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Higher-Order Partial Differentiation[1]

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Summary. In this article, we shall extend the formalization of [10] to discuss higher-order partial differentiation of real valued functions. The linearity of this operator is also proved (refer to [10], [12] and [13] for partial differentiation).

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The terminology and notation used here have been introduced in the following articles: [3], [8], [2], [4], [5], [15], [21], [17], [16], [20], [1], [6], [10], [12], [13], [18], [11], [9], [23], [7], [19], [14], and [22].

1. Preliminaries

We use the following convention: m, n denote non empty elements of $\mathbb{N}$, i, j denote elements of $\mathbb{N}$, and Z denotes a set.

One can prove the following propositions:

(1) Let S, T be real normed spaces, f be a point of the real norm space of bounded linear operators from S into T, and r be a real number. Suppose $0 \leq r$ and for every point x of S such that $\|x\| \leq 1$ holds $\|f(x)\| \leq r \cdot \|x\|$. Then $\|f\| \leq r$.

(2) Let S be a real normed space and f be a partial function from S to $\mathbb{R}$. Then f is continuous on Z if and only if the following conditions are satisfied:

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(i) $Z \subseteq \operatorname{dom} f$, and

(ii) for every sequence s_1 of S such that $\operatorname{rng} s_1 \subseteq Z$ and s_1 is convergent and $\lim s_1 \in Z$ holds $f_* s_1$ is convergent and $f_{\lim s_1} = \lim(f_* s_1)$.

(3) For every partial function f from $\mathcal{R}^i$ to $\mathbb{R}$ holds $\operatorname{dom}\langle f\rangle = \operatorname{dom} f$.

(4) For every partial function f from $\mathcal{R}^i$ to $\mathbb{R}$ such that $Z \subseteq \operatorname{dom} f$ holds $\operatorname{dom}(\langle f\rangle\restriction Z) = Z$.

(5) For every partial function f from $\mathcal{R}^i$ to $\mathbb{R}$ holds $\langle f\restriction Z\rangle = \langle f\rangle\restriction Z$.

(6) Let f be a partial function from $\mathcal{R}^i$ to $\mathbb{R}$ and x be an element of $\mathcal{R}^i$. If $x \in \operatorname{dom} f$, then $\langle f\rangle(x) = \langle f(x)\rangle$ and $\langle f\rangle_x = \langle f_x\rangle$.

(7) For all partial functions f, g from $\mathcal{R}^i$ to $\mathbb{R}$ holds $\langle f+g\rangle = \langle f\rangle + \langle g\rangle$ and $\langle f-g\rangle = \langle f\rangle - \langle g\rangle$.

(8) For every partial function f from $\mathcal{R}^i$ to $\mathbb{R}$ and for every real number r holds $\langle r\cdot f\rangle = r\cdot\langle f\rangle$.

(9) Let f be a partial function from $\mathcal{R}^i$ to $\mathbb{R}$ and g be a partial function from $\mathcal{R}^i$ to $\mathcal{R}^1$. If $\langle f\rangle = g$, then $|f| = |g|$.

(10) For every subset X of $\mathcal{R}^m$ and for every subset Y of $\langle\mathcal{E}^m, \|\cdot\|\rangle$ such that $X = Y$ holds X is open iff Y is open.

(11) For every element q of $\mathbb{R}$ such that $1 \leq i \leq j$ holds $|(\operatorname{reproj}(i, \underbrace{\langle 0, \ldots, 0\rangle}_{j}))(q)| = |q|$.

(12) For every element x of $\mathcal{R}^j$ holds $x = (\operatorname{reproj}(i, x))((\operatorname{proj}(i, j))(x))$.

2. Continuity and Differentiability

The following two propositions are true:

(13) Let X be a subset of $\mathcal{R}^m$ and f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$. If f is differentiable on X, then X is open.

(14) Let X be a subset of $\mathcal{R}^m$ and f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$. Suppose X is open. Then f is differentiable on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every element x of $\mathcal{R}^m$ such that $x \in X$ holds f is differentiable in x.

Let m, n be non empty elements of $\mathbb{N}$, let Z be a set, and let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$. Let us assume that $Z \subseteq \operatorname{dom} f$. The functor $f'_{\restriction Z}$ yields a partial function from $\mathcal{R}^m$ to $(\mathcal{R}^n)^{\mathcal{R}^m}$ and is defined by:

(Def. 1) $\operatorname{dom}(f'_{\restriction Z}) = Z$ and for every element x of $\mathcal{R}^m$ such that $x \in Z$ holds $(f'_{\restriction Z})_x = f'(x)$.

We now state a number of propositions:

(15) Let X be a subset of $\mathcal{R}^m$ and f, g be partial functions from $\mathcal{R}^m$ to $\mathcal{R}^n$. Suppose f is differentiable on X and g is differentiable on X. Then $f+g$ is differentiable on X and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $((f+g)'_{\restriction X})_x = f'(x) + g'(x)$.

(16) Let X be a subset of $\mathcal{R}^m$ and f, g be partial functions from $\mathcal{R}^m$ to $\mathcal{R}^n$. Suppose f is differentiable on X and g is differentiable on X. Then $f-g$ is differentiable on X and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $((f-g)'_{\restriction X})_x = f'(x) - g'(x)$.

(17) Let X be a subset of $\mathcal{R}^m$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and r be a real number. Suppose f is differentiable on X. Then $r \cdot f$ is differentiable on X and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $((r \cdot f)'_{\restriction X})_x = r \cdot f'(x)$.

(18) Let f be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^j, \|\cdot\| \rangle$. Then there exists a point p of $\langle \mathcal{E}^j, \|\cdot\| \rangle$ such that

(i) $p = f(\langle 1 \rangle)$,

(ii) for every real number r and for every point x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that $x = \langle r \rangle$ holds $f(x) = r \cdot p$, and

(iii) for every point x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ holds $\|f(x)\| = \|p\| \cdot \|x\|$.

(19) Let f be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^j, \|\cdot\| \rangle$. Then there exists a point p of $\langle \mathcal{E}^j, \|\cdot\| \rangle$ such that $p = f(\langle 1 \rangle)$ and $\|p\| = \|f\|$.

(20) Let f be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^j, \|\cdot\| \rangle$ and x be a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle$. Then $\|f(x)\| = \|f\| \cdot \|x\|$.

(21) Let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, X be a subset of $\mathcal{R}^m$, and Y be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $1 \leq i \leq m$ and X is open and $g = f$ and $X = Y$ and f is partially differentiable on X w.r.t. i. Let x be an element of $\mathcal{R}^m$ and y be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. If $x \in X$ and $x = y$, then $\mathrm{partdiff}(f, x, i) = (\mathrm{partdiff}(g, y, i))(\langle 1 \rangle)$.

(22) Let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, X be a subset of $\mathcal{R}^m$, and Y be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $1 \leq i \leq m$ and X is open and $g = f$ and $X = Y$ and f is partially differentiable on X w.r.t. i. Let x_0, x_1 be elements of $\mathcal{R}^m$ and y_0, y_1 be points of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. If $x_0 = y_0$ and $x_1 = y_1$ and $x_0, x_1 \in X$, then $|(f \restriction^i X)_{x_1} - (f \restriction^i X)_{x_0}| = \|(g \restriction^i Y)_{y_1} - (g \restriction^i Y)_{y_0}\|$.

(23) Let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, X be a subset of $\mathcal{R}^m$, and Y be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $1 \leq i \leq m$ and X is open and $g = f$ and $X = Y$. Then the following statements are equivalent

(i) f is partially differentiable on X w.r.t. i and $f \restriction^i X$ is continuous on X,

(ii) g is partially differentiable on Y w.r.t. i and $g{\upharpoonright}^i Y$ is continuous on Y.

(24) Let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, X be a subset of $\mathcal{R}^m$, and Y be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $X = Y$ and X is open and $f = g$. Then for every i such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f{\upharpoonright}^i X$ is continuous on X if and only if g is differentiable on Y and $g'_{\upharpoonright Y}$ is continuous on Y.

(25) Let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, X be a subset of $\mathcal{R}^m$, and Y be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose X is open and $X \subseteq \operatorname{dom} f$ and $g = f$ and $X = Y$. Then g is differentiable on Y and $g'_{\upharpoonright Y}$ is continuous on Y if and only if the following conditions are satisfied:

(i) f is differentiable on X, and

(ii) for every element x_0 of $\mathcal{R}^m$ and for every real number r such that $x_0 \in X$ and $0 < r$ there exists a real number s such that $0 < s$ and for every element x_1 of $\mathcal{R}^m$ such that $x_1 \in X$ and $|x_1 - x_0| < s$ and for every element v of $\mathcal{R}^m$ holds $|f'(x_1)(v) - f'(x_0)(v)| \leq r \cdot |v|$.

(26) Let X be a subset of $\mathcal{R}^m$ and f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$. Suppose X is open and $X \subseteq \operatorname{dom} f$. Then the following statements are equivalent

(i) for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f{\upharpoonright}^i X$ is continuous on X,

(ii) f is differentiable on X and for every element x_0 of $\mathcal{R}^m$ and for every real number r such that $x_0 \in X$ and $0 < r$ there exists a real number s such that $0 < s$ and for every element x_1 of $\mathcal{R}^m$ such that $x_1 \in X$ and $|x_1 - x_0| < s$ and for every element v of $\mathcal{R}^m$ holds $|f'(x_1)(v) - f'(x_0)(v)| \leq r \cdot |v|$.

(27) Let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$ and g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f = g$ and f is differentiable on Z, then $f'_{\upharpoonright Z} = g'_{\upharpoonright Z}$.

(28) Let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, X be a subset of $\mathcal{R}^m$, and Y be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $X = Y$ and X is open and $f = g$. Then for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f{\upharpoonright}^i X$ is continuous on X if and only if f is differentiable on X and $g'_{\upharpoonright Y}$ is continuous on Y.

(29) Let f, g be partial functions from $\mathcal{R}^m$ to $\mathcal{R}^n$ and x be an element of $\mathcal{R}^m$. Suppose f is continuous in x and g is continuous in x. Then $f + g$ is continuous in x and $f - g$ is continuous in x.

(30) Let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be an element of $\mathcal{R}^m$, and r be a real number. If f is continuous in x, then $r \cdot f$ is continuous in x.

(31) Let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$ and x be an element of $\mathcal{R}^m$. If f is continuous in x, then $-f$ is continuous in x.

(32) Let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$ and x be an element of $\mathcal{R}^m$. If f is continuous in x, then $|f|$ is continuous in x.

(33) Let Z be a set and f, g be partial functions from $\mathcal{R}^m$ to $\mathcal{R}^n$. Suppose f is continuous on Z and g is continuous on Z. Then $f+g$ is continuous on Z and $f-g$ is continuous on Z.

(34) Let r be a real number and f, g be partial functions from $\mathcal{R}^m$ to $\mathcal{R}^n$. If f is continuous on Z, then $r \cdot f$ is continuous on Z.

(35) For all partial functions f, g from $\mathcal{R}^m$ to $\mathcal{R}^n$ such that f is continuous on Z holds $-f$ is continuous on Z.

(36) Let f be a partial function from $\mathcal{R}^i$ to $\mathbb{R}$ and x_0 be an element of $\mathcal{R}^i$. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every element x of $\mathcal{R}^i$ such that $x \in \operatorname{dom} f$ and $|x - x_0| < s$ holds $|f_x - f_{x_0}| < r$.

(37) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$ and x_0 be an element of $\mathcal{R}^m$. Then f is continuous in x_0 if and only if $\langle f \rangle$ is continuous in x_0.

(38) Let f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$ and x_0 be an element of $\mathcal{R}^m$. Suppose f is continuous in x_0 and g is continuous in x_0. Then $f+g$ is continuous in x_0 and $f-g$ is continuous in x_0.

(39) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, x_0 be an element of $\mathcal{R}^m$, and r be a real number. If f is continuous in x_0, then $r \cdot f$ is continuous in x_0.

(40) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$ and x_0 be an element of $\mathcal{R}^m$. If f is continuous in x_0, then $|f|$ is continuous in x_0.

(41) Let f, g be partial functions from $\mathcal{R}^i$ to $\mathbb{R}$ and x be an element of $\mathcal{R}^i$. If f is continuous in x and g is continuous in x, then $f \cdot g$ is continuous in x.

Let m be a non empty element of $\mathbb{N}$, let Z be a set, and let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. We say that f is continuous on Z if and only if:

(Def. 2) For every element x_0 of $\mathcal{R}^m$ such that $x_0 \in Z$ holds $f \restriction Z$ is continuous in x_0.

We now state a number of propositions:

(42) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$ and g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\mathbb{R}$. Suppose $f = g$. Then $Z \subseteq \operatorname{dom} f$ and f is continuous on Z if and only if g is continuous on Z.

(43) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$ and g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\mathbb{R}$. Suppose $f = g$ and $Z \subseteq \operatorname{dom} f$. Then f is continuous on Z

if and only if for every sequence s of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $\operatorname{rng} s \subseteq Z$ and s is convergent and $\lim s \in Z$ holds $g_* s$ is convergent and $g_{\lim s} = \lim(g_* s)$.

(44) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$ and g be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^1$. Suppose $\langle f \rangle = g$. Then $Z \subseteq \operatorname{dom} f$ and f is continuous on Z if and only if g is continuous on Z.

(45) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose $Z \subseteq \operatorname{dom} f$. Then f is continuous on Z if and only if for every element x_0 of $\mathcal{R}^m$ and for every real number r such that $x_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every element x_1 of $\mathcal{R}^m$ such that $x_1 \in Z$ and $|x_1 - x_0| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(46) Let f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose f is continuous on Z and g is continuous on Z and $Z \subseteq \operatorname{dom} f$ and $Z \subseteq \operatorname{dom} g$. Then $f + g$ is continuous on Z and $f - g$ is continuous on Z.

(47) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$ and r be a real number. If $Z \subseteq \operatorname{dom} f$ and f is continuous on Z, then $r \cdot f$ is continuous on Z.

(48) Let f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose f is continuous on Z and g is continuous on Z and $Z \subseteq \operatorname{dom} f$ and $Z \subseteq \operatorname{dom} g$. Then $f \cdot g$ is continuous on Z.

(49) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$ and g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\mathbb{R}$. Suppose $f = g$. Then $Z \subseteq \operatorname{dom} f$ and f is continuous on Z if and only if g is continuous on Z.

(50) For all partial functions f, g from $\mathcal{R}^m$ to $\mathcal{R}^n$ such that f is continuous on Z holds $|f|$ is continuous on Z.

(51) Let f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$ and x be an element of $\mathcal{R}^m$. Suppose f is differentiable in x and g is differentiable in x. Then $f + g$ is differentiable in x and $(f+g)'(x) = f'(x) + g'(x)$ and $f - g$ is differentiable in x and $(f-g)'(x) = f'(x) - g'(x)$.

(52) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, r be a real number, and x be an element of $\mathcal{R}^m$. Suppose f is differentiable in x. Then $r \cdot f$ is differentiable in x and $(r \cdot f)'(x) = r \cdot f'(x)$.

Let Z be a set, let m be a non empty element of $\mathbb{N}$, and let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. We say that f is differentiable on Z if and only if:

(Def. 3) For every element x of $\mathcal{R}^m$ such that $x \in Z$ holds $f \restriction Z$ is differentiable in x.

Next we state three propositions:

(53) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$ and g be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^1$. Suppose $\langle f \rangle = g$. Then $Z \subseteq \operatorname{dom} f$ and f is differentiable on Z if and only if g is differentiable on Z.

(54) Let X be a subset of $\mathcal{R}^m$ and f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose $X \subseteq \operatorname{dom} f$ and X is open. Then f is differentiable on X if and

only if for every element x of $\mathcal{R}^m$ such that $x \in X$ holds f is differentiable in x.

(55) Let X be a subset of $\mathcal{R}^m$ and f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. If $X \subseteq \operatorname{dom} f$ and f is differentiable on X, then X is open.

Let m be a non empty element of $\mathbb{N}$, let Z be a set, and let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. Let us assume that $Z \subseteq \operatorname{dom} f$. The functor $f'_{\restriction Z}$ yields a partial function from $\mathcal{R}^m$ to $\mathbb{R}^{\mathcal{R}^m}$ and is defined by:

(Def. 4) $\operatorname{dom}(f'_{\restriction Z}) = Z$ and for every element x of $\mathcal{R}^m$ such that $x \in Z$ holds $(f'_{\restriction Z})_x = f'(x)$.

One can prove the following four propositions:

(56) Let X be a subset of $\mathcal{R}^m$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, and g be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^1$. Suppose $\langle f \rangle = g$ and $X \subseteq \operatorname{dom} f$ and f is differentiable on X. Then g is differentiable on X and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $(f'_{\restriction X})_x = \operatorname{proj}(1,1) \cdot (g'_{\restriction X})_x$.

(57) Let X be a subset of $\mathcal{R}^m$ and f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose $X \subseteq \operatorname{dom} f$ and $X \subseteq \operatorname{dom} g$ and f is differentiable on X and g is differentiable on X. Then $f + g$ is differentiable on X and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $((f+g)'_{\restriction X})_x = (f'_{\restriction X})_x + (g'_{\restriction X})_x$.

(58) Let X be a subset of $\mathcal{R}^m$ and f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose $X \subseteq \operatorname{dom} f$ and $X \subseteq \operatorname{dom} g$ and f is differentiable on X and g is differentiable on X. Then $f - g$ is differentiable on X and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $((f-g)'_{\restriction X})_x = (f'_{\restriction X})_x - (g'_{\restriction X})_x$.

(59) Let X be a subset of $\mathcal{R}^m$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, and r be a real number. Suppose $X \subseteq \operatorname{dom} f$ and f is differentiable on X. Then $r \cdot f$ is differentiable on X and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $((r \cdot f)'_{\restriction X})_x = r \cdot (f'_{\restriction X})_x$.

Let m be a non empty element of $\mathbb{N}$, let Z be a set, let i be an element of $\mathbb{N}$, and let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. We say that f is partially differentiable on Z w.r.t. i if and only if:

(Def. 5) $Z \subseteq \operatorname{dom} f$ and for every element x of $\mathcal{R}^m$ such that $x \in Z$ holds $f{\restriction}Z$ is partially differentiable in x w.r.t. i.

Let m be a non empty element of $\mathbb{N}$, let Z be a set, let i be an element of $\mathbb{N}$, and let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. Let us assume that f is partially differentiable on Z w.r.t. i. The functor $f{\restriction}^{i}Z$ yields a partial function from $\mathcal{R}^m$ to $\mathbb{R}$ and is defined as follows:

(Def. 6) $\operatorname{dom}(f{\restriction}^{i}Z) = Z$ and for every element x of $\mathcal{R}^m$ such that $x \in Z$ holds $(f{\restriction}^{i}Z)_x = \operatorname{partdiff}(f, x, i)$.

Next we state several propositions:

(60) Let X be a subset of $\mathcal{R}^m$ and f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose X is open and $1 \leq i \leq m$. Then f is partially differentiable on X

w.r.t. i if and only if $X \subseteq \operatorname{dom} f$ and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds f is partially differentiable in x w.r.t. i.

(61) Let X be a subset of $\mathcal{R}^m$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, and g be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^1$. Suppose $\langle f \rangle = g$ and X is open and $1 \leq i \leq m$. Then f is partially differentiable on X w.r.t. i if and only if g is partially differentiable on X w.r.t. i.

(62) Let X be a subset of $\mathcal{R}^m$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, and g be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^1$. Suppose $\langle f \rangle = g$ and X is open and $1 \leq i \leq m$ and f is partially differentiable on X w.r.t. i. Then $f{\restriction}^i X$ is continuous on X if and only if $g{\restriction}^i X$ is continuous on X.

(63) Let X be a subset of $\mathcal{R}^m$ and f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose X is open and $X \subseteq \operatorname{dom} f$. Then the following statements are equivalent

(i) for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f{\restriction}^i X$ is continuous on X,

(ii) f is differentiable on X and for every element x_0 of $\mathcal{R}^m$ and for every real number r such that $x_0 \in X$ and $0 < r$ there exists a real number s such that $0 < s$ and for every element x_1 of $\mathcal{R}^m$ such that $x_1 \in X$ and $|x_1 - x_0| < s$ and for every element v of $\mathcal{R}^m$ holds $|f'(x_1)(v) - f'(x_0)(v)| \leq r \cdot |v|$.

(64) Let f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$ and x be an element of $\mathcal{R}^m$. Suppose f is partially differentiable in x w.r.t. i and g is partially differentiable in x w.r.t. i. Then $f \cdot g$ is partially differentiable in x w.r.t. i and $\operatorname{partdiff}(f \cdot g, x, i) = \operatorname{partdiff}(f, x, i) \cdot g(x) + f(x) \cdot \operatorname{partdiff}(g, x, i)$.

(65) Let X be a subset of $\mathcal{R}^m$ and f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose that

(i) X is open,

(ii) $1 \leq i$,

(iii) $i \leq m$,

(iv) f is partially differentiable on X w.r.t. i, and

(v) g is partially differentiable on X w.r.t. i.

Then

(vi) $f + g$ is partially differentiable on X w.r.t. i,

(vii) $(f + g){\restriction}^i X = (f{\restriction}^i X) + (g{\restriction}^i X)$, and

(viii) for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $((f + g){\restriction}^i X)_x = \operatorname{partdiff}(f, x, i) + \operatorname{partdiff}(g, x, i)$.

(66) Let X be a subset of $\mathcal{R}^m$ and f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose that

(i) X is open,

(ii) $1 \leq i$,

(iii) $i \leq m$,

(iv) f is partially differentiable on X w.r.t. i, and
(v) g is partially differentiable on X w.r.t. i.
Then
(vi) $f - g$ is partially differentiable on X w.r.t. i,
(vii) $(f - g){\restriction^i}X = (f{\restriction^i}X) - (g{\restriction^i}X)$, and
(viii) for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $((f - g){\restriction^i}X)_x = \mathrm{partdiff}(f, x, i) - \mathrm{partdiff}(g, x, i)$.

(67) Let X be a subset of $\mathcal{R}^m$, r be a real number, and f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose X is open and $1 \le i \le m$ and f is partially differentiable on X w.r.t. i. Then
(i) $r \cdot f$ is partially differentiable on X w.r.t. i,
(ii) $r \cdot f{\restriction^i}X = r \cdot (f{\restriction^i}X)$, and
(iii) for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $(r \cdot f{\restriction^i}X)_x = r \cdot \mathrm{partdiff}(f, x, i)$.

(68) Let X be a subset of $\mathcal{R}^m$ and f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose that
(i) X is open,
(ii) $1 \le i$,
(iii) $i \le m$,
(iv) f is partially differentiable on X w.r.t. i, and
(v) g is partially differentiable on X w.r.t. i.
Then
(vi) $f \cdot g$ is partially differentiable on X w.r.t. i,
(vii) $f \cdot g{\restriction^i}X = (f{\restriction^i}X) \cdot g + f \cdot (g{\restriction^i}X)$, and
(viii) for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $(f \cdot g{\restriction^i}X)_x = \mathrm{partdiff}(f, x, i) \cdot g(x) + f(x) \cdot \mathrm{partdiff}(g, x, i)$.

3. Higher-Order Partial Differentiation

Let m be a non empty element of $\mathbb{N}$, let Z be a set, let I be a finite sequence of elements of $\mathbb{N}$, and let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. The functor $\mathrm{PartDiffSeq}(f, Z, I)$ yielding a sequence of partial functions from $\mathcal{R}^m$ into $\mathbb{R}$ is defined by:

(Def. 7) $(\mathrm{PartDiffSeq}(f, Z, I))(0) = f$ and for every natural number i holds $(\mathrm{PartDiffSeq}(f, Z, I))(i + 1) = (\mathrm{PartDiffSeq}(f, Z, I))(i){\restriction^{I_{i+1}}}Z$.

Let m be a non empty element of $\mathbb{N}$, let Z be a set, let I be a finite sequence of elements of $\mathbb{N}$, and let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. We say that f is partially differentiable on Z w.r.t. I if and only if:

(Def. 8) For every element i of $\mathbb{N}$ such that $i \le \mathrm{len}\, I - 1$ holds $(\mathrm{PartDiffSeq}(f, Z, I))(i)$ is partially differentiable on Z w.r.t. I_{i+1}.

Let m be a non empty element of $\mathbb{N}$, let Z be a set, let I be a finite sequence of elements of $\mathbb{N}$, and let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. The functor $f{\restriction}^I Z$ yielding a partial function from $\mathcal{R}^m$ to $\mathbb{R}$ is defined by:

(Def. 9) $f{\restriction}^I Z = (\mathrm{PartDiffSeq}(f, Z, I))(\mathrm{len}\, I)$.

The following propositions are true:

(69) Let X be a subset of $\mathcal{R}^m$, I be a non empty finite sequence of elements of $\mathbb{N}$, and f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose that
(i) X is open,
(ii) $\mathrm{rng}\, I \subseteq \mathrm{Seg}\, m$,
(iii) f is partially differentiable on X w.r.t. I, and
(iv) g is partially differentiable on X w.r.t. I.
Let given i. Suppose $i \leq \mathrm{len}\, I - 1$. Then $(\mathrm{PartDiffSeq}(f + g, X, I))(i)$ is partially differentiable on X w.r.t. I_{i+1} and $(\mathrm{PartDiffSeq}(f+g, X, I))(i) = (\mathrm{PartDiffSeq}(f, X, I))(i) + (\mathrm{PartDiffSeq}(g, X, I))(i)$.

(70) Let X be a subset of $\mathcal{R}^m$, I be a non empty finite sequence of elements of $\mathbb{N}$, and f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose that
(i) X is open,
(ii) $\mathrm{rng}\, I \subseteq \mathrm{Seg}\, m$,
(iii) f is partially differentiable on X w.r.t. I, and
(iv) g is partially differentiable on X w.r.t. I.
Then $f + g$ is partially differentiable on X w.r.t. I and $(f + g){\restriction}^I X = (f{\restriction}^I X) + (g{\restriction}^I X)$.

(71) Let X be a subset of $\mathcal{R}^m$, I be a non empty finite sequence of elements of $\mathbb{N}$, and f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose that
(i) X is open,
(ii) $\mathrm{rng}\, I \subseteq \mathrm{Seg}\, m$,
(iii) f is partially differentiable on X w.r.t. I, and
(iv) g is partially differentiable on X w.r.t. I.
Let given i. Suppose $i \leq \mathrm{len}\, I - 1$. Then $(\mathrm{PartDiffSeq}(f - g, X, I))(i)$ is partially differentiable on X w.r.t. I_{i+1} and $(\mathrm{PartDiffSeq}(f-g, X, I))(i) = (\mathrm{PartDiffSeq}(f, X, I))(i) - (\mathrm{PartDiffSeq}(g, X, I))(i)$.

(72) Let X be a subset of $\mathcal{R}^m$, I be a non empty finite sequence of elements of $\mathbb{N}$, and f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose that
(i) X is open,
(ii) $\mathrm{rng}\, I \subseteq \mathrm{Seg}\, m$,
(iii) f is partially differentiable on X w.r.t. I, and
(iv) g is partially differentiable on X w.r.t. I.
Then $f - g$ is partially differentiable on X w.r.t. I and $(f - g){\restriction}^I X = (f{\restriction}^I X) - (g{\restriction}^I X)$.

(73) Let X be a subset of $\mathcal{R}^m$, r be a real number, I be a non empty finite sequence of elements of $\mathbb{N}$, and f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$.

Suppose X is open and $\operatorname{rng} I \subseteq \operatorname{Seg} m$ and f is partially differentiable on X w.r.t. I. Let given i. Suppose $i \leq \operatorname{len} I - 1$. Then $(\operatorname{PartDiffSeq}(r \cdot f, X, I))(i)$ is partially differentiable on X w.r.t. I_{i+1} and $(\operatorname{PartDiffSeq}(r \cdot f, X, I))(i) = r \cdot (\operatorname{PartDiffSeq}(f, X, I))(i)$.

(74) Let X be a subset of $\mathcal{R}^m$, r be a real number, I be a non empty finite sequence of elements of $\mathbb{N}$, and f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose X is open and $\operatorname{rng} I \subseteq \operatorname{Seg} m$ and f is partially differentiable on X w.r.t. I. Then $r \cdot f$ is partially differentiable on X w.r.t. I and $r \cdot f{\restriction}^I X = r \cdot (f{\restriction}^I X)$.

Let m be a non empty element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, let k be an element of $\mathbb{N}$, and let Z be a set. We say that f is partial differentiable up to order k and Z if and only if the condition (Def. 10) is satisfied.

(Def. 10) Let I be a non empty finite sequence of elements of $\mathbb{N}$. If $\operatorname{len} I \leq k$ and $\operatorname{rng} I \subseteq \operatorname{Seg} m$, then f is partially differentiable on Z w.r.t. I.

The following proposition is true

(75) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$ and I, G be non empty finite sequences of elements of $\mathbb{N}$. Then f is partially differentiable on Z w.r.t. $G \frown I$ if and only if f is partially differentiable on Z w.r.t. G and $f{\restriction}^G Z$ is partially differentiable on Z w.r.t. I.

One can prove the following propositions:

(76) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. Then f is partially differentiable on Z w.r.t. $\langle i \rangle$ if and only if f is partially differentiable on Z w.r.t. i.

(77) For every partial function f from $\mathcal{R}^m$ to $\mathbb{R}$ holds $f{\restriction}^{\langle i \rangle} Z = f{\restriction}^i Z$.

(78) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$ and I be a non empty finite sequence of elements of $\mathbb{N}$. Suppose f is partial differentiable up to order $i + j$ and Z and $\operatorname{rng} I \subseteq \operatorname{Seg} m$ and $\operatorname{len} I = j$. Then $f{\restriction}^I Z$ is partial differentiable up to order i and Z.

(79) Let f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose f is partial differentiable up to order i and Z and $j \leq i$. Then f is partial differentiable up to order j and Z.

(80) Let X be a subset of $\mathcal{R}^m$ and f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose that

(i) X is open,

(ii) f is partial differentiable up to order i and X, and

(iii) g is partial differentiable up to order i and X.

Then $f + g$ is partial differentiable up to order i and X and $f - g$ is partial differentiable up to order i and X.

(81) Let X be a subset of $\mathcal{R}^m$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, and r be a real number. Suppose X is open and f is partial differentiable up to

order i and X. Then $r \cdot f$ is partial differentiable up to order i and X.

(82) Let X be a subset of $\mathcal{R}^m$. Suppose X is open. Let i be an element of $\mathbb{N}$ and f, g be partial functions from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose f is partial differentiable up to order i and X and g is partial differentiable up to order i and X. Then $f \cdot g$ is partial differentiable up to order i and X.

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