Modal Dependent Type Theory and Dependent Right Adjoints

Lars Birkedal[1], Ranald Clouston[2], Bassel Mannaa[3], Rasmus Ejlers Møgelberg[4], Andrew M. Pitts[5], and Bas Spitters[6]

1,6 Department of Computer Science and Engineering, Aarhus University, Aarhus, Denmark
`ranald.clouston@cs.au.dk, spitters@cs.au.dk`
2 Research School of Computer Science, Australian National University, Australia
3,4 Department of Computer Science, IT University of Copenhagen, Copenhagen, Denmark
`basm@itu.dk,mogel@itu.dk`
5 Department of Computer Science & Technology, University of Cambridge, Cambridge, United Kingdom
`Andrew.Pitts@cl.cam.ac.uk`

Abstract

In recent years we have seen several new models of dependent type theory extended with some form of modal necessity operator, including nominal type theory, guarded and clocked type theory, and spatial and cohesive type theory. In this paper we study *modal dependent type theory*: dependent type theory with an operator satisfying (a dependent version of) the K-axiom of modal logic. We investigate both semantics and syntax. For the semantics, we introduce categories with families with a *dependent right adjoint* (CwDRA) and show that the examples above can be presented as such. Indeed, we show that any finite limit category with an adjunction of endofunctors gives rise to a CwDRA via the local universe construction. For the syntax, we introduce a dependently typed extension of Fitch-style modal lambda-calculus, show that it can be interpreted in any CwDRA, and build a term model. We extend the syntax and semantics with universes.

1 Introduction

Dependent types are a powerful technology for both programming and formal proof. In recent years we have seen several new models of dependent type theory extended with a type-former resembling modal necessity, such as nominal type theory (Pitts, Matthiesen & Derikx 2015), guarded (Birkedal, Møgelberg, Schwinghammer & Støvring 2012, Bizjak, Grathwohl, Clouston, Møgelberg & Birkedal 2016, Bizjak & Møgelberg 2018, Birkedal, Bizjak, Clouston, Grathwohl, Spitters & Vezzosi 2018) and clocked (Mannaa & Møgelberg 2018) type theory, and spatial and cohesive type theory (Shulman 2017). These examples all satisfy the K axiom of modal logic

$$\Box(A \to B) \to \Box A \to \Box B$$

but are not all (co)monads, the more extensively studied construction in the context of dependent type theory (Krishnaswami, Pradic & Benton 2015, Shulman 2017, de Paiva & Ritter 2016). Motivated in part by these examples, in this paper we study *modal dependent type theory*: dependent type theory with an operator satisfying (a dependent generalisation of) the K-axiom of modal logic. We investigate both semantics and syntax.

For the semantics, we introduce categories with families with a *dependent right adjoint* (CwDRA) and show that this dependent right adjoint models the modality in the examples mentioned above. Indeed, we show that any finite limit category with an adjunction of endofunctors[1] gives rise to a CwDRA via the local universe construction (Lumsdaine & Warren 2015). In particular, by applying the local universe construction to a locally cartesian closed category with an adjunction of endofunctors, we get a model of modal dependent type theory with Π- and Σ-types.

For the syntax, we adapt the simply typed Fitch-style modal lambda-calculus introduced by Borghuis (1994) and Martini & Masini (1996), inspired by Fitch's proof theory for modal logic (Fitch 1952). In such a calculus □ is introduced by 'shutting' a strict subordinate proof and eliminating by 'opening' one. For example the axiom K is inhabited by the term

$$\lambda f.\lambda x.\mathsf{shut}((\mathsf{open}\, f)(\mathsf{open}\, x)) \qquad (1)$$

The nesting of subordinate proofs can be tracked in sequent style by a special symbol in the context which we call a *lock*, and write 🔓; the open lock symbol is intended to suggest we have access to the contents of a box. Following Clouston (2018), the lock can be understood as an operation on contexts *left adjoint* to □; hence Fitch-style modal λ-calculus has a model in any cartesian closed category equipped with an adjunction of endofunctors. Here we show, in work inspired by Clocked Type Theory (Bahr, Grathwohl & Møgelberg 2017), that Fitch-style λ-calculus lifts with a minimum of difficulty to dependent types. In particular the term (1), where f is a dependent function, has type

$$\Box(\Pi y : A.\, B) \to \Pi x : \Box A.\, \Box B[\mathsf{open}\, x/y]$$

This dependent version of the K axiom, not obviously expressible without the open construct of a Fitch-style calculus, allows modalised functions to be applied to modalised data even in the dependent case. This capability is known to be essential in at least one example, namely proofs about guarded recursion (Bizjak et al. 2016)[2]. We show that our calculus can be soundly interpreted in any CwDRA, and construct a term model.

We also extend the syntax and semantics of modal dependent type theory with universes. Here we restrict attention to models based on (pre)sheaves, for which Coquand has proposed a particularly simple formulation of universes (Coquand 2012). We show how to extend Coquand's notion of a category with universes with dependent right adjoints, and observe that a construction encoding the modality on the universe, introduced for guarded type theory by Bizjak et al. (2016), in fact arises for more general reasons.

Another motivation for the present work is that it can be understood as providing a notion of a *dependent* adjunction between endofunctors. An ordinary adjunction $\mathsf{L} \dashv \mathsf{R}$ on a category $\mathbf{C}$ is a natural bijective correspondence $\mathbf{C}(\mathsf{L}A, B) \cong \mathbf{C}(A, \mathsf{R}\,B)$. With dependent types one might consider *dependent* functions from $\mathsf{L}A$ to B, where B may depend on $\mathsf{L}A$, and similarly from A to $\mathsf{R}\,B$. Our notion of CwDRA then defines what it means to have an adjoint correspondence in this dependent case. Our Fitch-style modal dependent type theory can therefore also be understood as a term language for dependent adjoints.

Outline We introduce CwDRAs in Section 2, and present the syntax of modal dependent type theory in Section 3. In Section 4 we show how to construct a CwDRA from an adjunction on a category with finite limits. In Section 5 we show how various models in the

[1] This should not be confused with models where there are adjoint functors between different categories which can be composed to define a monad or comonad.

[2] This capability was achieved by Bizjak et al. (Bizjak et al. 2016) via *delayed substitutions*, but this construction does not straightforwardly support an operational semantics (Bahr et al. 2017).

literature can be presented as CwDRAs. The extension with universes is defined in Section 6. We end with a discussion of related and future work in Section 7.

2 Categorical Semantics of Modal Dependent Type Theory

The notion of *category with families* (CwF) (Dybjer 1995, Hofmann 1997) provides a semantics for the development of dependent type theory which elides some difficult aspects of syntax, such as variable binding, as well as the coherence problems of simpler notions of model. It can be connected to syntax by a soundness argument and term model construction, and to more intuitive models via 'strictification' constructions. In this section we extend this notion to introduce *categories with a dependent right adjoint* (CwDRA). We first recall the standard definition:

▶ **Definition 1 (category with families).** A **CwF** is specified by:

1. A category $\mathbf{C}$ with a terminal object $\top$. Given objects $\Gamma, \Delta \in \mathbf{C}$, write $\mathbf{C}(\Delta, \Gamma)$ for the set of morphisms from Δ to Γ in $\mathbf{C}$. The identity morphism on Γ is just written id with Γ implicit. The composition of $\gamma \in \mathbf{C}(\Delta, \Gamma)$ with $\delta \in \mathbf{C}(\Phi, \Delta)$ is written $\gamma \circ \delta$.
2. For each object $\Gamma \in \mathbf{C}$, a set $\mathbf{C}(\Gamma)$ of *families* over Γ.
3. For each object $\Gamma \in \mathbf{C}$ and family $A \in \mathbf{C}(\Gamma)$, a set $\mathbf{C}(\Gamma \vdash A)$ of *elements* of the family A over Γ.
4. For each morphism $\gamma \in \mathbf{C}(\Delta, \Gamma)$, *re-indexing* functions $A \in \mathbf{C}(\Gamma) \mapsto A[\gamma] \in \mathbf{C}(\Delta)$ and $a \in \mathbf{C}(\Gamma \vdash A) \mapsto a[\gamma] \in \mathbf{C}(\Delta \vdash A[\gamma])$, satisfying $A[\mathsf{id}] = A$, $A[\gamma \circ \delta] = A[\gamma][\delta]$, $a[\mathsf{id}] = a$ and $a[\gamma \circ \delta] = a[\gamma][\delta]$.
5. For each object $\Gamma \in \mathbf{C}$ and family $A \in \mathbf{C}(\Gamma)$, a *comprehension object* $\Gamma.A \in \mathbf{C}$ equipped with a *projection morphism* $\mathsf{p}_A \in \mathbf{C}(\Gamma.A, \Gamma)$, a *generic element* $\mathsf{q}_A \in \mathbf{C}(\Gamma.A \vdash A[\mathsf{p}_A])$ and a *pairing operation* $\gamma \in \mathbf{C}(\Delta, \Gamma), a \in \mathbf{C}(\Delta \vdash A[\gamma]) \mapsto (\gamma, a) \in \mathbf{C}(\Delta, \Gamma.A)$ satisfying $\mathsf{p}_A \circ (\gamma, a) = \gamma$, $\mathsf{q}_A[(\gamma, a)] = a$, $(\gamma, a) \circ \delta = (\gamma \circ \delta, a[\delta])$ and $(\mathsf{p}_A, \mathsf{q}_A) = \mathsf{id}$.

A *dependent right adjoint* then extends the definition of CwF with a functor on contexts L and an operation on families R, intuitively understood to be left and right adjoints:

▶ **Definition 2 (category with a dependent right adjoint).** A *CwDRA* is a CwF $\mathbf{C}$ equipped with the following extra structure:

1. An endofunctor $\mathsf{L} : \mathbf{C} \to \mathbf{C}$ on the underlying category of the CwF.
2. For each object $\Gamma \in \mathbf{C}$ and family $A \in \mathbf{C}(\mathsf{L}\Gamma)$, a family $\mathsf{R}_\Gamma A \in \mathbf{C}(\Gamma)$, stable under re-indexing in the sense that for all $\gamma \in \mathbf{C}(\Delta, \Gamma)$ we have

$$(\mathsf{R}_\Gamma A)[\gamma] = \mathsf{R}_\Delta(A[\mathsf{L}\gamma]) \in \mathbf{C}(\Delta) \tag{2}$$

3. For each object $\Gamma \in \mathbf{C}$ and family $A \in \mathbf{C}(\mathsf{L}\Gamma)$ a bijection

$$\mathbf{C}(\mathsf{L}\Gamma \vdash A) \cong \mathbf{C}(\Gamma \vdash \mathsf{R}_\Gamma A) \tag{3}$$

We write the effect of this bijection on $a \in \mathbf{C}(\mathsf{L}\Gamma \vdash A)$ as $\overline{a} \in \mathbf{C}(\Gamma \vdash \mathsf{R}_\Gamma A)$ and write the effect of its inverse on $b \in \mathbf{C}(\Gamma \vdash \mathsf{R}_\Gamma A)$ also as $\overline{b} \in \mathbf{C}(\mathsf{L}\Gamma \vdash A)$. Thus

$$\overline{\overline{a}} = a \qquad (a \in \mathbf{C}(\mathsf{L}\Gamma \vdash A)) \tag{4}$$

$$\overline{\overline{b}} = b \qquad (b \in \mathbf{C}(\Gamma \vdash \mathsf{R}_\Gamma A)) \tag{5}$$

The bijection is required to be stable under re-indexing in the sense that for all $\gamma \in \mathbf{C}(\Delta, \Gamma)$ we have

$$\overline{a}[\gamma] = \overline{a[\mathsf{L}\gamma]} \tag{6}$$

from which it follows that we also have

$$\overline{b}[\mathsf{L}\gamma] = \overline{b[\gamma]} \tag{7}$$

(since using (4)–(6), we have $\overline{b[\gamma]} = \overline{\overline{\overline{b}}[\gamma]} = \overline{\overline{\overline{b}[\mathsf{L}\gamma]}} = \overline{b}[\mathsf{L}\gamma]$).

3 Syntax of Modal Dependent Type Theory

In this section we extend Fitch-style modal λ-calculus (Borghuis 1994) to dependent types, and connect this to the notion of CwDRA via a soundness proof and term model construction. We define our dependent types broadly in the style of ECC (Luo 1989), as this is close to the implementation of some proof assistants (Norell 2007).

We define the raw syntax of contexts, types, and terms as follows:

$$\begin{aligned}
\Gamma &\triangleq \diamond \mid \Gamma, x : A \mid \Gamma, 🔓 \\
A &\triangleq \Pi x : A.\, B \mid \Box A \\
t &\triangleq x \mid \lambda x.t \mid t\,t \mid \mathsf{shut}\, t \mid \mathsf{open}\, t
\end{aligned}$$

We omit the leftmost '$\diamond$,' where the context is non-empty. Π-types are included in the grammar as an exemplar to show that standard constructions can be given standard definitions, without reference to the locks in the context. One could similarly add an empty type, unit type, booleans, Σ-types, W-types, universes (of which more in Section 6), and so forth.

Judgements have forms

$\Gamma \vdash$	'Γ is a well-formed context'
$\Gamma \vdash A$	'A is a well-formed type in context Γ'
$\Gamma \vdash A = B$	'A and B are equal types in context Γ'
$\Gamma \vdash t : A$	't is a term with type A in context Γ'
$\Gamma \vdash t = u : A$	't and u are equal terms with type A in context Γ'

Figure 1 presents the typing rules of the calculus. The syntactic results below follow easily by induction on these rules. We remark only that exchange of variables with locks, and weakening of locks, are not admissible, and that the (lock-free) weakening Γ' in the open rule is essential to proving variable weakening.

▶ **Lemma 3.** *Let $\mathcal{J}$ range over the possible strings to the right of a turnstile in a judgement.*

1. *If $\Gamma, x : A, y : B, \Gamma' \vdash \mathcal{J}$ and x is not free in B, then $\Gamma, y : B, x : A, \Gamma' \vdash \mathcal{J}$;*
2. *If $\Gamma, \Gamma' \vdash \mathcal{J}$, and $\Gamma \vdash A$, and x is a fresh variable, then $\Gamma, x : A, \Gamma' \vdash \mathcal{J}$;*
3. *If $\Gamma, x : A, \Gamma' \vdash \mathcal{J}$ and $\Gamma \vdash u : A$, then $\Gamma, \Gamma'[u/x] \vdash \mathcal{J}[u/x]$;*
4. *If $\Gamma \vdash t : A$ then $\Gamma \vdash A$;*
5. *If $\Gamma \vdash t = u : A$ then $\Gamma \vdash t : A$ and $\Gamma \vdash u : A$.*

Context formation rules:

$$\frac{}{\diamond \vdash} \qquad \frac{\Gamma \vdash \qquad \Gamma \vdash A}{\Gamma, x : A \vdash}\; x \notin \Gamma \qquad \frac{\Gamma \vdash}{\Gamma, \text{🔓} \vdash} \qquad \frac{\Gamma, x : A, y : B, \Gamma' \vdash}{\Gamma, y : B, x : A, \Gamma' \vdash}\; x \text{ NOT FREE IN } B$$

Type formation rules:

$$\frac{\Gamma \vdash A \qquad \Gamma, x : A \vdash B}{\Gamma \vdash \Pi x : A.\, B} \qquad \frac{\Gamma, \text{🔓} \vdash A}{\Gamma \vdash \Box A}$$

Type equality rules are just equivalence and congruence.
Term formation rules:

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash A = B}{\Gamma \vdash t : B} \qquad \frac{\Gamma, x : A, \Gamma' \vdash}{\Gamma, x : A, \Gamma' \vdash x : A}\; \text{🔓} \notin \Gamma' \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : \Pi x : A.\, B}$$

$$\frac{\Gamma \vdash t : \Pi x : A.\, B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\, u : B[u/x]}$$

$$\frac{\Gamma, \text{🔓} \vdash t : A}{\Gamma \vdash \mathsf{shut}\, t : \Box A} \qquad \frac{\Gamma \vdash t : \Box A \qquad \Gamma, \text{🔓}, \Gamma' \vdash}{\Gamma, \text{🔓}, \Gamma' \vdash \mathsf{open}\, t : A}\; \text{🔓} \notin \Gamma'$$

Term equality rules, omitting equivalence and congruence:

$$\frac{\Gamma \vdash (\lambda x.t)u : A}{\Gamma \vdash (\lambda x.t)u = t[u/x] : A} \qquad \frac{\Gamma \vdash \mathsf{open}\,\mathsf{shut}\, t : A}{\Gamma \vdash \mathsf{open}\,\mathsf{shut}\, t = t : A} \qquad \frac{\Gamma \vdash t : \Pi x : A.\, B}{\Gamma \vdash t = \lambda x.t\, x : \Pi x : A.\, B}\; x \notin \Gamma$$

$$\frac{\Gamma \vdash t : \Box A}{\Gamma \vdash t = \mathsf{shut}\,\mathsf{open}\, t : \Box A}$$

Figure 1 Typing rules for a dependent Fitch-style modal λ-calculus.

$$
\begin{array}{lcl}
\mathsf{P}(\Gamma; A; \diamond) & = & \mathsf{p}_{[\![A]\!]} \\
\mathsf{P}(\Gamma; A; \Gamma', y : B) & = & \big(\mathsf{P}(\Gamma; A; \Gamma') \circ \mathsf{p}_{[\![\Gamma, x:A, \Gamma' \vdash B]\!]}, \mathsf{q}_{[\![\Gamma, x:A, \Gamma' \vdash B]\!]}\big) \\
\mathsf{P}(\Gamma; A; \Gamma', \text{🔓}) & = & \mathsf{L}\,\mathsf{P}(\Gamma; A; \Gamma') \\
\mathsf{E}(\Gamma; A; B; \diamond) & = & \big((\mathsf{p}_{[\![\Gamma \vdash B]\!]} \circ \mathsf{p}_{[\![\Gamma, y:B \vdash A]\!]}, \mathsf{q}_{[\![\Gamma, y:B \vdash A]\!]}), \mathsf{q}_{[\![\Gamma \vdash B]\!]}[\mathsf{p}_{[\![\Gamma, y:B \vdash A]\!]}]\big) \\
\mathsf{E}(\Gamma; A; B; \Gamma', z : C) & = & (\mathsf{E}(\Gamma; A; B; \Gamma') \circ \mathsf{p}_{[\![\Gamma, y:B, x:A, \Gamma' \vdash C]\!]}, \mathsf{q}_{[\![\Gamma, y:B, x:A, \Gamma' \vdash C]\!]}) \\
\mathsf{E}(\Gamma; A; B; \Gamma', \text{🔓}) & = & \mathsf{L}\,\mathsf{E}(\Gamma; A; B; \Gamma') \\
\mathsf{S}(\Gamma; A; \diamond; t) & = & (\mathsf{id}, [\![t]\!]) \\
\mathsf{S}(\Gamma; A; \Gamma', y : B; t) & = & (\mathsf{S}(\Gamma; A; \Gamma'; t) \circ \mathsf{p}_{[\![\Gamma, \Gamma'[t/x] \vdash B[t/x]]\!]}, \mathsf{q}_{[\![\Gamma, \Gamma'[t/x] \vdash B[t/x]]\!]}) \\
\mathsf{S}(\Gamma; A; \Gamma', \text{🔓}; t) & = & \mathsf{L}\,\mathsf{S}(\Gamma; A; \Gamma'; t)
\end{array}
$$

Figure 2 **C**-morphisms corresponding to weakening, exchange, and substitution.

3.1 Sound interpretation in CwDRAs

In this section we show that the calculus of Figure 1 can be soundly interpreted in any CwDRA. We wish to give meaning to contexts, types, and terms, but (via the type conversion rule) these can have multiple derivations, so it is not possible to work by induction on the formation rules. Instead, following e.g. Hofmann (1997), we define a partial map from *raw syntax* to semantics by induction on the grammar, then prove this map is defined for well-formed syntax. By 'raw syntax' we mean contexts, types accompanied by a context, and terms accompanied by context and type, defined via the grammar. The *size* of a type or term is the number of connectives and variables used to define it, and the size of a context is the sum of the sizes of its types.

Well-defined contexts will be interpreted as objects in **C**, types in context $\Gamma \vdash A$ as families in $\mathbf{C}([\![\Gamma]\!])$, and typed terms in context $\Gamma \vdash t : A$ as elements in $\mathbf{C}([\![\Gamma]\!] \vdash [\![\Gamma \vdash A]\!])$. Where there is no confusion we write $[\![\Gamma \vdash A]\!]$ as $[\![A]\!]$ and $[\![\Gamma \vdash t : A]\!]$ as $[\![\Gamma \vdash t]\!]$ or $[\![t]\!]$.

The partial interpretation of raw syntax is as follows, following the convention that ill-formed expressions (for example, where a subexpression is undefined) are undefined. We omit the details for Π-types and other standard constructions, which are as usual.

- $[\![\diamond]\!] = \top$;
- $[\![\Gamma, x : A]\!] = [\![\Gamma]\!].[\![A]\!]$;
- $[\![\Gamma, \text{🔓}]\!] = \mathsf{L}[\![\Gamma]\!]$;
- $[\![\Gamma \vdash \Box A]\!] = R_{[\![\Gamma]\!]}([\![A]\!])$;
- $[\![\Gamma, x : A, x_1 : A_1, \ldots, x_n : A_n \vdash x : A]\!] = \mathsf{q}_{[\![A]\!]}[\mathsf{p}_{[\![A_1]\!]} \circ \cdots \circ \mathsf{p}_{[\![A_n]\!]}]$;
- $[\![\Gamma \vdash \mathsf{shut}\, t : \Box A]\!] = \overline{[\![t]\!]}$;
- $[\![\Gamma, \text{🔓}, x_1 : A_1, \ldots, x_n : A_n \vdash \mathsf{open}\, t : A]\!] = \overline{[\![t]\!]}[\mathsf{p}_{[\![A_1]\!]} \circ \cdots \circ \mathsf{p}_{[\![A_n]\!]}]$.

In Figure 2 we define expressions $\mathsf{P}(\Gamma; A; \Gamma')$, $\mathsf{E}(\Gamma; A; B; \Gamma')$, and $\mathsf{S}(\Gamma; A; \Gamma'; t)$ that, where defined, define **C**-morphisms corresponding respectively to weakening, exchange, and substitution in contexts.

▶ **Lemma 4.** *Suppose* $[\![\Gamma, \Gamma']\!]$ *and* $[\![\Gamma, x : A, \Gamma']\!]$ *are defined. Then the following properties hold:*

1. $[\![\Gamma, x : A, \Gamma' \vdash X]\!] \simeq [\![\Gamma, \Gamma' \vdash X]\!][\mathsf{P}(\Gamma; A; \Gamma')]$*, where* $\simeq$ *is Kleene equality, and* X *is a type or typed term;*
2. $\mathsf{P}(\Gamma; A; \Gamma')$ *is a well-defined morphism from* $[\![\Gamma, x : A, \Gamma']\!]$ *to* $[\![\Gamma, \Gamma']\!]$*;*

Proof. The proof proceeds by mutual induction on the size of Γ' (for statement 2) and the size of Γ' plus the size of X (for statement 1). We present only the cases particular to $\Box$.

We start with statement 1. We use the mutual induction with statement 2 at the smaller size of Γ' alone to ensure that $\mathsf{P}(\Gamma; A; \Gamma')$ is well-formed with the correct domain and codomain.

The $\Box$ case follows because $\llbracket \Gamma, x : A, \Gamma' \vdash \Box B \rrbracket = \mathsf{R}_{\llbracket \Gamma, x:A, \Gamma' \rrbracket} \llbracket \Gamma, x : A, \Gamma', \text{🔓} \vdash B \rrbracket = \mathsf{R}_{\llbracket \Gamma, x:A, \Gamma' \rrbracket}(\llbracket \Gamma, \Gamma', \text{🔓} \vdash B \rrbracket [\mathsf{P}(\Gamma; A; \Gamma', \text{🔓})])$ by induction, which is $\mathsf{R}_{\llbracket \Gamma, x:A, \Gamma' \rrbracket}(\llbracket \Gamma, \Gamma', \text{🔓} \vdash B \rrbracket [\mathsf{LP}(\Gamma; A; \Gamma')]) = (\mathsf{R}_{\llbracket \Gamma, \Gamma' \rrbracket} \llbracket \Gamma, \Gamma', \text{🔓} \vdash B \rrbracket)[\mathsf{P}(\Gamma; A; \Gamma')]$ by (2), which is then $\llbracket \Gamma, \Gamma' \vdash \Box B \rrbracket [\mathsf{P}(\Gamma; A; \Gamma')]$.

The shut case follows immediately from (6) and induction. For open, the case where the deleted variable x is to the right of the lock follows by Definition 1 part 5. Suppose instead it is to the left. Then $\llbracket \Gamma, \Gamma', \text{🔓}, y_1 : B_1, \ldots, y_n : B_n \vdash \mathsf{open}\, t \rrbracket [\mathsf{P}(\Gamma; A; \Gamma', \text{🔓}, y_1 : B_1, \ldots, y_n : B_n)] = \overline{\llbracket t \rrbracket}[\mathsf{p}_{\llbracket B_1 \rrbracket} \circ \cdots \circ \mathsf{p}_{\llbracket B_n \rrbracket} \circ \mathsf{P}(\Gamma; A; \Gamma', \text{🔓}, y_1 : B_1, \ldots, y_n : B_n)] = \overline{\llbracket t \rrbracket}[\mathsf{P}(\Gamma; A; \Gamma', \text{🔓}) \circ \mathsf{p}_{\llbracket B_1 \rrbracket} \circ \cdots \circ \mathsf{p}_{\llbracket B_n \rrbracket}]$ by Definition 1 part 5, which is $\overline{\llbracket t \rrbracket}[\mathsf{LP}(\Gamma; A; \Gamma')][\mathsf{p}_{\llbracket B_1 \rrbracket} \circ \cdots \circ \mathsf{p}_{\llbracket B_n \rrbracket}] = \overline{\llbracket t \rrbracket [\mathsf{P}(\Gamma; A; \Gamma')]}[\mathsf{p}_{\llbracket B_1 \rrbracket} \circ \cdots \circ \mathsf{p}_{\llbracket B_n \rrbracket}]$ by (7), which is $\llbracket \Gamma, x : A, \Gamma' \vdash t \rrbracket [\mathsf{p}_{\llbracket B_1 \rrbracket} \circ \cdots \circ \mathsf{p}_{\llbracket B_n \rrbracket}]$ by induction as required.

For statement 2, the lock case holds immediately by application of the functor L. ◻

▶ **Lemma 5.** *Suppose $\llbracket \Gamma, x : A, y : B, \Gamma' \rrbracket$ and $\llbracket \Gamma \vdash B \rrbracket$ are defined. Then the following properties hold:*

1. $\llbracket \Gamma, y : B, x : A, \Gamma' \vdash X \rrbracket \simeq \llbracket \Gamma, x : A, y : B, \Gamma' \vdash X \rrbracket [\mathsf{E}(\Gamma; A; B; \Gamma')]$*, where X is a type or typed term;*
2. $\mathsf{E}(\Gamma; A; B; \Gamma')$ *is a well-defined morphism from $\llbracket \Gamma, y : B, x : A, \Gamma' \rrbracket$ to $\llbracket \Gamma, x : A, y : B, \Gamma' \rrbracket$;*

Proof. The base case of statement 1 uses Lemma 4; the proof otherwise follows just as with Lemma 4. ◻

▶ **Lemma 6.** *Suppose $\llbracket \Gamma \vdash t : A \rrbracket$ and $\llbracket \Gamma, x : A, \Gamma' \rrbracket$ are defined. Then the following properties hold:*

1. $\llbracket \Gamma, \Gamma'[t/x] \vdash X[t/x] \rrbracket \simeq \llbracket \Gamma, x : A, \Gamma' \vdash X \rrbracket [\mathsf{S}(\Gamma; A; \Gamma'; t)]$*, where X is a type or typed term;*
2. $\mathsf{S}(\Gamma; A; \Gamma'; t)$ *is a well-defined morphism from $\llbracket \Gamma, \Gamma'[t/x] \rrbracket$ to $\llbracket \Gamma, x : A, \Gamma' \rrbracket$;*

Proof. As with Lemma 4. ◻

▶ **Theorem 7** (Soundness)**.** *Where a context, type, or term is well-formed, its denotation is well-defined, and all types and terms identified by equations have the same denotation.*

Proof. Most cases follow as usual, using Lemmas 4, 5, and 6 as needed. The well-definedness of the formation rules for $\Box$ are straightforward, so we present only the equations for $\Box$:

Starting with $\Gamma, \text{🔓} \vdash t : A$ we have $\Gamma, \text{🔓}, x_1 : A_1, \ldots, x_n : A_n \vdash \mathsf{open}\,\mathsf{shut}\, t : A$ and wish to prove its denotation is equal to that of t (with the weakening $x_1, \ldots, x_n$). Then $\llbracket \mathsf{open}\,\mathsf{shut}\, t \rrbracket = \overline{\overline{\llbracket t \rrbracket}}[\mathsf{p}_{\llbracket A_1 \rrbracket} \circ \cdots \circ \mathsf{p}_{\llbracket A_n \rrbracket}] = \llbracket t \rrbracket [\mathsf{p}_{\llbracket A_1 \rrbracket} \circ \cdots \circ \mathsf{p}_{\llbracket A_n \rrbracket}]$, which is the weakening of t by Lemma 4.

The equality of $\llbracket \mathsf{shut}\,\mathsf{open}\, t \rrbracket$ and $\llbracket t \rrbracket$ is straightforward. ◻

3.2 Term model

We now develop as our first example of a CwDRA, a term model built from the syntax of our calculus. The objects of this category are contexts modulo equality, which is defined pointwise via type equality. We define an arrow $\Delta \to \Gamma$ as a sequence of substitutions of an equivalence class of terms for each variable in Γ:

- the empty sequence is an arrow $\Delta \to \cdot$;
- Given $f : \Delta \to \Gamma$, type $\Gamma \vdash A$ and term $\Delta \vdash t : A f$, then $[t/x] \circ f$ modulo equality on t is an arrow $\Delta \to \Gamma, x : A$;

- Given $f : \Delta \to \Gamma$ and a well-formed context $\Delta, \text{🔓}, \Delta'$ with no locks in Δ', then f is also an arrow $\Delta, \text{🔓}, \Delta' \to \Gamma, \text{🔓}$.

We usually refer to the equivalence classes in arrows via representatives. Note that substitution respect these equivalence classes because of the congruence rules.

We next prove that this defines a category. Identity arrows are easily constructed:

▶ **Lemma 8.** *If $f : \Delta \to \Gamma$ then $f : \Delta, x : A \to \Gamma$.*

Proof. By induction on the construction on f. The base case is trivial.

Given $f : \Delta \to \Gamma$ and $\Delta \vdash t : B\,f$, by induction we have $f : \Delta, x : A \to \Gamma$ and by variable weakening we have $\Delta, x : A \vdash t : B\,f$ as required.

Supposing we have $f : \Delta \to \Gamma$ yielding $f : \Delta, \text{🔓}, \Delta' \to \Gamma$, we could similarly get $f : \Delta, \text{🔓}, \Delta', x : A \to \Gamma$. ◻

The identity on Γ simply replaces all variables by themselves.

▶ **Lemma 9.** *The identity on each Γ is well defined as an arrow.*

Proof. By induction on Γ. The identity on $\cdot$ is the empty sequence of substitutions. Given $id : \Gamma \to \Gamma$, we have $id : \Gamma, x : A \to \Gamma$ by Lemma 8, and $\Gamma, x : A \vdash x : A$ as required. $id : \Gamma \to \Gamma$ immediately yields $id : \Gamma, \text{🔓} \to \Gamma, \text{🔓}$. ◻

The composition case is slightly more interesting:

▶ **Lemma 10.** *Given $\Gamma, \Gamma' \vdash \mathcal{J}$ and $f : \Delta \to \Gamma$, we have $\Delta, \Gamma'\,f \vdash \mathcal{J}\,f$.*

Proof. By induction on the construction on f. The base case requires that $\Gamma' \vdash \mathcal{J}$ implies $\Delta, \Gamma' \vdash \mathcal{J}$; this *left weakening* property is easily proved by induction on the typing rules.

Given $f : \Delta \to \Gamma$, $\Delta \vdash t : A\,f$ and $\Gamma, x : A, \Gamma' \vdash \mathcal{J}$, by induction $\Delta, x : A\,f, \Gamma'\,f \vdash \mathcal{J}\,f$. Then by Lemma 3 part 3 we have $\Delta, (\Gamma'\,f)[t/x] \vdash (\mathcal{J}\,f)[t/x]$ as required. The lock case is trivial. ◻

The composition of $f : \Delta \to \Delta'$ and $g : \Delta' \to \Gamma$ involves replacing each $[t/x]$ in g with $[t\,f/x]$.

▶ **Lemma 11.** *The composition of two arrows $f : \Delta \to \Delta'$ and $g : \Delta' \to \Gamma$ is a well-defined arrow.*

Proof. By induction on the definition of g. The base case is trivial, and extension by a new substitution follows via Lemma 10.

Now suppose we have $g : \Delta' \to \Gamma$ yielding $g : \Delta', \text{🔓}, \Delta'' \to \Gamma, \text{🔓}$. Now if we have $f : \Delta \to \Delta', \text{🔓}, \Delta''$ this must have arisen via some $f' : \Delta_0 \to \Delta'$ generating $f' : \Delta_0, \text{🔓}, \Delta_1 \to \Delta', \text{🔓}$, where $\Delta = \Delta_0, \text{🔓}, \Delta_1$. By induction we have well-defined $g \circ f' : \Delta_0 \to \Gamma$. Hence $g \circ f' : \Delta \to \Gamma, \text{🔓}$. But $g \circ f' = g \circ f$ because the variables of Δ'' do not appear in g. ◻

Checking the category axioms is straightforward. The category definitions then extend to a CwF in the usual way: the terminal object is $\diamond$, the families over Γ are the types modulo equivalence well-defined in context Γ, the elements of any such type are the terms modulo equivalence, re-indexing is substitution, comprehension corresponds to extending a context with a new variable, the projection morphism is the replacement of variables by themselves, and the generic element is given by the variable rule.

Moving to the definition of a CwDRA, the endofunctor L acts by mapping $\Gamma \mapsto \Gamma, \text{🔓}$, and does not change arrows. The family $\mathsf{R}_\Gamma\,A$ is the type $\Gamma \vdash \Box A$, which is stable under re-indexing by Lemma 3 part 3. The bijections between families are supplied by the shut and open rules, with all equations following from the definitional equalities.

We do not attempt to prove that the term model is the *initial* CwDRA; such a result for dependent type theories appears to require syntax be written in a more verbose style than is appropriate for a paper introducing a new type theory (Castellan 2014). Nonetheless our type theory and notion of model are close enough that we conjecture that such a development is possible.

4 A general construction of CwDRAs

In this section we show how to construct a CwDRA from an adjunction of endofunctors on a category with finite limits. We will refer to categories with finite limits more briefly as *cartesian* categories. We will use this construction in Section 5 to prove that the examples mentioned in the introduction can indeed be presented as CwDRAs. Our construction is an extension of the local universe construction (Lumsdaine & Warren 2015), which maps cartesian categories to categories with families, and locally cartesian closed categories to categories with families with Π- and Σ-types. The local universe construction is one of the known solutions to the problem of constructing a strict model of type theory out of a locally cartesian closed category (see (Hofmann 1994, Lumsdaine & Warren 2015, Kapulkin & Lumsdaine 2016, Hofmann 1997) for discussions of alternative approaches to 'strictification').

We first recall the local universe construction. Since it can be traced back to Giraud's work on fibred categories (Giraud 1965), we refer to it as the Giraud CwF associated to a cartesian category.

▶ **Definition 12.** Let $\mathbf{C}$ be a cartesian category. The **Giraud CwF of C** ($\mathcal{G}\mathbf{C}$) is the CwF whose underlying category is $\mathbf{C}$, and where a family $A \in \mathcal{G}\mathbf{C}(\Gamma)$ is a pair of morphisms

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle v} \\ \Gamma & \xrightarrow{u} & U \end{array} \tag{8}$$

and an element of $\mathcal{G}\mathbf{C}(\Gamma \vdash A)$, for $A = (u, v) \in \mathcal{G}\mathbf{C}(\Gamma)$, is a map $a : \Gamma \to E$ such that $v \circ a = u$. Reindexing of $A = (u, v) \in \mathcal{G}\mathbf{C}(\Gamma)$ and $a \in \mathcal{G}\mathbf{C}(\Gamma \vdash A)$ along $\gamma \in \mathbf{C}(\Delta, \Gamma)$ are given by

$$A[\gamma] \triangleq (u \circ \gamma, v) \in \mathcal{G}\mathbf{C}(\Delta) \tag{9}$$

$$a[\gamma] \triangleq a \circ \gamma \in \mathcal{G}\mathbf{C}(\Delta \vdash A[\gamma]) \tag{10}$$

The comprehension $\Gamma.A \in \mathbf{C}$, for $A = (u, v) \in \mathcal{G}\mathbf{C}(\Gamma)$, is given by the pullback of diagram (8), and the pairing operation is obtained from the universal property of pullbacks.

Note that the local universe construction does indeed yield a category with families; in particular, reindexing in $\mathcal{G}\mathbf{C}$ is strict as required, simply because reindexing is given by composition.

▶ Remark. The name 'local universe' derives from the similarity to Voevodsky's use of a (global) universe U to construct strict models of type theory (Voevodsky 2014, Kapulkin & Lumsdaine 2016) in which types in a context Γ are modelled as morphisms $\Gamma \to U$. In the local universe construction, the universe varies from type to type.

In fact, the local universe construction is functorial; a precise statement requires a novel notion of CwF-morphism:

▶ **Definition 13.** A **weak CwF morphism** R between CwFs consists of a functor $\mathsf{R} : \mathbf{C} \to \mathbf{D}$ between the underlying categories preserving the terminal object, an operation on families mapping $A \in \mathbf{C}(\Gamma)$ to a family $\mathsf{R}\,A \in \mathbf{D}(\mathsf{R}\,\Gamma)$, an operation on elements mapping $a \in \mathbf{C}(\Gamma \vdash A)$ to an element $\mathsf{R}\,a \in \mathbf{D}(\mathsf{R}\,\Gamma \vdash \mathsf{R}\,A)$, and an isomorphism $\nu_{\Gamma,A} : \mathsf{R}\,\Gamma.\,\mathsf{R}\,A \to \mathsf{R}(\Gamma.A)$, inverse to $(\mathsf{R}\,\mathsf{p}_A, \mathsf{R}\,\mathsf{q}_A)$. These are required to commute with reindexing, in the sense that $\mathsf{R}\,A[\mathsf{R}\,\gamma] = \mathsf{R}(A[\gamma])$ and $\mathsf{R}\,t[\mathsf{R}\,\gamma] = \mathsf{R}(t[\gamma])$.

Note that a weak CwF morphism preserves comprehension and the terminal object only up to isomorphism instead of on the nose, as required by the stricter notion of morphism of Dybjer (Dybjer 1995, Definition 2). Weak CwF morphisms sit between strict CwF-morphisms and pseudo-CwF morphisms (Castellan, Clairambault & Dybjer 2017).

▶ **Theorem 14.** *$\mathcal{G}$ is a (fully faithful) functor from the category of cartesian categories and finite limit preserving functors, to the category of CwFs with weak morphisms.*

Proof. Let $\mathsf{R} : \mathbf{C} \to \mathbf{D}$ be a finite limit preserving functor. For each $\Gamma \in \mathbf{C}$ and $A = (u, v) \in \mathcal{G}\mathbf{C}(\Gamma)$, we simply let $\mathsf{R}\,A \triangleq (\mathsf{R}\,u, \mathsf{R}\,v)$. Likewise, for an element $a \in \mathcal{G}\mathbf{C}(\Gamma \vdash A)$, we let $\mathsf{R}\,a$ be the action of R on the morphism a. Finally, since comprehension is defined by pullback and R preserves pullbacks up to isomorphism, we obtain the required $\nu_{\Gamma,A}$. ◻

We now embark on showing that if we apply the local universe construction to a cartesian category $\mathbf{C}$ with a pair of adjoint endofunctors, then the resulting CwF $\mathcal{G}\mathbf{C}$ is in fact a CwDRA (Theorem 18). To this end, we introduce the intermediate notion of a category with families with an adjunction:

▶ **Definition 15.** A **CwF+A** consists of a CwF with an adjunction $\mathsf{L} \dashv \mathsf{R}$ on the category of contexts, such that R extends to a weak CwF endomorphism.

Note that the conditions for a CwF+A are stronger than those for a CwDRA; for instance, a CwDRA does not require R to be defined on the context category. We return to the relation between these constructions in Section 4.1

▶ **Lemma 16.** *If $\mathbf{C}$ is a cartesian category and $\mathsf{L} \dashv \mathsf{R}$ are adjoint endofunctors on $\mathbf{C}$, then $\mathcal{G}\mathbf{C}$ with the adjunction $\mathsf{L} \dashv \mathsf{R}$ is a CwF+A.*

Proof. We are already given an adjunction on the underlying category of $\mathcal{G}\mathbf{C}$. Theorem 14 constructs the weak CwF morphism. ◻

▶ **Lemma 17.** *If $\mathbf{C}$ with the adjunction $\mathsf{L} \dashv \mathsf{R}$ is a CwF+A, then there is a CwDRA structure on $\mathbf{C}$ with L as the required functor on $\mathbf{C}$.*

Proof. We write η for the unit of the adjunction. For a family $A \in \mathbf{C}(\mathsf{L}\Gamma)$, we define $\mathsf{R}_\Gamma\,A \in \mathbf{C}(\Gamma)$ to be $(\mathsf{R}\,A)[\eta]$. For an element $a \in \mathbf{C}(\mathsf{L}\Gamma \vdash A)$, we define its transpose $\overline{a} \in \mathbf{C}(\Gamma \vdash \mathsf{R}_\Gamma\,A)$ to be $(\mathsf{R}\,a)[\eta]$. For the opposite direction, suppose $b \in \mathbf{C}(\Gamma \vdash \mathsf{R}_\Gamma\,A)$. Since $(\eta, b) : \Gamma \to \mathsf{R}\,\mathsf{L}\Gamma.\,\mathsf{R}\,A$, we have that $\mathsf{L}(\nu \circ (\eta, t)) : \mathsf{L}\Gamma \to \mathsf{L}\,\mathsf{R}(\mathsf{L}\Gamma.A)$ and thus we can define $\overline{b} \in \mathbf{C}(\mathsf{L}\Gamma \vdash A)$ to be the element $q_A[\varepsilon \circ L(\nu \circ (\eta, t))]$. ◻

▶ **Theorem 18.** *If $\mathbf{C}$ is a cartesian category and $\mathsf{L} \dashv \mathsf{R}$ are adjoint endofunctors on $\mathbf{C}$, then $\mathcal{G}\mathbf{C}$ has the structure of a CwDRA.*

Proof. By Lemmas 16 and 17. ◻

The above Theorem 18 thus provides a general construction of CwDRAs. In Section 5 we use it to present examples from the literature. As mentioned earlier, the local universe construction interacts well with other type formers: If we start with a locally cartesian closed category $\mathbf{C}$ (with W-types, Id-types and a universe), then $\mathcal{G}\mathbf{C}$ also models dependent products Π and sums Σ (and W-types, Id-types and a universe); see Lumsdaine & Warren (2015). In Section 6 we consider universes.

4.1 CwF+A from a CwDRA

In this subsection we show how to produce a CwF+A from a CwDRA under the assumption that the CwF is *democratic*. Intuitively, a democratic CwF is one where every context comes from a type, and hence it is not surprising that for a democratic CwDRA one can use the action of the dependent right adjoint on families to define a right adjoint on contexts.

▶ **Definition 19.** A CwF is **democratic** (Clairambault & Dybjer 2014) if for every context Γ there is a family $\widehat{\Gamma} \in \mathbf{C}(\top)$ and an isomorphism $\zeta_\Gamma : \Gamma \to \top.\widehat{\Gamma}$.

▶ **Theorem 20.** *Let* $\mathbf{C}$ *be a democratic CwDRA. The endofunctor* $\mathsf{L} : \mathbf{C} \to \mathbf{C}$*, part of the CwDRA structure, has a right adjoint* R*.*

Proof. For $\Gamma \in \mathbf{C}$, we define $\mathsf{R}\,\Gamma \in \mathbf{C}$ by $\mathsf{R}\,\Gamma \triangleq \top.\,\mathsf{R}_\top(\widehat{\Gamma}[!_{\mathsf{L}\top}])$.

$$\mathsf{R}\,\Gamma \triangleq \top.\,\mathsf{R}_\top(\widehat{\Gamma}[!_{\mathsf{L}\top}]) \tag{11}$$

We have a bijection, natural in Δ

$$\begin{aligned}
\mathbf{C}(\Delta, \mathsf{R}\,\Gamma) &\cong \mathbf{C}(\Delta \vdash \mathsf{R}_\top(\widehat{\Gamma}[!_{\mathsf{L}\top}]))[!_\Delta])) \\
&\cong \mathbf{C}(\Delta \vdash \mathsf{R}_\Delta(\widehat{\Gamma}[!_{\mathsf{L}\Delta}]))) \\
&\cong \mathbf{C}(\mathsf{L}\Delta \vdash \widehat{\Gamma}[!_{\mathsf{L}\Delta}])) \\
&\cong \mathbf{C}(\mathsf{L}\Delta, \top.\widehat{\Gamma}) \\
&\cong \mathbf{C}(\mathsf{L}\Delta, \Gamma)
\end{aligned}$$

The last of the above bijections follows by composition with ζ_Γ^{-1}.

Let $\gamma : \Gamma' \to \Gamma$ we have then an action $\gamma^* : \mathbf{C}(-, \mathsf{R}\,\Gamma') \to \mathbf{C}(-, \mathsf{R}\,\Gamma)$ given by

$$\mathbf{C}(-, \mathsf{R}\,\Gamma') \cong \mathbf{C}(\mathsf{L}-, \Gamma') \xrightarrow{-\circ\gamma} \mathbf{C}(\mathsf{L}-, \Gamma) \cong \mathbf{C}(-, \mathsf{R}\,\Gamma)$$

Define $\mathsf{R}\,\gamma = \gamma^*_{\mathsf{R}\,\Gamma'}(\mathsf{id}_{\mathsf{R}\,\Gamma'})$. Then the correspondence $\mathbf{C}(\Delta, \mathsf{R}\,\Gamma) \cong \mathbf{C}(\mathsf{L}\Delta, \Gamma)$ is natural in Γ, proving that R is a right adjoint to L. ☐

Consider a democratic CwDRA, with $\mathbf{C}$ as the underlying category, and $\mathsf{L} \dashv \mathsf{R}$ the adjunction obtained from the above theorem. We then extend R to a weak CwF morphism by defining, for a family $A \in \mathbf{C}(\Gamma)$ and an element $a \in \mathbf{C}(\Gamma \vdash A)$,

$$\mathsf{R}\,A \triangleq \mathsf{R}_{\mathsf{R}\,\Gamma}(A[\varepsilon]) \qquad\qquad \mathsf{R}\,a \triangleq \overline{a[\varepsilon]}$$

where $\varepsilon : \mathsf{L}\,\mathsf{R}\,\Gamma \to \Gamma$ is the counit of the adjunction.

▶ **Lemma 21.** R *as defined above is a weak CwF morphism. In particular, for* $A \in \mathbf{C}(\Gamma)$ *we have an isomorphism* $\nu_{\Gamma,A} : \mathsf{R}\,\Gamma.\,\mathsf{R}\,A \to \mathsf{R}(\Gamma.A)$*, inverse to* $(\mathsf{R}\,\mathsf{p}_A, \mathsf{R}\,\mathsf{q}_A)$*.*

Proof. We will show a bijection $\mathbf{C}(\Delta, \mathsf{R}\,\Gamma.\,\mathsf{R}\,A) \cong \mathbf{C}(\Delta, \mathsf{R}(\Gamma.A))$ natural in Δ. We have

$$\mathbf{C}(\Delta, \mathsf{R}\,\Gamma.\,\mathsf{R}\,A) \cong \prod_{\gamma:\mathbf{C}(\Delta,\mathsf{R}\,\Gamma)} \mathbf{C}(\Delta \vdash (\mathsf{R}\,A)[\gamma])$$

We have a bijection $-^{\top} : \mathbf{C}(\Delta, \mathsf{R}\,\Gamma) \cong \mathbf{C}(\mathsf{L}\Delta, \Gamma)$. But

$$(\mathsf{R}\,A)[\gamma] = (\mathsf{R}_{\mathsf{R}\,\Gamma}\,A[\varepsilon])[\gamma] = \mathsf{R}_{\Delta}(A[\varepsilon \circ \mathsf{L}\gamma]) = \mathsf{R}_{\Delta}(A[\gamma^{\top}])$$

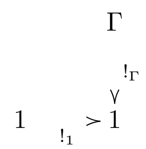
Hence we have a bijection $\mathbf{C}(\Delta \vdash (\mathsf{R}\,A)[\gamma]) \cong \mathbf{C}(\mathsf{L}\Delta \vdash A[\gamma^{\top}])$. So

$$\begin{aligned}\mathbf{C}(\Delta, \mathsf{R}\,\Gamma.\,\mathsf{R}\,A) &\cong \prod_{\gamma:\mathbf{C}(\Delta,\mathsf{R}\,\Gamma)} \mathbf{C}(\Delta \vdash (\mathsf{R}\,A)[\gamma]) \\ &\cong \prod_{\gamma':\mathbf{C}(\mathsf{L}\Delta,\Gamma)} \mathbf{C}(\mathsf{L}\Delta \vdash A[\gamma']) \\ &\cong \mathbf{C}(\mathsf{L}\Delta, \Gamma.A) \\ &\cong \mathbf{C}(\Delta, \mathsf{R}(\Gamma.A))\end{aligned}$$

By the Yoneda lemma, this implies $\mathsf{R}\,\Gamma.\,\mathsf{R}\,A \cong \mathbf{C}(\Delta, \mathsf{R}(\Gamma.A))$, and it is easy to check that the direction $\mathbf{C}(\Delta, \mathsf{R}(\Gamma.A)) \to \mathsf{R}\,\Gamma.\,\mathsf{R}\,A$ is given by $(\mathsf{R}\,\mathsf{p}_A, \mathsf{R}\,\mathsf{q}_A)$. ☐

▶ **Corollary 22.** *A democratic CwDRA has the structure of CwF+A*

▶ Remark. For a category $\mathbf{C}$ with a terminal object, the CwF $\mathcal{G}\mathbf{C}$ is democratic with $\widehat{\Gamma}$ given by the diagram:



▶ Remark. For ordinary dependent type theory, the term model is a democratic CwF (Castellan et al. 2017, Section 4). However, the term model for our modal dependent type theory is *not* democratic, since there is, for example, no type corresponding to the context 🔓 consisting of just one lock.

5 Examples

We now present concrete examples of CWDRAs generated from cartesian categories with an adjunction of endofunctors, including those mentioned in the introduction.

Π type with closed domain Consider a CwF where the underlying category of contexts $\mathbf{C}$ is cartesian closed, and let A be a closed type. We have then an adjunction of endofunctors $- \times \top.A \dashv -^{\top.A}$ on $\mathbf{C}$, and suppose that the right adjoint extends to a weak CwF endomorphism, giving the structure of a CwF+A. As we saw above, this happens e.g. when the CwF is of the form $\mathcal{G}\mathbf{C}$. In this case $\mathsf{R}_\Gamma\,B$ behaves as a type of the form $\Pi(x : A)B$ since $\mathbf{C}(\Gamma \vdash \mathsf{R}_\Gamma\,B) \cong \mathbf{C}(\Gamma \times \top.A \vdash B) \cong \mathbf{C}(\Gamma.(A[!_\Gamma]) \vdash B)$.

Thus, the notion of dependent right adjoint generalises Π types with closed domain. This generalises to the setting where $\mathbf{C}$ carries the structure of a monoidal closed category, in which case the adjunction $- \otimes \top.A \dashv \top.A \multimap (-)$ extends to give a dependent notion of linear function space with closed domain. The next example is an instance of this.

Dependent name abstraction The notion of *dependent name abstraction* for families of nominal sets was introduced by Pitts et al. (Pitts et al. 2015, Section 3.6) to give a semantics for an extension of Martin-Löf Type Theory with names and constructs for freshness and name-abstraction. It provides an example of a CwDRA that can be presented via Theorem 18. In this case $\mathbf{C}$ is the category $\mathbf{Nom}$ of nominal sets and equivariant functions (Pitts 2013). Its objects are sets Γ equipped with an action of finite permutations of a fixed infinite set of atomic names $\mathbb{A}$, with respect to which the elements of Γ are finitely supported, and its morphisms are functions that preserve the action of name permutations. $\mathbf{Nom}$ is a topos (it is equivalent to the Schanuel topos (Pitts 2013, Section 6.3)) and hence in particular is cartesian. We take the functor $\mathsf{L} : \mathbf{Nom} \to \mathbf{Nom}$ to be separated product (Pitts 2013, Section 3.4) with the nominal set of atomic names. This has a right adjoint R that sends each $\Gamma \in \mathbf{Nom}$ to the nominal set of name abstractions $[\mathbb{A}]\Gamma$ (Pitts 2013, Section 4.2) whose elements are a generic form of α-equivalence class in the case that Γ is a nominal set of syntax trees for some language.

Applying Theorem 18, we get a CwDRA structure on $\mathcal{G}\mathbf{Nom}$. In fact the CwF $\mathcal{G}\mathbf{Nom}$ has an equivalent, more concrete description in this case, in terms of *families of nominal sets* (Pitts et al. 2015, Section 3.1). Under this equivalence, the value $\mathsf{R}_\Gamma\, A \in \mathcal{G}\mathbf{Nom}(\Gamma)$ of the dependent right adjoint at $A \in \mathcal{G}(\mathsf{L}\Gamma)$ corresponds to the family of *dependent name abstractions* defined by Pitts et al. (2015, Section 3.6). The bijection (3) is given in one direction by the name abstraction operation (Pitts et al. 2015, (40)) and in the other by concretion at a fresh name (Pitts et al. 2015, (42)).

Guarded and Clocked Type Theory Guarded recursion (Nakano 2000) is an extension of type theory with a modal *later* operator, denoted $\triangleright$, on types, an operation $\mathsf{next} : A \to \triangleright A$ and a guarded fixed point operator $\mathsf{fix} : (\triangleright A \to A) \to A$ mapping f to a fixed point for $f \circ \mathsf{next}$. The standard model of guarded recursion is the topos of trees (Birkedal et al. 2012), i.e., the category of presheaves on ω, with $\triangleright X(n+1) = X(n)$, $\triangleright X(0) = 1$. The later operator has a left adjoint $\triangleleft$, called *earlier*, given by $\triangleleft X(n) = X(n+1)$, so $\triangleright$ yields a dependent right adjoint on the induced CwDRA.

Birkedal et al. (Birkedal et al. 2012, Section 6.1) show that $\triangleleft$ in a dependently typed setting does not commute with reindexing. However it *does* have a left adjoint, namely the 'stutter' functor ! with $!X(0) = X(0)$ and $!X(n+1) = X(n)$, so $\triangleleft$ does give rise to a well-behaved modality in the setting of this paper. This apparent contradiction is resolved by the use of locks in the context: $\Gamma \vdash A$ does not give rise to a well-behaved $\Gamma \vdash \triangleleft A$, but $\Gamma,$ 🔓 $\vdash A$ does. This is an intriguing example of the Fitch-style approach increasing expressivity.

Guarded recursion can be used to encode coinduction given a *constant* modality (Clouston, Bizjak, Grathwohl & Birkedal 2015), denoted $\Box$, on the topos of trees, defined as $\Box X(n) = \lim_k X(k)$. The $\Box$ functor is the right adjoint of the essential geometric morphism on $\hat{\omega}$ induced by $0 : \omega \to \omega$, the constant map to 0, and hence it also yields a dependent right adjoint. In Clouston et al. (2015), $\Box$ was used in a simple type theory, employing 'explicit substitutions' following Bierman & de Paiva (2000). As we will discuss in Section 7 this approach proved difficult to extend to dependent types, and we wish to use the modal dependent type theory of the present paper to study $\Box$ in dependent type theory.

An alternative to the constant modality are the *clock quantifiers* of Atkey & McBride (2013), which unlike the constant modality have already been combined succesfully with dependent types (Møgelberg 2014, Bizjak et al. 2016). They are also slightly more general than the constant modality, as multiple clocks allow coinductive data structures that unroll in multiple dimensions, such as infinitely-wide infinitely-deep trees. The denotational semantics (Bizjak & Møgelberg 2018), however, are more complicated, consisting of presheaves over a category of 'time objects', restricted to those fulfilling an 'orthogonality' condition. Nevertheless the $\lhd \dashv \rhd$ adjunction of the topos of trees lifts to this category, and so once again we may construct a CwDRA.

Clocked Type Theory (CloTT) (Bahr et al. 2017) is a recent type theory for guarded recursion that has strongly normalising reduction semantics, and has been shown to have semantics in the category discussed above (Mannaa & Møgelberg 2018). The operator $\rhd$ is refined to a form of dependent function type $\rhd(\alpha : \kappa).A$ over ticks α on clock κ. Ticks can appear in contexts as $\Gamma, \alpha : \kappa$; these are similar to the locks of Fitch-style contexts, except that ticks have names, and can be weakened. The names of ticks play a crucial role in controlling fixed point unfoldings.

Finally, the modal operator $\rhd$ on the topos of trees can be generalized to the presheaf topos $\widehat{\mathbf{C} \times \omega}$ for any category $\mathbf{C}$, simply by using the identity on $\mathbf{C}$ to extend the underlying functor (which generates the essential geometric morphism) on ω to $\mathbf{C} \times \omega$. In Birkedal et al. (2018) this topos, with $\mathbf{C}$ the cube category, is used to model guarded *cubical* type theory; an extension of cubical type theory (Cohen, Coquand, Huber & Mörtberg 2016). In more detail, one uses a CwF where families are certain *fibrations*, and since $\rhd$ preserves fibrations, it does indeed extend to a CwDRA.

Cohesive Toposes Cohesive toposes have also recently been considered as models of a form of modal type theory (Shulman 2017, Rijke, Shulman & Spitters 2018). Cohesive toposes carry a triple adjunction $\int \dashv \flat \dashv \sharp$ and hence induce two dependent right adjoints. Examples of cohesive toposes include simplicial sets $\hat{\Delta}$ and cubical sets $\hat{\square}$; since these are presheaf toposes they also model universes. For example, for simplicial sets, the triple of adjoints are given by the essential geometric morphism induced by the constant functor $0 : \Delta \to \Delta$. In the category of cubical sets $\sharp$ has a further right adjoint, used by Nuyts, Vezzosi & Devriese (2017) to reason about parametricity.

Tiny objects Licata, Orton, Pitts & Spitters (2018) use a tiny object $\mathbb{I}$ to construct the fibrant universe in the cubical model of homotopy type theory. An object $\mathbb{I}$ is *tiny* if exponentation by it has a right-adjoint. The corresponding *dependent* right adjoint plays an important part in the construction of the fibrant universe. Like $\lhd$ above, this right-adjoint is not available in the internal logic of a topos, but our present framework is still applicable.

6 Universes

In this section, we extend our modal dependent type theory with universes. For the semantics, we start from Coquand's notion of a category with universes (Coquand 2012), which covers all presheaf models of dependent type theory with universes. The notion of *category with universes* rests on the observation that in presheaf models one can interpret an inverse $\ulcorner\urcorner$ to the usual function El from codes to types, and hence obtain a simpler notion of universe than usual (such as in Hofmann 1997, section 2.1.6).

▶ **Definition 23** (category with universes)**.** A CwU is specified by:

1. A category $\mathbf{C}$ with a terminal object $\top$.
2. For each object $\Gamma \in \mathbf{C}$ and natural number $n \in \mathbb{N}$, a set $\mathbf{C}(\Gamma, n)$ of *families at universe level* n over Γ.
3. For each object $\Gamma \in \mathbf{C}$, natural number n, and family $A \in \mathbf{C}(\Gamma, n)$, a set $\mathbf{C}(\Gamma \vdash A)$ of *elements* (at some level) of the family A over Γ.

4. For each morphism $\gamma \in \mathbf{C}(\Delta, \Gamma)$, *re-indexing* functions $A \in \mathbf{C}(\Gamma, n) \mapsto A[\gamma] \in \mathbf{C}(\Delta, n)$ and $a \in \mathbf{C}(\Gamma \vdash A) \mapsto a[\gamma] \in \mathbf{C}(\Delta \vdash A[\gamma])$, satisfying equations for associativity and identity as in a CwF.
5. For each object $\Gamma \in \mathbf{C}$, number n and family $A \in \mathbf{C}(\Gamma, n)$, a *comprehension object* $\Gamma.A \in \mathbf{C}$ equipped with projections and generic elements satisfying equations as in a CwF.
6. For each number n, a family $\mathsf{U}_n \in \mathbf{C}(\top, n+1)$, the *universe at level* n.
7. For each object $\Gamma \in \mathbf{C}$ and number n, a *code* function $A \in \mathbf{C}(\Gamma, n) \mapsto \ulcorner A \urcorner \in \mathbf{C}(\Gamma \vdash \mathsf{U}_n[!_\Gamma])$, and an *element* function $u \in \mathbf{C}(\Gamma \vdash \mathsf{U}_n[!_\Gamma]) \mapsto \mathsf{E}\,u \in \mathbf{C}(\Gamma, n)$, satisfying $\ulcorner A \urcorner[\gamma] = \ulcorner A[\gamma] \urcorner$, $\mathsf{E} \ulcorner A \urcorner = A$, and $\ulcorner \mathsf{E}\,u \urcorner = u$.

We will of course want the universes to be closed under various type-forming operations, but in this formalisation of universes these definitions are just as for CwFs, without having to explicitly reflect them into the universes.

▶ **Lemma 24.** *The element function is stable under re-indexing:* $(\mathsf{E}\,u)[\gamma] = \mathsf{E}(u[\gamma])$.

Proof. $(\mathsf{E}\,u)[\gamma] = \mathsf{E}\ulcorner(\mathsf{E}\,u)[\gamma]\urcorner = \mathsf{E}(\ulcorner \mathsf{E}\,u \urcorner[\gamma]) = \mathsf{E}(u[\gamma])$. ◻

▶ **Corollary 25.** *In a CwU there is a* generic family $\mathsf{El} \in \mathbf{C}(\top.\mathsf{U}_n, n)$ *of types of level* n *(for each* $n \in \mathbb{N}$*), with the property that* $\mathsf{El}\,[(!_\Gamma, \ulcorner A \urcorner)] = A$*, for all* $A \in \mathbf{C}(\Gamma, n)$*.*

Proof. Since $\mathsf{p} = ! : \top.\mathsf{U}_n \to \top$, we have $\mathsf{q} \in \mathbf{C}(\top.\mathsf{U}_n \vdash \mathsf{U}_n)$ and thus we can define El to be $\mathsf{E}\,\mathsf{q}$, and then the required property follows by Lemma 24. ◻

For a CwU, there is an underlying CwF with families over Γ given as $\mathbf{C}(\Gamma) = \bigcup_n \mathbf{C}(\Gamma, n)$. Using this we can extend the definition of CwDRA to categories with universes in the obvious way, as follows:

▶ **Definition 26** (CwUDRA)**.** A **category with universe and dependent right adjoint (CwUDRA)** is a CwU with the structure of a CwDRA such that operation on types preserves universe levels in the sense that $A \in \mathbf{C}(\mathsf{L}\Gamma, n)$ implies $\mathsf{R}_\Gamma\,A \in \mathbf{C}(\Gamma, n)$.

Such structure is often obtained in practice by showing that there is a right adjoint at the level of contexts (as in our notion of CwF+A from Definition 15) that preserves universes in the following sense:

▶ **Definition 27.** A **universe endomorphism** on a CwU is a finite limit preserving functor R on the category of contexts together with, for each n, a family $\mathsf{Rl} \in \mathbf{C}(\mathsf{R}(\top.\mathsf{U}_n), n)$ and an isomorphism

$$\begin{array}{ccc} \mathsf{R}(\top.\mathsf{U}_n).\mathsf{Rl} & \xrightarrow{\cong} & \mathsf{R}(\top.\mathsf{U}_n.\mathsf{El}) \\ \big\downarrow \mathsf{p} & \swarrow_{\mathsf{R}\,\mathsf{p}} & \\ \mathsf{R}(\top.\mathsf{U}_n) & & \end{array} \tag{12}$$

In other words, there is a morphism $\ell : \mathsf{R}(\top.\mathsf{U}_n).\mathsf{Rl} \to \mathsf{R}(\top.\mathsf{U}_n.\mathsf{El})$ and an element $\mathsf{r} \in \mathbf{C}(\mathsf{R}(\top.\mathsf{U}_n.\mathsf{El}) \vdash \mathsf{Rl}[\mathsf{R}\,\mathsf{p}])$ satisfying $\ell \circ (\mathsf{R}\,\mathsf{p}, \mathsf{r}) = \mathsf{id}$ and $(\mathsf{R}\,\mathsf{p}, \mathsf{r}) \circ \ell = \mathsf{id}$.

Arrow-theoretically the above definition means that the universe $\mathsf{R}(\top.\mathsf{U}_n.\mathsf{El}) \to \mathsf{R}(\top.\mathsf{U}_n)$ is a pullback of $\top.\mathsf{U}_n.\mathsf{El} \to \top.\mathsf{U}_n$, that is, we have a universe category endomorphism in the sense of Voevodsky (2014). To see how such universe endomorphisms give rise to CwUDRA structure (Theorem 30) we need to extend the notion of CwF+A from Definition 15 to the setting of universes:

▶ **Definition 28** (CwU+A)**.** A **weak CwU morphism** R is a weak CwF morphism on the underlying CwFs preserving size in the sense that $A \in \mathbf{C}(\Gamma, n)$ implies $\mathsf{R}\,A \in \mathbf{C}(\mathsf{R}\,\Gamma, n)$. A **CwU+A** consists of a CwU with an adjunction $\mathsf{L} \dashv \mathsf{R}$ on the category of contexts, such that R extends to a weak CwU morphism.

Given a CwU with a weak CwU morphism R, then clearly R is a universe endomorphism, with $\mathsf{Rl} \triangleq \mathsf{R}(\mathsf{El}\,)$, $\mathsf{r} \triangleq \mathsf{R}\,\mathsf{q}$ and $\ell \triangleq \nu$. Conversely:

▶ **Lemma 29.** *Any CwU with a universe endomorphism $\mathsf{R} : \mathbf{C} \to \mathbf{D}$ extends to a weak CwU morphism.*

Proof. Given $\Gamma \vdash_n A$, since we have $(!_\Gamma, \ulcorner A \urcorner) : \Gamma \to \top.\mathsf{U}_n$, we can define $\mathsf{R}\,\Gamma \vdash_n \mathsf{R}\,A$ by:

$$\mathsf{R}\,A \triangleq \mathsf{Rl}[\mathsf{R}\,(!_\Gamma, \ulcorner A \urcorner)] \tag{13}$$

This is stable under re-indexing, since for $\gamma : \Delta \to \Gamma$

$$\begin{aligned}
\mathsf{R}(A\,\gamma) &\triangleq \mathsf{Rl}[\mathsf{R}\,(!_\Delta, \ulcorner A\,\gamma \urcorner)] \\
&= \mathsf{Rl}[\mathsf{R}\,(!_\Delta, \ulcorner A \urcorner \gamma)] \\
&= \mathsf{Rl}[\mathsf{R}((!_\Gamma, \ulcorner A \urcorner) \circ \gamma)] \\
&= \mathsf{Rl}[\mathsf{R}\,(!_\Gamma, \ulcorner A \urcorner) \circ \mathsf{R}\,\gamma] \\
&= (\mathsf{Rl}[\mathsf{R}\,(!_\Gamma, \ulcorner A \urcorner)])[\mathsf{R}\,\gamma] \\
&\triangleq (\mathsf{R}\,A)[\mathsf{R}\,\gamma]
\end{aligned}$$

Given $\Gamma \vdash a : A$, by Corollary 25 we have $\Gamma \vdash a : \mathsf{El}\,[(!_\Gamma, \ulcorner A \urcorner)]$ and hence

$$((!_\Gamma, \ulcorner A \urcorner)\,, a) : \Gamma \to \top.\mathsf{U}_n.\mathsf{El}$$

Therefore

$$\mathsf{R}\,\Gamma \vdash \mathsf{r}[\mathsf{R}\,((!_\Gamma, \ulcorner A \urcorner)\,, a)] : (\mathsf{Rl}[\mathsf{R}\,\mathsf{p}])[\mathsf{R}\,((!_\Gamma, \ulcorner A \urcorner)\,, a)]$$

But $(\mathsf{Rl}[\mathsf{R}\,\mathsf{p}])[\mathsf{R}\,((!_\Gamma, \ulcorner A \urcorner)\,, a)] = \mathsf{Rl}[\mathsf{R}(\mathsf{p} \circ ((!_\Gamma, \ulcorner A \urcorner)\,, a))] = \mathsf{Rl}[\mathsf{R}\,(!_\Gamma, \ulcorner A \urcorner)) \triangleq \mathsf{R}\,A$. Therefore we get $\mathsf{R}\,\Gamma \vdash \mathsf{R}\,a : \mathsf{R}\,A$ by defining

$$\mathsf{R}\,a \triangleq \mathsf{r}[\mathsf{R}\,((!_\Gamma, \ulcorner A \urcorner)\,, a)] \tag{14}$$

and this is stable under re-indexing, since for $\gamma : \Delta \to \Gamma$

$$\begin{aligned}
(\mathsf{R}\,a)[\mathsf{R}\,\gamma] &\triangleq \mathsf{r}[\mathsf{R}\,((!_\Gamma, \ulcorner A \urcorner)\,, a)][\mathsf{R}\,\gamma] \\
&= \mathsf{r}[\mathsf{R}\,((!_\Delta, \ulcorner A \urcorner[\gamma])\,, a[\gamma])] \\
&= \mathsf{r}[\mathsf{R}\,((!_\Delta, \ulcorner A[\gamma] \urcorner)\,, a[\gamma])] \\
&\triangleq \mathsf{R}(a[\gamma])
\end{aligned}$$

Finally we must show that R commutes with comprehension. For this, note that there are pullback squares

$$\begin{array}{ccc}
\mathsf{R}(\Gamma.A) & \xrightarrow{\mathsf{R}((!_\Gamma, \ulcorner A \urcorner), \mathsf{q})} & \mathsf{R}(\top.\mathsf{U}_n.\mathsf{El}\,) \\
{\scriptstyle \mathsf{R}\,\mathsf{p}} \downarrow & & \downarrow {\scriptstyle \mathsf{R}\,\mathsf{p}} \\
\mathsf{R}\,\Gamma & \xrightarrow[\mathsf{R}(!_\Gamma, \ulcorner A \urcorner)]{} & \mathsf{R}(\top.\mathsf{U}_n)
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{R}\,\Gamma.\,\mathsf{R}\,A & \xrightarrow{(\mathsf{R}(!_\Gamma, \ulcorner A \urcorner), \mathsf{q})} & \mathsf{R}(\top.\mathsf{U}_n).\mathsf{Rl} \\
{\scriptstyle \mathsf{p}} \downarrow & & \downarrow {\scriptstyle \mathsf{p}} \\
\mathsf{R}\,\Gamma & \xrightarrow[\mathsf{R}(!_\Gamma, \ulcorner A \urcorner)]{} & \mathsf{R}(\top.\mathsf{U}_n)
\end{array}$$

(the former because the functor R has a left adjoint and hence preserves pullbacks and the latter by definition of $\mathsf{R}\,A$). Thus the pullback of the isomorphism in (12) gives an isomorphism $\mathsf{pr} : \mathsf{R}\,\Gamma.\,\mathsf{R}\,A \cong \mathsf{R}(\Gamma.A)$ whose inverse is $(\mathsf{R}\,\mathsf{p}, \mathsf{R}\,\mathsf{q})$. ◻

▶ Remark. We observe that for R as constructed above, the image under R of maps with U_n-small fibers is classified by $\mathsf{RI} \in \mathbf{C}(\mathsf{R}(\top.\mathsf{U}_n), n)$. That is to say that $(!_{\mathsf{R}\,\Gamma}, \ulcorner \mathsf{R}\,A \urcorner) = (!_{\mathsf{R}\,\Gamma}, \ulcorner \mathsf{RI} \urcorner) \circ \mathsf{R}\,(!_\Gamma, \ulcorner A \urcorner)$ which is true by our choice of $\mathsf{R}\,A = \mathsf{RI}[\mathsf{R}\,(!_\Gamma, \ulcorner A \urcorner)]$. Hence, the type of codes for such fibers is R applied to the codes for types. The same situation occurred for $\triangleright$ in Birkedal & Møgelberg (2013, V.5), but was not observed at the time.

▶ **Theorem 30.** *Any CwU equipped with an adjunction on the category of contexts whose right adjoint is a universe endomorphism can be given the structure of a CwUDRA.*

Proof. Combine Lemmas 17 and 29. ◻

For most of the presheaf examples considered in Section 5, the dependent right adjoint is obtained as the direct image of an essential geometric morphism arising from a functor on the category on which the presheaves are defined. We show that in this case, the right adjoint preserves universe levels and hence gives a CwUDRA. For simplicity, we will restrict to one universe and show that the right adjoint preserves smallness with respect to this.

Let U be a universe in an ambient set theory. We call the elements of U, U-sets. A *U-small category* is one where both the sets of objects and the set of morphisms are U-small. Let us assume that U is U-complete — it is closed under limits of U-small diagrams. A Grothendieck universe in ZFC would satisfy these conditions.

▶ Proposition 31. Let C, D be a U-small categories and $f : C \to D$ a functor between them. The direct image f_* of the induced geometric morphism preserves size. In particular, for each endofunctor f, the direct image is a weak CwU morphism.

Proof. Since f_* is a right adjoint, we know that it induces a weak CwF morphism, and we just need to show that it maps U-small families to U-small families. Recall first that the direct image f_* is the (pointwise) right Kan extension (Johnstone 2002, A4.1.4) defined on objects by the limit of the diagram

$$(\mathrm{Ran}_f\, F)d \triangleq \lim(f \downarrow d) \xrightarrow{\pi_1} C^{op} \xrightarrow{F} Uset,$$

for $F \in \widehat{C}$ and $d \in D$. Here $(f \downarrow d)$ denotes the *comma category* consisting of pairs $(c; g : f(c) \to d)$.

A family $\alpha : F \to G$, for $F, G \in \hat{C}$ is U-small if for each c and each $x \in G(c)$ the set $\alpha_c^{-1}(x)$ is in U. Given $(x_g)_{g \in f\downarrow d} \in f_*G(d) = (\mathrm{Ran}_f\, G)d$, the preimage $(f_*\alpha)^{-1}((x_g)_{g\in f\downarrow d})$ is the set

$$\{(y_g)_{g\in f\downarrow d} \in (\mathrm{Ran}_f\, G)d \mid \forall g.\alpha_c(y_g) = x_g\}$$

which is the limit of the diagram associating to each g the set $\alpha_c^{-1}(x_g)$. Since each of these sets are in U by assumption and since also $f \downarrow d$ is in U, by the assumption of U being closed under limits, also $(f_*\alpha)^{-1}((x_g)_{g\in f\downarrow d})$ is in U as desired. ◻

Syntax At this stage it should hopefully be clear that one can refine and extend the syntax of modal dependent type theory from Section 3 so that the resulting syntactic type theory can be modelled in a CwU+A. The idea is, of course, to refine the judgement for well-formed types and to include a level n, so that it has the form $\Gamma \vdash_n A$, and likewise for type equality judgements. For example,

$$\frac{\Gamma, 🔓 \vdash_n A}{\Gamma \vdash_n \Box A}$$

In addition to the existing rules for types (indexed with a level) and terms, we then also include:

$$\frac{}{\diamond \vdash_{n+1} \mathsf{U}_n} \qquad \frac{\Gamma \vdash_n A}{\Gamma \vdash \ulcorner A \urcorner : \mathsf{U}_n} \qquad \frac{\Gamma \vdash u : \mathsf{U}_n}{\Gamma \vdash_n \mathsf{E}\, u}$$

Finally, we add the following type and term equality rules:

$$\frac{\Gamma \vdash_n A}{\Gamma \vdash_n \mathsf{E}\ulcorner A \urcorner = A} \qquad \frac{\Gamma \vdash u : \mathsf{U}_n}{\Gamma \vdash \ulcorner \mathsf{E}\, u \urcorner = u : \mathsf{U}_n}$$

As an example, there is a term

$$\widehat{\Box} \triangleq \lambda x. \ulcorner \Box\, \mathsf{E}(\mathsf{open}\, x) \urcorner \ : \ \Box \mathsf{U}_n \to \mathsf{U}_n$$

which encodes the $\Box$ type constructor on the universe in the sense that

$$\mathsf{E}(\widehat{\Box}(\mathsf{shut}\, u)) = \Box(\mathsf{E}\, u)$$

This is similar to the $\widehat{\triangleright}$ operator of Guarded Dependent Type Theory (Bizjak et al. 2016), which is essential to defining guarded recursive types. Thus, $\widehat{\triangleright}$ arises for general reasons quite unconnected to the specifics of guarded recursion.

7 Discussion

7.1 Related Work

Modal dependent type theory builds on work on the computational interpretation of modal logic with *simple* types. Some of this work involves a standard notion of context; most relevantly to this paper, the calculus for Intuitionistic K of Bellin, De Paiva & Ritter (2001), which employs *explicit substitutions* in terms. Departing from standard contexts, Fitch-style calculi were introduced independently by Borghuis (1994) and Martini & Masini (1996). Recent work by Clouston (2018) argued that Fitch-style calculus can be extended to a variety of different modal logics, and gave a sound categorical interpretation by modelling the modality as a right adjoint. Another non-standard notion of context are the *dual contexts* introduced by Davies & Pfenning (2001) for the modal logic Intuituionistic S4 of comonads. Here a context $\Delta; \Gamma$ has semantics $\Box\Delta \wedge \Gamma$, so the structure in the context is modelled by the modality itself, not its left adjoint. Recent work by Kavvos (2017) has extended this approach to a variety of modal logics, including Intuitionistic K.

There exists recent work employing variants of dual contexts for modal *dependent* type theory, all involving (co)monads rather than the more basic logic of this paper. Indeed we do not know how to combine the K axiom alone with dependent types via dual contexts; it is not obvious how to extend Kavvos's simply-typed calculus. This should be compared to the ease of extending the simply-typed Fitch-style calculus with dependent types. We hope that Fitch-style calculi continue to provide a relatively simple setting for modal dependent type theory as we explore the extensions discussed in the next subsection.

The first work on modal dependent types with dual contexts, spatial type theory (Shulman 2017), designed for applications in homotopy type theory (see also (Wellen 2017, Licata et al. 2018)), extends the Davies-Pfenning calculus for a comonad with both dependent types and a second modality, a monad right adjoint to the comonad. Second, the calculus for parametricity of Nuyts et al. (2017) uses *three* zones to extend Davies-Pfenning with a monad *left* adjoint to the comonad. They focus on Π- and Σ-types with modalised arguments, but a more standard modality can be extracted by taking the second argument of a modalised Σ-type to be the unit type. In both the above works the leftmost modality is intended to itself be a right adjoint, so they potentially could also be captured by a Fitch-style calculus. Third, de Paiva & Ritter (2016) suggest a generalisation of Davies-Pfenning with some unusual properties, as $\Box$ types carry an auxiliary typed variable and Π-types may only draw their argument from the modal context. We finally note the dual contexts approach has inspired the mode theories of Licata, Shulman & Riley (2017), but this line of work as yet does not support a term calculus.

We are not aware of any successful extensions of the explicit substitution approach to dependent types; our own experiments with this while developing Guarded Dependent Type Theory (Clouston et al. 2015) suggests this is probably possible but becomes unwieldy with real examples. Far more succesful was the Clocked Type Theory (Bahr et al. 2017) discussed in Section 5, which can now be seen to have rediscovered the Fitch-style framework, albeit with the innovation of named locks to control fixed-point unfoldings. That work provides the inspiration for the more foundational developments of this paper.

7.2 Future work

We wish to develop operational semantics for dependent Fitch-style calculi, and conjecture that standard techniques for sound normalisation and canonicity can be extended, as was possible for simply-typed Fitch-style calculi (Borghuis 1994, Clouston 2018), and for Clocked Type Theory (Bahr et al. 2017). Such results should then lead to practical implementation.

The modal axiom Intuitionistic K was used in this paper because it provides a basic notion of modal necessity and holds of many useful models. Nonetheless for particular applications we will want to develop Fitch-style calculi corresponding to more particular logics. There can be no algorithm for converting additional axioms to well-behaved calculi, but we know that Fitch-style calculi are extremely versatile in the simply typed case (Clouston 2018), and Clocked Type Theory provides one example of this with dependent types. In particular we are interested in Fitch-style calculi with multiple interacting modalities, each of which is assigned its own lock; we hope to develop guarded type theory with both $\triangleright$ and $\Box$ modalities in this style.

The notion of CwF with a weak CwF endomorphism (Definition 13) is more general than our CwF+A as it does not require a left adjoint, but because such a morphism must preserve products it appears to be a rival candidate for a notion of model of modal dependent type theory. However we do not know how to capture this class of models in syntax. Understanding this would be valuable because *truncation* (Awodey & Bauer 2004), considered as an endofunctor for example on sets, defines such a morphism but is not a right adjoint. Truncation allows one to move between general types and propositions. For example combining it with guarded types would allow us to formalise work in this field that makes that distinction (Birkedal et al. 2012, Clouston et al. 2015).

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