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The Cauchy-Riemann Differential Equations of Complex Functions

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Summary. In this article we prove Cauchy-Riemann differential equations of complex functions. These theorems give necessary and sufficient condition for differentiable function.

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The articles [20], [21], [6], [7], [22], [8], [3], [1], [4], [14], [13], [19], [16], [9], [2], [5], [10], [17], [11], [18], [12], and [15] provide the notation and terminology for this paper.

Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$. The functor $\Re(f)$ yielding a partial function from $\mathbb{C}$ to $\mathbb{R}$ is defined as follows:

(Def. 1) $\operatorname{dom} f = \operatorname{dom} \Re(f)$ and for every complex number z such that $z \in \operatorname{dom} \Re(f)$ holds $\Re(f)(z) = \Re(f_z)$.

Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$. The functor $\Im(f)$ yields a partial function from $\mathbb{C}$ to $\mathbb{R}$ and is defined as follows:

(Def. 2) $\operatorname{dom} f = \operatorname{dom} \Im(f)$ and for every complex number z such that $z \in \operatorname{dom} \Im(f)$ holds $\Im(f)(z) = \Im(f_z)$.

One can prove the following propositions:

(1) For every partial function f from $\mathbb{C}$ to $\mathbb{C}$ such that f is total holds $\operatorname{dom} \Re(f) = \mathbb{C}$ and $\operatorname{dom} \Im(f) = \mathbb{C}$.

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(2) Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$, u, v be partial functions from $\mathcal{R}^2$ to $\mathbb{R}$, z_0 be a complex number, x_0, y_0 be real numbers, and x_1 be an element of $\mathcal{R}^2$. Suppose that

(i) for all real numbers x, y such that $x+y\cdot i \in \operatorname{dom} f$ holds $\langle x, y\rangle \in \operatorname{dom} u$ and $u(\langle x, y\rangle) = \Re(f)(x+y\cdot i)$,

(ii) for all real numbers x, y such that $x+y\cdot i \in \operatorname{dom} f$ holds $\langle x, y\rangle \in \operatorname{dom} v$ and $v(\langle x, y\rangle) = \Im(f)(x+y\cdot i)$,

(iii) $z_0 = x_0 + y_0 \cdot i$,

(iv) $x_1 = \langle x_0, y_0\rangle$, and

(v) f is differentiable in z_0.

Then

(vi) u is partially differentiable in x_1 w.r.t. coordinate 1 and partially differentiable in x_1 w.r.t. coordinate 2,

(vii) v is partially differentiable in x_1 w.r.t. coordinate 1 and partially differentiable in x_1 w.r.t. coordinate 2,

(viii) $\Re(f'(z_0)) = \operatorname{partdiff}(u, x_1, 1)$,

(ix) $\Re(f'(z_0)) = \operatorname{partdiff}(v, x_1, 2)$,

(x) $\Im(f'(z_0)) = -\operatorname{partdiff}(u, x_1, 2)$, and

(xi) $\Im(f'(z_0)) = \operatorname{partdiff}(v, x_1, 1)$.

(3) For every sequence s of real numbers holds s is convergent and $\lim s = 0$ iff $|s|$ is convergent and $\lim|s| = 0$.

(4) Let X be a real normed space and s be a sequence of X. Then s is convergent and $\lim s = 0_X$ if and only if $\|s\|$ is convergent and $\lim\|s\| = 0$.

(5) Let u be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$, x_0, y_0 be real numbers, and x_1 be an element of $\mathcal{R}^2$. Suppose $x_1 = \langle x_0, y_0\rangle$ and $\langle u\rangle$ is differentiable in x_1. Then

(i) u is partially differentiable in x_1 w.r.t. coordinate 1 and partially differentiable in x_1 w.r.t. coordinate 2,

(ii) $\langle\operatorname{partdiff}(u, x_1, 1)\rangle = \langle u\rangle'(x_1)(\langle 1, 0\rangle)$, and

(iii) $\langle\operatorname{partdiff}(u, x_1, 2)\rangle = \langle u\rangle'(x_1)(\langle 0, 1\rangle)$.

(6) Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$, u, v be partial functions from $\mathcal{R}^2$ to $\mathbb{R}$, z_0 be a complex number, x_0, y_0 be real numbers, and x_1 be an element of $\mathcal{R}^2$. Suppose that for all real numbers x, y such that $\langle x, y\rangle \in \operatorname{dom} v$ holds $x+y\cdot i \in \operatorname{dom} f$ and for all real numbers x, y such that $x+y\cdot i \in \operatorname{dom} f$ holds $\langle x, y\rangle \in \operatorname{dom} u$ and $u(\langle x, y\rangle) = \Re(f)(x+y\cdot i)$ and for all real numbers x, y such that $x+y\cdot i \in \operatorname{dom} f$ holds $\langle x, y\rangle \in \operatorname{dom} v$ and $v(\langle x, y\rangle) = \Im(f)(x+y\cdot i)$ and $z_0 = x_0 + y_0 \cdot i$ and $x_1 = \langle x_0, y_0\rangle$ and $\langle u\rangle$ is differentiable in x_1 and $\langle v\rangle$ is differentiable in x_1 and $\operatorname{partdiff}(u, x_1, 1) = \operatorname{partdiff}(v, x_1, 2)$ and $\operatorname{partdiff}(u, x_1, 2) = -\operatorname{partdiff}(v, x_1, 1)$. Then f is differentiable in z_0 and u is partially differentiable in x_1 w.r.t. coordinate 1 and partially differentiable in x_1 w.r.t. coordinate 2 and v is partially differentiable in

x_1 w.r.t. coordinate 1 and partially differentiable in x_1 w.r.t. coordinate 2 and $\Re(f'(z_0)) = \mathrm{partdiff}(u, x_1, 1)$ and $\Re(f'(z_0)) = \mathrm{partdiff}(v, x_1, 2)$ and $\Im(f'(z_0)) = -\mathrm{partdiff}(u, x_1, 2)$ and $\Im(f'(z_0)) = \mathrm{partdiff}(v, x_1, 1)$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[4] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[9] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[10] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[11] Noboru Endou and Yasunari Shidama. Completeness of the real Euclidean space. *Formalized Mathematics*, 13(**4**):577–580, 2005.
[12] Noboru Endou, Yasunari Shidama, and Keiichi Miyajima. Partial differentiation on normed linear spaces $\mathcal{R}^n$. *Formalized Mathematics*, 15(**2**):65–72, 2007, doi:10.2478/v10037-007-0008-5.
[13] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[14] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[15] Chanapat Pacharapokin, Hiroshi Yamazaki, Yasunari Shidama, and Yatsuka Nakamura. Complex function differentiability. *Formalized Mathematics*, 17(**2**):67–72, 2009, doi: 10.2478/v10037-009-0007-9.
[16] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[17] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[18] Yasunari Shidama. Banach space of bounded linear operators. *Formalized Mathematics*, 12(**1**):39–48, 2004.
[19] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[20] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[21] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[22] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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