{\square_T}_{\mathcal{S}_\chi,N} \to A$ and $P^{\square_T}_{\mathcal{S}_\chi,N} \to A$. We write P^{loc} for the kernel of the map $R^{\mathrm{loc}}_{\mathcal{S}_\chi,T} \to A$. We define

$$\widetilde{R}^{\square_T}_{\mathcal{S}_\chi,N} = R^{\square_T}_{\mathcal{S}_\chi,N} \otimes_\Lambda \widetilde{\Lambda},\ \widetilde{P}^{\square_T}_{\mathcal{S}_\chi,N} = P^{\square_T}_{\mathcal{S}_\chi,N} \otimes_\Lambda \widetilde{\Lambda},\ \widetilde{R}^{\mathrm{loc}}_{\mathcal{S}_\chi,T} = R^{\mathrm{loc}}_{\mathcal{S}_\chi,T} \otimes_\Lambda \widetilde{\Lambda},$$

and let $\widetilde{\mathfrak{p}}_N, \widetilde{\mathfrak{q}}_N$, and $\widetilde{P}^{\mathrm{loc}}$ denote the natural extensions of the ideals $\mathfrak{p}_N, \mathfrak{q}_N$, and P^{loc} to prime ideals of these rings.

Theorem 4.19. *Suppose that $\mathfrak{p}$ satisfies the following hypotheses.*

1. *There exists an integer $q \geq [L^+ : \mathbb{Q}]n(n-1)/2$ and for each positive integer N, a choice of Taylor–Wiles data of order q and level N as above such that there is an isomorphism of A-modules*
$$\widetilde{\mathfrak{p}}_N/(\widetilde{P}^{loc} + \widetilde{\mathfrak{p}}_N^2) \cong A^{(q-[L^+:\mathbb{Q}]n(n-1)/2)} \oplus T(N),$$
where $T(N)$ is finite of cardinality bounded independently of N.

2. *$\overline{r}_{\mathfrak{m}}|_{G_{F,S}} = \overline{\rho}_1 \oplus \overline{\rho}_2$ is a direct sum of two absolutely irreducible representations. The representation $r_{\mathfrak{p}}|_{G_{F,S}} \otimes_A E : G_{F,S} \to \mathrm{GL}_n(E)$ is absolutely irreducible.*

3. *For each $v|l$, the pushforwards of the universal characters $\psi^v_1, \dots, \psi^v_n$ to A are pairwise distinct and $[L_{\widetilde{v}} : \mathbb{Q}_l] > n(n-1)/2 + 1$.*

4. *For each $v \in R$, $r_{\mathfrak{p}}|_{G_{L_{\widetilde{v}}}}$ is the trivial representation, and if $l^N||q_v - 1$ then $l^N > n$. For each $v \in S(B)$, $r_{\mathfrak{p}}|_{G_{L_{\widetilde{v}}}}$ is unramified and $r_{\mathfrak{p}}(\mathrm{Frob}_{\widetilde{v}})$ is a scalar matrix. The field K is sufficiently large, in the sense of Proposition 3.15 and Proposition 3.17.*

5. $\mathrm{Frac}\, P_{\mathcal{S}_\chi}/\mathfrak{q} = \mathrm{Frac}\, R_{\mathcal{S}_\chi}/\mathfrak{p}$.

Then the map $\widetilde{P}_{\mathcal{S}_1,\widetilde{\mathfrak{q}}} \to \widetilde{\mathbb{T}}_{1,\widetilde{\mathfrak{q}}}$ has nilpotent kernel.

Before giving the proof of Theorem 4.19, we discuss the role played by each of the above hypotheses.

1. The first condition asserts the existence of sufficiently many Taylor–Wiles data. We will show the existence of these data under some additional conditions on the image of $r_{\mathfrak{p}}$ in §5 below.

2. The second condition that $r_{\mathfrak{p}}|_{G_{F,S}}$ be absolutely irreducible implies (cf. Proposition 3.37) that for each choice of χ, the relative tangent space of the morphism $P_{\mathcal{S}_\chi} \to R^{\text{univ}}_{\mathcal{S}_\chi}$ vanishes after localization at $\mathfrak{p}$. In particular, we will be able to control the rings $P_{\mathcal{S}_\chi}$ (and by extension, their Hecke algebra quotients) locally at $\mathfrak{p}$ by controlling $R^{\text{univ}}_{\mathcal{S}_\chi}$ locally at $\mathfrak{p}$, and this latter problem can be attacked using the usual Galois-cohomological methods.

   The condition that $\overline{r}_{\mathfrak{m}}|_{G_{F,S}}$ has only 2 irreducible constituents is imposed because we have carried out a full analysis of the maps $P_{\mathcal{S}_\chi} \to R^{\text{univ}}_{\mathcal{S}_\chi}$ (cf. Proposition 3.29) only in this case.

3. The third set of conditions on $r_{\mathfrak{p}}$ and L locally at the primes above l are necessary to use the results of §3.3.2 about the rings $R_v^{\triangle}$.

4. The fourth set of conditions is imposed so that we can apply Lemma 3.40. As is usual in the application of the Taylor–Wiles–Kisin method, we need to control the irreducible components of the completed tensor product of the local lifting rings; in our special situtation, we need to control the irreducible components even after localization and completion at $\mathfrak{p}$, and we can do this only under these extra assumptions on the local behavior of the reprsentation $r_{\mathfrak{p}}$ at the primes of $R \cup S(B)$.

5. The fifth condition is imposed so that we can apply the second part of Proposition 3.37, which is convenient for technical reasons.

We also pause to mention the role played by the variable choice of characters χ. In order to prove automorphy lifting theorems without restriction on the local behavior of the representations considered, we must allow arbitrary unipotent ramification (corresponding to the choice $\chi_v = 1$ for each $v \in R$). However, in this case the local lifting rings R_v^1 ($v \in R$) have many irreducible components, which causes problems when one wants to apply the Taylor–Wiles–Kisin method. Taylor [Tay08] introduced a beautiful trick to get around this issue, by allowing a variable choice of characters χ which are all trivial modulo λ. We have adapted this method to our purposes here.

Corollary 4.20. *With hypotheses as in Theorem 4.19, let $Q \subset \mathfrak{p}$ be a minimal prime of $R^{univ}_{\mathcal{S}_1}$. Then $J_{\mathcal{S}_1} R^{univ}_{\mathcal{S}_1} \subset Q$.*

Proof of Corollary 4.20. The theorem shows that the ideals $J_{\mathcal{S}_1}\widetilde{P}_{\mathcal{S}_1,\widetilde{\mathfrak{q}}}$ and hence $J_{\mathcal{S}_1}\widetilde{R}^{\text{univ}}_{\mathcal{S}_1,\widetilde{\mathfrak{p}}}$ are nilpotent. Since the map $R^{\text{univ}}_{\mathcal{S}_1,(\mathfrak{p})} \to (\widetilde{R}^{\text{univ}}_{\mathcal{S}_1})_{\widetilde{\mathfrak{p}}}$ is faithfully flat, it follows that $J_{\mathcal{S}_1} R^{\text{univ}}_{\mathcal{S}_1,(\mathfrak{p})}$ is nilpotent, hence $J_{\mathcal{S}_1} R^{\text{univ}}_{\mathcal{S}_1,(Q)}$ is nilpotent. This implies $J_{\mathcal{S}_1} R^{\text{univ}}_{\mathcal{S}_1} \subset Q$. ☐

The remainder of this section is now devoted to the proof of Theorem 4.19. We fix an integer q and for each integer $N \geq 1$ a choice of Taylor–Wiles data as in the statement of the theorem. We define $\widetilde{H}_\chi = H_\chi \otimes_\Lambda \widetilde{\Lambda}$, and introduce auxiliary Hecke modules

$$\widetilde{H}^{\square}_\chi = \widetilde{H}_\chi \otimes_{\widetilde{P}_{\mathcal{S}_\chi}} \widetilde{P}^{\square_T}_{\mathcal{S}_\chi},\ \widetilde{H}^{\square}_{\chi,N} = \widetilde{H}_{\chi,N,1} \otimes_{\widetilde{P}_{\mathcal{S}_{\chi,N}}} \widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}$$

Then $\widetilde{H}^{\square}_{\chi,N}$ is a free $\widetilde{\mathcal{T}}[\Delta_N]$-module, with Δ_N-covariants isomorphic to $\widetilde{H}^{\square}_\chi$, by Theorem 4.15. We now set $S_\infty = \widetilde{\mathcal{T}}[\![S_1, \ldots, S_q]\!]$, and write $\mathfrak{a} = \ker(S_\infty \to \widetilde{\Lambda})$, $P_\infty = \ker(S_\infty \to \widetilde{\Lambda} \to A)$. We choose for every N a surjection $S_\infty \to \widetilde{\mathcal{T}}[\Delta_N]$, and write $\mathfrak{c}_N = \ker(S_\infty \to \widetilde{\mathcal{T}}[\Delta_N])$. With these choices $\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}$, $\widetilde{R}^{\square_T}_{\mathcal{S}_{\chi,N}}$ become S_∞-algebras for every N, and $\widetilde{H}^{\square}_{\chi,N}$ is a free $S_\infty/\mathfrak{c}_N$-module (by Theorem 4.15). We have isomorphisms

$$\widetilde{H}^{\square}_{\chi,N}/(\mathfrak{a}) \cong \widetilde{H}_\chi \text{ and } \widetilde{R}^{\square_T}_{\mathcal{S}_{\chi,N}}/(\mathfrak{a}) \cong \widetilde{R}^{\text{univ}}_{\mathcal{S}_\chi}$$

by Lemma 3.19. We write $\mathfrak{b}_N \subset S_\infty$ for the ideal $(\mathfrak{m}^N_{\widetilde{\Lambda}}, (S_1+1)^{l^N} - 1, \ldots, (S_q+1)^{l^N} - 1, X_1^{l^N}, \ldots, X_{n^2t}^{l^N})$. Thus $\mathfrak{c}_N \subset \mathfrak{b}_N$ and $\widetilde{H}^{\square}_{\chi,N}/\mathfrak{b}_N$ is a free $S_\infty/\mathfrak{b}_N$-module.

Let $q' = (q - [L^+ : \mathbb{Q}]n(n-1)/2)$. We define

$$R^\infty_\chi = \widetilde{R}^{\text{loc}}_{\mathcal{S}_\chi,T}[\![y_1, \ldots, y_{q'}]\!].$$

We fix for every χ and $N \geq 1$ a homomorphism of $\widetilde{R}^{\mathrm{loc}}_{\mathcal{S}_\chi,T}$-algebras

$$R^\infty_\chi \to \widetilde{R}^{\square_T}_{\mathcal{S}_{\chi,N}}$$

mapping $y_1, \ldots, y_{q'}$ into $\widetilde{\mathfrak{p}}_N$ and onto a basis of the maximal free quotient of the A-module $\widetilde{\mathfrak{p}}_N/(\widetilde{P}^{\mathrm{loc}} + \widetilde{\mathfrak{p}}_N^2)$. We suppose that these are chosen to be identified upon reduction modulo λ. We write P^∞ for the kernel of the surjective homomorphism $R^\infty_\chi \to A$. Thus P^∞ is the pullback of $\widetilde{P}^{\mathrm{loc}}$ along the natural augmentation $R^\infty_\chi \to \widetilde{R}^{\mathrm{loc}}_{\mathcal{S}_\chi,T}$.

In §3.7, we have defined an action of the group $\mu_2 \times \mu_2$ on the rings $R^{\mathrm{univ}}_{\mathcal{S}_\chi}$ and $R^{\mathrm{univ}}_{\mathcal{S}_{\chi,N}}$, fixing pointwise the subrings $P_{\mathcal{S}_\chi} \subset R^{\mathrm{univ}}_{\mathcal{S}_\chi}$ and $P_{\mathcal{S}_{\chi,N}} \subset R^{\mathrm{univ}}_{\mathcal{S}_{\chi,N}}$. We make $\mu_2 \times \mu_2$ act on the rings

$$P^{\square_T}_{\mathcal{S}_\chi} \cong P_{\mathcal{S}_\chi} \widehat{\otimes}_\Lambda \mathcal{T} \subset R^{\square_T}_{\mathcal{S}_\chi} \cong R^{\mathrm{univ}}_{\mathcal{S}_\chi} \widehat{\otimes}_\Lambda \mathcal{T}$$

and

$$P^{\square_T}_{\mathcal{S}_{\chi,N}} \cong P_{\mathcal{S}_{\chi,N}} \widehat{\otimes}_\Lambda \mathcal{T} \subset R^{\square_T}_{\mathcal{S}_{\chi,N}} \cong R^{\mathrm{univ}}_{\mathcal{S}_{\chi,N}} \widehat{\otimes}_\Lambda \mathcal{T},$$

by this action on the first factor and the trivial action on $\mathcal{T}$. Similarly, we make $\mu_2 \times \mu_2$ act on the rings $\widetilde{P}^{\square_T}_{\mathcal{S}_\chi,N}$, $\widetilde{R}^{\square_T}_{\mathcal{S}_\chi,N}$, $\widetilde{P}_{\mathcal{S}_\chi}$, and $\widetilde{R}^{\mathrm{univ}}_{\mathcal{S}_\chi}$ by giving $\widetilde{\Lambda}$ the trivial action. These actions are compatible with the maps between these objects and identifications modulo λ. (We note that we do not define an action of $\mu_2 \times \mu_2$ on R^∞_χ.)

Let $r_M = M(q + n^2 t)sl^M$, where $s = \dim_k \widetilde{H}_\chi/\mathfrak{m}_{\widetilde{\Lambda}}$. (Note that s is independent of the choice of χ.) For any integer $M \geq 1$ we define a patching datum $(A_{\chi,M}, B_{\chi,M}, \mathcal{M}_{\chi,M})_\chi$ of level M to be for each choice of χ a commutative diagram of complete Noetherian local $\widetilde{\Lambda}$-algebras with residue field k:

$$\begin{tikzcd}
S_\infty \arrow[r] & A_{\chi,M} \arrow[r, two heads, "\phi_P"] \arrow[d, "\phi_M"] & \widetilde{P}_{\mathcal{S}_\chi}/(\mathfrak{m}^{r_M}_{\widetilde{P}_{\mathcal{S}_\chi}} + \mathfrak{b}_M) \arrow[d] \\
R^\infty_\chi \arrow[r] & B_{\chi,M} \arrow[r, two heads, "\phi_R"] & \widetilde{R}^{\mathrm{univ}}_{\mathcal{S}_\chi}/(\mathfrak{m}^{r_M}_{\widetilde{P}_{\mathcal{S}_\chi}} + \mathfrak{b}_M)
\end{tikzcd}$$

together with an $A_{\chi,M}$-module $\mathcal{M}_{\chi,M}$ killed by $\mathfrak{b}_M$ and a homomorphism $\psi_H : \mathcal{M}_{\chi,M} \to \widetilde{H}_\chi/(\mathfrak{b}_M)$ inducing an isomorphism $\mathcal{M}_{\chi,M}/(\mathfrak{a}) \cong \widetilde{H}_\chi/(\mathfrak{b}_M)$. We fix also the data of identifications between these diagrams and modules mod λ for varying χ. We require further that $\mathcal{M}_{\chi,M}$ be finite free as an $S_\infty/\mathfrak{b}_M$-module. We also fix the data of an action of the group $\mu_2 \times \mu_2$ on the rings $A_{\chi,M}$ and $B_{\chi,M}$ such that the arrows in the right-hand square of the above diagram are equivariant for the action of this group.

A morphism $(A_{\chi,M}, B_{\chi,M}, \mathcal{M}_{\chi,M})_\chi \to (A'_{\chi,M}, B'_{\chi,M}, \mathcal{M}'_{\chi,M})_\chi$ of patching data of level M is, by definition, the data for each χ of isomorphisms $A_{\chi,M} \to A'_{\chi,M}$, $B_{\chi,M} \to B'_{\chi,M}$ compatible with the action of $\mu_2 \times \mu_2$ and a compatible isomorphism of modules $\mathcal{M}_{\chi,M} \to \mathcal{M}'_{\chi,M}$, making the diagram

$$\begin{tikzcd}
S_\infty \arrow[r] \arrow[drr] & A_{\chi,M} \arrow[rr, two heads] \arrow[dr] \arrow[dd] & & \widetilde{P}_{\mathcal{S}_\chi}/(\mathfrak{m}^{r_M}_{\widetilde{P}_{\mathcal{S}_\chi}} + \mathfrak{b}_M) \arrow[dd] \\
& & A'_{\chi,M} \arrow[ur, two heads] \arrow[dd] & \\
R^\infty_\chi \arrow[r] \arrow[drr] & B_{\chi,M} \arrow[rr, two heads] \arrow[dr] & & \widetilde{R}^{\mathrm{univ}}_{\mathcal{S}_\chi}/(\mathfrak{m}^{r_M}_{\widetilde{P}_{\mathcal{S}_\chi}} + \mathfrak{b}_M) \\
& & B'_{\chi,M} \arrow[ur, two heads] &
\end{tikzcd}$$

commutative, and such that the identifications mod λ are preserved.

For each pair of positive integers $M \leq N$ we can construct a patching datum $D(M,N)$ of level M by taking for each χ the diagram

$$\begin{tikzcd}
S_\infty \arrow[r] & \widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}/(\mathfrak{m}^{r_M}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}} + \mathfrak{b}_M) \arrow[r, two heads] \arrow[d] & \widetilde{P}_{\mathcal{S}_\chi}/(\mathfrak{m}^{r_M}_{\widetilde{P}_{\mathcal{S}_\chi}} + \mathfrak{b}_M) \arrow[d] \\
R^\infty_\chi \arrow[r] & \widetilde{R}^{\square_T}_{\mathcal{S}_{\chi,N}}/(\mathfrak{m}^{r_M}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}} + \mathfrak{b}_M) \arrow[r, two heads] & \widetilde{R}^{\mathrm{univ}}_{\mathcal{S}_\chi}/(\mathfrak{m}^{r_M}_{\widetilde{P}_{\mathcal{S}_\chi}} + \mathfrak{b}_M),
\end{tikzcd}$$

with Hecke modules $\widetilde{H}^\square_{\chi,N}/(\mathfrak{b}_M) \to \widetilde{H}_\chi/(\mathfrak{b}_M)$. The action of the group $\mu_2 \times \mu_2$ on these rings is induced from its action on $\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}} \cong \widetilde{P}_{\mathcal{S}_{\chi,N}} \widehat{\otimes}_{\widetilde{\Lambda}} \widetilde{\mathcal{T}}$ and $\widetilde{R}^{\square_T}_{\mathcal{S}_{\chi,N}} \cong \widetilde{R}_{\mathcal{S}_{\chi,N}} \widehat{\otimes}_{\widetilde{\Lambda}} \widetilde{\mathcal{T}}$.

Lemma 4.21. *1. The ring $\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}/(\mathfrak{m}^{r_M}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}} + \mathfrak{b}_M)$ acts on $\widetilde{H}^\square_{\chi,N}/(\mathfrak{b}_M)$. In particular, this does indeed define a patching datum of level M.*

2. Fix M. Then as $N \geq M$ varies, the patching data $D(M,N)$ fall into finitely many isomorphism classes.

Proof. For the first part it suffices to show that $\mathfrak{m}^{r_M}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}} \widetilde{H}^\square_{\chi,N} \subset \mathfrak{b}_M \widetilde{H}^\square_{\chi,N}$. Suppose that $x \in \mathfrak{m}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}}$. Then x is nilpotent on the s-dimensional k-vector space

$$\widetilde{H}^\square_{\chi,N}/(\mathfrak{m}_{\widetilde{\Lambda}} + \mathfrak{a}) \cong H_\chi/(\mathfrak{m}_{\widetilde{\Lambda}}),$$

so x^s acts as the zero map. It follows that

$$x^s \widetilde{H}^\square_{\chi,N} \subset (\mathfrak{m}_{\widetilde{\Lambda}}, S_1, \ldots, S_q, X_1, \ldots, X_{n^2t}) \widetilde{H}^\square_{\chi,N}$$

and hence

$$x^{(q+n^2t)sl^M} \widetilde{H}^\square_{\chi,N} \subset (\mathfrak{m}_{\widetilde{\Lambda}}, S_1^{l^M}, \ldots, S_q^{l^M}, X_1^{l^M}, \ldots, X_{n^2t}^{l^M}) \widetilde{H}^\square_{\chi,N} =$$
$$(\mathfrak{m}_{\widetilde{\Lambda}}, (S_1+1)^{l^M} - 1, \ldots, (S_q+1)^{l^M} - 1, X_1^{l^M}, \ldots, X_{n^2t}^{l^M}) \widetilde{H}^\square_{\chi,N}$$

Then we have

$$x^{M(q+n^2t)sl^M} \widetilde{H}^\square_{\chi,N} \subset (\mathfrak{m}^M_{\widetilde{\Lambda}}, (S_1+1)^{l^M} - 1, \ldots, (S_q+1)^{l^M} - 1, X_1^{l^M}, \ldots, X_{n^2t}^{l^M}) \widetilde{H}^\square_{\chi,N},$$

as required.

For the second part, it suffices to show that the orders of the rings $\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}/(\mathfrak{m}^{r_M}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}} + \mathfrak{b}_M)$ and $\widetilde{R}^{\square_T}_{\mathcal{S}_{\chi,N}}/(\mathfrak{m}^{r_M}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}} + \mathfrak{b}_M)$ can be bounded solely in terms of M, this being clear for all other objects in the diagrams above. For the quotient of $\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}$, this is an immediate consequence of Lemma 3.28. On the other hand, note that $\widetilde{R}^{\square_T}_{\mathcal{S}_{\chi,N}}/(\mathfrak{m}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}}) \cong \widetilde{R}^{\mathrm{univ}}_{\mathcal{S}_\chi}/(\mathfrak{m}_{\widetilde{P}_{\mathcal{S}_\chi}})$, hence for any $N \geq 1$ $\widetilde{R}^{\square_T}_{\mathcal{S}_{\chi,N}}$ is generated as a $\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N}}$-module by $\dim_k \widetilde{R}^{\mathrm{univ}}_{\mathcal{S}_\chi}/(\mathfrak{m}_{\widetilde{P}_{\mathcal{S}_\chi}})$ elements. The result follows. □

We now patch to obtain objects at 'infinite level'. For every fixed M, the patching data $D(M,N)$ for $N = M, M+1, \ldots$ fall into a finite number of isomorphism classes. Thus we can find an infinite sequence $1 \leq N_1 < N_2 < \ldots$ of integers such that for all fixed M, the patching data $D(M, N_j)$ for $j \geq M$ are all equivalent. Choosing for each M an isomorphism of patching data $D(M, N_{M+1}) \cong D(M, N_M)$ we obtain an inverse system and can therefore pass to the limit to obtain for each χ a diagram of rings

$$\begin{tikzcd}
S_\infty \arrow[r] & A_{\chi,\infty} \arrow[r, two heads] \arrow[d] & \widetilde{P}_{\mathcal{S}_\chi} \arrow[d] \\
R^\infty_\chi \arrow[r] & B_{\chi,\infty} \arrow[r, two heads] & \widetilde{R}^{\mathrm{univ}}_{\mathcal{S}_\chi}
\end{tikzcd}$$

and a module $\mathcal{M}_{\chi,\infty}$ for $A_{\chi,\infty}$ which is free over S_∞ of rank s. Moreover for varying χ these diagrams are identified modulo λ, and $\mu_2 \times \mu_2$ acts on $A_{\chi,\infty}$ and $B_{\chi,\infty}$ in such a way that the arrows in the right-hand square are equivariant for this action.

Let $\mathfrak{q}_\infty$, $\mathfrak{p}_\infty$ denote respectively the pullback of $\widetilde{\mathfrak{q}}$ and $\widetilde{\mathfrak{p}}$ to $A_{\chi,\infty}$ and $B_{\chi,\infty}$. It follows from Proposition 3.37 that $\mathfrak{p}_\infty/(\mathfrak{q}_\infty + \mathfrak{p}_\infty^2)$ is a finite torsion A-module. On the other hand, there is by construction an isomorphism $B_{\chi,\infty}/\mathfrak{q}_\infty \cong \widetilde{R}^{\mathrm{univ}}_{\mathcal{S}_\chi}/\widetilde{\mathfrak{q}}$, compatible with the action of $\mu_2 \times \mu_2$. By Proposition 3.29, this group acts transitively on the set of primes of $\widetilde{R}^{\mathrm{univ}}_{\mathcal{S}_\chi}$ above $\widetilde{\mathfrak{q}}$.

Lemma 4.22. *1. $B_{\chi,\infty}$ is a finite $A_{\chi,\infty}$-algebra, and the map $A_{\chi,\infty} \to B_{\chi,\infty}$ has nilpotent kernel.*

2. The map $A_{\chi,\infty,\mathfrak{q}_\infty} \to B_{\chi,\infty,\mathfrak{p}_\infty}$ is surjective, with nilpotent kernel.

Proof. For the first part, the finiteness follows from the corresponding fact at finite level, and the completed version of Nakayama's lemma. To calculate the kernel we use Fitting ideals. Recall that if R is a Noetherian ring and M is an R-module that can be generated by r elements, we have $(\operatorname{Ann}_R M)^r \subset \operatorname{Fitt}_R M \subset \operatorname{Ann}_R M$, and for any homomorphism $R \to S$ we have $\operatorname{Fitt}_S M \otimes_R S = \operatorname{Fitt}_R M \cdot S$. For each $M \geq 1$ we have $\operatorname{Ann}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N_M}}} \widetilde{R}^{\square_T}_{\mathcal{S}_{\chi,N_M}} = 0$, hence $\operatorname{Fitt}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N_M}}} \widetilde{R}^{\square_T}_{\mathcal{S}_{\chi,N_M}} = 0$, hence

$$\operatorname{Fitt}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N_M}}/(\mathfrak{m}^{r_M}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N_M}}} + \mathfrak{b}_M)} \widetilde{R}^{\square_T}_{\mathcal{S}_{\chi,N_M}}/(\mathfrak{m}^{r_M}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N_M}}} + \mathfrak{b}_M) = 0,$$

hence

$$\operatorname{Fitt}_{A_{\chi,\infty}} B_{\chi,\infty} = \varprojlim_M \operatorname{Fitt}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N_M}}/(\mathfrak{m}^{r_M}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N_M}}} + \mathfrak{b}_M)} \widetilde{R}^{\square_T}_{\mathcal{S}_{\chi,N_M}}/(\mathfrak{m}^{r_M}_{\widetilde{P}^{\square_T}_{\mathcal{S}_{\chi,N_M}}} + \mathfrak{b}_M) = 0.$$

For the second part, the surjectivity follows from the vanishing of the relative tangent space. Indeed, this tangent space is $\mathfrak{p}_\infty/(\mathfrak{q}_\infty + \mathfrak{p}_\infty^2) \otimes_A E = 0$, since $\mathfrak{p}_\infty/(\mathfrak{q}_\infty + \mathfrak{p}_\infty^2)$ is a torsion A-module. To see that the kernel is nilpotent, we note that the map $A_{\chi,\infty,\mathfrak{q}_\infty} \to B_{\chi,\infty,\mathfrak{q}_\infty} \cong \prod_{\mathfrak{r}_\infty} B_{\chi,\infty,\mathfrak{r}_\infty}$ has nilpotent kernel, the product being over primes $\mathfrak{r}_\infty$ of $B_{\chi,\infty}$ above $\mathfrak{q}_\infty$. However, for any such prime $\mathfrak{r}_\infty$, the rings $B_{\chi,\infty,\mathfrak{p}_\infty}$ and $B_{\chi,\infty,\mathfrak{r}_\infty}$ are isomorphic $A_{\chi,\infty,\mathfrak{q}_\infty}$-algebras, by the $\mu_2 \times \mu_2$-action. The result follows. □

Lemma 4.23. *1. $\widetilde{H}_{\chi,\widetilde{\mathfrak{q}}}$ is a direct factor of $\widetilde{H}_{\chi,\widetilde{P}}$ and is a non-zero free $\widetilde{\Lambda}_{\widetilde{P}}$-module.*

2. $\mathcal{M}_{\chi,\infty,\mathfrak{q}_\infty}$ is a direct factor of $\mathcal{M}_{\chi,\infty,P_\infty}$, and is a non-zero free S_{∞,P_∞}-module, with $\mathcal{M}_{\chi,\infty,\mathfrak{q}_\infty}/\mathfrak{a} \cong \widetilde{H}_{\chi,\widetilde{\mathfrak{q}}}$ compatibly with the action of $A_{\chi,\infty,\mathfrak{q}_\infty} \to \widetilde{P}_{\mathcal{S}_\chi,\widetilde{\mathfrak{q}}}$.

3. The induced map $R^\infty_{\chi,P^\infty} \to B_{\chi,\infty,\mathfrak{p}_\infty}$ is surjective.

Proof. For the first part, $\widetilde{H}_{\chi,\widetilde{P}} = H_\chi \otimes_\Lambda \widetilde{\Lambda}_{\widetilde{P}}$. The action of $\widetilde{P}_{\mathcal{S}_\chi}$ on $\widetilde{H}_\chi$ factors through a quotient $\widetilde{\mathbb{T}}_\chi$ which is finite over $\widetilde{\Lambda}$, hence $\widetilde{\mathbb{T}}_\chi \otimes_{\widetilde{\Lambda}} \widetilde{\Lambda}_{\widetilde{P}}$ has $\widetilde{\mathbb{T}}_{\chi,\widetilde{\mathfrak{q}}}$ as a direct factor. To see that $\widetilde{H}_{\chi,\widetilde{\mathfrak{q}}}$ is non-zero, note that $\widetilde{\mathbb{T}}_{\chi,\widetilde{\mathfrak{q}}}$ is non-zero and acts faithfully on $\widetilde{H}_{\chi,\widetilde{\mathfrak{q}}}$. The second part can be proved in a similar way.

For the third part, we must show that the A-module $\mathfrak{p}_\infty/(P^\infty + \mathfrak{p}_\infty^2)$ vanishes after tensoring with E. This this A-module is the cokernel of an inverse limit of maps whose cokernels are finite torsion A-modules of uniformly bounded cardinality (by hypothesis 1 of the theorem). □

By Lemma 3.40, we know the following:

- Suppose that for each $v \in R$, the characters $\chi_{v,1}, \ldots, \chi_{v,n}$ are pairwise distinct. Let $Q \subset \Lambda$ be a minimal prime. Then there is a unique minimal prime $\mathfrak{p} \subset R^\infty_{\chi,P^\infty}/(Q)$, and $R^\infty_{\chi,P^\infty}/\mathfrak{p}$ is $\mathcal{O}$-flat of dimension $n(n+1)[L^+ : \mathbb{Q}]/2 + n^2|T| + q'$.
- Suppose instead that for each $v \in R$, we have $\chi_{v,1} = \cdots = \chi_{v,n} = 1$. Let $Q \subset \Lambda$ be a minimal prime. Then for each minimal prime $\mathfrak{p} \subset R^\infty_{\chi,P^\infty}/(Q)$, the quotient $R^\infty_{\chi,P^\infty}/\mathfrak{p}$ is $\mathcal{O}$-flat of dimension $n(n+1)[L^+ : \mathbb{Q}]/2 + n^2|T| + q'$. Each minimal prime of $R^\infty_{\chi,P^\infty}/(Q, \lambda)$ contains a unique minimal prime of $R^\infty_{\chi,P^\infty}/(Q)$.

In either case, for each minimal prime $\mathfrak{p} \subset R^{\infty}_{\chi,P^\infty}/(Q)$ we have

$$\begin{aligned}\dim R^{\infty}_{\chi,P^\infty}/(\mathfrak{p}) &= n(n+1)[L^+ : \mathbb{Q}]/2 + n^2|T| + q' \\ &= n(n+1)[L^+ : \mathbb{Q}]/2 + n^2|T| + q - n(n-1)[L^+ : \mathbb{Q}]/2 \\ &= n[L^+ : \mathbb{Q}] + n^2|T| + q = \dim S_{\infty,P_\infty}/(Q).\end{aligned}$$

We can now complete the proof of Theorem 4.19. We recall that we have constructed for each choice of χ a commutative diagram

$$\begin{tikzcd} S_\infty \arrow[r] & A_{\chi,\infty} \arrow[r, two heads] \arrow[d] & \widetilde{P}_{\mathcal{S}_\chi} \arrow[d] \\ R^{\infty}_{\chi} \arrow[r] & B_{\chi,\infty} \arrow[r, two heads] & \widetilde{R}^{\mathrm{univ}}_{\mathcal{S}_\chi} \end{tikzcd}$$

and a module $\mathcal{M}_{\chi,\infty}$ for $A_{\chi,\infty}$ which is free over S_∞ of rank s. For varying χ these diagrams are identified modulo λ. The map $R^{\infty}_{\chi,P^\infty} \to B_{\chi,\infty,\mathfrak{p}_\infty}$ is surjective, and the map $A_{\chi,\infty,\mathfrak{q}_\infty} \to B_{\chi,\infty,\mathfrak{p}_\infty}$ is surjective with nilpotent kernel. We can therefore identify $\operatorname{Spec} A_{\chi,\infty,\mathfrak{q}_\infty}$ with a closed subspace of $\operatorname{Spec} R^{\infty}_{\chi,P^\infty}$.

We now suppose either that $\chi = 1$ or that for each $v \in R$, the characters $\chi_{v,1}, \dots, \chi_{v,n}$ are pairwise distinct. For any minimal prime Q of Λ, $\mathcal{M}_{\chi,\infty,\mathfrak{q}_\infty}/(Q)$ is a free $S_{\infty,P_\infty}/(Q)$-module, and $S_{\infty,P_\infty}/(Q)$ is a regular local ring. It follows that

$$\begin{aligned}\operatorname{depth}_{A_{\chi,\infty,\mathfrak{q}_\infty}/(Q)} \mathcal{M}_{\chi,\infty,\mathfrak{q}_\infty}/(Q) \geq \operatorname{depth}_{S_{\infty,P_\infty}/(Q)} \mathcal{M}_{\chi,\infty,\mathfrak{q}_\infty}/(Q) &= \dim S_{\infty,P_\infty}/(Q) \\ &= \dim R^{\infty}_{\chi,P^\infty}/(Q) \geq \dim A_{\chi,\infty,\mathfrak{q}_\infty}/(Q).\end{aligned}$$

It now follows from the first part of Lemma 1.10 that $\operatorname{Supp}_{A_{\chi,\infty,\mathfrak{q}_\infty}/(Q)} \mathcal{M}_{\chi,\infty,\mathfrak{q}_\infty}/(Q)$ is a union of irreducible components of $\operatorname{Spec} A_{\chi,\infty,\mathfrak{q}_\infty}$ of dimension equal to $\dim R^{\infty}_{\chi,P^\infty}$. Viewing this support as a closed subspace of $\operatorname{Spec} R^{\infty}_{\chi,P^\infty}/(Q)$, we see that it is a union of irreducible components of $\operatorname{Spec} R^{\infty}_{\chi,P^\infty}/(Q)$, which are necessarily all of characteristic zero.

Suppose now that $\chi \neq 1$. Then $\operatorname{Spec} R^{\infty}_{\chi,P^\infty}/(Q)$ is irreducible, and hence

$$\operatorname{Supp}_{A_{\chi,\infty,\mathfrak{q}_\infty}/(Q)} \mathcal{M}_{\chi,\infty,\mathfrak{q}_\infty}/(Q) = \operatorname{Spec} A_{\chi,\infty,\mathfrak{q}_\infty}/(Q) = \operatorname{Spec} R^{\infty}_{\chi,P^\infty}/(Q). \tag{4.1}$$

Because everything is identified modulo λ as χ varies, it follows from (4.1) and the second part of Lemma 1.10 that

$$\begin{aligned}\operatorname{Supp}_{A_{1,\infty,\mathfrak{q}_\infty}/(Q,\lambda)} \mathcal{M}_{1,\infty,\mathfrak{q}_\infty}/(Q,\lambda) &= \operatorname{Supp}_{A_{\chi,\infty,\mathfrak{q}_\infty}/(Q,\lambda)} \mathcal{M}_{\chi,\infty,\mathfrak{q}_\infty}/(Q,\lambda) \\ &= \operatorname{Spec} R^{\infty}_{\chi,P^\infty}/(Q,\lambda) \\ &= \operatorname{Spec} R^{\infty}_{1,P^\infty}/(Q,\lambda) \\ &= \operatorname{Spec} A_{1,\infty,\mathfrak{q}_\infty}/(Q,\lambda).\end{aligned}$$

We now argue directly that

$$\operatorname{Supp}_{A_{1,\infty,\mathfrak{q}_\infty}/(Q)} \mathcal{M}_{1,\infty,\mathfrak{q}_\infty}/(Q) = \operatorname{Spec} A_{1,\infty,\mathfrak{q}_\infty}/(Q) = \operatorname{Spec} R^{\infty}_{1,P^\infty}/(Q). \tag{4.2}$$

It suffices to show that each generic point of $\operatorname{Spec} R^{\infty}_{1,P^\infty}/(Q)$ is contained in $\operatorname{Supp}_{A_{1,\infty,\mathfrak{q}_\infty}/(Q)} \mathcal{M}_{1,\infty,\mathfrak{q}_\infty}/(Q)$. Let $\mathfrak{P} \subset \operatorname{Spec} R^{\infty}_{1,P^\infty}/(Q)$ be a minimal prime (necessarily of characteristic 0), and let $\wp \subset \operatorname{Spec} R^{\infty}_{1,P^\infty}/(\mathfrak{P},\lambda)$ be a minimal prime. Then $\wp$ is minimal in $\operatorname{Supp}_{A_{1,\infty,\mathfrak{q}_\infty}/(Q,\lambda)} \mathcal{M}_{1,\infty,\mathfrak{q}_\infty}/(Q,\lambda)$. It follows from the third part of Lemma 1.10 that $\wp$ is not minimal in $\operatorname{Supp}_{A_{1,\infty,\mathfrak{q}_\infty}/(Q)} \mathcal{M}_{1,\infty,\mathfrak{q}_\infty}/(Q)$. (Note that $\mathcal{M}_{1,\infty,\mathfrak{q}_\infty}/(Q)$ is $\mathcal{O}$-flat, since it is free over $S_{\infty,P_\infty}/(Q)$ and this last ring is itself $\mathcal{O}$-flat.) Since $\mathfrak{P}$ is the unique prime of $\operatorname{Spec} R^{\infty}_{1,P^\infty}/(Q)$ properly contained in $\wp$, we deduce that $\mathfrak{P}$ lies in $\operatorname{Supp}_{A_{1,\infty,\mathfrak{q}_\infty}/(Q)} \mathcal{M}_{1,\infty,\mathfrak{q}_\infty}/(Q)$. Since $\mathfrak{P}$ was arbitrary, this shows the equality (4.2).

We deduce that $A_{1,\infty,\mathfrak{q}_\infty}/(Q)$ acts nearly faithfully on $\mathcal{M}_{1,\infty,\mathfrak{q}_\infty}/(Q)$, and hence that $A_{1,\infty,\mathfrak{q}_\infty}/(Q,\mathfrak{a})$ acts nearly faithfully on $\mathcal{M}_{1,\infty,\mathfrak{q}_\infty}/(Q,\mathfrak{a}) \cong \widetilde{H}_{1,\widetilde{\mathfrak{q}}}/(Q)$. This action factors through the surjective homomorphism

$$A_{1,\infty,\mathfrak{q}_\infty}/(Q,\mathfrak{a}) \to \widetilde{P}_{\mathcal{S}_1,\widetilde{\mathfrak{q}}}/(Q),$$

so we finally deduce that $\widetilde{P}_{\mathcal{S}_1,\widetilde{\mathfrak{q}}}/(Q)$ acts nearly faithfully on $\widetilde{H}_{1,\widetilde{\mathfrak{q}}}/(Q)$. Since Q was arbitrary, it follows that $\widetilde{H}_{1,\widetilde{\mathfrak{q}}}$ is a nearly faithful $\widetilde{P}_{\mathcal{S}_1,\widetilde{\mathfrak{q}}}$-module. This concludes the proof.

5 Taylor–Wiles systems

5.1 Group theory

Let k be a finite field of characteristic $l > 3$, $A = k[\![T]\!]$, and $E = \operatorname{Frac} A$. If M is an A-module and $x \in M$ is killed by a power of T, we will call the least integer $m \geq 0$ such that $T^m x = 0$ the order of x.

Let $\Gamma = \Delta \rtimes \{1, c\}$ be a profinite group. We suppose given a continuous representation $r : \Gamma \to \mathcal{G}_n(A)$ such that $\Delta = r^{-1}(\mathcal{G}_n^0(A))$. This section is devoted to some group theoretical results about such representations, which will allow us to show the first hypothesis of Theorem 4.19 in certain situations. We assume:

1. For each open subgroup $N \subset \Delta$, $r|_N \otimes_A E$ is absolutely irreducible.
2. There exists $\sigma_0 \in \Delta$ such that $r(\sigma_0) \in \mathrm{GL}_n(A)$ is regular semisimple, and its eigenvalues lie in $A^\times$ and do not satisfy any non-trivial $\mathbb{Z}$-linear relation in $A^\times$. (It may be helpful to recall that there is an isomorphism $A^\times \cong k^\times \times \prod_{i=1}^\infty \mathbb{Z}_l$.)
3. The integer n is not divisible by l.
4. Let $\mu = \nu \circ r$. Then $\mu(c) = -1$. Equivalently, writing $r(c) = (J, -\mu(c))$, J is a symmetric matrix.

Proposition 5.1. *Let $G = r(\Delta) \subset \mathrm{GL}_n(A) \times \mathrm{GL}_1(A)$, and let $\overline{G}$ denote the image of G in $\mathrm{PGL}_n(A)$. There exists a closed subfield $K \subset E$ and a descent H of PGL_n to K such that $\overline{G}$ is identified with an open compact subgroup of $H(K)$.*

Proof. By [Pin98, Theorem 0.7], it suffices to show that $\overline{G}$ is Zariski dense in PGL_n, viewed as algebraic group over E. Write J for the connected component of the Zariski closure; this is reductive. Since $\sigma_0 \in \Delta$, J contains a maximal torus of PGL_n. In other words, $J \subset \mathrm{PGL}_n$ is a subgroup of maximal rank. We claim that J has trivial center. Indeed, $J(E) \cap \overline{G} \subset \overline{G}$ is a finite index subgroup. If the center of J is non-trivial, then the inverse image of $J(E)$ in $\mathrm{GL}_n(E)$ centralizes a finite index subgroup of $r|_\Delta(\Delta) \subset \mathrm{GL}_n(E)$, which contradicts our assumption that for each open subgroup $N \subset \Delta$, $r|_N \otimes_A E$ is absolutely irreducible. By the Borel–de Siebenthal theorem (the naïve generalization which holds here, since $l > 3$; see [Gil]), it follows that $J = \mathrm{PGL}_n$. ☐

Lemma 5.2. *Suppose that $\Delta' \subset \Delta$ is a closed normal subgroup such that Δ/Δ' is abelian. Let $G' = r(\Delta')$, and let $\overline{G}'$ denote the image of G' in $\mathrm{PGL}_n(E)$. Then:*

1. *The group $\overline{G}'$ has finite index in $\overline{G}$.*
2. *Let $\operatorname{ad}^0 r \subset \operatorname{ad} r$ denote the subspace of trace 0 endomorphisms. There exists an integer $K_0 \geq 1$ such that for any integers $a, m \geq 0$ and any $A[G']$-submodule $M \subset \operatorname{ad}^0 r \otimes_A A/T^m$ containing an element of exact order $m - a$, we have*
$$T^{a+K_0} \operatorname{ad}^0 r \otimes_A A/T^m \subset M.$$
In particular, $T^{K_0} H^0(G', \operatorname{ad}^0 r \otimes_A A/T^m) = 0$.
3. *Suppose further that $\mu|_{\Delta'} = 1$ and Δ' is normalized by c. Then there exists an integer $K_1 \geq 1$ such that*
$$T^{K_1} H^1(G', \operatorname{ad}^0 r \otimes_A A/T^m)^{c=-1} = 0$$
for all $m \geq 0$.

Proof. For the first part, it is enough to show that the maximal Hausdorff abelian quotient of $\overline{G}$ is finite. The triple $(K, H, \overline{G})$ is minimal, in the sense of [Pin98, Definition 0.1]. It follows from [Pin98, Theorem 0.2, (c)] and the fact that the universal covering $H' \to H$ of H has non-vanishing derivative that the closed subgroup of $\overline{G}$ generated by commutators is open in $\overline{G}$. Since $\overline{G}$ is profinite, this implies the desired finiteness. The proof of the second part of the lemma is elementary, using that the adjoint representation of PGL_n is irreducible and $\overline{G}$ is Zariski dense in H.

We now come to the third part of the lemma. Let $[\phi] \in H^1(G', \mathrm{ad}^0 r \otimes_A A/T^m)^{c=-1}$ be a cohomology class. Since $l \neq 2$, we can assume that $\phi^c = -\phi$, where by definition we have

$$\phi^c(\sigma) = \mathrm{ad}(c)\phi(\sigma^c) = -J^t \phi(\sigma^c) J^{-1}.$$

For $\sigma \in G'$, we have $\phi(\sigma^c) = J^t \phi(\sigma) J^{-1}$ and $\sigma^c = J^t \sigma^{-1} J^{-1}$ (since $\mu|_{\Delta'} = 1$). Let us write $\rho : G' \to \mathrm{GL}_n(A)$ for the representation induced by the composite $G' \subset \mathrm{GL}_n(A) \times \mathrm{GL}_1(A) \to \mathrm{GL}_n(A)$. We can view ϕ as attached to a representation $\rho_\phi : G' \to \mathrm{GL}_n(A \oplus \epsilon A/T^m)$ via the formula $\rho_\phi(\sigma) = (1 + \epsilon\phi(\sigma))\rho(\sigma)$. The cocycle ϕ represents the trivial cohomology class if and only if this representation is $1 + \epsilon M_{n\times n}(A/T^m)$-conjugate to ρ. For each $\sigma \in G'$, we have

$$\mathrm{tr}\, \rho_\phi(\sigma^{-1}) = \mathrm{tr}\, \rho(\sigma^{-1}) + \epsilon\, \mathrm{tr}\, \phi(\sigma^{-1})\rho(\sigma^{-1}),$$

$$\mathrm{tr}\, \rho_\phi(\sigma^c) = \mathrm{tr}\, \rho(\sigma^c) + \epsilon\, \mathrm{tr}\, \phi(\sigma^c)\rho(\sigma^c) = \mathrm{tr}\, \rho(\sigma^c) + \epsilon\, \mathrm{tr}\, \phi(\sigma)\rho(\sigma^{-1}) = \mathrm{tr}\, \rho(\sigma^c) - \epsilon\, \mathrm{tr}\, \phi(\sigma^{-1})\rho(\sigma^{-1}).$$

Thus if σ^c and σ^{-1} are G'-conjugate, then $\mathrm{tr}\, \rho_\phi(\sigma^{-1}) = \mathrm{tr}\, \rho(\sigma^{-1})$. Let X denote the closure of the set of elements $\sigma \in G'$ such that σ^c and σ^{-1} are G'-conjugate. Then X is stable under conjugation by G', and if $\sigma \in X$, then $\mathrm{tr}\, \rho_\phi(\sigma^m) = \mathrm{tr}\, \rho(\sigma^m)$ for each $m \in \mathbb{Z}$. Writing the characteristic polynomials of $\rho(\sigma)$ and $\rho_\phi(\sigma)$ for $\sigma \in G'$ as

$$\det(x1_n - \rho(\sigma)) = \sum_{i=0}^{n} (-1)^i \Lambda_i(\sigma) x^{n-i},$$

$$\det(x1_n - \rho_\phi(\sigma)) = \sum_{i=0}^{n} (-1)^i \Lambda_{\phi,i}(\sigma) x^{n-i},$$

we obtain for each $\sigma \in X$ the equalities

$$\begin{gathered} \mathrm{tr}\, \rho(\sigma) = \Lambda_1(\sigma) = \Lambda_{\phi,1}(\sigma) = \mathrm{tr}\, \rho_\phi(\sigma), \\ \Lambda_2(\sigma) = \Lambda_{\phi,2}(\sigma),\ \Lambda_3(\sigma) = \Lambda_{\phi,3}(\sigma). \end{gathered} \tag{5.1}$$

Indeed, $\Lambda_1(\sigma)$, $\Lambda_2(\sigma)$ and $\Lambda_3(\sigma)$ can be expressed in terms of $\mathrm{tr}\, \rho(\sigma), \mathrm{tr}\, \rho(\sigma^2)$ and $\mathrm{tr}\, \rho(\sigma^3)$. (We use here that the characteristic is $l > 3$.)

We claim that X has positive Haar measure in G'. Conjugation by c induces an involution of $\overline{G}'$, which by [Pin98, Corollary 0.3] is induced by an involution θ of H. (We apply *loc. cit.* to the inverse image of $\overline{G}$ in $H'(K)$; this is valid since the isogeny $H' \to H$ is separable and H' is simply connected. Of course, after extending scalars to E, θ is given by the map $g \mapsto J^t g^{-1} J^{-1}$.) Since K is infinite, we can choose a regular semisimple element x of $\mathfrak{h}^{\theta=-1}$; then the centralizer of x in H is a maximal torus T on which θ acts by $h \mapsto h^{-1}$ (cf. [Lev07], Lemma 2.4). (This uses that the matrix J is symmetric.) If $g \in G'$ maps to an element of $T(K) \cap \overline{G}' \subset \overline{G}'$, and $\mathrm{tr}\, g^{-1} \neq 0$, then $g \in X$. Indeed, there is a scalar $\lambda \in E^\times$ such that $g^c = \lambda g^{-1}$, hence (by comparing traces) $\lambda = 1$ and $g^c = g^{-1}$. Let $T(K)^0 \subset T(K)$ denote the subset of elements $t \in T(K)$ with $\mathrm{tr}\, t^{-1} \neq 0$. It follows from the above remarks that

$$(T(K)^0 \cap \overline{G}')^{G'} = \{gtg^{-1} \mid t^{-1} \in T(K)^0 \cap \overline{G}', g \in G'\}$$

in G' is contained in X. Moreover, this pre-image will have positive Haar measure in G' if $(T(K)^0 \cap \overline{G}')^{G'}$ has positive Haar measure in $\overline{G}'$. To prove that X has positive Haar measure in G', it therefore suffices to show that $(T(K)^0 \cap \overline{G}')^{G'}$ has positive measure in $\overline{G}'$, and this is true since $(T(K)^0 \cap \overline{G}')^{G'}$ contains an open subset of $\overline{G}'$. Indeed, the adjoint map $H \times T \to H$ is smooth in the neighborhood of an element $(1, t)$ with $t \in T(K)^0$ regular; now apply [Ser06, Part II, Ch. III, §10].

For any proper Zariski closed subset $C \subset \mathrm{PGL}_n$ over E, $C(E) \cap \overline{G}'$ has Haar measure zero. This can be deduced easily from the argument of [Tay93, §2, Lemma 2]. In particular, given countably many Zariski-closed subsets $C_1, C_2, \dots$ of PGL_n, we can find elements of X whose image in $\mathrm{PGL}_n(E)$ is not contained in any $C_i(E), i \geq 1$. Using this observation, we choose $\sigma \in X$ whose eigenvalues $\lambda_1, \dots, \lambda_n$ satisfy the following conditions:

1. For each sequence of integers $r_1 = 0 < r_2 < \cdots < r_n$, the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1^{r_1} & \lambda_2^{r_1} & \cdots & \lambda_n^{r_1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{r_n} & \lambda_2^{r_n} & \cdots & \lambda_n^{r_n} \end{pmatrix}$$

   is not zero.

2. Write $M = \binom{n}{2}$. For each sequence of integers $r_1 = 0 < r_2 < \cdots < r_M$, the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ (\lambda_1\lambda_2)^{r_2} & (\lambda_1\lambda_3)^{r_2} & \cdots & (\lambda_{n-1}\lambda_n)^{r_2} \\ (\lambda_1\lambda_2)^{r_3} & (\lambda_1\lambda_3)^{r_3} & \cdots & (\lambda_{n-1}\lambda_n)^{r_3} \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_1\lambda_2)^{r_M} & (\lambda_1\lambda_3)^{r_M} & \cdots & (\lambda_{n-1}\lambda_n)^{r_M} \end{pmatrix}$$

   is not zero.

3. Write instead $M = \binom{n}{3}$. For each sequence of integers $r_1 = 0 < r_2 < \cdots < r_M$, the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ (\lambda_1\lambda_2\lambda_3)^{r_2} & (\lambda_1\lambda_2\lambda_4)^{r_2} & \cdots & (\lambda_{n-2}\lambda_{n-1}\lambda_n)^{r_2} \\ (\lambda_1\lambda_2\lambda_3)^{r_3} & (\lambda_1\lambda_2\lambda_4)^{r_3} & \cdots & (\lambda_{n-2}\lambda_{n-1}\lambda_n)^{r_3} \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_1\lambda_2\lambda_3)^{r_M} & (\lambda_1\lambda_2\lambda_4)^{r_M} & \cdots & (\lambda_{n-2}\lambda_{n-1}\lambda_n)^{r_M} \end{pmatrix}$$

   is not zero.

Moreover, since the definition of X does not depend on ϕ, the element σ be chosen independently of ϕ. After possibly enlarging E and replacing ϕ by $T^N\phi$ for some N depending only on σ, we may assume that $\rho(\sigma) = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. For each integer $t \geq 0$ we have

$$\mathrm{tr}\,\rho_\phi(\sigma^t) - \mathrm{tr}\,\rho(\sigma^t) = \epsilon \sum_{i=1}^{n} t\lambda_i^t \phi(\sigma)_{i,i} = 0.$$

Multiplying ϕ by the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1^{l+1} & \lambda_2^{l+1} & \cdots & \lambda_n^{l+1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{(n-1)l+1} & \lambda_2^{(n-1)l+1} & \cdots & \lambda_n^{(n-1)l+1} \end{pmatrix}$$

we can thus suppose that $\phi(\sigma)_{i,i} = 0$ for each i. Further multiplying ϕ by the element $\lambda_i/\lambda_j - 1$ with highest valuation, $i \neq j$, we can alter ϕ by a coboundary to assume that $\phi(\sigma) = 0$ and hence $\rho_\phi(\sigma) = \rho(\sigma)$. Indeed, for any $y \in \operatorname{ad} r \otimes_A A/T^m$, we have $(\sigma y \sigma^{-1} - y)_{i,j} = (\lambda_i/\lambda_j - 1)y_{i,j}$, so after scaling ϕ we can find y with $(\sigma y \sigma^{-1} - y) = \phi(\sigma)$.

Since X has positive measure we can find integers $r_1 = 0 < r_2 < \cdots < r_{\binom{n}{3}}$ such that $Y = X \cap \sigma^{-r_2} X \cap \cdots \cap \sigma^{-r_{\binom{n}{3}}} X$ has positive measure. If $\gamma \in Y$, then for each $M = 1, \ldots, n$ we have

$$\operatorname{tr} \rho_\phi(\sigma^{r_M}\gamma) = \sum_{i=1}^{n} \lambda_i^{r_M} \rho_\phi(\gamma)_{i,i} \in A.$$

Thus after multiplying ϕ by the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1^{r_2} & \lambda_2^{r_2} & \cdots & \lambda_n^{r_2} \\ \lambda_1^{r_3} & \lambda_2^{r_3} & \cdots & \lambda_n^{r_3} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{r_n} & \lambda_2^{r_n} & \cdots & \lambda_n^{r_n} \end{pmatrix}$$

the diagonal entries of $\rho_\phi(\gamma)$ must lie in A.

Now since Y has positive measure, we can choose $\tau \in Y$ satisfying the following conditions:

1. Write $\rho(\tau) = x$. For all integers $1 \leq i, j \leq n$, $x_{i,j}$ is not zero.
2. For all integers $1 \leq i < j < k \leq n$, the determinant of the matrix

   $$\begin{pmatrix} x_{i,j} & x_{j,i} \\ x_{k,k}x_{i,j} - x_{i,k}x_{k,j} & x_{j,i}x_{k,k} - x_{k,i}x_{j,k} \end{pmatrix}$$

   is not zero.
3. For each sequence $s_1 = 0 < s_2 < \cdots < s_n$, and for each integer $t = 1, \ldots, n$ define a matrix $A(t)_{i,j} = \rho(\tau^{s_i})_{t,j}$. Then $\det A(t) \neq 0$.

We claim that after changing ϕ by a coboundary and multiplying by a power of T depending only on σ and τ, we shall have $\rho_\phi(\tau) = \rho(\tau)$. To see this, first note that $\rho_\phi(\tau)_{i,i} = \rho(\tau)_{i,i}$ for each $i = 1, \ldots n$. After multiplying ϕ by the element $\rho(\tau)_{j,j+1}$ with largest valuation and scaling the basis elements of A^n by elements of $1 + \epsilon A/T^m$ (equivalently, changing ϕ by a coboundary), we can assume that $\rho_\phi(\tau)_{j,j+1} = \rho(\tau)_{j,j+1}$ for each $j = 1, \ldots, n-1$ and $\rho_\phi(\sigma) = \rho(\sigma)$.

We now use the equalities (5.1), i.e. that for $i = 1, 2, 3$ and for each $g \in X$, we have $\Lambda_i(g) = \Lambda_{\phi,i}(g)$. We consider first $\Lambda_3(g)$. This is the sum, up to signs, of the determinants of the 3×3 submatrices obtained by fixing $1 \leq i < j < k \leq n$ and taking the intersection of the i, j, k rows and the i, j, k columns. Comparing these determinants for $\rho(\tau)$ and $\rho(\sigma^m\tau)$, we see that they differ by $(\lambda_i\lambda_j\lambda_k)^m$. Thus, multiplying ϕ by the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ (\lambda_1\lambda_2\lambda_3)^{r_2} & (\lambda_1\lambda_2\lambda_4)^{r_2} & \cdots & (\lambda_{n-2}\lambda_{n-1}\lambda_n)^{r_2} \\ (\lambda_1\lambda_2\lambda_3)^{r_3} & (\lambda_1\lambda_2\lambda_4)^{r_3} & \cdots & (\lambda_{n-2}\lambda_{n-1}\lambda_n)^{r_3} \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_1\lambda_2\lambda_3)^{r_{\binom{n}{3}}} & (\lambda_1\lambda_2\lambda_4)^{r_{\binom{n}{3}}} & \cdots & (\lambda_{n-2}\lambda_{n-1}\lambda_n)^{r_{\binom{n}{3}}} \end{pmatrix},$$

the determinants of these 3×3 submatrices agree for $\rho(\tau)$ and $\rho_\phi(\tau)$. Multiplying by the determinant of a similar $\binom{n}{2} \times \binom{n}{2}$ matrix, we obtain the same result for the 2×2 submatrices obtained by deleting all but the i, j rows and columns for fixed $i \neq j$.

We now show by induction on $|i - j|$ that, modifying ϕ at each step in a way depending only on $\rho(\tau)$, we can assume that $\rho(\tau)_{i,j} = \rho_\phi(\tau)_{i,j}$ for each i, j. The cases of $i = j$ and $i = j - 1$ have already been solved. To conserve notation let us temporarily write $\rho(\tau) = x$ and $\rho_\phi(\tau) = x + \epsilon X$. To take care of the remaining case when $j = i - 1$, note that determinants

$$\det \begin{pmatrix} x_{i,i} & x_{i,j} \\ x_{j,i} & x_{j,j} \end{pmatrix} = x_{i,i}x_{j,j} - x_{i,j}x_{j,i},$$

$$\det \begin{pmatrix} x_{i,i} & x_{i,j} \\ x_{j,i} + \epsilon X_{j,i} & x_{j,j} \end{pmatrix} = x_{i,i}x_{j,j} - x_{i,j}x_{j,i} - \epsilon X_{j,i}x_{i,j}$$

have been shown to be equal; multiplying ϕ by $x_{i,j}$ kills $X_{j,i}$.

For the induction step, fix $i < k < j$ and consider the determinants

$$\det \begin{pmatrix} x_{i,i} & x_{i,j} \\ x_{j,i} & x_{j,j} \end{pmatrix} = \det \begin{pmatrix} x_{i,i} & x_{i,j} + \epsilon X_{i,j} \\ x_{j,i} + \epsilon X_{j,i} & x_{j,j} \end{pmatrix},$$

$$\det \begin{pmatrix} x_{i,i} & x_{i,k} & x_{i,j} \\ x_{k,i} & x_{k,k} & x_{k,j} \\ x_{j,i} & x_{j,k} & x_{j,j} \end{pmatrix} = \det \begin{pmatrix} x_{i,i} & x_{i,k} & x_{i,j} + \epsilon X_{i,j} \\ x_{k,i} & x_{k,k} & x_{k,j} \\ x_{j,i} + \epsilon X_{j,i} & x_{j,k} & x_{j,j} \end{pmatrix},$$

resulting in equations

$$x_{i,j}X_{j,i} + x_{j,i}X_{i,j} = 0$$

and

$$(x_{k,k}x_{i,j} - x_{i,k}x_{k,j})X_{j,i} + (x_{j,i}x_{k,k} - x_{k,i}x_{j,k})X_{i,j} = 0.$$

Multiplying ϕ by the determinant of the matrix

$$\begin{pmatrix} x_{i,j} & x_{j,i} \\ x_{k,k}x_{i,j} - x_{i,k}x_{k,j} & x_{j,i}x_{k,k} - x_{k,i}x_{j,k} \end{pmatrix}$$

completes the induction step.

Since Y has positive measure, we can find integers $s_1 = 0 < s_2 < \cdots < s_n$ such that $Z = Y \cap \tau^{-s_2} Y \cap \cdots \cap \tau^{-s_n} Y$ has positive measure. If $\zeta \in Z$ then for each $M = 1, \ldots, n$ we have $\tau^{s_M}\zeta \in Y$ and hence for each $i = 1, \ldots, n$

$$\rho_\phi(\tau^{s_M}\zeta)_{i,i} = \sum_{k=1}^{n} \rho(\tau^{s_M})_{i,k}\rho_\phi(\zeta)_{k,i} = \rho(\tau^{s_M}\zeta)_{i,i} = \sum_{k=1}^{n} \rho(\tau^{s_M})_{i,k}\rho(\zeta)_{k,i}.$$

After multiplying ϕ by the quantity $\det A(i)$ with largest valuation, we have that $\rho_\phi(\zeta) = \rho(\zeta)$ for all $\zeta \in Z$, and hence for all ζ in the closed subgroup generated by Z. Being of positive measure, this subgroup is open of finite index in G, and contains an open normal subgroup N. We note that the definition of Z depends only on G' and not on ϕ, and N can therefore be chosen to depend only on G' and not on ϕ. We now use the inflation restriction exact sequence for N:

$$0 \to H^1(G'/N, (\mathrm{ad}^0\, r \otimes_A A/T^m)^N) \to H^1(G', \mathrm{ad}^0\, r \otimes_A A/T^m) \to H^1(N, \mathrm{ad}^0\, r \otimes_A A/T^m).$$

We have shown that for any cocycle ϕ for G' with $\phi^c = -\phi$, the restriction of the cohomology class of ϕ to N is annihilated by T^{K_2} for some integer K_2 depending on G' but not on ϕ. By the second part of the lemma applied to N, the first group in this sequence is annihilated by some T^{K_0} depending only on N. The third part of the lemma now follows on taking $K_1 = K_0 + K_2$. □

5.2 Galois theory

We now return to the notation of the beginning of §3. Thus F is a CM field with maximal totally real subfield F^+. We fix a finite set of places S of F^+ which split in F, and write $F(S)$ for the maximal extension of F unramified outside S. We write $G_{F^+,S} = \mathrm{Gal}(F(S)/F^+)$ and $G_{F,S} \subset G_{F^+,S}$ for the subgroup fixing F. For each $v \in S$ we choose a place $\widetilde{v}$ of F above it, and write $\widetilde{S}$ for the set of these places. We choose a complex conjugation $c \in G_{F^+,S}$. We suppose that S contains all places dividing l. We fix a finite extension $K/\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$ with ring of integers $\mathcal{O}$, residue field k, and maximal ideal λ, and define as in the previous section $A = k[\![T]\!]$, $E = \operatorname{Frac} E$. We assume that $l > 3$.

We now suppose given a representation $r : G_{F^+,S} \to \mathcal{G}_n(A)$ satisfying the following conditions:

1. $r|_{G_{F,S}} \otimes_A E$ is absolutely irreducible.
2. $\zeta_l \notin F$, $\overline{r}|_{G_{F^+(\zeta_l)}}$ is Schur, and $\overline{r}|_{G_F}$ is primitive (i.e. not induced from any proper subgroup of G_F).
3. The image of $\overline{r}|_{G_{F(\zeta_l)}}$ has no non-trivial quotients of l-power order. This will be the case if, for example, the irreducible constituents of $\overline{r}|_{G_{F(\zeta_l)}}$ are adequate in the sense of [Tho12].
4. There exists $\sigma_0 \in G_{F,S}$ such that $r(\sigma_0) \in \mathrm{GL}_n(A)$ is regular semisimple, and its eigenvalues lie in $A^\times$ and do not satisfy any non-trivial $\mathbb{Z}$-linear relation in $A^\times$.
5. The integer n is not divisible by l.
6. Let $\mu = \nu \circ r$. Then $\mu(c) = -1$. Equivalently, writing $r(c) = (J, -\mu(c))$, J is a symmetric matrix.

We set $\Delta = G_F$ and $\Gamma = G_{F^+} = G_F \rtimes \{1, c\}$.

Proposition 5.3. *For every open subgroup $N \subset \Delta$, $r|_N$ is absolutely irreducible.*

We can therefore apply the results of §5.1.

Proof. Suppose not. After replacing N by an open subgroup, we can assume that N is normal in Δ. We have $\sigma_0^a \in N$ for some $a \geq 1$, and hence the representation $r|_N \otimes_A E$ is multiplicity-free (being already semisimple). Let $\rho \subset r|_N \otimes_A E$ be a simple subrepresentation. By [Kar89, Ch. 2, Theorem 2.2] (i.e. Clifford theory), it follows that the action of N on ρ extends to an action of a subgroup $N' \subset \Delta$ and that there is an isomorphism $r|_\Delta \cong \operatorname{Ind}_{N'}^{\Delta} \rho$ over E. Then $\overline{r}|_\Delta^{\mathrm{ss}} \cong \operatorname{Ind}_{N'}^{\Delta} \overline{\rho}^{\mathrm{ss}}$. Since we have assumed that $\overline{r}|_\Delta$ is primitive, it follows that $N' = \Delta$ and $\rho = r|_N \otimes_A E$, and hence this representation is irreducible. Enlarging the field E does not affect our hypotheses, so we see that the representation is even absolutely irreducible. □

We now introduce a slight variant of the cohomology groups defined in [CHT08, §2.2]. Fix a Λ-algebra structure on A and a deformation problem

$$\mathcal{S} = \left(F/F^+, S, \widetilde{S}, \Lambda, \overline{r}, \epsilon^{1-n}\delta_{F/F^+}^n, \{\mathcal{D}_v\}_{v \in S} \right),$$

such that r is of type $\mathcal{S}$. We fix $T = S$, $\widetilde{T} = \widetilde{S}$. (Thus T denotes both a set of places of F^+ and an element of the base ring A, but we hope that this will not cause confusion.) Fix also a choice of Taylor–Wiles data $(Q_N, \widetilde{Q}_N, \{\overline{\alpha}_v\}_{v \in Q_N})$ of order q and level N. (We allow the case $q = 0$.) If n_v is the multiplicity of $\overline{\alpha}_v$ as an eigenvalue of $\overline{r}(\mathrm{Frob}_{\widetilde{v}})$ then we suppose n_v prime to l. We write $S_N = S \cup Q_N$, $\widetilde{S}_N = \widetilde{S} \cup \widetilde{Q}_N$. This induces an auxiliary deformation problem

$$\mathcal{S}_N = \left(F/F^+, S_N, \widetilde{S}_N, \Lambda, \overline{r}, \epsilon^{1-n}\delta_{F/F^+}^n, \{\mathcal{D}_v\}_{v \in S} \cup \{\mathcal{D}_v^{\mathrm{TW}}(\overline{\alpha}_v)\}_{v \in Q_N} \right).$$

Then for each integer $m \geq 1$ and place $v \in Q_N$ there are associated submodules $L_{v,m} \subset H^1(G_{F_{\widetilde{v}}}, \operatorname{ad} r \otimes_A A/T^m)$. To define these, we note that the choice of Taylor–Wiles data induces a direct sum decomposition $r|_{G_{F_{\widetilde{v}}}} = s_v \oplus \psi_v$. We have a natural map

$$H^1(G_{F_{\widetilde{v}}}, \operatorname{ad} r \otimes_A A/T^m) \to H^1(I_{\widetilde{v}}, \operatorname{ad} r \otimes_A A/T^m)^{G_{F_{\widetilde{v}}}}$$

$$\hookrightarrow H^1(I_{\widetilde{v}}, \operatorname{ad} s_v \otimes_A A/T^m) \oplus H^1(I_{\widetilde{v}}, \operatorname{ad} \psi_v \otimes_A A/T^m).$$

We define $L_{v,m}$ to be the pre-image in $H^1(G_{F_{\widetilde{v}}}, \operatorname{ad} r \otimes_A A/T^m)$ of the submodule $H^1(I_{\widetilde{v}}, Z(\psi_v) \otimes_A A/T^m)$, where $Z(\psi_v) \subset \operatorname{ad} \psi_v$ denotes the submodule of diagonal matrices. We write $L^1_{v,m}$ for the pre-image in the cochain group $C^1(G_{F_{\widetilde{v}}}, \operatorname{ad} r \otimes_A A/T^m)$ of $L_{v,m}$. Then we define

$$C^i_{\mathcal{S}_N,T}(G_{F^+,S_N}, \operatorname{ad} r \otimes_A A/T^m) =$$

$$C^i(G_{F^+,S_N}, \operatorname{ad} r \otimes_A A/T^m) \oplus \bigoplus_{v \in S_N} C^{i-1}(G_{F_{\widetilde{v}}}, \operatorname{ad} r \otimes_A A/T^m)/M^{i-1}_{v,m},$$

where $M^i_{v,m} = 0$ unless $v \in Q_N$ and $i = 0$, in which case we set $M^0_{v,m} = C^0(G_{F_{\widetilde{v}}}, \operatorname{ad} r \otimes_A A/T^m)$ or $v \in Q_N$ and $i = 1$, in which case we set $M^1_{v,m} = L^1_{v,m}$. The boundary map is given by the formula

$$\partial(\phi, (\psi_{\widetilde{v}})_{v \in S_N}) = (\partial\phi, (\phi|_{F_{\widetilde{v}}} - \partial\psi_{\widetilde{v}})_{v \in S_N}).$$

The groups $H^*_{\mathcal{S}_N,T}(G_{F^+,S_N}, \operatorname{ad} r \otimes_A A/T^m)$ are then by definition the cohomology groups of this complex.

We are also given dual Selmer conditions $L^\perp_{v,m} \subset H^1(G_{F_{\widetilde{v}}}, \operatorname{ad} r(1) \otimes_A A/T^m)$ for $v \in Q_N$, defined to be the annihilator of $L_{v,m}$ under the local duality pairing. We define a group

$$H^1_{\mathcal{S}_N^\perp,T}(G_{F^+,S_N}, \operatorname{ad} r(1) \otimes_A A/T^m) =$$

$$\ker H^1(G_{F^+,S_N}, \operatorname{ad} r(1) \otimes_A A/T^m) \to \bigoplus_{v \in Q_N} H^1(G_{F_{\widetilde{v}}}, \operatorname{ad} r(1) \otimes_A A/T^m)/L^\perp_{v,m}.$$

Finally, we write

$$H^1_{\mathcal{S}_N,T}(G_{F^+,S_N}, \operatorname{ad} r \otimes_A E/A) = \varinjlim_m H^1_{\mathcal{S}_N,T}(G_{F^+,S_N}, \operatorname{ad} r \otimes_A A/T^m),$$

and similarly for $H^1_{\mathcal{S}_N^\perp,T}(G_{F^+,S_N}, \operatorname{ad} r(1) \otimes_A E/A)$.

Proposition 5.4. *1. For each $m \geq 0$, we have*

$$H^1_{\mathcal{S}_N,T}(G_{F^+,S_N}, \operatorname{ad} r \otimes_A A/T^m) = H^1_{\mathcal{S}_N,T}(G_{F^+,S_N}, \operatorname{ad} r \otimes_A E/A)[T^m].$$

2. For each $m \geq 0$, we have

$$H^1_{\mathcal{S}_N^\perp,T}(G_{F^+,S_N}, \operatorname{ad} r(1) \otimes_A A/T^m) = H^1_{\mathcal{S}_N^\perp,T}(G_{F^+,S_N}, \operatorname{ad} r(1) \otimes_A E/A)[T^m].$$

3. For each $m \geq 0$, we have

$$|H^1_{\mathcal{S}_N,T}(G_{F^+,S_N}, \operatorname{ad} r \otimes_A A/T^m)| =$$

$$|H^1_{\mathcal{S}_N^\perp,T}(G_{F^+,S_N}, \operatorname{ad} r(1) \otimes_A A/T^m)| \times |A/T^m|^{q-n(n-1)/2[F^+:\mathbb{Q}]}.$$

Proof. Write $\mathcal{M}_m = \operatorname{ad} r \otimes_A A/T^m$, $\mathcal{M} = \varinjlim_m \mathcal{M}_m = \operatorname{ad} r \otimes_A E/A$. We have exact sequences for every $m' \geq m$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M}_m \arrow[r] & \mathcal{M}_{m'} \arrow[r] & \mathcal{M}_{m'-m} \arrow[r] & 0. \end{tikzcd}$$

Since $\bar{r}$ is Schur, we have $H^0(G_{F^+,S}, \mathcal{M}_1) = 0$ and hence $H^0(G_{F^+,S}, \mathcal{M}_m) = 0$ for each $m \geq 1$. It follows that we have exact sequences

$$\begin{tikzcd} 0 \arrow[r] & H^1(G_{F^+,S}, \mathcal{M}_m) \arrow[r] & H^1(G_{F^+,S}, \mathcal{M}_{m'}) \arrow[r] & H^1(G_{F^+,S}, \mathcal{M}_{m'-m}) \end{tikzcd}$$

for each $m' \geq m$. Since the multiplication by T^m map on $\mathcal{M}_{m'}$ factors $\mathcal{M}_{m'} \to \mathcal{M}_{m'-m} \hookrightarrow \mathcal{M}_{m'}$ and this last inclusion also induces an injection on H^1, we find that we can identify

$$H^1(G_{F^+,S}, \mathcal{M}_m) = H^1(G_{F^+,S}, \mathcal{M}_{m'})[T^m].$$

The second part now follows on noting that for every $m' \geq m$ and $v \in Q_N$, the natural maps

$$H^1(G_{F_{\widetilde{v}}}, \mathcal{M}_m(1))/L_{v,m}^{\perp} \longrightarrow H^1(G_{F_{\widetilde{v}}}, \mathcal{M}_{m'}(1))/L_{v,m'}^{\perp}$$

are injective, and moreover that $H^0(G_{F^+,S}, \mathcal{M}_m(1)) = H^0(G_{F^+,S}, \mathcal{M}_m) = 0$ (since $\overline{r}|_{G_{F^+(\zeta_l)}}$ is Schur; cf. Lemma 3.3).

For the first part, we note (cf. the discussion after [CHT08, Definition 2.2.7]) that $H^1_{\mathcal{S}_N,T}(G_{F^+,S_N}, \mathcal{M}_m)$ fits into an exact sequence

$$0 \longrightarrow \oplus_{v \in T} H^0(G_{F_{\widetilde{v}}}, \mathcal{M}_m) \longrightarrow H^1_{\mathcal{S}_N,T}(G_{F^+,S_N}, \mathcal{M}_m) \longrightarrow H^1(G_{F^+,S_N}, \mathcal{M}_m).$$

It follows that we have inclusions for every $m' \geq m$

$$H^1_{\mathcal{S}_N,T}(G_{F^+,S_N}, \mathcal{M}_m) \subset H^1_{\mathcal{S}_N,T}(G_{F^+,S_N}, \mathcal{M}_{m'})[T^m].$$

We show equality. Suppose that $(\phi, (\alpha_{\widetilde{v}})_{v \in S})$ represents a cohomology class in the group on the right. By the above reasoning, we can assume that $\phi \in Z^1(G_{F^+,S}, \mathcal{M}_m)$. Then we have

$$T^m[(\phi, (\alpha_{\widetilde{v}})_{v \in S})] = [(0, (T^m \alpha_{\widetilde{v}})_{v \in S})] = 0,$$

and so there exists $Q \in \mathcal{M}_{m'}$ such that $(0, (T^m \alpha_{\widetilde{v}})_{v \in S}) = \partial Q$. Thus $Q \in H^0(G_{F^+,S}, \mathcal{M}_{m'}) = 0$ and hence $T^m \alpha_v = 0$, as desired.

The third part is proved exactly as in [CHT08, Lemma 2.3.4]. (We use here our assumption that the conjugate self-duality of r is symmetric; in the notation of *loc. cit.*, this means that $\chi(c_v) = -1$ for every choice of complex conjugation $c_v \in G_{F^+,S}$. We also use the calculation of the length of the finite A-module $L_{v,m}$ for $v \in Q_N$; if $m = 1$, then $\mathcal{M}_m = \operatorname{ad} \overline{r}$ and we have

$$\dim_k L_{v,1} - \dim_k H^0(G_{F_{\widetilde{v}}}, \operatorname{ad} \overline{r}) = 1.)$$

□

We now define certain field extensions. Let F^+_∞ be the extension of F^+ obtained by adjoining all l-power roots of unity. For m, $N \geq 1$, let $L_{m,N}$ be the extension of $F^+(\zeta_{l^N})$ cut out by the representation

$$r \bmod T^m : G_{F^+,S} \to \mathcal{G}_n(A/T^m).$$

Let L_∞ be the extension of F^+_∞ cut out by r.

Lemma 5.5. *Let* $\operatorname{ad} r = \operatorname{ad}^0 r \oplus Z$ *denote the natural decomposition of* $\operatorname{ad} r$ *into its trace 0 and diagonal parts.*

1. *For every* $N, m \geq 1$,
$$H^1(\operatorname{Gal}(F(\zeta_{l^N})/F^+), Z(1) \otimes_A A/T^m) = 0.$$

2. *There exists an integer* $K_0 > 0$, *not depending on* N *or* m, *such that for every* $N, m \geq 1$ *and* $a \geq 0$ *and for any* $G_{F^+(\zeta_{l^N})}$*-submodule* $M \subset \operatorname{ad}^0 r \otimes_A A/T^m$ *containing an element of exact order* $m - a$, *we have*
$$T^{a+K_0} \operatorname{ad}^0 r \otimes_A A/T^m \subset M.$$
In particular, we have $T^{K_0} H^0(G_{F^+(\zeta_{l^N})}, \operatorname{ad}^0 r \otimes_A A/T^m) = 0$.

3. *There exists an integer* $K_1 > 0$, *not depending on* N *or* m, *such that every* N, $m \geq 1$,
$$T^{K_1} H^1(\operatorname{Gal}(L_{m,N}/F^+), \operatorname{ad}^0 r(1) \otimes_A A/T^m) = 0.$$

Proof. There is an isomorphism $Z(1) \otimes_A A/T^m \cong k(\epsilon\delta_{F/F^+}) \otimes_k A/T^m$ of $G_{F^+,S}$-modules. By restriction, we have an isomorphism

$$H^1(\mathrm{Gal}(F(\zeta_{l^N})/F^+), k(\epsilon\delta_{F/F^+})) \cong H^1(\mathrm{Gal}(F(\zeta_{l^N})/F(\zeta_l)), k)^{\epsilon\delta_{F/F^+}}.$$

This latter group is identified with the set of homomorphisms $f : \mathrm{Gal}(F(\zeta_{l^N})/F(\zeta_l)) \to k$ such that for all $x \in \mathrm{Gal}(F(\zeta_{l^N})/F(\zeta_l))$, $y \in G_{F^+,S}$, we have $f(yxy^{-1}) = \epsilon\delta_{F/F^+}(y)f(x)$. Since the conjugation action of $G_{F^+,S}$ is trivial, but the character $\epsilon\delta_{F/F^+}$ is non-trivial, this group is in fact 0. This shows the first part of the lemma. The second part follows from the second part of Lemma 5.2, applied with $\Delta' = G_{F \cdot F^+_\infty}$. For the third part, we use the inflation-restriction exact sequence. First, we have an exact sequence

$$\begin{aligned} 0 \to H^1(\mathrm{Gal}(F^+_\infty/F^+), (\mathrm{ad}^0\, r(1) \otimes_A A/T^m)^{G_{F^+_\infty}}) \to H^1(\mathrm{Gal}(L_\infty/F^+), \mathrm{ad}^0\, r(1) \otimes_A A/T^m) \to \\ \to H^1(\mathrm{Gal}(L_\infty/F^+_\infty), \mathrm{ad}^0\, r(1) \otimes_A A/T^m)^{\mathrm{Gal}(F^+_\infty/F^+)}. \end{aligned} \tag{5.2}$$

The first term in the sequence is killed by a power of T independent of m, by the second part of Lemma 5.2. Moreover, inflation gives an injection of A-modules

$$H^1(\mathrm{Gal}(L_{m,N}/F^+), \mathrm{ad}^0\, r(1) \otimes_A A/T^m) \hookrightarrow H^1(\mathrm{Gal}(L_\infty/F^+), \mathrm{ad}^0\, r(1) \otimes_A A/T^m).$$

To complete the proof of the lemma, it therefore enough to show that the last term of the sequence (5.2) is killed by a power of T independent of m. Restriction gives an isomorphism

$$H^1(\mathrm{Gal}(L_\infty/F^+_\infty), \mathrm{ad}^0\, r(1) \otimes_A A/T^m)^{\mathrm{Gal}(F^+_\infty/F^+)} \cong H^1(\mathrm{Gal}(L_\infty/F \cdot F^+_\infty), \mathrm{ad}^0\, r(1) \otimes_A A/T^m)^{\mathrm{Gal}(F \cdot F^+_\infty/F^+)},$$

and this last term is a submodule of

$$H^1(\mathrm{Gal}(L_\infty/F \cdot F^+_\infty), \mathrm{ad}^0\, r(1) \otimes_A A/T^m)^{c=1} \cong H^1(\mathrm{Gal}(L_\infty/F \cdot F^+_\infty), \mathrm{ad}^0\, r \otimes_A A/T^m)^{c=-1}$$

(since $\epsilon(c) = -1$). The desired result then follows from the third part of Lemma 5.2 applied with $\Delta' = G_{F \cdot F^+_\infty}$. ☐

Lemma 5.6. *We can find a constant $C > 0$, an integer $q \geq n(n-1)/2[F^+ : \mathbb{Q}]$ and for each $N \geq 1$ a choice of Taylor–Wiles data $(Q_N, \widetilde{Q}_N, \{\overline{\alpha}_v\}_{v \in Q_N})$ of order q and level N such that:*

1. *For all $N, m \geq 1$,*
$$H^1_{\mathcal{S}_N^\perp, T}(G_{F^+,S_N}, \mathrm{ad}\, r(1) \otimes_A A/T^m)$$
is a finite A-module of cardinality bounded by C.
2. *For each $N \geq 1$, there is an isomorphism of A-modules*
$$H^1_{\mathcal{S}_N, T}(G_{F^+,S_N}, \mathrm{ad}\, r \otimes_A E/A) \cong (E/A)^{q - n(n-1)/2[F^+:\mathbb{Q}]} \oplus T(N),$$
where $T(N)$ is a finite A-module of cardinality bounded by C.

Proof. By Proposition 5.4, the second part of the lemma will follow from the first. We prove the first. Suppose given a tuple $(Q_N, \widetilde{Q}_N, \{\overline{\alpha}_v\}_{v \in Q_N})$ of Taylor–Wiles data of level N, and consider adding an extra place u to Q_N to obtain $Q'_N = Q_N \cup \{u\}$, $\widetilde{Q}'_N = \widetilde{Q}_N \cup \{\widetilde{u}\}$ for some place $\widetilde{u}$ of F above u, and choosing an eigenvalue $\overline{\alpha}_u$ of $\overline{r}|_{G_{F_{\widetilde{u}}}}(\mathrm{Frob}_{\widetilde{u}})$ such that $\overline{r}|_{G_{F_{\widetilde{u}}}}(\mathrm{Frob}_{\widetilde{u}})$ acts semisimply on its $\overline{\alpha}_u$-generalized eigenspace, to obtain a new choice of Taylor–Wiles data:

$$(Q'_N, \widetilde{Q}'_N, \{\overline{\alpha}_v\}_{v \in Q_N} \cup \{\overline{\alpha}_u\}).$$

Then, writing $\mathcal{S}_N$ and $\mathcal{S}'_N$ for the respective augmented deformation problems, we have for each $m \geq 1$ a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathcal{S}'_N{}^\perp, T}(G_{F^+,S'_N}, \mathrm{ad}\, r(1) \otimes_A A/T^m) \arrow[r] \arrow[d] & H^1_{\mathcal{S}_N^\perp, T}(G_{F^+,S_N}, \mathrm{ad}\, r(1) \otimes_A A/T^m) \arrow[r] \arrow[d] & A/T^m \arrow[d] \\
0 \arrow[r] & H^1_{\mathcal{S}'_N{}^\perp, T}(G_{F^+,S'_N}, \mathrm{ad}\, r(1) \otimes_A E/A) \arrow[r] & H^1_{\mathcal{S}_N^\perp, T}(G_{F^+,S_N}, \mathrm{ad}\, r(1) \otimes_A E/A) \arrow[r] & E/A
\end{tikzcd}$$

The last arrow in each row is given on cocycles by the map $\phi \mapsto \operatorname{tr} e_{\mathrm{Frob}_{\widetilde{u}},\overline{\alpha}_u}\phi(\mathrm{Frob}_{\widetilde{u}})$, where $e_{\mathrm{Frob}_{\widetilde{u}},\overline{\alpha}_u}$ is by definition the unique $r|_{G_{F_{\widetilde{u}}}}$-equivariant projection of A^n onto a direct summand A-module lifting the $\overline{\alpha}_u$-eigenspace of $\overline{r}|_{G_{F_{\widetilde{u}}}}(\mathrm{Frob}_{\widetilde{u}})$.

Suppose that $H^1_{\mathcal{S}_N^{\perp},T}(G_{F^+,S_N}, \operatorname{ad} r(1) \otimes_A E/A) \cong (E/A)^r \oplus X$, where X is a finite A-module, annihilated by T^M. Suppose that there exist integers $K, m \geq 0$ and a cohomology class $[\phi] \in H^1_{\mathcal{S}_N^{\perp},T}(G_{F^+,S_N}, \operatorname{ad} r(1) \otimes_A A/T^m)$ of exact order m such that the image of this class in A/T^m has exact order at least $m - K$, with $m - K > M$. Then there is an isomorphism $H^1_{\mathcal{S}'_N{}^{\perp},T}(G_{F^+,S'_N}, \operatorname{ad} r(1) \otimes_A E/A) \cong (E/A)^{r-1} \oplus X'$, where $|X'| \leq |X| \times |A/T^K|$.

To prove the lemma it therefore suffices to show that there is an integer $K \geq 0$, such that for any Q_N as above, and for any integer m and cohomology class $[\phi] \in H^1_{\mathcal{S}_N^{\perp},T}(G_{F^+,S_N}, \operatorname{ad} r(1) \otimes_A A/T^m)$ of exact order m, we can find data $(u, \widetilde{u}, \widetilde{\alpha}_u)$ such that $\operatorname{tr} e_{\mathrm{Frob}_{\widetilde{u}},\overline{\alpha}_u}\phi(\mathrm{Frob}_{\widetilde{u}}) \in A/T^m$ has exact order at least $m - K$. (We can then use induction on the size of Q_N.) We will show that we can choose $K = K_0 + K_1$, where K_0 and K_1 are as defined in Lemma 5.5. Let us therefore fix a choice of Q_N, an integer m, and a cohomology class $[\phi] \in H^1_{\mathcal{S}_N^{\perp},T}(G_{F^+,S_N}, \operatorname{ad} r(1) \otimes_A A/T^m)$ of exact order m.

Using the decomposition $\operatorname{ad} r(1) = \operatorname{ad}^0 r(1) \oplus Z(1)$, we can suppose that ϕ is valued either in $\operatorname{ad}^0 r(1) \otimes_A A/T^m$ or $Z(1) \otimes_A A/T^m$. Suppose first that ϕ is valued in $\operatorname{ad}^0 r(1) \otimes_A A/T^m$. By the third part of Lemma 5.5, the image of $[\phi]$ in $H^1(G_{L_{m,N}}, \operatorname{ad}^0 r(1) \otimes_A A/T^m)^{G_{F^+}}$ has order at least $m - K_1$. Let us write f for this restriction, which can be viewed as a homomorphism $f : G_{L_{m,N}} \to \operatorname{ad}^0 r \otimes_A A/T^m$ whose image is invariant under the action of $G_{F^+(\zeta_l)}$. Moreover, the image of f contains an element of exact order $m - K_1$.

By the second part of Lemma 5.5, we have

$$\mathrm{image} f \supset T^{K_0+K_1} \operatorname{ad}^0 r(1) \otimes_A A/T^m.$$

Choose $\sigma \in G_{F(\zeta_{l^N})}$ such that $\overline{r}|_{G_F}(\sigma)$ has an eigenvalue $\overline{\alpha}$ of multiplicity $1 \leq p \leq n-1$, with $(p, l) = 1$, and such that $\overline{r}|_{G_F}(\sigma)$ acts semisimply on its $\overline{\alpha}$-generalized eigenspace. (It is easy to see that such a σ exists.) If $\operatorname{tr} e_{\sigma,\overline{\alpha}}\phi(\sigma)$ has exact order at least $m - K_0 - K_1$, then let $\sigma_0 = \sigma$. Otherwise, we can find $\tau \in G_{L_{m,N}}$ such that $\operatorname{tr} e_{\sigma,\overline{\alpha}} f(\tau)$ has exact order at least $m - K_0 - K_1$. Now set $\sigma_0 = \tau\sigma$. In this case we have $r(\sigma_0) \bmod T^m = r(\sigma) \bmod T^m$ and $\phi(\sigma_0) = \phi(\tau) + \phi(\sigma)$, so that

$$\operatorname{tr} e_{\sigma_0,\overline{\alpha}}\phi(\sigma_0) = \operatorname{tr} e_{\sigma,\overline{\alpha}}\phi(\sigma) + \operatorname{tr} e_{\sigma,\overline{\alpha}} f(\tau)$$

also has order at least $m - K_0 - K_1$. In either case, we see that $\operatorname{tr} e_{\sigma_0,\overline{\alpha}}\phi(\sigma_0)$ has exact order at least $m - K_0 - K_1$. By the Chebotarev density theorem, we can now find a place u of F^+ with extension $\widetilde{u}$ to F, split in $F^+(\zeta_{l^N})$, such that $\operatorname{tr} e_{\mathrm{Frob}_{\widetilde{u}},\overline{\alpha}}\phi(\mathrm{Frob}_{\widetilde{u}})$ has exact order at least $m - K_0 - K_1$. This completes the proof in this case.

Now suppose instead that ϕ is valued in $Z(1)$. By the first part of Lemma 5.5, the image of $[\phi]$ in $H^1(G_{F(\zeta_{l^N})}, Z(1) \otimes_A A/T^m)$ has exact order m. We can view this image as a homomorphism

$$\phi : G_{F(\zeta_{l^N})} \to Z \otimes_A A/T^m.$$

Choose $\sigma \in G_{F(\zeta_{l^N})}$ such that $\phi(\sigma)$ has exact order m. Now, $\ker\phi$ projects surjectively onto $\overline{r}|_{G_F}(G_{F(\zeta_{l^N})})$, since this latter group has no non-trivial quotients of l-power order. In particular, we can choose $\tau \in \ker\phi$ such that $\overline{r}|_{G_F}(\tau\sigma)$ has an eigenvalue $\overline{\alpha}$ of multiplicity p prime to l, and acts semisimply on its $\overline{\alpha}$-generalized eigenspace. Taking $\sigma_0 = \tau\sigma$, we have $\operatorname{tr} e_{\sigma_0,\overline{\alpha}}\phi(\sigma_0) = p\phi(\sigma)$, which is therefore of exact order m. Applying the Chebotarev density theorem once more now completes the proof. ☐

Corollary 5.7. *Suppose that we are in the situation of §4.6, and that the hypotheses of this section hold for* $r = r_{\mathfrak{p}}$ *and* $\mathcal{S} = \mathcal{S}_\chi$. *Then hypothesis 1 of Theorem 4.19 holds.*

Proof. With notation in §4.6, we must construct an isomorphism of A-modules

$$\operatorname{Hom}_A(\widetilde{\mathfrak{p}}_N/(\widetilde{P}^{\mathrm{loc}} + \widetilde{\mathfrak{p}}_N^2), E/A) \cong H^1_{\mathcal{S}_N,T}(G_{F^+,S_N}, \operatorname{ad} r \otimes_A E/A).$$

The first term here is naturally isomorphic to

$$\mathrm{Hom}_A(\widetilde{R}^{\square_T}_{\mathcal{S}_{\chi,N}}/(\widetilde{\mathfrak{p}}_N^2, \widetilde{P}^{\mathrm{loc}}), A \oplus \epsilon E/A).$$

Let $(r; \alpha_v)_{v \in T}$ denote a representative of the T-framed deformation of $\overline{r}$ over A corresponding to the ideal $\widetilde{\mathfrak{p}}_N$. The above group is in bijection with the set of equivalence classes of T-framed liftings $(\widetilde{r}; \widetilde{\alpha}_v)_{v \in T}$ of $\overline{r}$ to $A \oplus \epsilon E/A$ which are equivalent to $(r; \alpha_v)_{v \in T}$ after reduction modulo ϵ, are of type $\mathcal{S}_N$, and such that for each $v \in T$, $\widetilde{\alpha}_v^{-1} \widetilde{r}|_{G_{F_{\widetilde{v}}}} \widetilde{\alpha}_v$ is equal to $r|_{G_{F_{\widetilde{v}}}}$. This set is itself in bijection with the set of T-framed liftings $(\widetilde{r}; \widetilde{\alpha}_v)_{v \in T}$ over $A \oplus \epsilon E/A$ which are equal to $(r; \alpha_v)_{v \in T}$ after reduction modulo ϵ, are of type $\mathcal{S}_N$, and such that for all $v \in T$, $\widetilde{\alpha}_v^{-1} \widetilde{r}|_{G_{F_{\widetilde{v}}}} \widetilde{\alpha}_v = r|_{G_{F_{\widetilde{v}}}}$, taken up to $1 + \epsilon M_n(E/A)$-conjugation. (It is an abuse of language to speak of liftings to $A \oplus \epsilon E/A$, since this ring does not lie in $\mathcal{C}_\Lambda$; however, this does not cause problems, cf. the discussion before [CHT08, Definition 2.2.2].)

Given such a T-framed lifting $(\widetilde{r}; \widetilde{\alpha}_v)_{v \in T}$, we write $\widetilde{r} = (1 + \epsilon \phi) r$, with $\phi \in Z^1(G_{F^+, S_N}, \operatorname{ad} r \otimes_A E/A)$, and $\widetilde{\alpha}_v = \alpha_v + \epsilon \psi_{\widetilde{v}}$, $\psi_{\widetilde{v}} \in M_n(E/A)$. The cohomology class

$$[(\phi, (\psi_{\widetilde{v}})_{v \in T})] \in H^1_{\mathcal{S}_N, T}(G_{F^+, S_N}, \operatorname{ad} r \otimes_A E/A)$$

then depends only on $(\widetilde{r}; \widetilde{\alpha}_v)_{v \in T}$ up to $1 + \epsilon M_n(E/A)$-conjugation, and it is now easy to check (following [CHT08, Proposition 2.2.9]) that this assignment gives the desired isomorphism of A-modules. $\square$

6 The main argument

In this section we combine the results of the previous two sections to prove the analogue of an $R = \mathbb{T}$ theorem in our context. We take up the notations of the beginning of §4.3. Thus L is a CM field with maximal totally real subfield L^+, G is a unitary group over L^+ of dimension n, and $S = T = S_l \cup R \cup S(B) \cup S_a$ is a set of primes of L^+ split in L. We fix an open compact subgroup $U = \prod_v U_v$ of $G(\mathbb{A}^\infty_{L^+})$ having the following form:

- For v inert in L, $U_v \subset G(L_v^+)$ is a hyperspecial maximal compact subgroup.
- For $v \notin T$ split in L, $U_v = G(\mathcal{O}_{L_v^+})$.
- For $v \in S(B), U_v$ is the unique maximal compact subgroup.
- For $v \in S_l, U_v = G(\mathcal{O}_{L_v^+})$.
- For $v \in S_a, U_v = \iota_{\widetilde{v}}^{-1} \ker(\mathrm{GL}_n(\mathcal{O}_{L_{\widetilde{v}}}) \to \mathrm{GL}_n(k(\widetilde{v})))$.
- For $v \in R, U_v = \iota_{\widetilde{v}}^{-1} \operatorname{Iw}(\widetilde{v})$.

We suppose that $\mathfrak{m} \subset \mathbb{T}_1^T(U(\mathfrak{l}^\infty), \mathcal{O})$ is a residually Schur maximal ideal, giving rise to a residual Galois representation $\overline{r}_\mathfrak{m} : G_{L^+, S} \to \mathcal{G}_n(k)$. We suppose that $\overline{r}_\mathfrak{m}$ satisfies the following hypotheses:

1. $\overline{r}_\mathfrak{m}(G_{L(\zeta_l)})$ has no non-trivial quotients of l-power order. This will be the case if the irreducible constituents of $\overline{r}_\mathfrak{m}|_{G_{L(\zeta_l)}}$ are adequate, in the sense of [Tho12].
2. $\overline{r}_\mathfrak{m}|_{G_{L,S}} = \overline{\rho}_1 \oplus \overline{\rho}_2$ is a direct sum of two absolutely irreducible representations. If $\dim \overline{\rho}_i = n_i$ then $n_1 n_2 (n_1 + n_2)$ is coprime to l.
3. $\overline{r}_\mathfrak{m}|_{G_{L,S}}$ is primitive, i.e. not induced from a proper subgroup of $G_{L,S}$.

We suppose that S_a is non-empty and that for every $v \in S_a$, v is absolutely unramified, $\overline{r}_\mathfrak{m}$ is unramified above v, $\operatorname{ad} \overline{r}(\mathrm{Frob}_{\widetilde{v}}) = 1$ and v does not split in $L(\zeta_l)$. Then $H^0(G_{L_{\widetilde{v}}}, \operatorname{ad} \overline{r}(1)) = 0$, and U is sufficiently small. Suppose also that for each $v \in S_l \cup R \cup S(B)$, $\overline{r}_\mathfrak{m}|_{G_{L_{\widetilde{v}}}}$ is trivial, and that for each $v \in R \cup S(B)$, $q_v \equiv 1 \bmod l$. We suppose as well that K is sufficiently large in the sense that it contains the image of every embedding $L \hookrightarrow \overline{\mathbb{Q}}_l$, and the conclusions of Proposition 3.15 and Proposition 3.17 hold for $v \in R \cup S(B)$. Under these assumptions we have defined a global deformation problem

$$\mathcal{S}_1 = \left(L/L^+, T, \widetilde{T}, \Lambda, \overline{r}_\mathfrak{m}, \epsilon^{1-n} \delta^n_{L/L^+}, \{R_v^\triangle\}_{v \in S_l} \cup \{R_v^{\mathrm{St}}\}_{v \in S(B)} \cup \{R_v^\square\}_{v \in S_a} \cup \{R_v^1\}_{v \in R} \right).$$

This section is devoted to the proof of the following theorem.

Theorem 6.1. *With assumptions as above, suppose further that:*

1. *The quotient $R^{red}_{\mathcal{S}_1}$ of $R^{univ}_{\mathcal{S}_1}$ classifying reducible deformations is finite over Λ and of dimension bounded above by $n[L^+ : \mathbb{Q}] - rn(n+1) - 5$, where $r = |R|$.*
2. *The prime l is strictly greater than 3, and for each $v \in R$, the highest power of l dividing $q_v - 1$ is strictly greater than n.*
3. *For each $v \in S_l$, we have $[L_{\widetilde{v}} : \mathbb{Q}_l] > \sup(rn(n+1) + 5, n(n-1)/2 + 1)$.*

Let $r : G_{L^+,S} \to \mathcal{G}_n(\mathcal{O})$ be a lifting of $\overline{r}_{\mathfrak{m}}$ of type $\mathcal{S}_1$ such that $r|_{G_L}$ is ordinary of weight λ, for some $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(L,\overline{\mathbb{Q}}_l)}$. Then r is automorphic of weight λ.

Let us say that a soluble CM extension M/L is good if it is linearly disjoint from the extension of $L(\zeta_l)$ cut out by $\overline{r}_{\mathfrak{m}}|_{G_{L(\zeta_l)}}$ and every prime above $S_l \cup S_a \cup R$ splits in M. If M/L is a good extension we have constructed in §4.5 a deformation problem $\mathcal{S}_{1,M}$ and a diagram of Λ_M-algebras

$$\begin{tikzcd}
R^{\mathrm{univ}}_{\mathcal{S}_1} & P_{\mathcal{S}_1} \arrow[l] \arrow[r] & \mathbb{T}^T_1(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}} \\
R^{\mathrm{univ}}_{\mathcal{S}_{1,M}} \arrow[u] & P_{\mathcal{S}_{1,M}} \arrow[l] \arrow[r] \arrow[u] & \mathbb{T}^{T_M}_1(U_M(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}_M} \arrow[u]
\end{tikzcd}$$

We have defined an ideal

$$J_{\mathcal{S}_1,M} = \ker\left(P_{\mathcal{S}_{1,M}} \to \mathbb{T}^{T_M}_1(U_M(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}_M}\right).$$

We write $J_M = J_{\mathcal{S}_1,M} P_{\mathcal{S}_1}$. Let $\mathfrak{p} \subset R^{\mathrm{univ}}_{\mathcal{S}_1}$ be a prime ideal. We say that $\mathfrak{p}$ is *potentially pro-automorphic* if there exists a good extension M/L such that $J_M \subset \mathfrak{p}$.

Let $\mathfrak{p} \subset R^{\mathrm{univ}}_{\mathcal{S}_1}$ be a prime ideal of dimension 1 and characteristic l. For each $v \in S_l$ there are universal characters $\psi^v_1, \ldots, \psi^v_n : I^{\mathrm{ab}}_{L_{\widetilde{v}}}(l) \to \Lambda^\times$ (cf. §3.3.2). Let A denote the normalization of $R^{\mathrm{univ}}_{\mathcal{S}}/\mathfrak{p}$, and $E = \operatorname{Frac} A$. We say that $\mathfrak{p}$ is generic if it satisfies the following properties:

- The representation $r_{\mathfrak{p}}|_{G_L} \otimes_A E$ is absolutely irreducible.
- For each $v \in S_l$, the characters $\psi^v_1, \ldots, \psi^v_n$ are distinct modulo $\mathfrak{p}$.
- There exists $v \in S_l$ and $\sigma \in I^{\mathrm{ab}}_{L_{\widetilde{v}}}(l)$ such that the elements $\psi^v_1(\sigma) \bmod \mathfrak{p}, \ldots, \psi^v_n(\sigma) \bmod \mathfrak{p} \in A^\times$ satisfy no non-trivial $\mathbb{Z}$-linear relation.

The interest of these concepts is the following consequence of our work so far.

Proposition 6.2. *Let $\mathfrak{p} \subset R^{univ}_{\mathcal{S}_1}$ be a prime which is potentially pro-automorphic and generic. Suppose further that for each $v \in R$, the restriction $r_{\mathfrak{p}}|_{G_{L_{\widetilde{v}}}}$ is trivial. Then every minimal prime $Q \subset \mathfrak{p}$ is potentially pro-automorphic.*

Proof. By hypothesis, there exists a good extension M_0/L such that $J_{M_0} \subset \mathfrak{p}$. By making a further soluble extension, we can find a good extension M_1/L containing M_0 such that for every prime $\widetilde{w}$ of M_1 above a prime of $S(B)$, $r_{\mathfrak{p}}|_{G_{M_{1,\widetilde{w}}}}$ is unramified and $r_{\mathfrak{p}}(\mathrm{Frob}_{\widetilde{w}})$ is scalar. In fact, if $t_{\widetilde{w}}$ denotes a generator of the l-part of tame inertia at the place $\widetilde{w}$, $r_{\mathfrak{p}}(t_{\widetilde{w}})$ is a unipotent matrix in $\mathrm{GL}_n(A)$, hence of finite (and l-power) order. After making a finite local extension to kill off the image of inertia, Frobenius is mapped to a unipotent element times a scalar matrix (since $q_v \equiv 1 \bmod l$). A further l-power extension now gives a local representation of the desired form.

Then $J_{M_1} \subset J_{M_0}$, by Proposition 4.18. Let $\mathfrak{p}_{M_1} \subset R^{\mathrm{univ}}_{\mathcal{S}_{1,M_1}}$ denote the pullback of $\mathfrak{p}$. Then $J_{\mathcal{S}_{1,M_1}} \subset \mathfrak{p}_{M_1}$. Let Q be as in the proposition, and let Q_{M_1} denote its pullback to $R^{\mathrm{univ}}_{\mathcal{S}_{1,M_1}}$. We will show that $J_{\mathcal{S}_{1,M_1}} \subset Q_{M_1}$. This will imply $J_{M_1} \subset Q$, which is what we need to prove. By Lemma 3.30 and Proposition 5.3, $r_{\mathfrak{p}}|_{G_{M_1}} \otimes_A E$ is absolutely irreducible and so $\mathfrak{p}_{M_1}$ is generic. Arguing as in the proof of Lemma 3.38, we can find a character $\psi : G_{M_1,S_{M_1}} \to 1 + \mathfrak{m}_A$ such that $\mathfrak{p}_{M_1,\psi}$ is defined and satisfies hypotheses 2–5 of Theorem 4.19. In particular, we have $J_{\mathcal{S}_1,M_1} \subset \mathfrak{p}_{M_1,\psi}$ by Corollary 4.14, and we can choose the character ψ so that $\psi|_{G_{M_1,\tilde{v}}}$ is trivial for each $v \in R_{M_1}$, because

$$[M_1^+ : \mathbb{Q}] = [M_1^+ : L^+][L^+ : \mathbb{Q}] > |R_{M_1}| = [M_1^+ : L^+]|R|,$$

by hypothesis 3 of Theorem 6.1. By Corollary 5.7, $\mathfrak{p}_{M_1,\psi}$ also satisfies hypothesis 1. Let $Q' \subset Q_{M_1}$ be a minimal prime of $R^{\mathrm{univ}}_{\mathcal{S}_{1,M_1}}$. Then $Q' \subset \mathfrak{p}_{M_1}$, so $Q' \subset \mathfrak{p}_{M_1,\psi}$, by Lemma 3.38. Corollary 4.20 now implies that $J_{\mathcal{S}_{1,M_1}} \subset Q' \subset Q_{M_1}$. This completes the proof of the proposition. (In order to make sure that the hypotheses of Theorem 4.19 are satisfied, we use here assumptions 2 and 3 of Theorem 6.1.) ☐

Proof of Theorem 6.1. Let $J^{\mathrm{red}} \subset R^{\mathrm{univ}}_{\mathcal{S}_1}$ denote the ideal cutting out the subspace of reducible deformations, and let $I^{\mathrm{red}} \subset \Lambda$ denote the pullback of J^{red} to Λ. Hypothesis 1 of Theorem 6.1 implies that the dimension of Λ/I^{red} is at most $n[L^+ : \mathbb{Q}] - rn(n+1) - 5$.

For each $v \in l$, let $\sigma_1^v, \dots, \sigma_{d_v}^v$ denote a basis of a maximal free $\mathbb{Z}_l$-summand of $\mathcal{O}^\times_{L_{\tilde{v}}}(l)$, where $d_v = [L_{\tilde{v}} : \mathbb{Q}_l]$. For each $i, j, v \in S_l$, define an ideal

$$I(i,j,v) = (\lambda, \{\psi_i^v(\sigma_k) - \psi_j^v(\sigma_k)\}_{k=1,\dots,d_v}) \subset \Lambda.$$

Then $\Lambda/I(i,j,v)$ has dimension $n[L^+ : \mathbb{Q}] - d_v$. On the other hand, suppose given for each $v \in S_l$ an $n \times d_v$ matrix of integers $a^v_{i,j}$ such that each column contains a non-zero entry. Let $J(a^v_{i,j})$ denote the ideal of Λ generated by λ and the elements

$$\left(\prod_{i=1}^{n} \psi_i^v(\sigma_j)^{a^v_{i,j}}\right) - 1 \text{ as } j = 1, \dots, d_v \text{ and } v \in S_l.$$

Then $\Lambda/J(a^v_{i,j})$ has dimension $(n-1)[L^+ : \mathbb{Q}]$. (These ideals are related to the notion of being generic. If $\mathfrak{p} \subset \Lambda/(\lambda)$ is a prime and $\mathfrak{p}$ contains no ideal $I(i,j,v)$, then for each $v \in S_l$, the characters ψ_v^i mod $\mathfrak{p}$ and ψ_v^j mod $\mathfrak{p}$ are distinct if $i \neq j$. If $\mathfrak{p}$ contains no ideal $J(a^v_{i,j})$, then there exists $v \in S_l$ and $\sigma \in I_{L_{\tilde{v}}}(l)$ such that the elements $\psi_v^1(\sigma)$ mod $\mathfrak{p}, \dots, \psi_v^n(\sigma)$ mod $\mathfrak{p} \in (\Lambda/\mathfrak{p})^\times$ satisfy no non-trivial $\mathbb{Z}$-linear relation.)

Together I^{red}, $I(i,j,v)$ and $J(a^v_{i,j})$ define a countable collection of ideals of Λ whose quotients have dimension bounded above by $n[L^+ : \mathbb{Q}] - rn(n+1) - 5$. (This uses assumption 3 of Theorem 6.1.) It follows from Lemma 1.9 that for any good extension M/L, any quotient of $R^{\mathrm{univ}}_{\mathcal{S}_1}/(\lambda, J_M)$ of dimension at least $n[L^+ : \mathbb{Q}] - rn(n+1) - 4$ contains a generic potentially pro-automorphic prime $\mathfrak{p}$. (Note that $R^{\mathrm{univ}}_{\mathcal{S}_1}/J_M$ is finite over Λ, since it is finite over $R^{\mathrm{univ}}_{\mathcal{S}_{1,M}}/J_{\mathcal{S}_{1,M}}$ (by Proposition 4.17), hence over $P_{\mathcal{S}_{1,M}}/J_{\mathcal{S}_{1,M}} = \mathbb{T}_1^{T_M}(U_M(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}_M}$ (by Proposition 3.29), hence over Λ_M (by Proposition 4.3).)

Fix a choice of lifting $r^{\mathrm{univ}}_{\mathcal{S}_1}$ representing the universal deformation. This induces for each $v \in R$ a homomorphism $R_v^1 \to R^{\mathrm{univ}}_{\mathcal{S}_1}$, and we let J_R denote the ideal generated by the images of $\mathfrak{m}_{R_v^1}, v \in R$. This ideal is independent of the choice of lifting, and for any quotient $R^{\mathrm{univ}}_{\mathcal{S}_1}/I$ of characteristic l, we have $\dim R^{\mathrm{univ}}_{\mathcal{S}_1}/(J_R, I) \geq \dim R^{\mathrm{univ}}_{\mathcal{S}_1}/I - rn^2$, by [Mat89, Theorem 15.1]. It follows that there exists a generic prime $\mathfrak{p} \subset R^{\mathrm{univ}}_{\mathcal{S}_1}/(J_R, J_L)$, since $\dim R^{\mathrm{univ}}_{\mathcal{S}_1}/J_L \geq \dim \Lambda = 1 + n[L^+ : \mathbb{Q}]$. (Indeed, there is a finite ring map $\mathbb{T}_1^T(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}} \to R^{\mathrm{univ}}_{\mathcal{S}_1}/J_L$ with nilpotent kernel, and we have $\dim \mathbb{T}_1^T(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}} = \Lambda$, by Proposition 4.3.) By Proposition 6.2, any minimal prime $Q \subset \mathfrak{p}$ of $R^{\mathrm{univ}}_{\mathcal{S}_1}$ is potentially pro-automorphic.

We now consider the partition of the set of minimal primes of $R^{\mathrm{univ}}_{\mathcal{S}_1}$ into two sets $\mathcal{C}_1, \mathcal{C}_2$, consisting of those primes which respectively are and are not potentially pro-automorphic. We have shown that $\mathcal{C}_1$ is non-empty. We claim that $\mathcal{C}_2$ is empty. Otherwise, it follows from Lemma 3.21 and Definition 1.7 that we can find minimal primes $Q_1 \in \mathcal{C}_1, Q_2 \in \mathcal{C}_2$ such that

$$\dim R^{\mathrm{univ}}_{\mathcal{S}_1}/(Q_1, Q_2) \geq c(R^{\mathrm{univ}}_{\mathcal{S}_1}) \geq n[L^+ : \mathbb{Q}] - rn - 2,$$

and hence

$$\dim R^{\mathrm{univ}}_{\mathcal{S}_1}/(Q_1, Q_2, J_R) \geq n[L^+ : \mathbb{Q}] - rn - rn^2 - 3 = n[L^+ : \mathbb{Q}] - rn(n+1) - 3.$$

In particular, this ring contains a generic potentially pro-automorphic prime $\mathfrak{p}$. Applying Proposition 6.2 once more, we deduce that Q_2 is potentially pro-automorphic, a contradiction.

Now let $r : G_{L^+,S} \to \mathcal{G}_n(\mathcal{O})$ be a lifting of $\overline{r}_\mathfrak{m}$ which is ordinary of weight λ and of type $\mathcal{S}_1$, as in the statement of the theorem. This induces a homomorphism $R^{\text{univ}}_{\mathcal{S}_1} \to \mathcal{O}$. Let Q be a minimal prime contained inside the kernel of this homomorphism. Then there is a good extension M/L such that $J_M \subset Q$, and so the induced homomorphism $R^{\text{univ}}_{\mathcal{S}_{1,M}} \to \mathcal{O}$ kills $J_{\mathcal{S}_{1,M}}$, and the map $P_{\mathcal{S}_{1,M}} \to \mathcal{O}$ induced by r factors through $\mathbb{T}_1^{T_M}(U_M(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}_M}$. It now follows from [Ger, Lemma 2.6.4] and [CHT08, Proposition 3.3.2] that $r|_{G_M}$ is automorphic. The automorphy of $r|_{G_L}$ then follows from Lemma 2.7. (The representation $r|_{G_L}$ is irreducible because the set $S(B)$ is non-empty.) $\square$

7 The main theorem

Let $l > 3$ be a prime. Let K be a finite extension of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$, with ring of integers $\mathcal{O}$ and residue field k. In this section we prove the following result.

Theorem 7.1. *Let F be an imaginary CM number field with maximal totally real subfield F^+, and let $n \geq 2$ be an integer. Suppose that $\rho : G_F \to \mathrm{GL}_n(K)$ is a continuous semisimple representation satisfying the following hypotheses.*

1. *$\rho^c \cong \rho^\vee \epsilon^{1-n}$.*
2. *ρ is ramified at only finitely many places.*
3. *ρ is ordinary of weight λ, for some $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_l)}$.*
4. *$F(\zeta_l)$ is not contained in $\overline{F}^{\ker \mathrm{ad}(\overline{\rho}^{ss})}$.*
5. *$\overline{\rho}^{ss} \cong \overline{\rho}_1 \oplus \overline{\rho}_2$, where $\overline{\rho}_1|_{G_{F(\zeta_l)}}$ and $\overline{\rho}_2|_{G_{F(\zeta_l)}}$ are adequate, in the sense of [Tho12, §2]. (In particular, they are each absolutely irreducible, and each $n_i = \dim \overline{\rho}_i$ is not divisible by l.) Moreover, $\overline{\rho}^{ss}$ is primitive, in the sense that it is not induced from any proper subgroup of G_F, and n is not divisible by l.*
6. *$\overline{\rho}_1 \not\cong \overline{\rho}_2$ and $\epsilon^{1-n}\overline{\rho}_1^\vee \not\cong \overline{\rho}_2^c$.*
7. *There exists a finite place $\widetilde{v}_0$ of F, prime to l, such that $\rho|^{ss}_{G_{F_{\widetilde{v}_0}}} \cong \oplus_{i=1}^n \psi\epsilon^{n-i}$ for some unramified character $\psi : G_{F_{\widetilde{v}_0}} \to K^\times$.*
8. *There exists a RACSDC representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$ such that:*
   - *(a) π is ι-ordinary.*
   - *(b) $\overline{r_\iota(\pi)}^{ss} \cong \overline{\rho}^{ss}$.*
   - *(c) $\pi_{\widetilde{v}_0}$ is an unramified twist of the Steinberg representation.*
9. *There exists a CM extension F_0/F linearly disjoint from the extension of $F(\zeta_l)$ cut out by $\overline{\rho}^{ss}|_{G_{F(\zeta_l)}}$ and RAECSDC representations $(\pi_1, \chi_1), (\pi_2, \chi_2)$ of $\mathrm{GL}_{n_1}(\mathbb{A}_{F_0})$ and $\mathrm{GL}_{n_2}(\mathbb{A}_{F_0})$, respectively, such that π_1, π_2 are ι-ordinary and*
$$\overline{r_\iota(\pi_i)} \cong \overline{\rho}_i|_{G_{F_0}} \text{ for } i = 1, 2.$$

Then ρ is automorphic.

Before giving the proof of Theorem 7.1, we discuss the role played by some of the assumptions. Assumptions 1–3 assert that ρ is 'geometric' and related to the ordinary automorphic forms on unitary groups of the type considered in §4. Assumption 4 is of a technical nature, and is used to ensure in §4 to ensure that the level subgroup U can be chosen to be 'sufficiently small'. The assumption 5 of residual reducibility of ρ is the main novelty of the above theorem; the assumption that the irreducible constituents $\overline{\rho}_1$ and $\overline{\rho}_2$ of $\overline{\rho}$ are adequate is required so that we can apply pre-existing automorphy lifting theorems to $\overline{\rho}_1$ and $\overline{\rho}_2$. The assumption that $\overline{\rho}^{\mathrm{ss}}$ is primitive is used in §5.2, and holds automatically e.g. if n_1 and n_2 are coprime. Together with this, assumption 6 implies that any extension of $\overline{\rho}$ to a $\mathcal{G}_n$-valued representation will be Schur, in the sense of §3.1.

Assumption 7 asserts that $\rho|_{G_{F_{\widetilde{v}_0}}}$ corresponds, under the local Langlands correspondence, to a twist of the Steinberg representation; this is the lever we use to control the space of reducible deformations of $\overline{\rho}$. Assumption 8 is the usual residual automorphy hypothesis for $\overline{\rho}$. Finally, assumption 9 asserts that $\overline{\rho}_1$ and $\overline{\rho}_2$ are 'potentially automorphic', which is often known to be the case, cf. [BLGGT]; this condition is used, together with the method of Khare–Wintenberger, to control the dimension of the universal deformation rings of $\overline{\rho}_1$ and $\overline{\rho}_2$.

Proof of Theorem 7.1. After possibly enlarging the field K, we can find a self-dual $\mathcal{O}$-lattice for ρ inside K, so we can view ρ as a representation $G_F \to \mathrm{GL}_n(\mathcal{O})$ such that $\overline{\rho} : G_F \to \mathrm{GL}_n(k)$ is conjugate-self-dual (see [CHT08, Lemma 2.1.5]). We do the same for $r_\iota(\pi)$ to obtain a homomorphism $\rho' : G_F \to \mathrm{GL}_n(\mathcal{O})$. Then $\overline{\rho}$ is semisimple, and we can assume after conjugating ρ' by an element of $\mathrm{GL}_n(\mathcal{O})$ that $\overline{\rho} = \overline{\rho}'$.

After replacing F by a preliminary soluble extension, and π by its base change (Lemma 2.7), we can assume that F, ρ and π satisfy the following additional conditions:

10. Every finite place of F which divides l or at which ρ or π is ramified splits in F/F^+, and F/F^+ is unramified at all finite places.
11. $[F^+ : \mathbb{Q}]$ is divisible by 4.
12. The place $\widetilde{v}_0$ is split over F^+. We write v_0 for the place of F^+ below it.
13. For each place w of F above a place at which ρ or π is ramified, or dividing l, $\overline{\rho}|_{G_{F_w}}$ is trivial, and, if $v \nmid l$, then $q_v \equiv 1 \bmod l$, the highest power of l dividing $q_v - 1$ is strictly greater than n, and $\rho|_{G_{F_w}}$ and $\rho'|_{G_{F_w}}$ are unipotently ramified.

(This reduction is valid since we will show that ρ is irreducible and automorphic after restriction to the Galois group of this soluble extension. Automorphy of ρ over the original choice of F will then follow by soluble descent, by Lemma 2.7.) We can find a set $\widetilde{X}_0$ of finite places of F satisfying the following conditions:

- $\widetilde{X}_0$ does not contain any place of F at which ρ or π is ramified, or which divides v_0 or l.
- Let $E/F(\zeta_l)$ denote the extension cut out by $\overline{\rho}|_{G_{F(\zeta_l)}}$. Then for any Galois subextension $E/E'/F$ with $\mathrm{Gal}(E'/F)$ simple and non-trivial, there exists a place of $\widetilde{X}_0$ which does not split in E'.

Fix a choice of $\widetilde{X}_0$ satisfying these conditions, and let X_0 denote the set of places of F^+ below $\widetilde{X}_0$. It is easy to see if that L/F is any Galois CM $\widetilde{X}_0$-split extension, then L satisfies the following conditions:

- There exists a place $\widetilde{v}_1$ of L split over L^+, absolutely unramified, such that $q_{\widetilde{v}_1} \not\equiv 1 \bmod l$ and $\overline{\rho}(\mathrm{Frob}_{\widetilde{v}_1})$ is a scalar. (Use hypothesis 4 of the theorem.)
- For each $i = 1, 2$, $\overline{\rho}_i|_{G_{L(\zeta_l)}}$ is adequate and $\overline{\rho}_1|_{G_L} \not\cong \overline{\rho}_2|_{G_L}$, $\overline{\rho}_1|^c_{G_L} \not\cong \overline{\rho}_2|^\vee_{G_L}\epsilon^{1-n}$.
- The representation $\overline{\rho}|_{G_L}$ is primitive. (Use that $\overline{\rho}$ is primitive, and $\overline{\rho}(G_F) = \overline{\rho}(G_L)$.)

If L^+/F^+ is a Galois X_0-split totally real extension and $L = L^+ \cdot F$, then L is CM and $\widetilde{X}_0$-split. We claim that we can find a soluble $\widetilde{X}_0$-split CM extension L/F satisfying the following conditions:

- Let R denote the set of places v of L^+ such that $\rho|_{G_L}$ or π_L is ramified above v, but v does not divide l or v_0. Let $r = |R|$. Then for each prime $w|l$ of L, $[L_w : \mathbb{Q}_l] > \sup(rn(n+1) + 5, n(n-1)/2 + 1)$.

- Let $S(B)$ denote the set of places of L^+ dividing v_0, let Δ denote the Galois group of the maximal abelian l-extension of L unramified outside l, and let Δ_0 denote the Galois group of the maximal abelian l-extension of L unramified outside l in which every place above $S(B)$ splits completely. Let $c \in \mathrm{Gal}(L/L^+)$ be complex conjugation. Then we have
$$\dim_{\mathbb{Q}_l} \ker(\Delta \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \to \Delta_0 \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)^{c=-1} > 6 + rn(n+1).$$
- The cardinality of $S(B)$ is even.

Let $\widetilde{Y}_0$ denote the set of places $\widetilde{v} \neq \widetilde{v}_0, \widetilde{v}_0^c$ of F dividing l or at which ρ or π is ramified, and let Y_0 denote the set of places of F^+ below a place of $\widetilde{Y}_0$. (By construction, every place of Y_0 splits in F, and $Y_0 \cap (X_0 \cup \{v_0\}) = \emptyset$.) For any odd integer $d \geq 1$, we can choose (by [AT09, §X.2, Theorem 5]) a cyclic totally real extension M_0 of F^+ of degree d and satisfying the following conditions:

- The extension M_0/F^+ is $X_0 \cup \{v_0\}$-split.
- If $v \in Y_0$ then v is totally inert in M_0.

Let M_1 be a totally real quadratic extension of F^+ which is $X_0 \cup \{v_0\} \cup Y_0$-split. We will take $L^+ = F^+ \cdot M_0 \cdot M_1$ and $L = L^+ \cdot F$. We claim that if d is chosen appropriately, then L will indeed satisfy the above requirements. For this we recall that
$$\dim_{\mathbb{Q}_l} \ker(\Delta \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \to \Delta_0 \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)^{c=-1} = \mathrm{rank}_{\mathbb{Z}_l} \left(\overline{\mathcal{O}_{L,S(B)}^\times} \right)^{c=-1}.$$

(Here the overline denotes the closure of the image of the units inside $\prod_{w|l} \mathcal{O}_{L_w}^\times(l)$.) It follows from [Jau85, Théorème 3] that the latter quantity is equal to $2d$. (We note that in [Jau85] this theorem is stated only for an abelian extension $K/\mathbb{Q}$, but the same proof gives the result relative to any abelian extension of number fields, cf. [Mai02, Proposition 19]. We apply this result of Jaulent to the finitely generated $\mathbb{Q}_l[\mathrm{Gal}(L/F^+)]$-submodule of $\mathcal{O}_{L,S(B)}^\times \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ on which c acts by -1, which has dimension $2d$ as $\mathbb{Q}_l$-vector space. We can calculate its decomposition into simple $\mathbb{Q}_l[\mathrm{Gal}(L/F^+)]$-submodules using the generalized Dirichlet unit theorem, cf. [Gra03, Part I, 3.7, Theorem].)

If we choose d to be prime to the absolute residue degrees of all elements of Y_0, then each place of M_1 above a place of Y_0 will be totally inert in the cyclic degree d extension L^+/M_1. Thus $|R| \leq 2|Y_0|$ in this case, and each place $w|l$ of L has absolute residue degree at least d. It follows that L will have the desired properties provided that d is chosen to be prime to the absolute residue degrees of all elements of Y_0 and strictly larger than
$$\sup(2|Y_0|n(n+1) + 6, n(n-1)/2 + 1).$$

We now fix such a choice. Let $S(B)$, R be as above, and let S_l denote the set of places of L^+ dividing l. Let $\widetilde{v}_1$ be a place of L which is absolutely unramified, split over L^+, not split in $L(\zeta_l)$, and such that $\overline{\rho}(\mathrm{Frob}_{\widetilde{v}_1})$ is a scalar, and let $S_a = \{v_1\}$, where v_1 is the place of L^+ below $\widetilde{v}_1$. Let $T = S = S(B) \cup S_l \cup S_a \cup R$. We choose lifts of these sets to sets $\widetilde{S(B)}$, $\widetilde{S}_l$, $\widetilde{S}_a$ and $\widetilde{R}$ of places of L, and set $\widetilde{T} = \widetilde{S} = \widetilde{S(B)} \cup \widetilde{S}_l \cup \widetilde{S}_a \cup \widetilde{R}$. With the above hypotheses, we can choose a definite unitary group G over L^+ as in §4.

By Lemma 3.1 and the discussion in the first paragraph of the proof of this theorem, we can choose an extension of $\rho|_{G_{L,S}}$ to a homomorphism $r : G_{L^+,S} \to \mathcal{G}_n(\mathcal{O})$ with $r(c) \notin \mathcal{G}_n^0(k)$ and $\nu \circ r = \epsilon^{1-n}\delta_{L/L^+}^n$. Similarly, we can choose an extension of $\rho'|_{G_L}$ to a homomorphism $r' : G_{L^+,S} \to \mathcal{G}_n(\mathcal{O})$ with $\overline{r} = \overline{r}'$ and $\nu \circ r' = \nu \circ r$.

We now have a deformation problem
$$\mathcal{S} = \left(L/L^+, T, \widetilde{T}, \Lambda, \overline{r}, \epsilon^{1-n}\delta_{L/L^+}^n, \{R_v^\triangle\}_{v \in S_l} \cup \{R_v^{\mathrm{St}}\}_{v \in S(B)} \cup \{R_v^\square\}_{v \in S_a} \cup \{R_v^1\}_{v \in R} \right),$$

and both r and r' are of type $\mathcal{S}$. If we can show that the hypotheses of Theorem 6.1 are satisfied, then it will follow that $\rho|_{G_L}$ is automorphic; the automorphy of ρ itself will then follow by Lemma 2.7. It thus remains to show that the quotient $R_{\mathcal{S}}^{\mathrm{red}}$ of $R_{\mathcal{S}}^{\mathrm{univ}}$ is finite over Λ and of dimension at most $n[L^+ : \mathbb{Q}] - rn(n+1) - 5$.

In fact, it suffices to show that for any minimal prime $Q \subset R_{\mathcal{S}}^{\mathrm{red}}$, $R_{\mathcal{S}}^{\mathrm{red}}/Q$ is finite over Λ and of dimension at most $n[L^+ : \mathbb{Q}] - rn(n+1) - 5$.

To show this we write $\overline{r} = \overline{r}_1 \oplus \overline{r}_2$ and $\Lambda_i = \widehat{\otimes}_{v \in S_l} \Lambda_{v,i} = \widehat{\otimes}_{v \in S_l} \mathcal{O}[\![I_{L_{\widetilde{v}}}^{\mathrm{ab}}(l)^{n_i}]\!]$, as in §3.5, and introduce the auxiliary deformation problems

$$\mathcal{S}_1 = \left(L/L^+, T, \widetilde{T}, \Lambda_1, \overline{r}_1, \epsilon^{1-n}\delta_{L/L^+}^n, \{R_v^{\triangle}\}_{v \in S_l} \cup \{R_v^{\square}\}_{v \in S_a} \cup \{R_v^1\}_{v \in R \cup S(B)} \right),$$

$$\mathcal{S}_2 = \left(L/L^+, T, \widetilde{T}, \Lambda_2, \overline{r}_2, \epsilon^{1-n}\delta_{L/L^+}^n, \{R_v^{\triangle}\}_{v \in S_l} \cup \{R_v^{\square}\}_{v \in S_a} \cup \{R_v^1\}_{v \in R \cup S(B)} \right),$$

where the local deformation problems are now taken with respect to the $\overline{r}_i$. (We are abusing notation here by writing the same symbols $R_v^{\triangle}$, $R_v^{\square}$, and R_v^1 for the local liftings rings of the n_1– and n_2–dimensional Galois representations $\overline{r}_i|_{G_{L_{\widetilde{v}}}}$.) It follows from [Tho12, Corollary 8.7] and [BLGGT, Lemma 1.2.3] that for $i = 1, 2$, $R_{\mathcal{S}_i}^{\mathrm{univ}}$ is finite over Λ_i of dimension $1 + n_i[L^+ : \mathbb{Q}]$. (This is where we use hypothesis 9 of Theorem 7.1, as well as the assertion of hypothesis 5 that $\overline{\rho}_1$ and $\overline{\rho}_2$ are adequate.)

Let us write $R = R_{\mathcal{S}}^{\mathrm{red}}/Q$, and $r_Q : G_{L^+} \to \mathcal{G}_n(R)$ for a lifting representing the induced deformation over R. We may choose r_Q to be of the form $r_Q = r_1 \oplus r_2$, where r_i is a lift of $\overline{r}_i$. Let $E = \mathrm{Frac}(R)$, and choose an algebraic closure $\overline{E}$. For each place $v \in S_l$, we have the unrestricted lifting ring $R_v^{\square}$, a Λ_v-algebra, and its quotient $R_v^{\triangle}$. We recall (cf. §3.3.2) that there is a projective morphism $\mathcal{G}_v \to R_v^{\square}$, where $\mathcal{G}_v$ is $\mathcal{O}$-flat and reduced, and that $R_v^{\triangle}$ is defined as the scheme-theoretic image of this morphism. In particular, the induced map $R_v^{\triangle} \to R \to \overline{E}$ lifts to an $\overline{E}$-point of $\mathcal{G}_v$, and hence there exists an increasing filtration $0 = \mathrm{Fil}_v^0 \subset \mathrm{Fil}_v^1 \subset \cdots \subset \mathrm{Fil}_v^n = r_Q|_{G_{L_{\widetilde{v}}}} \otimes_R \overline{E}$ of $r_Q|_{G_{L_{\widetilde{v}}}} \otimes_R \overline{E}$ with the property that the action of $I_{L_{\widetilde{v}}}$ on $\mathrm{gr}_v^i = \mathrm{Fil}_v^i / \mathrm{Fil}_v^{i-1}$ is given by the specialization of the universal character $\psi_i^v : I_{L_{\widetilde{v}}} \to \Lambda_v^\times$ via the morphism $\Lambda_v \to R_v^{\square} \to \overline{E}$. We set $F_v^i = \mathrm{Fil}_v^i \cap (r_1|_{G_{L_{\widetilde{v}}}} \otimes_R \overline{E})$ and $G_v^i = \mathrm{Fil}_v^i \cap (r_2|_{G_{L_{\widetilde{v}}}} \otimes_R \overline{E})$. Then $F_v^\bullet$ and $G_v^\bullet$ are increasing filtrations with graded pieces of dimension at most one, and $F_v^n \oplus G_v^n = \mathrm{Fil}_v^n = r_Q|_{G_{L_{\widetilde{v}}}} \otimes_R \overline{E}$. We write $\alpha_1^v, \ldots, \alpha_{n_1}^v$ for the characters $I_{L_{\widetilde{v}}} \to \overline{E}^\times$ afforded by the non-trivial graded pieces $\mathrm{gr}^i F_v^\bullet$, $i = 1, \ldots, n$, and $\beta_1^v, \ldots, \beta_{n_2}^v$ for the characters afforded by the non-trivial graded pieces $\mathrm{gr}^i G_v^\bullet$. Let us write $\gamma_1^v, \ldots, \gamma_n^v$ for the characters $\alpha_1^v, \ldots, \alpha_{n_1}^v, \beta_1^v, \ldots, \beta_{n_2}^v$. There exists a unique permutation σ_v, increasing on $\{1, \ldots, n_1\}$ and $\{n_1+1, \ldots, n_1+n_2\}$, such that γ_i^v is the specialization of the universal character $\psi_{\sigma_v(i)}^v$ via the morphism $\Lambda_v \to \overline{E}$.

The permutations σ_v define isomorphisms $\Lambda_{v,1} \widehat{\otimes}_{\mathcal{O}} \Lambda_{v,2} \cong \Lambda_v$ and $\Lambda_1 \widehat{\otimes}_{\mathcal{O}} \Lambda_2 \cong \Lambda$ in an obvious manner. Moreover, via these isomorphisms, R obtains the structure of Λ_1- and Λ_2-algebra, and it makes sense to ask whether the liftings r_1, r_2 over R are of type $\mathcal{S}_1$ or $\mathcal{S}_2$, respectively. Let us introduce the further auxiliary deformation problems

$$\mathcal{S}_1' = \left(L/L^+, T, \widetilde{T}, \Lambda_1, \overline{r}_1, \epsilon^{1-n}\delta_{L/L^+}^n, \{R_v^{\square}\}_{v \in S_l} \cup \{R_v^{\square}\}_{v \in S_a} \cup \{R_v^1\}_{v \in R \cup S(B)} \right),$$

$$\mathcal{S}_2' = \left(L/L^+, T, \widetilde{T}, \Lambda_2, \overline{r}_2, \epsilon^{1-n}\delta_{L/L^+}^n, \{R_v^{\square}\}_{v \in S_l} \cup \{R_v^{\square}\}_{v \in S_a} \cup \{R_v^1\}_{v \in R \cup S(B)} \right).$$

It is clear that r_1 and r_2 are of type $\mathcal{S}_1'$ and $\mathcal{S}_2'$, respectively. To show that they are of type $\mathcal{S}_1$ and $\mathcal{S}_2$, it remains to show that for each prime $v \in S_l$, the restrictions $r_i|_{G_{L_{\widetilde{v}}}}$ in fact define points of the quotients $R_v^{\triangle}$ of $R_v^{\square}$ classifying ordinary liftings of dimension n_i. However, this follows from Corollary 3.12. The induced homomorphism $R_{\mathcal{S}_1}^{\mathrm{univ}} \widehat{\otimes}_{\mathcal{O}} R_{\mathcal{S}_2}^{\mathrm{univ}} \to R$ is surjective, by universality, and is a homomorphism of Λ-algebras, by construction. Since the former ring here is a finite Λ-algebra, by the above, we deduce that R is also a finite Λ-algebra.

It remains to bound the dimension of R. For $i = 1, 2$ let $\psi_i : G_{L,S} \to \mathcal{O}^\times$ be the Teichmüller lift of $\det \overline{r}_i|_{G_{L,S}}$. Write $R_{\mathcal{S}_i,\psi_i}^{\mathrm{univ}}$ for the quotient of $R_{\mathcal{S}_i}^{\mathrm{univ}}$ where the determinant of the universal deformation restricted to $G_{L,S}$ is equal to ψ_i. By Lemma 3.36, we have $R_{\mathcal{S}_i}^{\mathrm{univ}} \cong R_{\mathcal{S}_i,\psi_i}^{\mathrm{univ}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Delta/(c+1)]\!]$, and

$$R_{\mathcal{S}_1}^{\mathrm{univ}} \widehat{\otimes}_{\mathcal{O}} R_{\mathcal{S}_2}^{\mathrm{univ}} / \lambda \cong R_{\mathcal{S}_1,\psi_1}^{\mathrm{univ}} \widehat{\otimes}_{\mathcal{O}} R_{\mathcal{S}_2,\psi_2}^{\mathrm{univ}} / \lambda \widehat{\otimes}_k k[\![\Delta/(c+1)]\!] \widehat{\otimes}_k k[\![\Delta/(c+1)]\!], \tag{7.1}$$

and the ring on the left hand side is flat over $k[\![\Delta/(c+1)]\!] \widehat{\otimes}_k k[\![\Delta/(c+1)]\!]$. Write $\Psi_1, \Psi_2 : \Delta/(c+1) \to k[\![\Delta/(c+1)]\!] \widehat{\otimes}_k k[\![\Delta/(c+1)]\!]^\times$ for the universal characters valued in this ring.

If $\mathfrak{p} \subset R/(\lambda)$ is a prime ideal and $v \in S(B)$, then there exists $\alpha \in R/\mathfrak{p}$ such that $r_1(\mathrm{Frob}_{\widetilde{v}})$ has characteristic polynomial $(X-\alpha)^{n_1}$ and $r_2(\mathrm{Frob}_{\widetilde{v}})$ has characteristic polynomial $(X-\alpha)^{n_2}$. Since $\overline{r}|_{G_{L_{\widetilde{v}}}}$ is trivial, by assumption, we have $\psi_1(\mathrm{Frob}_{\widetilde{v}}) = \psi_2(\mathrm{Frob}_{\widetilde{v}}) = 1$ and $\alpha \equiv 1 \bmod \mathfrak{m}_R$. Comparing the determinants of r_1, r_2 we obtain the relation

$$\Psi_1(\mathrm{Frob}_{\widetilde{v}})^{n_1 n_2} = \det r_{\mathcal{S}_1}|_{G_{L_{\widetilde{v}}}}(\mathrm{Frob}_{\widetilde{v}})^{n_2} \equiv \det r_{\mathcal{S}_2}|_{G_{L_{\widetilde{v}}}}(\mathrm{Frob}_{\widetilde{v}})^{n_1} = \Psi_2(\mathrm{Frob}_{\widetilde{v}})^{n_1 n_2} \bmod \mathfrak{p},$$

hence

$$\Psi_1(\mathrm{Frob}_{\widetilde{v}})^{n_1 n_2} \equiv \Psi_2(\mathrm{Frob}_{\widetilde{v}})^{n_1 n_2} \bmod \mathfrak{p}. \tag{7.2}$$

Since $\mathfrak{p} \subset R/(\lambda)$ was arbitrary, the relation (7.2) holds in the underlying reduced subring of $R/(\lambda)$ for each $v \in S(B)$. The quotient of the ring $k[\![\Delta/(c+1)]\!]\widehat{\otimes}_k k[\![\Delta/(c+1)]\!]$ defined by these relations for $v \in S(B)$ has codimension at least $6+rn(n+1)$, by the choice of L. Since the ring (7.1) is flat over $k[\![\Delta/(c+1)]\!]\widehat{\otimes}_k k[\![\Delta/(c+1)]\!]$, we obtain

$$\dim R \leq 1 + \dim R_{\mathcal{S}_1}^{\mathrm{univ}} \widehat{\otimes}_{\mathcal{O}} R_{\mathcal{S}_2}^{\mathrm{univ}}/(\lambda) - (6 + rn(n+1)) \leq n[L^+ : \mathbb{Q}] - rn(n+1)/2 - 5,$$

as required. This completes the proof. □

8 An application to the Fontaine–Mazur conjecture

We now combine our main theorem with Serre's conjecture for $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ to deduce the following result.

Theorem 8.1. *Let $E/\mathbb{Q}$ be a quadratic imaginary extension and let l be a prime. Suppose that $\rho : G_E \to \mathrm{GL}_3(\overline{\mathbb{Q}}_l)$ is a continuous irreducible representation satisfying the following hypotheses:*

1. *ρ is ramified at only finitely many places.*
2. *$\rho^c \cong \rho^\vee \epsilon^{-2}$.*
3. *$\overline{\rho}^{ss} = \overline{\rho}_1 \oplus \overline{\rho}_2$, where:*
   (a) *$\dim \overline{\rho}_1 = 2$ and $\overline{\rho}|_{G_{E(\zeta_l)}}$ is irreducible.*
   (b) *$\dim \overline{\rho}_2 = 1$.*
4. *ρ is crystalline ordinary of weight λ for some $\lambda \in (\mathbb{Z}_+^3)^{\mathrm{Hom}(E,\overline{\mathbb{Q}}_l)}$. Moreover, l splits in E and for each embedding $\tau : E \hookrightarrow \overline{\mathbb{Q}}_l$, we have $\lambda_{\tau,1} > \lambda_{\tau,2} > \lambda_{\tau,3}$ and $6 + \sum_{j=1}^3 (\lambda_{\tau,j} - \lambda_{\tau,3}) < l/2$.*
5. *There exists a place v_0 of E split over $\mathbb{Q}$ and not dividing l and an unramified character $\psi_0 : G_{E_{v_0}} \to \overline{\mathbb{Q}}_l^\times$ such that $\overline{\rho}^{ss}$ is ramified at v_0 and $(\rho|_{G_{E_{v_0}}})^{ss} \cong \psi_0 \oplus \epsilon\psi_0 \oplus \epsilon^2\psi_0$. Moreover, l does not divide $\prod_{j=1}^3 (q_{v_0}^j - 1)$.*

Then ρ is automorphic, in the sense that it arises from a RACSDC automorphic representation of $\mathrm{GL}_3(\mathbb{A}_E)$.

Proof of Theorem 8.1. We fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$. Let $K \subset \overline{\mathbb{Q}}_l$ be a finite extension of $\mathbb{Q}_l$ over which ρ is defined. We write, as usual, $\mathcal{O}$ for the ring of integers of K and k for its residue field. After possibly conjugating ρ and enlarging K, we can extend ρ to a continuous representation $r : G_\mathbb{Q} \to \mathcal{G}_3(\mathcal{O})$. Let $\mu = \nu \circ r : G_\mathbb{Q} \to \mathcal{O}^\times$. If $c \in G_\mathbb{Q}$ is a complex conjugation and $r(c) = (A, -\mu(c))j$, then ${}^tA = -\mu(c)A$. Since 3 is odd, A must be symmetric and therefore $\mu(c) = -1$.

Reducing modulo the maximal ideal of $\mathcal{O}$, we see that there is an extension of $\overline{\rho}_1$ to a homomorphism $r_1 : G_\mathbb{Q} \to \mathcal{G}_2(k)$ such that if $\mu_1 = \nu \circ r_1$ then $\mu_1(c) = -1$. Let $\overline{\chi} = \det \overline{\rho}_1$. Then $\overline{\chi\chi^c} = \overline{\epsilon}^{-4}$. Since $\dim \overline{\rho}_1 = 2$, we have $\overline{\rho}_1 \cong \overline{\rho}_1^\vee \overline{\chi}$. Since $(\overline{\chi\epsilon}^2)(\overline{\chi\epsilon}^2)^c = 1$, we can find a character $\overline{\psi} : G_E \to \overline{\mathbb{F}}_l^\times$ such that $\overline{\psi}^c/\overline{\psi} = \overline{\chi\epsilon}^2$. (Indeed, it suffices to note that the group $H^1(\mathrm{Gal}(E/\mathbb{Q}), H^1(E, \mathbb{Q}/\mathbb{Z}))$ vanishes, since E is totally complex.) The representation $\overline{\rho}_1\overline{\psi}$ now satisfies

$$(\overline{\rho}_1\overline{\psi})^c \cong \overline{\rho}_1^\vee \overline{\epsilon}^{-2}\overline{\psi}^c \cong \overline{\rho}_1 \overline{\chi}^{-1}\overline{\epsilon}^{-2}\overline{\psi}^c \cong \overline{\rho}_1\overline{\psi}.$$

Thus the representation $\overline{\rho}_1\overline{\psi}$ extends to a continuous representation $\overline{R}_1 : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$. In fact, this representation is odd ($\det \overline{R}_1(c) = -1$): the self-duality of $\overline{R}_1$ is symplectic, the conjugate self-duality of $\overline{\rho}_1\overline{\psi}$ is orthogonal, and an easy calculation shows that the difference of these signs is given by $\det \overline{R}_1(c)$. (We learned this observation from Frank Calegari. Compare [Cal11].)

Fix an embedding $\tau : E \hookrightarrow \overline{\mathbb{Q}}_l$, and let v denote the induced place of E above l. Since ρ is crystalline of weight λ, we have

$$\rho|_{I_{E_v}} \cong \begin{pmatrix} \epsilon^{-\lambda_3} & * & * \\ 0 & \epsilon^{-(\lambda_2+1)} & * \\ 0 & 0 & \epsilon^{-(\lambda_1+2)} \end{pmatrix},$$

and hence

$$\overline{R}_1|_{I_{E_v}} \cong \overline{\psi} \begin{pmatrix} \overline{\epsilon}^{-a} & * \\ 0 & \overline{\epsilon}^{-b} \end{pmatrix},$$

for some $a < b$ in $\{\lambda_3, \lambda_2 + 1, \lambda_1 + 2\}$. Let c denote the other element of $\{\lambda_3, \lambda_2 + 1, \lambda_1 + 2\}$. By Serre's conjecture [KW09], $\overline{R}_1$ is residually automorphic.

Let us write p for the prime of $\mathbb{Q}$ below v_0. Let E_0 be an imaginary quadratic field in which l splits, p is inert, and in which every other prime below a place of E at which ρ or E is ramified splits, and such that $\overline{\rho}_1|_{G_{E_0 \cdot E(\zeta_l)}}$ remains absolutely irreducible. Let $E_1 = E_0 \cdot E$. Then E_1/E_1^+ is an everywhere unramified quadratic extension, split at every prime at which ρ is ramified, and p is inert in E_1^+. Let w_0 be the unique place of E_1 above v_0. By [BLGG, Theorem A] and [BLGGT, Theorem 4.4.1], we can find an ι-ordinary RAECSDC automorphic representation π_1 of $\mathrm{GL}_2(\mathbb{A}_{E_1})$ such that $\pi_1^c \cong \pi_1^\vee|\cdot|^{-1}$ and $r_\iota(\pi_1)$ is crystalline with $\mathrm{HT}_{\tau_1}(r_\iota(\pi_1)) = \{a, b\}$, for any embedding $\tau_1 : E_1 \hookrightarrow \overline{\mathbb{Q}}_l$ such that $\tau_1|_E = \tau$. (We note that the latter reference requires the assumption $l > 2(n + 1) = 8$; this is clearly implied by hypothesis 4 of the theorem.) We may further suppose that π_{1,w_0} is an unramified twist of the Steinberg representation. Let $\rho_1 = r_\iota(\pi_1)$.

We can also (cf. [CHT08, Lemma 4.1.6]) choose a lift of $\overline{\rho}_2$ to a character $\rho_2 : G_E \to \overline{\mathbb{Q}}_l^\times$ satisfying $\rho_2^c = \rho_2^\vee \epsilon^{-2}$, unramified at v_0, and such that ρ_2 is crystalline with $\mathrm{HT}_\tau(\rho_2) = \{c\}$. Let π_2 denote the RAECSDC automorphic representation of $\mathrm{GL}_1(\mathbb{A}_{E_1})$ corresponding under ι to $\rho_2|_{G_{E_1}}$. Then $\pi_2^c \cong \pi_2^\vee|\cdot|^{-2}$.

The automorphic representation $\Pi = \pi_1|\cdot|^{1/2} \boxplus \pi_2|\cdot|$ is regular algebraic and conjugate self-dual, and satisfies $r_\iota(\Pi) = \rho_1 \oplus \rho_2$. (We have not defined here the Galois representation associated to a regular algebraic and conjugate self-dual but not necessarily cuspidal automorphic representation, but it exists and satisfies the analogous properties to Theorem 2.2; see [Tho, Theorem 2.1].) One can now check that the hypotheses of [Tho, Theorem 7.1] apply to Π. Indeed, hypothesis 5 implies that l is a banal characteristic for $\mathrm{GL}_3(E_{1,w_0})$, and that the image under $\overline{r_\iota(\Pi)}^{\mathrm{ss}}$ of a generator of the l-part of tame inertia at w_0 has 2 Jordan blocks, and hypothesis 4 implies the required conditions on the weight of Π. It follows that there exists an ι-ordinary RACSDC automorphic representation π of $\mathrm{GL}_3(\mathbb{A}_{E_1})$ such that $\overline{r_\iota(\pi)}^{\mathrm{ss}} \cong \overline{\rho}^{\mathrm{ss}}|_{G_{E_1}}$ and π_{w_0} is an unramified twist of the Steinberg representation.

We claim that Theorem 7.1 now applies to $\rho|_{G_{E_1}}$. Conditions 1–3 are immediate. Condition 4 holds because l splits in E_1 and $\overline{\rho}_1|_{G_{E_1}}$ is irreducible. Indeed, it follows from the classification of finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_l)$ that the abelianization of the projective image of $\overline{\rho}_1|_{G_{E_1}}$ has order strictly less that $l - 1 = [E_1(\zeta_l) : E_1]$. Condition 5 holds by [Tho12, Theorem A.9]. (The representation $\overline{\rho}|_{G_{E_1}}$ is primitive because its irreducible constituents have coprime dimension.) Condition 6 is automatic. Condition 7 holds by hypothesis, and conditions 8 and 9 hold by construction. Theorem 7.1 therefore implies that $\rho|_{G_{E_1}}$ is automorphic, and it follows by Lemma 2.7 that ρ itself is automorphic. This completes the proof. □

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