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Representation Theorem for Stacks

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Summary. In the paper the concept of stacks is formalized. As the main result the Theorem of Representation for Stacks is given. Formalization is done according to [13].

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The papers [6], [15], [14], [2], [4], [7], [16], [8], [9], [10], [5], [1], [17], [11], [19], [21], [20], [3], [18], and [12] provide the terminology and notation for this paper.

1. Introductions

In this paper i is a natural number and x is a set.

Let A be a set and let s_1, s_2 be finite sequences of elements of A. Then $s_1 \frown s_2$ is an element of A^*.

Let A be a set, let i be a natural number, and let s be a finite sequence of elements of A. Then $s_{\restriction i}$ is an element of A^*.

The following two propositions are true:

(1) $\emptyset_{\restriction i} = \emptyset$.

(2) Let D be a non empty set and s be a finite sequence of elements of D. Suppose $s \neq \emptyset$. Then there exists a finite sequence w of elements of D and there exists an element n of D such that $s = \langle n \rangle \frown w$.

The scheme *IndSeqD* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every finite sequence p of elements of $\mathcal{A}$ holds $\mathcal{P}[p]$

provided the following conditions are met:

- $\mathcal{P}[\varepsilon_{\mathcal{A}}]$, and

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- For every finite sequence p of elements of $\mathcal{A}$ and for every element x of $\mathcal{A}$ such that $\mathcal{P}[p]$ holds $\mathcal{P}[\langle x \rangle \frown p]$.

Let C, D be non empty sets and let R be a binary relation. A function from $C \times D$ into D is said to be a binary operation of C and D being congruence w.r.t. R if:

(Def. 1) For every element x of C and for all elements y_1, y_2 of D such that $\langle y_1, y_2 \rangle \in R$ holds $\langle \mathrm{it}(x, y_1), \mathrm{it}(x, y_2) \rangle \in R$.

The scheme *LambdaD2* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and states that:

There exists a function M from $\mathcal{A} \times \mathcal{B}$ into $\mathcal{C}$ such that for every element i of $\mathcal{A}$ and for every element j of $\mathcal{B}$ holds $M(i, j) = \mathcal{F}(i, j)$

for all values of the parameters.

Let C, D be non empty sets, let R be an equivalence relation of D, and let b be a function from $C \times D$ into D. Let us assume that b is a binary operation of C and D being congruence w.r.t. R. The functor $b_{/R}$ yielding a function from $C \times \mathrm{Classes}\, R$ into $\mathrm{Classes}\, R$ is defined as follows:

(Def. 2) For all sets x, y, y_1 such that $x \in C$ and $y \in \mathrm{Classes}\, R$ and $y_1 \in y$ holds $b_{/R}(x, y) = [b(x, y_1)]_R$.

Let A, B be non empty sets, let C be a subset of A, let D be a subset of B, let f be a function from A into B, and let g be a function from C into D. Then $f{+}\!\cdot g$ is a function from A into B.

2. Stack Algebra

We introduce stack systems which are extensions of 2-sorted and are systems

⟨ a carrier, a carrier’, empty stacks, a push function, a pop function, a top function ⟩,

where the carrier is a set, the carrier’ is a set, the empty stacks constitute a subset of the carrier’, the push function is a function from the carrier×the carrier’ into the carrier’, the pop function is a function from the carrier’ into the carrier’, and the top function is a function from the carrier’ into the carrier.

Let a_1 be a non empty set, let a_2 be a set, let a_3 be a subset of a_2, let a_4 be a function from $a_1 \times a_2$ into a_2, let a_5 be a function from a_2 into a_2, and let a_6 be a function from a_2 into a_1. Observe that stack system$\langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle$ is non empty.

Let a_1 be a set, let a_2 be a non empty set, let a_3 be a subset of a_2, let a_4 be a function from $a_1 \times a_2$ into a_2, let a_5 be a function from a_2 into a_2, and let a_6 be a function from a_2 into a_1. One can verify that stack system$\langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle$ is non void.

Let us note that there exists a stack system which is non empty, non void, and strict.

Let X be a stack system. A stack of X is an element of the carrier' of X.

Let X be a non empty non void stack system and let s be a stack of X. The predicate empty(s) is defined by:

(Def. 3) $s \in$ the empty stacks of X.

The functor pop s yields a stack of X and is defined by:

(Def. 4) $\mathrm{pop}\, s =$ (the pop function of X)(s).

The functor top s yields an element of X and is defined by:

(Def. 5) $\mathrm{top}\, s =$ (the top function of X)(s).

Let e be an element of X. The functor push(e, s) yields a stack of X and is defined by:

(Def. 6) $\mathrm{push}(e, s) =$ (the push function of X)(e,s).

Let A be a non empty set. Standard stack system over A yielding a non empty non void strict stack system is defined by the conditions (Def. 7).

(Def. 7)(i) The carrier of standard stack system over $A = A$,

(ii) the carrier' of standard stack system over $A = A^*$, and

(iii) for every stack s of standard stack system over A holds empty(s) iff s is empty and for every finite sequence g such that $g = s$ holds if not empty(s), then $\mathrm{top}\, s = g(1)$ and $\mathrm{pop}\, s = g_{\restriction 1}$ and if empty(s), then $\mathrm{top}\, s =$ the element of standard stack system over A and $\mathrm{pop}\, s = \emptyset$ and for every element e of standard stack system over A holds $\mathrm{push}(e, s) = \langle e \rangle ^\frown g$.

In the sequel A denotes a non empty set, c denotes an element of standard stack system over A, and m denotes a stack of standard stack system over A.

Let us consider A. Note that every stack of standard stack system over A is relation-like and function-like.

Let us consider A. Observe that every stack of standard stack system over A is finite sequence-like.

We adopt the following convention: X denotes a non empty non void stack system, s, s_1 denote stacks of X, and e, e_1, e_2 denote elements of X.

Let us consider X. We say that X is pop-finite if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let f be a function from $\mathbb{N}$ into the carrier' of X. Then there exists a natural number i and there exists s such that $f(i) = s$ and if not empty(s), then $f(i+1) \neq \mathrm{pop}\, s$.

We say that X is push-pop if and only if:

(Def. 9) If not empty(s), then $s = \mathrm{push}(\mathrm{top}\, s, \mathrm{pop}\, s)$.

We say that X is top-push if and only if:

(Def. 10) $e = \mathrm{top}\, \mathrm{push}(e, s)$.

We say that X is pop-push if and only if:

(Def. 11) $s = \mathrm{pop}\, \mathrm{push}(e, s)$.

We say that X is push-non-empty if and only if:

(Def. 12) not empty(push(e, s)).

Let A be a non empty set. One can verify the following observations:

* standard stack system over A is pop-finite,
* standard stack system over A is push-pop,
* standard stack system over A is top-push,
* standard stack system over A is pop-push, and
* standard stack system over A is push-non-empty.

Let us observe that there exists a non empty non void stack system which is pop-finite, push-pop, top-push, pop-push, push-non-empty, and strict.

A stack algebra is a pop-finite push-pop top-push pop-push push-non-empty non empty non void stack system.

Next we state the proposition

(3) For every non empty non void stack system X such that X is pop-finite there exists a stack s of X such that empty(s).

Let X be a pop-finite non empty non void stack system. Note that the empty stacks of X is non empty.

We now state two propositions:

(4) If X is top-push and pop-push and push(e_1, s_1) = push(e_2, s_2), then $e_1 = e_2$ and $s_1 = s_2$.

(5) If X is push-pop and not empty(s_1) and not empty(s_2) and pop s_1 = pop s_2 and top s_1 = top s_2, then $s_1 = s_2$.

3. Schemes of Induction

Now we present three schemes. The scheme *INDsch* deals with a stack algebra $\mathcal{A}$, a stack $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the following conditions are satisfied:

- For every stack s of $\mathcal{A}$ such that empty(s) holds $\mathcal{P}[s]$, and
- For every stack s of $\mathcal{A}$ and for every element e of $\mathcal{A}$ such that $\mathcal{P}[s]$ holds $\mathcal{P}[\text{push}(e, s)]$.

The scheme *EXsch* deals with a stack algebra $\mathcal{A}$, a stack $\mathcal{B}$ of $\mathcal{A}$, a non empty set $\mathcal{C}$, an element $\mathcal{D}$ of $\mathcal{C}$, and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and states that:

There exists an element a of $\mathcal{C}$ and there exists a function F from the carrier' of $\mathcal{A}$ into $\mathcal{C}$ such that

(i) $a = F(\mathcal{B})$,

(ii) for every stack s_1 of $\mathcal{A}$ such that empty(s_1) holds $F(s_1) = \mathcal{D}$, and

(iii) for every stack s_1 of $\mathcal{A}$ and for every element e of $\mathcal{A}$ holds $F(\mathrm{push}(e, s_1)) = \mathcal{F}(e, F(s_1))$

for all values of the parameters.

The scheme *UNIQsch* deals with a stack algebra $\mathcal{A}$, a stack $\mathcal{B}$ of $\mathcal{A}$, a non empty set $\mathcal{C}$, an element $\mathcal{D}$ of $\mathcal{C}$, and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and states that:

Let a_1, a_2 be elements of $\mathcal{C}$. Suppose that

(i) there exists a function F from the carrier' of $\mathcal{A}$ into $\mathcal{C}$ such that $a_1 = F(\mathcal{B})$ and for every stack s_1 of $\mathcal{A}$ such that empty(s_1) holds $F(s_1) = \mathcal{D}$ and for every stack s_1 of $\mathcal{A}$ and for every element e of $\mathcal{A}$ holds $F(\mathrm{push}(e, s_1)) = \mathcal{F}(e, F(s_1))$, and

(ii) there exists a function F from the carrier' of $\mathcal{A}$ into $\mathcal{C}$ such that $a_2 = F(\mathcal{B})$ and for every stack s_1 of $\mathcal{A}$ such that empty(s_1) holds $F(s_1) = \mathcal{D}$ and for every stack s_1 of $\mathcal{A}$ and for every element e of $\mathcal{A}$ holds $F(\mathrm{push}(e, s_1)) = \mathcal{F}(e, F(s_1))$.

Then $a_1 = a_2$

for all values of the parameters.

4. Stack Congruence

We adopt the following rules: X is a stack algebra, s, s_1, s_2, s_3 are stacks of X, and e, e_1, e_2, e_3 are elements of X.

Let us consider X, s. The functor $|s|$ yielding an element of (the carrier of X)* is defined by the condition (Def. 13).

(Def. 13) There exists a function F from the carrier' of X into (the carrier of X)* such that $|s| = F(s)$ and for every s_1 such that empty(s_1) holds $F(s_1) = \emptyset$ and for all s_1, e holds $F(\mathrm{push}(e, s_1)) = \langle e \rangle \frown F(s_1)$.

Next we state several propositions:

(6) If empty(s), then $|s| = \emptyset$.

(7) If not empty(s), then $|s| = \langle \mathrm{top}\, s \rangle \frown |\,\mathrm{pop}\, s|$.

(8) If not empty(s), then $|\,\mathrm{pop}\, s| = |s|_{\restriction 1}$.

(9) $|\,\mathrm{push}(e, s)| = \langle e \rangle \frown |s|$.

(10) If not empty(s), then $\mathrm{top}\, s = |s|(1)$.

(11) If $|s| = \emptyset$, then empty(s).

(12) For every stack s of standard stack system over A holds $|s| = s$.

(13) For every element x of (the carrier of X)* there exists s such that $|s| = x$.

Let us consider X, s_1, s_2. The predicate $s_1 =_G s_2$ is defined as follows:

(Def. 14) $|s_1| = |s_2|$.

Let us notice that the predicate $s_1 =_G s_2$ is reflexive and symmetric.

The following propositions are true:

(14) If $s_1 =_G s_2$ and $s_2 =_G s_3$, then $s_1 =_G s_3$.

(15) If $s_1 =_G s_2$ and $\mathrm{empty}(s_1)$, then $\mathrm{empty}(s_2)$.

(16) If $\mathrm{empty}(s_1)$ and $\mathrm{empty}(s_2)$, then $s_1 =_G s_2$.

(17) If $s_1 =_G s_2$, then $\mathrm{push}(e, s_1) =_G \mathrm{push}(e, s_2)$.

(18) If $s_1 =_G s_2$ and not $\mathrm{empty}(s_1)$, then $\mathrm{pop}\, s_1 =_G \mathrm{pop}\, s_2$.

(19) If $s_1 =_G s_2$ and not $\mathrm{empty}(s_1)$, then $\mathrm{top}\, s_1 = \mathrm{top}\, s_2$.

Let us consider X. We say that X is proper for identity if and only if:

(Def. 15) For all s_1, s_2 such that $s_1 =_G s_2$ holds $s_1 = s_2$.

Let us consider A. Observe that standard stack system over A is proper for identity.

Let us consider X. The functor $==_X$ yields a binary relation on the carrier' of X and is defined as follows:

(Def. 16) $\langle s_1, s_2 \rangle \in ==_X$ iff $s_1 =_G s_2$.

Let us consider X. Observe that $==_X$ is total, symmetric, and transitive.

One can prove the following proposition

(20) If $\mathrm{empty}(s)$, then $[s]_{==_X}$ = the empty stacks of X.

Let us consider X, s. The functor $\mathrm{coset}\, s$ yielding a subset of the carrier' of X is defined by the conditions (Def. 17).

(Def. 17)(i) $s \in \mathrm{coset}\, s$,

(ii) for all e, s_1 such that $s_1 \in \mathrm{coset}\, s$ holds $\mathrm{push}(e, s_1) \in \mathrm{coset}\, s$ and if not $\mathrm{empty}(s_1)$, then $\mathrm{pop}\, s_1 \in \mathrm{coset}\, s$, and

(iii) for every subset A of the carrier' of X such that $s \in A$ and for all e, s_1 such that $s_1 \in A$ holds $\mathrm{push}(e, s_1) \in A$ and if not $\mathrm{empty}(s_1)$, then $\mathrm{pop}\, s_1 \in A$ holds $\mathrm{coset}\, s \subseteq A$.

Next we state three propositions:

(21) If $\mathrm{push}(e, s) \in \mathrm{coset}\, s_1$, then $s \in \mathrm{coset}\, s_1$ and if not $\mathrm{empty}(s)$ and $\mathrm{pop}\, s \in \mathrm{coset}\, s_1$, then $s \in \mathrm{coset}\, s_1$.

(22) $s \in \mathrm{coset}\, \mathrm{push}(e, s)$ and if not $\mathrm{empty}(s)$, then $s \in \mathrm{coset}\, \mathrm{pop}\, s$.

(23) There exists s_1 such that $\mathrm{empty}(s_1)$ and $s_1 \in \mathrm{coset}\, s$.

Let us consider A and let R be a binary relation on A. Note that there exists a reduction sequence w.r.t. R which is A-valued.

Let us consider X. The construction reduction X yielding a binary relation on the carrier' of X is defined as follows:

(Def. 18) $\langle s_1, s_2 \rangle \in$ the construction reduction X iff not $\mathrm{empty}(s_1)$ and $s_2 = \mathrm{pop}\, s_1$ or there exists e such that $s_2 = \mathrm{push}(e, s_1)$.

Next we state the proposition

(24) Let R be a binary relation on A and t be a reduction sequence w.r.t. R. Then $t(1) \in A$ if and only if t is A-valued.

The scheme *PathIND* deals with a non empty set $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, a binary relation $\mathcal{D}$ on $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{C}]$

provided the parameters meet the following conditions:

- $\mathcal{P}[\mathcal{B}]$,
- $\mathcal{D}$ reduces $\mathcal{B}$ to $\mathcal{C}$, and
- For all elements x, y of $\mathcal{A}$ such that $\mathcal{D}$ reduces $\mathcal{B}$ to x and $\langle x, y\rangle \in \mathcal{D}$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$.

One can prove the following propositions:

(25) For every reduction sequence t w.r.t. the construction reduction X such that $s = t(1)$ holds $\operatorname{rng} t \subseteq \operatorname{coset} s$.

(26) $\operatorname{coset} s = \{s_1 :$ the construction reduction Xreduces s to $s_1\}$.

Let us consider X, s. The functor $\operatorname{core} s$ yields a stack of X and is defined by the conditions (Def. 19).

(Def. 19)(i) empty(core s), and

(ii) there exists a the carrier' of X-valued reduction sequence t w.r.t. the construction reduction X such that $t(1) = s$ and $t(\operatorname{len} t) = \operatorname{core} s$ and for every i such that $1 \leq i < \operatorname{len} t$ holds not empty(t_i) and $t_{i+1} = \operatorname{pop}(t_i)$.

The following propositions are true:

(27) If empty(s), then $\operatorname{core} s = s$.

(28) $\operatorname{core} \operatorname{push}(e, s) = \operatorname{core} s$.

(29) If not empty(s), then $\operatorname{core} \operatorname{pop} s = \operatorname{core} s$.

(30) $\operatorname{core} s \in \operatorname{coset} s$.

(31) For every element x of (the carrier of X)* there exists s_1 such that $|s_1| = x$ and $s_1 \in \operatorname{coset} s$.

(32) If $s_1 \in \operatorname{coset} s$, then $\operatorname{core} s_1 = \operatorname{core} s$.

(33) If s_1, $s_2 \in \operatorname{coset} s$ and $|s_1| = |s_2|$, then $s_1 = s_2$.

(34) There exists s such that $\operatorname{coset} s_1 \cap [s_2]_{==_X} = \{s\}$.

5. Quotient Stack System

Let us consider X. The functor $X_{/==}$ yields a strict stack system and is defined by the conditions (Def. 20).

(Def. 20)(i) The carrier of $X_{/==}$ = the carrier of X,

(ii) the carrier' of $X_{/==}$ = Classes $==_X$,

(iii) the empty stacks of $X_{/==}$ = {the empty stacks of X},

(iv) the push function of $X_{/==}$ = (the push function of X)$_{/==_X}$,

(v) the pop function of $X_{/==}$ =
((the pop function of X)$+\cdot\mathrm{id}_{\text{the empty stacks of } X}$)$_{/==_X}$, and

(vi) for every choice function f of Classes $==_X$ holds the top function of $X_{/==}$ = (the top function of X) $\cdot f +\cdot$ (the empty stacks of X, the element of the carrier of X).

Let us consider X. One can verify that $X_{/==}$ is non empty and non void.

The following propositions are true:

(35) For every stack S of $X_{/==}$ there exists s such that $S = [s]_{==_X}$.

(36) $[s]_{==_X}$ is a stack of $X_{/==}$.

(37) For every stack S of $X_{/==}$ such that $S = [s]_{==_X}$ holds empty(s) iff empty(S).

(38) For every stack S of $X_{/==}$ holds empty(S) iff S = the empty stacks of X.

(39) For every stack S of $X_{/==}$ and for every element E of $X_{/==}$ such that $S = [s]_{==_X}$ and $E = e$ holds $\text{push}(e, s) \in \text{push}(E, S)$ and $[\text{push}(e, s)]_{==_X} = \text{push}(E, S)$.

(40) For every stack S of $X_{/==}$ such that $S = [s]_{==_X}$ and not empty(s) holds $\text{pop}\, s \in \text{pop}\, S$ and $[\text{pop}\, s]_{==_X} = \text{pop}\, S$.

(41) For every stack S of $X_{/==}$ such that $S = [s]_{==_X}$ and not empty(s) holds $\text{top}\, S = \text{top}\, s$.

Let us consider X. One can verify the following observations:

* $X_{/==}$ is pop-finite,
* $X_{/==}$ is push-pop,
* $X_{/==}$ is top-push,
* $X_{/==}$ is pop-push, and
* $X_{/==}$ is push-non-empty.

Next we state the proposition

(42) For every stack S of $X_{/==}$ such that $S = [s]_{==_X}$ holds $|S| = |s|$.

Let us consider X. Note that $X_{/==}$ is proper for identity.

Let us note that there exists a stack algebra which is proper for identity.

6. Representation Theorem for Stacks

Let X_1, X_2 be stack algebras and let F, G be functions. We say that F and G form isomorphism between X_1 and X_2 if and only if the conditions (Def. 21) are satisfied.

(Def. 21) $\text{dom}\, F$ = the carrier of X_1 and $\text{rng}\, F$ = the carrier of X_2 and F is one-to-one and $\text{dom}\, G$ = the carrier' of X_1 and $\text{rng}\, G$ = the carrier' of X_2 and G is one-to-one and for every stack s_1 of X_1 and for every stack s_2 of X_2 such that $s_2 = G(s_1)$ holds empty(s_1) iff empty(s_2) and if not empty(s_1), then $\text{pop}\, s_2 = G(\text{pop}\, s_1)$ and $\text{top}\, s_2 = F(\text{top}\, s_1)$ and for every element

e_1 of X_1 and for every element e_2 of X_2 such that $e_2 = F(e_1)$ holds $\mathrm{push}(e_2, s_2) = G(\mathrm{push}(e_1, s_1))$.

We use the following convention: X_1, X_2, X_3 are stack algebras and F, F_1, F_2, G, G_1, G_2 are functions.

The following propositions are true:

(43) $\mathrm{id}_{\text{the carrier of } X}$ and $\mathrm{id}_{\text{the carrier' of } X}$ form isomorphism between X and X.

(44) If F and G form isomorphism between X_1 and X_2, then F^{-1} and G^{-1} form isomorphism between X_2 and X_1.

(45) Suppose F_1 and G_1 form isomorphism between X_1 and X_2 and F_2 and G_2 form isomorphism between X_2 and X_3. Then $F_2 \cdot F_1$ and $G_2 \cdot G_1$ form isomorphism between X_1 and X_3.

(46) Suppose F and G form isomorphism between X_1 and X_2. Let s_1 be a stack of X_1 and s_2 be a stack of X_2. If $s_2 = G(s_1)$, then $|s_2| = F \cdot |s_1|$.

Let X_1, X_2 be stack algebras. We say that X_1 and X_2 are isomorphic if and only if:

(Def. 22) There exist functions F, G such that F and G form isomorphism between X_1 and X_2.

Let us notice that the predicate X_1 and X_2 are isomorphic is reflexive and symmetric.

We now state four propositions:

(47) If X_1 and X_2 are isomorphic and X_2 and X_3 are isomorphic, then X_1 and X_3 are isomorphic.

(48) If X_1 and X_2 are isomorphic and X_1 is proper for identity, then X_2 is proper for identity.

(49) Let X be a proper for identity stack algebra. Then there exists G such that

(i) for every stack s of X holds $G(s) = |s|$, and

(ii) $\mathrm{id}_{\text{the carrier of } X}$ and G form isomorphism between X and standard stack system over the carrier of X.

(50) Let X be a proper for identity stack algebra. Then X and standard stack system over the carrier of X are isomorphic.

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