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Convergent Filter Bases

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Summary. We are inspired by the work of Henri Cartan [16], Bourbaki [10] (TG. I Filtres) and Claude Wagschal [34]. We define the base of filter, image filter, convergent filter bases, limit filter and the filter base of tails (fr: *filtre des sections*).

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The notation and terminology used in this paper have been introduced in the following articles: [24], [1], [2], [33], [20], [18], [28], [11], [12], [13], [29], [3], [37], [25], [26], [4], [17], [30], [5], [14], [23], [35], [36], [22], [31], [6], [7], [9], [19], [27], and [15].

1. Filters – Set-Theoretical Approach

From now on X denotes a non empty set, $\mathcal{F}$ denotes a filter of X, and S denotes a family of subsets of X.

Let X be a set and S be a family of subsets of X. We say that S is upper if and only if

(Def. 1) for every subsets Y_1, Y_2 of X such that $Y_1 \in S$ and $Y_1 \subseteq Y_2$ holds $Y_2 \in S$.

Let us note that there exists a $\cap$-closed family of subsets of X which is non empty and there exists a non empty, $\cap$-closed family of subsets of X which is upper.

Let X be a non empty set. Let us note that there exists a non empty, upper, $\cap$-closed family of subsets of X which has non empty elements.

Now we state the propositions:

(1) S is a non empty, upper, $\cap$-closed family of subsets of X with non empty elements if and only if S is a filter of X.

(2) Let us consider non empty sets X_1, X_2, a filter $\mathcal{F}_1$ of X_1, and a filter $\mathcal{F}_2$ of X_2. Then the set of all $f_1 \times f_2$ where f_1 is an element of $\mathcal{F}_1$, f_2 is an element of $\mathcal{F}_2$ is a non empty family of subsets of $X_1 \times X_2$.

Let X be a non empty set. We say that X is $\cap$-finite closed if and only if

(Def. 2) for every finite, non empty subset S_1 of X, $\bigcap S_1 \in X$.

One can check that there exists a non empty set which is $\cap$-finite closed.

Now we state the proposition:

(3) Let us consider a non empty set X. If X is $\cap$-finite closed, then X is $\cap$-closed.

Note that every non empty set which is $\cap$-finite closed is also $\cap$-closed.

(4) Let us consider a set X, and a family S of subsets of X. Then S is $\cap$-closed and $X \in S$ if and only if $\mathrm{FinMeetCl}(S) \subseteq S$.

(5) Let us consider a non empty set X, and a non empty subset A of X. Then $\{B$, where B is a subset of $X : A \subseteq B\}$ is a filter of X.

Let X be a non empty set. Note that every filter of X is $\cap$-closed.

(6) Let us consider a set X, and a family B of subsets of X. If $B = \{X\}$, then B is upper.

(7) Let us consider a non empty set X, and a filter $\mathcal{F}'$ of X. Then $\mathcal{F}' \neq 2^X$.

Let X be a non empty set. The functor $\mathrm{Filt}(X)$ yielding a non empty set is defined by the term

(Def. 3) the set of all $\mathcal{F}'$ where $\mathcal{F}'$ is a filter of X.

Let I be a non empty set and M be a $(\mathrm{Filt}(X))$-valued many sorted set indexed by I. The intersection of the family of filters M yielding a filter of X is defined by the term

(Def. 4) $\bigcap \mathrm{rng}\, M$.

Let $\mathcal{F}_1$, $\mathcal{F}_2$ be filters of X. We say that $\mathcal{F}_1$ is coarser than $\mathcal{F}_2$ if and only if

(Def. 5) $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

One can verify that the predicate is reflexive. We say that $\mathcal{F}_1$ is finer than $\mathcal{F}_2$ if and only if

(Def. 6) $\mathcal{F}_2 \subseteq \mathcal{F}_1$.

Observe that the predicate is reflexive.

Now we state the propositions:

(8) Let us consider a non empty set X, a filter $\mathcal{F}'$ of X, and a filter $\mathcal{F}$ of X. Suppose $\mathcal{F} = \{X\}$. Then $\mathcal{F}$ is coarser than $\mathcal{F}'$.

(9) Let us consider a non empty set X, a non empty set I, a $(\mathrm{Filt}(X))$-valued many sorted set M indexed by I, an element i of I, and a filter $\mathcal{F}'$ of X. Suppose $\mathcal{F}' = M(i)$. Then the intersection of the family of filters M is coarser than $\mathcal{F}'$.

(10) Let us consider a set X, and a family S of subsets of X. Suppose $\mathrm{FinMeetCl}(S)$ has non empty elements. Then S has non empty elements.

(11) Let us consider a non empty set X, a family G of subsets of X, and a filter $\mathcal{F}'$ of X. Suppose $G \subseteq \mathcal{F}'$. Then

(i) $\mathrm{FinMeetCl}(G) \subseteq \mathcal{F}'$, and

(ii) $\mathrm{FinMeetCl}(G)$ has non empty elements.

The theorem is a consequence of (4).

Let X be a non empty set, $\mathcal{F}'$ be a filter of X, and B be a non empty subset of $\mathcal{F}'$. We say that B is filter basis if and only if

(Def. 7) for every element f of $\mathcal{F}'$, there exists an element b of B such that $b \subseteq f$.

Now we state the proposition:

(12) Let us consider a non empty set X, a filter $\mathcal{F}'$ of X, and a non empty subset B of $\mathcal{F}'$. Then $\mathcal{F}'$ is coarser than B if and only if B is filter basis.

Let X be a non empty set and $\mathcal{F}'$ be a filter of X. Observe that there exists a non empty subset of $\mathcal{F}'$ which is filter basis.

A generalized basis of $\mathcal{F}'$ is a filter basis, non empty subset of $\mathcal{F}'$. Now we state the proposition:

(13) Let us consider a non empty set X. Then every filter of X is a generalized basis of $\mathcal{F}'$.

Let X be a set and B be a family of subsets of X. The functor $[B]$ yielding a family of subsets of X is defined by

(Def. 8) for every subset x of X, $x \in it$ iff there exists an element b of B such that $b \subseteq x$.

Now we state the propositions:

(14) Let us consider a set X, and a family S of subsets of X. Then $[S] = \{x$, where x is a subset of X : there exists an element b of S such that $b \subseteq x\}$.

(15) Let us consider a set X, and an empty family B of subsets of X. Then $[B] = 2^X$.

(16) Let us consider a set X, and a family B of subsets of X. If $\emptyset \in B$, then $[B] = 2^X$.

2. Filters – Lattice-Theoretical Approach

Now we state the propositions:

(17) Let us consider a set X, a non empty family B of subsets of X, and a subset L of $2^X_\subseteq$. If $B = L$, then $[B] = \uparrow L$.

(18) Let us consider a set X, and a family B of subsets of X. Then $B \subseteq [B]$.

Let X be a set and B_1, B_2 be families of subsets of X. We say that B_1 and B_2 are equivalent generators if and only if

(Def. 9) for every element b_1 of B_1, there exists an element b_2 of B_2 such that $b_2 \subseteq b_1$ and for every element b_2 of B_2, there exists an element b_1 of B_1 such that $b_1 \subseteq b_2$.

Let us note that the predicate is reflexive and symmetric.

Let us consider a set X and families B_1, B_2 of subsets of X.

Let us assume that B_1 and B_2 are equivalent generators. Now we state the propositions:

(19) $[B_1] \subseteq [B_2]$.

(20) $[B_1] = [B_2]$.

Let X be a non empty set, $\mathcal{F}'$ be a filter of X, and B be a non empty subset of $\mathcal{F}'$. The functor $\# B$ yielding a non empty family of subsets of X is defined by the term

(Def. 10) B.

Now we state the propositions:

(21) Let us consider a non empty set X, a filter $\mathcal{F}'$ of X, and a generalized basis B of $\mathcal{F}'$. Then $\mathcal{F}' = [\# B]$.

(22) Let us consider a non empty set X, a filter $\mathcal{F}'$ of X, and a family B of subsets of X. If $\mathcal{F}' = [B]$, then B is a generalized basis of $\mathcal{F}'$.

(23) Let us consider a non empty set X, a filter $\mathcal{F}'$ of X, a generalized basis B of $\mathcal{F}'$, a family S of subsets of X, and a subset S_1 of $\mathcal{F}'$. Suppose $S = S_1$ and $\# B$ and S are equivalent generators. Then S_1 is a generalized basis of $\mathcal{F}'$. The theorem is a consequence of (19), (21), and (22).

(24) Let us consider a non empty set X, a filter $\mathcal{F}'$ of X, and generalized bases $\mathcal{B}_1$, $\mathcal{B}_2$ of $\mathcal{F}'$. Then $\# \mathcal{B}_1$ and $\# \mathcal{B}_2$ are equivalent generators. The theorem is a consequence of (21).

Let X be a set and B be a family of subsets of X. We say that B is quasi basis if and only if

(Def. 11) for every elements b_1, b_2 of B, there exists an element b of B such that $b \subseteq b_1 \cap b_2$.

Let X be a non empty set. Let us note that there exists a non empty family of subsets of X which is quasi basis and there exists a non empty, quasi basis family of subsets of X which has non empty elements.

A filter base of X is a non empty, quasi basis family of subsets of X with non empty elements. Now we state the proposition:

(25) Let us consider a non empty set X, and a filter base B of X. Then $[B]$ is a filter of X.

Let X be a non empty set and B be a filter base of X. The functor $[B)$ yielding a filter of X is defined by the term

(Def. 12) $[B]$.

Now we state the propositions:

(26) Let us consider a non empty set X, and filter bases $\mathcal{B}_1$, $\mathcal{B}_2$ of X. Suppose $[\mathcal{B}_1) = [\mathcal{B}_2)$. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent generators.

(27) Let us consider a non empty set X, a filter base $\mathcal{F}$ of X, and a filter $\mathcal{F}'$ of X. Suppose $\mathcal{F} \subseteq \mathcal{F}'$. Then $[\mathcal{F})$ is coarser than $\mathcal{F}'$.

(28) Let us consider a non empty set X, and a family G of subsets of X. Suppose $\mathrm{FinMeetCl}(G)$ has non empty elements. Then

(i) $\mathrm{FinMeetCl}(G)$ is a filter base of X, and

(ii) there exists a filter $\mathcal{F}'$ of X such that $\mathrm{FinMeetCl}(G) \subseteq \mathcal{F}'$.

The theorem is a consequence of (4).

(29) Let us consider a non empty set X, and a filter $\mathcal{F}'$ of X. Then every generalized basis of $\mathcal{F}'$ is a filter base of X.

(30) Let us consider a non empty set X. Then every filter base of X is a generalized basis of $[B)$.

(31) Let us consider a non empty set X, a filter $\mathcal{F}'$ of X, a generalized basis B of $\mathcal{F}'$, and a subset L of $2^X_\subseteq$. If $L = \# B$, then $\mathcal{F}' = \uparrow L$. The theorem is a consequence of (21) and (17).

(32) Let us consider a non empty set X, a filter base B of X, and a subset L of $2^X_\subseteq$. If $L = B$, then $[B) = \uparrow L$.

(33) Let us consider a non empty set X, filters $\mathcal{F}_1$, $\mathcal{F}_2$ of X, a generalized basis $\mathcal{B}_1$ of $\mathcal{F}_1$, and a generalized basis $\mathcal{B}_2$ of $\mathcal{F}_2$. Then $\mathcal{F}_1$ is coarser than $\mathcal{F}_2$ if and only if $\mathcal{B}_1$ is coarser than $\mathcal{B}_2$. The theorem is a consequence of (21).

(34) Let us consider non empty sets X, Y, a function f from X into Y, a filter $\mathcal{F}'$ of X, and a generalized basis B of $\mathcal{F}'$. Then

(i) $f^\circ(\# B)$ is a filter base of Y, and

(ii) $[f^\circ(\# B)]$ is a filter of Y, and

(iii) $[f^\circ(\# B)] = \{M$, where M is a subset of $Y : f^{-1}(M) \in \mathcal{F}'\}$.

Proof: Set $\mathcal{F} = f^\circ(\# B)$. $\mathcal{F}$ is a quasi basis, non empty family of subsets of Y by (29), [35, (123), (121)]. $\mathcal{F}$ has non empty elements by [35, (118)]. $[\mathcal{F}] = \{M$, where M is a subset of $Y : f^{-1}(M) \in \mathcal{F}'\}$ by [35, (143)], [12, (42)], (21), [35, (123)]. □

Let X, Y be non empty sets, f be a function from X into Y, and $\mathcal{F}'$ be a filter of X. The image of filter $\mathcal{F}'$ under f yielding a filter of Y is defined by the term

(Def. 13) $\{M$, where M is a subset of $Y : f^{-1}(M) \in \mathcal{F}'\}$.

Now we state the propositions:

(35) Let us consider non empty sets X, Y, a function f from X into Y, and a filter $\mathcal{F}'$ of X. Then

(i) $f^\circ\mathcal{F}'$ is a filter base of Y, and

(ii) $[f^\circ\mathcal{F}']$ = the image of filter $\mathcal{F}'$ under f.

The theorem is a consequence of (13) and (34).

(36) Let us consider a non empty set X, and a filter base B of X. If $B = [B)$, then B is a filter of X.

(37) Let us consider non empty sets X, Y, a function f from X into Y, a filter $\mathcal{F}'$ of X, and a generalized basis B of $\mathcal{F}'$. Then

(i) $f^\circ(\# B)$ is a generalized basis of the image of filter $\mathcal{F}'$ under f, and

(ii) $[f^\circ(\# B)]$ = the image of filter $\mathcal{F}'$ under f.

The theorem is a consequence of (34) and (30).

(38) Let us consider non empty sets X, Y, a function f from X into Y, and filter bases $\mathcal{B}_1$, $\mathcal{B}_2$ of X. Suppose $\mathcal{B}_1$ is coarser than $\mathcal{B}_2$. Then $[\mathcal{B}_1)$ is coarser than $[\mathcal{B}_2)$. The theorem is a consequence of (30) and (33).

(39) Let us consider non empty sets X, Y, a function f from X into Y, and a filter $\mathcal{F}'$ of X. Then $f^\circ\mathcal{F}'$ is a filter of Y if and only if $Y = \operatorname{rng} f$.
Proof: Reconsider $f_3 = f^\circ\mathcal{F}'$ as a filter base of Y. $[f_3) \subseteq f_3$ by [35, (143)], [11, (76), (77)]. □

(40) Let us consider a non empty set X, a non empty subset A of X, a filter $\mathcal{F}'$ of A, and a generalized basis B of $\mathcal{F}'$. Then

(i) $(\overset{A}{\hookrightarrow})^\circ(\# B)$ is a filter base of X, and

(ii) $[(\overset{A}{\hookrightarrow})^\circ(\# B)]$ is a filter of X, and

(iii) $[(\overset{A}{\hookrightarrow})^\circ(\# B)] = \{M$, where M is a subset of $X : (\overset{A}{\hookrightarrow})^{-1}(M) \in \mathcal{F}'\}$.

Let L be a non empty relational structure. The functor $\operatorname{Tails}(L)$ yielding a non empty family of subsets of L is defined by the term

(Def. 14) the set of all $\uparrow i$ where i is an element of L.

Now we state the proposition:

(41) Let us consider a non empty, transitive, reflexive relational structure L. Suppose Ω_L is directed. Then $[\mathrm{Tails}(L)]$ is a filter of Ω_L.
PROOF: $\mathrm{Tails}(L)$ is non empty family of subsets of L and quasi basis and has non empty elements by [6, (22)]. $\square$

Let L be a non empty, transitive, reflexive relational structure. Assume Ω_L is directed. The functor $\mathrm{TailsFilter}\,L$ yielding a filter of Ω_L is defined by the term

(Def. 15) $[\mathrm{Tails}(L)]$.

Now we state the proposition:

(42) Let us consider a non empty, transitive, reflexive relational structure L. Suppose Ω_L is directed. Then $\mathrm{Tails}(L)$ is a generalized basis of $\mathrm{TailsFilter}\,L$. The theorem is a consequence of (22).

Let L be a relational structure and x be a family of subsets of L. The functor $\#\,x$ yielding a family of subsets of Ω_L is defined by the term

(Def. 16) x.

Now we state the proposition:

(43) Let us consider a non empty set X, a non empty, transitive, reflexive relational structure L, and a function f from Ω_L into X. Suppose Ω_L is directed. Then $f^{\circ}(\#\,\mathrm{Tails}(L))$ is a generalized basis of the image of filter $\mathrm{TailsFilter}\,L$ under f. The theorem is a consequence of (42) and (37).

Let us consider a non empty set X, a non empty, transitive, reflexive relational structure L, a function f from Ω_L into X, and a subset x of X. Now we state the propositions:

(44) Suppose Ω_L is directed and $x \in f^{\circ}(\#\,\mathrm{Tails}(L))$. Then there exists an element j of L such that for every element i of L such that $i \geqslant j$ holds $f(i) \in x$.

(45) Suppose Ω_L is directed and there exists an element j of L such that for every element i of L such that $i \geqslant j$ holds $f(i) \in x$. Then there exists an element b of $\mathrm{Tails}(L)$ such that $f^{\circ}b \subseteq x$.

(46) Let us consider a non empty set X, a non empty, transitive, reflexive relational structure L, a function f from Ω_L into X, a filter $\mathcal{F}'$ of X, and a generalized basis B of $\mathcal{F}'$. Suppose Ω_L is directed. Then $\mathcal{F}'$ is coarser than the image of filter $\mathrm{TailsFilter}\,L$ under f if and only if B is coarser than $f^{\circ}(\#\,\mathrm{Tails}(L))$. The theorem is a consequence of (43) and (33).

(47) Let us consider a non empty set X, a non empty, transitive, reflexive relational structure L, a function f from Ω_L into X, and a filter base B of

X. Suppose Ω_L is directed. Then B is coarser than $f^\circ(\#\,\mathrm{Tails}(L))$ if and only if for every element b of B, there exists an element i of L such that for every element j of L such that $i \leqslant j$ holds $f(j) \in b$. The theorem is a consequence of (44) and (45).

Let X be a non empty set and s be a sequence of X. The elementary filter of s yielding a filter of X is defined by the term

(Def. 17) the image of filter FrechetFilter($\mathbb{N}$) under s.

Now we state the propositions:

(48) There exists a sequence $\mathcal{F}'$ of $2^{\mathbb{N}}$ such that for every element x of $\mathbb{N}$, $\mathcal{F}'(x) = \{y$, where y is an element of $\mathbb{N} : x \leqslant y\}$.

PROOF: Define $\mathcal{F}(\text{object}) = \{y$, where y is an element of $\mathbb{N}$: there exists an element x_0 of $\mathbb{N}$ such that $x_0 = \$_1$ and $x_0 \leqslant y\}$. There exists a function f from $\mathbb{N}$ into $2^{\mathbb{N}}$ such that for every object x such that $x \in \mathbb{N}$ holds $f(x) = \mathcal{F}(x)$ from [12, Sch. 2]. Consider $\mathcal{F}'$ being a function from $\mathbb{N}$ into $2^{\mathbb{N}}$ such that for every object x such that $x \in \mathbb{N}$ holds $\mathcal{F}'(x) = \mathcal{F}(x)$. For every element x of $\mathbb{N}$, $\mathcal{F}'(x) = \{y$, where y is an element of $\mathbb{N} : x \leqslant y\}$. □

(49) Let us consider a natural number n. Then $\mathbb{N} \setminus \{t$, where t is an element of $\mathbb{N} : n \leqslant t\}$ is finite.

PROOF: $\mathbb{N} \setminus \{t$, where t is an element of $\mathbb{N} : n \leqslant t\} \subseteq n+1$ by [8, (3), (5)], [32, (4)]. □

(50) Let us consider an element p of the ordered $\mathbb{N}$. Then $\{x$, where x is an element of $\mathbb{N}$: there exists an element p_0 of $\mathbb{N}$ such that $p = p_0$ and $p_0 \leqslant x\} = \uparrow p$.

PROOF: For every element p of the carrier of the ordered $\mathbb{N}$, $\{x$, where x is an element of the carrier of the ordered $\mathbb{N} : p \leqslant x\} = \uparrow p$ by [6, (18)]. □

Observe that $\Omega_{\text{the ordered } \mathbb{N}}$ is directed and the ordered $\mathbb{N}$ is reflexive.

Now we state the proposition:

(51) Let us consider a denumerable set X. Then FrechetFilter(X) = the set of all $X \setminus A$ where A is a finite subset of X.

Let us consider a sequence $\mathcal{F}'$ of $2^{\mathbb{N}}$.

Let us assume that for every element x of $\mathbb{N}$, $\mathcal{F}'(x) = \{y$, where y is an element of $\mathbb{N} : x \leqslant y\}$. Now we state the propositions:

(52) $\operatorname{rng} \mathcal{F}'$ is a generalized basis of FrechetFilter($\mathbb{N}$).

PROOF: FrechetFilter($\mathbb{N}$) = the set of all $\mathbb{N} \setminus A$ where A is a finite subset of $\mathbb{N}$. For every object t such that $t \in \operatorname{rng} \mathcal{F}'$ holds $t \in$ FrechetFilter($\mathbb{N}$). Reconsider $\mathcal{F}_1 = \operatorname{rng} \mathcal{F}'$ as a non empty subset of FrechetFilter($\mathbb{N}$). $\mathcal{F}_1$ is filter basis by [21, (2)], [4, (44)], [11, (3)]. □

(53) $\#$ Tails(the ordered $\mathbb{N}$) $=$ rng $\mathcal{F}'$. The theorem is a consequence of (50).

Now we state the proposition:

(54) (i) $\#$ Tails(the ordered $\mathbb{N}$) is a generalized basis of FrechetFilter($\mathbb{N}$), and

(ii) TailsFilter the ordered $\mathbb{N} =$ FrechetFilter($\mathbb{N}$).

The theorem is a consequence of (48), (53), (52), and (21).

The base of Frechet filter yielding a filter base of $\mathbb{N}$ is defined by the term

(Def. 18) $\#$ Tails(the ordered $\mathbb{N}$).

Now we state the propositions:

(55) $\mathbb{N} \in$ the base of Frechet filter.

(56) The base of Frechet filter is a generalized basis of FrechetFilter($\mathbb{N}$).

(57) Let us consider a non empty set X, filters $\mathcal{F}_1$, $\mathcal{F}_2$ of X, and a filter $\mathcal{F}'$ of X. Suppose $\mathcal{F}'$ is finer than $\mathcal{F}_1$ and $\mathcal{F}'$ is finer than $\mathcal{F}_2$. Let us consider an element M_1 of $\mathcal{F}_1$, and an element M_2 of $\mathcal{F}_2$. Then $M_1 \cap M_2$ is not empty.

(58) Let us consider a non empty set X, and filters $\mathcal{F}_1$, $\mathcal{F}_2$ of X. Suppose for every element M_1 of $\mathcal{F}_1$ for every element M_2 of $\mathcal{F}_2$, $M_1 \cap M_2$ is not empty. Then there exists a filter $\mathcal{F}'$ of X such that

(i) $\mathcal{F}'$ is finer than $\mathcal{F}_1$, and

(ii) $\mathcal{F}'$ is finer than $\mathcal{F}_2$.

Let X be a set and x be a subset of X. The functor SubsetToBooleSubset x yielding an element of $2^X_\subseteq$ is defined by the term

(Def. 19) x.

Now we state the propositions:

(59) Let us consider an infinite set X. Then $X \in$ the set of all $X \setminus A$ where A is a finite subset of X.

(60) Let us consider a set X, and a subset A of X. Then $\{B$, where B is an element of $2^X_\subseteq : A \subseteq B\} = \{B$, where B is a subset of $X : A \subseteq B\}$.

(61) Let us consider a set X, and an element a of $2^X_\subseteq$. Then $\uparrow a = \{Y$, where Y is a subset of $X : a \subseteq Y\}$.

(62) Let us consider a set X, and a subset A of X. Then $\{B$, where B is an element of $2^X_\subseteq : A \subseteq B\} = \uparrow$SubsetToBooleSubset A. The theorem is a consequence of (60).

(63) Let us consider a non empty set X, and a filter $\mathcal{F}'$ of X. Then $\bigcup \mathcal{F}' = X$.

(64) Let us consider an infinite set X. Then the set of all $X \setminus A$ where A is a finite subset of X is a filter of X. The theorem is a consequence of (59).

Let us consider a set X. Now we state the propositions:

(65) 2^X is a filter of $2^X_\subseteq$.

(66) $\{X\}$ is a filter of $2^X_\subseteq$.

(67) Let us consider a non empty set X. Then $\{X\}$ is a filter of X.

Let us consider an element A of $2^X_\subseteq$. Now we state the propositions:

(68) $\{Y$, where Y is a subset of $X : A \subseteq Y\}$ is a filter of $2^X_\subseteq$.

(69) $\{B$, where B is an element of $2^X_\subseteq : A \subseteq B\}$ is a filter of $2^X_\subseteq$. The theorem is a consequence of (60) and (68).

Now we state the proposition:

(70) Let us consider a non empty set X, and a non empty subset B of $2^X_\subseteq$. Then for every elements x, y of B, there exists an element z of B such that $z \subseteq x \cap y$ if and only if B is filtered.

PROOF: For every elements x, y of B, there exists an element z of B such that $z \subseteq x \cap y$ by [19, (2)]. $\square$

Let us consider a non empty set X and a non empty subset $\mathcal{F}'$ of the lattice of subsets of X. Now we state the propositions:

(71) $\mathcal{F}'$ is a filter of the lattice of subsets of X if and only if for every elements p, q of $\mathcal{F}'$, $p \cap q \in \mathcal{F}'$ and for every element p of $\mathcal{F}'$ and for every element q of the lattice of subsets of X such that $p \subseteq q$ holds $q \in \mathcal{F}'$.

(72) $\mathcal{F}'$ is a filter of the lattice of subsets of X if and only if for every subsets Y_1, Y_2 of X, if Y_1, $Y_2 \in \mathcal{F}'$, then $Y_1 \cap Y_2 \in \mathcal{F}'$ and if $Y_1 \in \mathcal{F}'$ and $Y_1 \subseteq Y_2$, then $Y_2 \in \mathcal{F}'$. The theorem is a consequence of (71).

Now we state the propositions:

(73) Let us consider a non empty set X, and a non empty family $\mathcal{F}$ of subsets of X. Suppose $\mathcal{F}$ is a filter of the lattice of subsets of X. Then $\mathcal{F}$ is a filter of $2^X_\subseteq$. The theorem is a consequence of (71).

(74) Let us consider a non empty set X. Then every filter of $2^X_\subseteq$ is a filter of the lattice of subsets of X. The theorem is a consequence of (72).

(75) Let us consider a non empty set X, and a non empty subset $\mathcal{F}'$ of the lattice of subsets of X. Then $\mathcal{F}'$ is filter of the lattice of subsets of X and has non empty elements if and only if $\mathcal{F}'$ is a filter of X. The theorem is a consequence of (72).

(76) Let us consider a non empty set X. Then every proper filter of $2^X_\subseteq$ is a filter of X.

PROOF: $\mathcal{F}'$ has non empty elements by [19, (18)], [7, (4)]. $\square$

(77) Let us consider a non empty topological space T, and a point x of T. Then the neighborhood system of x is a filter of the carrier of T.

Let T be a non empty topological space and $\mathcal{F}'$ be a proper filter of $2_{\subseteq}^{\Omega_T}$. The functor BooleanFilterToFilter($\mathcal{F}'$) yielding a filter of the carrier of T is defined by the term

(Def. 20) $\mathcal{F}'$.

Let $\mathcal{F}_1$ be a filter of the carrier of T and $\mathcal{F}_2$ be a proper filter of $2_{\subseteq}^{\Omega_T}$. We say that $\mathcal{F}_1$ is finer than $\mathcal{F}_2$ if and only if

(Def. 21) BooleanFilterToFilter($\mathcal{F}_2$) $\subseteq \mathcal{F}_1$.

3. Limit of a Filter

Let T be a non empty topological space and $\mathcal{F}'$ be a filter of the carrier of T. The functor LimFilter($\mathcal{F}'$) yielding a subset of T is defined by the term

(Def. 22) $\{x$, where x is a point of T : $\mathcal{F}'$ is finer than the neighborhood system of $x\}$.

Let B be a filter base of the carrier of T. The functor Lim B yielding a subset of T is defined by the term

(Def. 23) LimFilter($[B)$).

Now we state the proposition:

(78) Let us consider a non empty topological space T, and a filter $\mathcal{F}'$ of the carrier of T. Then there exists a proper filter $\mathcal{F}_1$ of $2_{\subseteq}^{\alpha}$ such that $\mathcal{F}' = \mathcal{F}_1$, where α is the carrier of T. The theorem is a consequence of (73) and (75).

Let T be a non empty topological space and $\mathcal{F}'$ be a filter of the carrier of T. The functor FilterToBooleanFilter($\mathcal{F}', T$) yielding a proper filter of $2_{\subseteq}^{\Omega_T}$ is defined by the term

(Def. 24) $\mathcal{F}'$.

Let us consider a non empty topological space T, a point x of T, and a filter $\mathcal{F}'$ of the carrier of T. Now we state the propositions:

(79) x is a convergence point of $\mathcal{F}'$ and T if and only if x is a convergence point of FilterToBooleanFilter($\mathcal{F}', T$) and T.

(80) x is a convergence point of $\mathcal{F}'$ and T if and only if $x \in$ LimFilter($\mathcal{F}'$). The theorem is a consequence of (78).

Let T be a non empty topological space and $\mathcal{F}'$ be a filter of $2_{\subseteq}^{\Omega_T}$. The functor LimFilterB($\mathcal{F}'$) yielding a subset of T is defined by the term

(Def. 25) $\{x$, where x is a point of T : the neighborhood system of $x \subseteq \mathcal{F}'\}$.

Let us consider a non empty topological space T and a filter $\mathcal{F}'$ of the carrier of T. Now we state the propositions:

(81) $\mathrm{LimFilter}(\mathcal{F}') = \mathrm{LimFilterB}(\mathrm{FilterToBooleanFilter}(\mathcal{F}', T))$.

(82) $\mathrm{Lim}(\text{the net of } \mathrm{FilterToBooleanFilter}(\mathcal{F}', T)) = \mathrm{LimFilter}(\mathcal{F}')$.

(83) Let us consider a Hausdorff, non empty topological space T, a filter $\mathcal{F}'$ of the carrier of T, and points p, q of T. If p, $q \in \mathrm{LimFilter}(\mathcal{F}')$, then $p = q$.

Let T be a Hausdorff, non empty topological space and $\mathcal{F}'$ be a filter of the carrier of T. Note that $\mathrm{LimFilter}(\mathcal{F}')$ is trivial.

Let X be a non empty set, T be a non empty topological space, f be a function from X into the carrier of T, and $\mathcal{F}'$ be a filter of X. The functor $\lim_{\mathcal{F}'} f$ yielding a subset of Ω_T is defined by the term

(Def. 26) LimFilter(the image of filter $\mathcal{F}'$ under f).

Let L be a non empty, transitive, reflexive relational structure and f be a function from Ω_L into the carrier of T. The functor $\mathrm{LimF}(f)$ yielding a subset of Ω_T is defined by the term

(Def. 27) LimFilter(the image of filter TailsFilter L under f).

Now we state the proposition:

(84) Let us consider a non empty topological space T, a non empty, transitive, reflexive relational structure L, a function f from Ω_L into the carrier of T, a point x of T, and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x). Suppose Ω_L is directed. Then $x \in \mathrm{LimF}(f)$ if and only if for every element b of B, there exists an element i of L such that for every element j of L such that $i \leqslant j$ holds $f(j) \in b$. The theorem is a consequence of (46), (29), and (47).

Let T be a non empty topological space and s be a sequence of T. The functor $\mathrm{LimF}(s)$ yielding a subset of T is defined by the term

(Def. 28) LimFilter(the elementary filter of s).

Now we state the proposition:

(85) Let us consider a non empty topological space T, and a sequence s of T. Then $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} s = \mathrm{LimF}(s)$.

Let us consider a non empty topological space T and a point x of T.

(86) The neighborhood system of x is a filter base of Ω_T. The theorem is a consequence of (76), (13), and (29).

(87) Every generalized basis of BooleanFilterToFilter(the neighborhood system of x) is a filter base of Ω_T.

(88) Let us consider a non empty set X, a sequence s of X, and a filter base B of X. Then B is coarser than s°(the base of Frechet filter) if and only if for every element b of B, there exists an element i of the ordered $\mathbb{N}$ such that for every element j of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$.

(89) Let us consider a non empty topological space T, a sequence s of T, a point x of T, and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x). Then $x \in \lim_{\mathrm{FrechetFilter}(\mathbb{N})} s$ if and only if B is coarser than s°(the base of Frechet filter). The theorem is a consequence of (46) and (54).

(90) Let us consider a non empty topological space T, a sequence s of Ω_T, a point x of T, and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x). Then B is coarser than s°(the base of Frechet filter) if and only if for every element b of B, there exists an element i of the ordered $\mathbb{N}$ such that for every element j of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (29) and (47).

Let us consider a non empty topological space T, a sequence s of the carrier of T, a point x of T, and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x).

(91) $x \in \lim_{\mathrm{FrechetFilter}(\mathbb{N})} s$ if and only if for every element b of B, there exists an element i of the ordered $\mathbb{N}$ such that for every element j of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (89) and (90).

(92) $x \in \mathrm{LimF}(s)$ if and only if for every element b of B, there exists an element i of the ordered $\mathbb{N}$ such that for every element j of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (91).

4. Nets

Let L be a 1-sorted structure and s be a sequence of the carrier of L. The net of s yielding a non empty, strict net structure over L is defined by the term

(Def. 29) $\langle \mathbb{N}, \leqslant_{\mathbb{N}}, s \rangle$.

Let L be a non empty 1-sorted structure. Let us note that the net of s is non empty.

Now we state the proposition:

(93) Let us consider a non empty 1-sorted structure L, a set B, and a sequence s of the carrier of L. Then the net of s is eventually in B if and only if there exists an element i of the net of s such that for every element j of the net of s such that $i \leqslant j$ holds (the net of s)$(j) \in B$.

Let us consider a non empty topological space T, a sequence s of the carrier of T, a point x of T, and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x). Now we state the propositions:

(94) for every element b of B, there exists an element i of the ordered $\mathbb{N}$ such that for every element j of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$ if

and only if for every element b of B, there exists an element i of the net of s such that for every element j of the net of s such that $i \leqslant j$ holds (the net of s)$(j) \in b$.

(95) $x \in \mathrm{LimF}(s)$ if and only if for every element b of B, the net of s is eventually in b. The theorem is a consequence of (92), (94), and (93).

(96) $x \in \mathrm{LimF}(s)$ if and only if for every element b of B, there exists an element i of $\mathbb{N}$ such that for every element j of $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (91).

(97) $x \in \mathrm{LimF}(s)$ if and only if for every element b of B, there exists a natural number i such that for every natural number j such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (96).

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