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Semiring of Sets

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Summary. Schmets [22] has developed a measure theory from a generalized notion of a semiring of sets. Goguadze [15] has introduced another generalized notion of semiring of sets and proved that all known properties that semiring have according to the old definitions are preserved. We show that this two notions are almost equivalent. We note that Patriota [20] has defined this quasi-semiring. We propose the formalization of some properties developed by the authors.

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The notation and terminology used in this paper have been introduced in the following articles: [1], [3], [4], [21], [6], [12], [24], [8], [9], [25], [13], [23], [11], [5], [17], [18], [27], [28], [19], [26], [14], [16], and [10].

1. Preliminaries

From now on X denotes a set and S denotes a family of subsets of X.

Now we state the proposition:

(1) Let us consider sets X, Y. Then $(X \cup Y) \setminus (Y \setminus X) = X$.

Let us consider X and S. Let S_1, S_2 be finite subsets of S. Let us note that $S_1 \Cap S_2$ is finite.

Now we state the proposition:

(2) Let us consider a family S of subsets of X and an element A of S. Then $\{x$, where x is an element of $S : x \in \bigcup(\mathrm{PARTITIONS}(A) \cap \mathrm{Fin}\, S)\} = \bigcup(\mathrm{PARTITIONS}(A) \cap \mathrm{Fin}\, S)$.

Let us consider X and S. Note that $\bigcup(\mathrm{PARTITIONS}(\emptyset) \cap \mathrm{Fin}\, S)$ is empty. Note that 2^X_* has empty element. Now we state the proposition:

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(3) Let us consider a set X. Suppose X is $\cap$-closed and $\cup$-closed. Then X is a ring of sets.

2. The Existence of Partitions

Let X be a set and S be a family of subsets of X. We say that S is $\cap_{fp}$-closed if and only if

(Def. 1) Let us consider elements S_1, S_2 of S. Suppose $S_1 \cap S_2$ is not empty. Then there exists a finite subset x of S such that x is a partition of $S_1 \cap S_2$.

Let us observe that 2^X_* is $\cap_{fp}$-closed.

Observe that there exists a family of subsets of X which is $\cap_{fp}$-closed.

One can verify that every family of subsets of X which is $\cap$-closed is also $\cap_{fp}$-closed.

Now we state the propositions:

(4) Let us consider a non empty set A, a $\cap_{fp}$-closed family S of subsets of X, and partitions P_1, P_2 of A. Suppose

(i) P_1 is a finite subset of S, and

(ii) P_2 is a finite subset of S.

Then there exists a partition P of A such that

(iii) P is a finite subset of S, and

(iv) $P \Subset P_1 \wedge P_2$.

Proof: Define $\mathcal{F}[\text{object}, \text{object}] \equiv \$_1 \in P_1 \wedge P_2$ and $\$_2$ is a finite subset of S and there exists a set A such that $A = \$_1$ and $\$_2$ is a partition of A. Set $F_1 = \{y$, where y is a finite subset of S : there exists a set t such that $t \in P_1 \wedge P_2$ and y is a partition of $t\}$. $F_1 \subseteq 2^{2^x}$ by [10, (67)]. For every object u such that $u \in P_1 \wedge P_2$ there exists an object v such that $v \in F_1$ and $\mathcal{F}[u, v]$. Consider f being a function such that $\operatorname{dom} f = P_1 \wedge P_2$ and $\operatorname{rng} f \subseteq F_1$ and for every object x such that $x \in P_1 \wedge P_2$ holds $\mathcal{F}[x, f(x)]$ from [8, Sch. 6]. $\bigcup f$ is a finite subset of S by [2, (88)]. $\bigcup f$ is a partition of x by [10, (77), (81), (74)]. $\bigcup f \Subset P_1 \wedge P_2$. $\square$

(5) Let us consider a $\cap_{fp}$-closed family S of subsets of X and finite subsets A, B of S. Suppose

(i) A is mutually-disjoint, and

(ii) B is mutually-disjoint.

Then there exists a finite subset P of S such that P is a partition of $\bigcup A \cap \bigcup B$.

(6) Let us consider a $\cap_{fp}$-closed family S of subsets of X and a finite subset W of S. Then there exists a finite subset P of S such that P is a partition of $\bigcap W$.

(7) Let us consider a $\cap_{fp}$-closed family S of subsets of X. Then $\{\bigcup x$, where x is a finite subset of $S : x$ is mutually-disjoint$\}$ is $\cap$-closed. The theorem is a consequence of (5).

Let X be a set and S be a family of subsets of X. We say that S is $\setminus_{fp}$-closed if and only if

(Def. 2) Let us consider elements S_1, S_2 of S. Suppose $S_1 \setminus S_2$ is not empty. Then there exists a finite subset x of S such that x is a partition of $S_1 \setminus S_2$.

Let us note that 2^X_* is $\setminus_{fp}$-closed.

Note that there exists a family of subsets of X which is $\setminus_{fp}$-closed.

Observe that every family of subsets of X which is diff-closed is also $\setminus_{fp}$-closed. Now we state the proposition:

(8) Let us consider a $\setminus_{fp}$-closed family S of subsets of X, an element S_1 of S, and a finite subset T of S. Then there exists a finite subset P of S such that P is a partition of $S_1 \setminus \bigcup T$. Proof: Consider p_0 being a finite sequence such that $T = \operatorname{rng} p_0$. Define $\mathcal{P}[\text{finite sequence}] \equiv$ there exists a finite subset p_1 of S such that p_1 is a partition of $S_1 \setminus \bigcup \operatorname{rng} \$_1$. For every finite sequence p of elements of S and for every element x of S such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \frown \langle x \rangle]$ by [6, (31)], [10, (78)], [6, (38)], [12, (8), (7)]. $\mathcal{P}[\varepsilon_S]$ by [26, (1)], [21, (45)], [26, (41)], [21, (39)]. For every finite sequence p of elements of S, $\mathcal{P}[p]$ from [7, Sch. 2]. □

3. Partitions in a Difference of Sets

Let X be a set and S be a family of subsets of X. We say that S is $\setminus^{\subseteq}_{fp}$-closed if and only if

(Def. 3) Let us consider elements S_1, S_2 of S. Suppose $S_2 \subseteq S_1$. Then there exists a finite subset x of S such that x is a partition of $S_1 \setminus S_2$.

Now we state the proposition:

(9) Let us consider a family S of subsets of X. Suppose S is $\setminus_{fp}$-closed. Then S is $\setminus^{\subseteq}_{fp}$-closed.

Let us consider X. Note that every family of subsets of X which is $\setminus_{fp}$-closed is also $\setminus^{\subseteq}_{fp}$-closed.

Observe that 2^X_* is $\setminus^{\subseteq}_{fp}$-closed. Observe that there exists a family of subsets of X which is $\setminus^{\subseteq}_{fp}$-closed, $\setminus_{fp}$-closed, and $\cap_{fp}$-closed and has empty element.

Now we state the propositions:

(10) Let us consider a $\setminus_{fp}$-closed family S of subsets of X. Then $\{\bigcup x$, where x is a finite subset of $S : x$ is mutually-disjoint$\}$ is diff-closed. Proof: Set $Y = \{\bigcup x$, where x is a finite subset of $S : x$ is mutually-disjoint$\}$. For every sets A, B such that $A, B \in Y$ holds $A \setminus B \in Y$ by [6, (52)], (8), (5), [12, (8), (7)]. □

(11) Let us consider a $\cap_{fp}$-closed $\setminus_{fp}^{\subseteq}$-closed family S of subsets of X, an element A of S, and a finite subset Q of S. Suppose

(i) $\bigcup Q \subseteq A$, and

(ii) Q is a partition of $\bigcup Q$.

Then there exists a finite subset R of S such that

(iii) $\bigcup R$ misses $\bigcup Q$, and

(iv) $Q \cup R$ is a partition of A.

(12) Every $\setminus_{fp}^{\subseteq}$-closed $\cap_{fp}$-closed family of subsets of X is $\setminus_{fp}$-closed. Proof: For every elements S_1, S_2 of S such that $S_1 \setminus S_2$ is not empty there exists a finite subset P_0 of S such that P_0 is a partition of $S_1 \setminus S_2$ by (11), [10, (77), (81)]. □

Let X be a set. Let us observe that every $\cap_{fp}$-closed family of subsets of X which is $\setminus_{fp}^{\subseteq}$-closed is also $\setminus_{fp}$-closed. Now we state the propositions:

(13) Let us consider a $\cap_{fp}$-closed $\setminus_{fp}^{\subseteq}$-closed family S of subsets of X and finite subsets W, T of S. Then there exists a finite subset P of S such that P is a partition of $\bigcap W \setminus \bigcup T$.

(14) Let us consider a $\cap_{fp}$-closed $\setminus_{fp}^{\subseteq}$-closed family S of subsets of X and a finite subset W of S. Then there exists a finite subset P of S such that

(i) P is a partition of $\bigcup W$, and

(ii) for every element Y of W, $Y = \bigcup\{s$, where s is an element of $S : s \in P$ and $s \subseteq Y\}$.

(15) Let us consider a $\cap_{fp}$-closed $\setminus_{fp}^{\subseteq}$-closed family S of subsets of X and a function W from $\mathbb{N}^+$ into S. Then there exists a countable subset P of S such that

(i) P is a partition of $\bigcup W$, and

(ii) for every positive natural number n, $\bigcup(W \restriction \operatorname{Seg} n) = \bigcup\{s$, where s is an element of $S : s \in P$ and $s \subseteq \bigcup(W \restriction \operatorname{Seg} n)\}$.

The theorem is a consequence of (8).

4. Countable Covers

Let X be a set and S be a family of subsets of X. We say that S has countable cover if and only if

(Def. 4) There exists a countable subset X_1 of S such that X_1 is a cover of X.

Let us consider X. One can check that 2_*^X has countable cover.

One can check that there exists a family of subsets of X which is $\setminus_{fp}^{\subseteq}$-closed, $\setminus_{fp}$-closed, and $\cap_{fp}$-closed and has empty element and countable cover.

Now we state the proposition:

(16) Let us consider a $\cap_{fp}$-closed $\setminus_{fp}^{\subseteq}$-closed family S of subsets of X. Suppose S has countable cover. Then there exists a countable subset P of S such that P is a partition of X. The theorem is a consequence of (15).

5. Semiring of Sets

Let X be a set. A semiring of sets of X is a $\cap_{fp}$-closed $\setminus_{fp}^{\subseteq}$-closed family of subsets of X with empty element.

Let us consider a $\cap_{fp}$-closed family S of subsets of X and an element A of S. Now we state the propositions:

(17) $\{x$, where x is an element of $S : x \in \bigcup(\text{PARTITIONS}(A) \cap \text{Fin}\, S)\}$ is a $\cap_{fp}$-closed family of subsets of A. The theorem is a consequence of (4).

(18) $\{x$, where x is an element of $S : x \in \bigcup(\text{PARTITIONS}(A) \cap \text{Fin}\, S)\}$ is a $\setminus_{fp}^{\subseteq}$-closed family of subsets of A. The theorem is a consequence of (4).

(19) $\bigcup(\text{PARTITIONS}(A) \cap \text{Fin}\, S)$ is $\cap_{fp}$-closed $\setminus_{fp}$-closed family of subsets of A and has non empty elements. The theorem is a consequence of (2), (17), and (18).

(20) $\{\emptyset\} \cup \bigcup(\text{PARTITIONS}(A) \cap \text{Fin}\, S)$ is a semiring of sets of A. PROOF: Set $A_1 = \bigcup(\text{PARTITIONS}(A) \cap \text{Fin}\, S)$. Set $B = \bigcup(\text{PARTITIONS}(A) \cap \text{Fin}\, S) \cup \{\emptyset\}$. A_1 is a $\cap_{fp}$-closed $\setminus_{fp}$-closed family of subsets of A. $B \subseteq 2^A$. B is $\cap_{fp}$-closed. B is $\setminus_{fp}$-closed by (19), [21, (39)]. □

6. A Ring of Sets

Let us consider a $\cap_{fp}$-closed $\setminus_{fp}$-closed family S of subsets of X. Now we state the propositions:

(21) $\{\bigcup x$, where x is a finite subset of $S : x$ is mutually-disjoint$\}$ is $\cup$-closed. The theorem is a consequence of (14).

(22) $\{\bigcup x$, where x is a finite subset of $S : x$ is mutually-disjoint$\}$ is a ring of sets. The theorem is a consequence of (7), (21), and (3).

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