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Summable Family in a Commutative Group

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Summary. Hölzl et al. showed that it was possible to build "a generic theory of limits based on filters" in Isabelle/HOL [22], [7]. In this paper we present our formalization of this theory in Mizar [6].

First, we compare the notions of the limit of a family indexed by a directed set, or a sequence, in a metric space [30], a real normed linear space [29] and a linear topological space [14] with the concept of the limit of an image filter [16].

Then, following Bourbaki [9], [10] (TG.III, §5.1 *Familles sommables dans un groupe commutatif*), we conclude by defining the summable families in a commutative group ("additive notation" in [17]), using the notion of filters.

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The notation and terminology used in this paper have been introduced in the following articles: [26], [16], [1], [27], [4], [18], [34], [32], [30], [11], [12], [35], [17], [23], [29], [20], [37], [2], [13], [8], [28], [39], [14], [36], [19], [31], [38], [24], [3], [25], [5], [21], and [15].

1. Preliminaries

Now we state the propositions:

(1) Let us consider a set I. Then $\emptyset$ is an element of $\operatorname{Fin} I$.

(2) Let us consider sets I, J. Suppose $J \in \operatorname{Fin} I$. Then there exists a finite sequence p of elements of I such that

(i) $J = \operatorname{rng} p$, and

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(ii) p is one-to-one.

(3) Let us consider a set I, a non empty set Y, a Y-valued many sorted set x indexed by I, and a finite sequence p of elements of I. Then $p \cdot x$ is a finite sequence of elements of Y.

(4) Let us consider non empty sets I, X, an X-valued many sorted set x indexed by I, and finite sequences p, q of elements of I. Then $(p \frown q) \cdot x = p \cdot x \frown (q \cdot x)$.

PROOF: For every object t such that $t \in \operatorname{dom}((p \frown q) \cdot x)$ holds $((p \frown q) \cdot x)(t) = (p \cdot x \frown (q \cdot x))(t)$ by [33, (120)], [11, (13)], [4, (25)]. □

Let I be a set, Y be a non empty set, x be a Y-valued many sorted set indexed by I, and p be a finite sequence of elements of I. The functor $\#_x^p$ yielding a finite sequence of elements of Y is defined by the term

(Def. 1) $p \cdot x$.

The functor $\mathcal{F}(I)$ yielding a non empty, transitive, reflexive relational structure is defined by the term

(Def. 2) $\langle \operatorname{Fin} I, \subseteq \rangle$.

Now we state the proposition:

(5) Let us consider a set I. Then $\Omega_{\mathcal{F}(I)}$ is directed.

2. Convergence in Metric Spaces

Now we state the propositions:

(6) Let us consider a non empty metric space M, and a point x of M_{top}. Then $\operatorname{Balls} x$ is a generalized basis of BooleanFilterToFilter(the neighborhood system of x).

(7) Let us consider a non empty metric space M, a non empty, transitive, reflexive relational structure L, a function f from Ω_L into the carrier of M_{top}, a point x of M_{top}, and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x). Suppose Ω_L is directed. Then $x \in \operatorname{LimF}(f)$ if and only if for every element b of B, there exists an element i of L such that for every element j of L such that $i \leqslant j$ holds $f(j) \in b$.

(8) Let us consider a non empty metric space M, a non empty, transitive, reflexive relational structure L, a function f from Ω_L into the carrier of M_{top}, and a point x of M_{top}. Suppose Ω_L is directed. Then $x \in \operatorname{LimF}(f)$ if and only if for every element b of $\operatorname{Balls} x$, there exists an element n of L such that for every element m of L such that $n \leqslant m$ holds $f(m) \in b$. The theorem is a consequence of (6).

(9) Let us consider a non empty metric space M, a sequence s of the carrier of $M_{\rm top}$, and a point x of $M_{\rm top}$. Then $x \in \mathrm{LimF}(s)$ if and only if for every element b of $\mathrm{Balls}\,x$, there exists a natural number i such that for every natural number j such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (6).

(10) Let us consider a non empty topological structure T, a sequence s of T, and a point x of T. Then $x \in \mathrm{Lim}\,s$ if and only if for every subset U_1 of T such that U_1 is open and $x \in U_1$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $s(m) \in U_1$.

Let us consider a non empty metric space M, a sequence s of the carrier of $M_{\rm top}$, and a point x of $M_{\rm top}$. Now we state the propositions:

(11) $x \in \mathrm{Lim}\,s$ if and only if for every element b of $\mathrm{Balls}\,x$, there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $s(m) \in b$. The theorem is a consequence of (6) and (10).

(12) $x \in \mathrm{LimF}(s)$ if and only if $x \in \mathrm{Lim}\,s$. The theorem is a consequence of (9) and (11).

3. Filter and Limit of a Sequence in Real Normed Space

Now we state the propositions:

(13) Let us consider a real normed space N, a non empty, transitive, reflexive relational structure L, a function f from Ω_L into the carrier of $(\mathrm{MetricSpaceNorm}\,N)_{\rm top}$, a point x of $(\mathrm{MetricSpaceNorm}\,N)_{\rm top}$, and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x). Suppose Ω_L is directed. Then $x \in \mathrm{LimF}(f)$ if and only if for every element b of B, there exists an element i of L such that for every element j of L such that $i \leqslant j$ holds $f(j) \in b$.

(14) Let us consider a real normed space N, and a point x of $(\mathrm{MetricSpaceNorm}\,N)_{\rm top}$. Then $\mathrm{Balls}\,x$ is a generalized basis of BooleanFilterToFilter(the neighborhood system of x).

(15) Let us consider a real normed space N, a sequence s of the carrier of $(\mathrm{MetricSpaceNorm}\,N)_{\rm top}$, and a point x of $(\mathrm{MetricSpaceNorm}\,N)_{\rm top}$. Then $x \in \mathrm{LimF}(s)$ if and only if for every element b of $\mathrm{Balls}\,x$, there exists a natural number i such that for every natural number j such that $i \leqslant j$ holds $s(j) \in b$.

(16) Let us consider a real normed space N, and a point x of $(\mathrm{MetricSpaceNorm}\,N)_{\rm top}$. Then there exists a point y of $\mathrm{MetricSpaceNorm}\,N$ such that

(i) $y = x$, and

(ii) $\mathrm{Balls}\,x = \{\mathrm{Ball}(y, \frac{1}{n})$, where n is a natural number : $n \neq 0\}$.

(17) Let us consider a real normed space N, a point x of $(\mathrm{MetricSpaceNorm}\,N)_{\mathrm{top}}$, a point y of $\mathrm{MetricSpaceNorm}\,N$, and a positive natural number n. If $x = y$, then $\mathrm{Ball}(y, \frac{1}{n}) \in \mathrm{Balls}\,x$.

(18) Let us consider a real normed space N, a point x of $\mathrm{MetricSpaceNorm}\,N$, and a natural number n. Suppose $n \neq 0$. Then $\mathrm{Ball}(x, \frac{1}{n}) = \{q$, where q is an element of $\mathrm{MetricSpaceNorm}\,N : \rho(x, q) < \frac{1}{n}\}$.

(19) Let us consider a real normed space N, an element x of $\mathrm{MetricSpaceNorm}\,N$, and a natural number n. Suppose $n \neq 0$. Then there exists a point y of N such that

(i) $x = y$, and

(ii) $\mathrm{Ball}(x, \frac{1}{n}) = \{q$, where q is a point of $N : \|y - q\| < \frac{1}{n}\}$.

Let us consider a metric structure P_1. Now we state the propositions:

(20) $P_{1\mathrm{top}} = \langle$the carrier of P_1, the open set family of $P_1\rangle$.

(21) The carrier of $\langle$the carrier of P_1, the open set family of $P_1\rangle$ = the carrier of P_1.

(22) The carrier of $P_{1\mathrm{top}}$ = the carrier of $\langle$the carrier of P_1, the open set family of $P_1\rangle$.

(23) The carrier of $P_{1\mathrm{top}}$ = the carrier of P_1.

Now we state the proposition:

(24) Let us consider a real normed space N, a sequence s of the carrier of $(\mathrm{MetricSpaceNorm}\,N)_{\mathrm{top}}$, and a natural number j. Then $s(j)$ is an element of the carrier of $(\mathrm{MetricSpaceNorm}\,N)_{\mathrm{top}}$.

Let N be a real normed space and x be a point of $(\mathrm{MetricSpaceNorm}\,N)_{\mathrm{top}}$. The functor $\#\,x$ yielding a point of N is defined by the term

(Def. 3) x.

Now we state the proposition:

(25) Let us consider a real normed space N, a sequence s of the carrier of $(\mathrm{MetricSpaceNorm}\,N)_{\mathrm{top}}$, and a point x of $(\mathrm{MetricSpaceNorm}\,N)_{\mathrm{top}}$. Then $x \in \mathrm{LimF}(s)$ if and only if for every positive natural number n, there exists a natural number i such that for every natural number j such that $i \leqslant j$ holds $\|\#\,x - \#\,s(j)\| < \frac{1}{n}$.

Proof: Reconsider $x_1 = x$ as a point of $(\mathrm{MetricSpaceNorm}\,N)_{\mathrm{top}}$. Consider y_0 being a point of $\mathrm{MetricSpaceNorm}\,N$ such that $y_0 = x_1$ and $\mathrm{Balls}\,x_1 = \{\mathrm{Ball}(y_0, \frac{1}{n})$, where n is a natural number : $n \neq 0\}$. If $x \in \mathrm{LimF}(s)$, then for every positive natural number n, there exists a natural number i such that for every natural number j such that $i \leqslant j$ holds

$\|\# x - \# s(j)\| < \frac{1}{n}$ by (9), [20, (2)]. If for every positive natural number n, there exists a natural number i such that for every natural number j such that $i \leqslant j$ holds $\|\# x - \# s(j)\| < \frac{1}{n}$, then $x \in \mathrm{LimF}(s)$ by [20, (2)], (9). □

4. Filter and Limit of a Sequence in Linear Topological Space

Now we state the propositions:

(26) Let us consider a non empty linear topological space X. Then the neighborhood system of 0_X is a local base of X.

(27) Let us consider a linear topological space X, a local base O of X, a point a of X, and a family P of subsets of X. Suppose $P = \{a+U$, where U is a subset of $X : U \in O\}$. Then P is a generalized basis of a.

(28) Let us consider a non empty linear topological space X, a point x of X, and a local base O of X. Then $\{x + U$, where U is a subset of X : $U \in O$ and U is a neighbourhood of $0_X\} = \{x + U$, where U is a subset of $X : U \in O$ and $U \in$ the neighborhood system of $0_X\}$.

(29) Let us consider a non empty linear topological space X, a point x of X, a local base O of X, and a family B of subsets of X. Suppose $B = \{x + U$, where U is a subset of X : $U \in O$ and U is a neighbourhood of $0_X\}$. Then B is a generalized basis of BooleanFilterToFilter(the neighborhood system of x).

PROOF: Set F = BooleanFilterToFilter(the neighborhood system of x). $F \subseteq [B]$ by [14, (9)], [27, (3)], [14, (8), (6)]. $[B] \subseteq F$ by [14, (37)]. □

(30) Let us consider a non empty linear topological space X, a sequence s of the carrier of X, a point x of X, a local base V of X, and a family B of subsets of X. Suppose $B = \{x+U$, where U is a subset of $X : U \in V$ and U is a neighbourhood of $0_X\}$. Then $x \in \mathrm{LimF}(s)$ if and only if for every element v of B, there exists a natural number i such that for every natural number j such that $i \leqslant j$ holds $s(j) \in v$. The theorem is a consequence of (29).

(31) Let us consider a non empty linear topological space X, a sequence s of the carrier of X, a point x of X, and a local base V of X. Then $x \in \mathrm{LimF}(s)$ if and only if for every subset v of X such that $v \in V \cap$(the neighborhood system of 0_X) there exists a natural number i such that for every natural number j such that $i \leqslant j$ holds $s(j) \in x + v$.

PROOF: Set $B = \{x + U$, where U is a subset of $X : U \in V$ and U is a neighbourhood of $0_X\}$. B is a generalized basis of BooleanFilterToFilter (the neighborhood system of x). For every element b of B, there exists a natural number i such that for every natural number j such that $i \leqslant j$ holds $s(j) \in b$ by [5, (2)]. □

(32) Let us consider a non empty linear topological space T, a non empty, transitive, reflexive relational structure L, a function f from Ω_L into the carrier of T, a point x of T, and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x). Suppose Ω_L is directed. Then $x \in \mathrm{LimF}(f)$ if and only if for every element b of B, there exists an element i of L such that for every element j of L such that $i \leqslant j$ holds $f(j) \in b$.

(33) Let us consider a non empty linear topological space T, a non empty, transitive, reflexive relational structure L, a function f from Ω_L into the carrier of T, a point x of T, and a local base V of T. Suppose Ω_L is directed. Then $x \in \mathrm{LimF}(f)$ if and only if for every subset v of T such that $v \in V \cap$ (the neighborhood system of 0_T) there exists an element i of L such that for every element j of L such that $i \leqslant j$ holds $f(j) \in x + v$.

5. Series in Abelian Group: a Definition

Let I be a non empty set, L be an Abelian group, x be a (the carrier of L)-valued many sorted set indexed by I, and J be an element of $\mathrm{Fin}\, I$. The functor $\sum_{\kappa=0}^{J} x(\kappa)$ yielding an element of L is defined by

(Def. 4) there exists a one-to-one finite sequence p of elements of I such that $\mathrm{rng}\, p = J$ and $it =$ (the addition of L) $\odot \#_x^p$.

Now we state the proposition:

(34) Let us consider a non empty set I, an Abelian group L, a (the carrier of L)-valued many sorted set x indexed by I, an element J of $\mathrm{Fin}\, I$, and an element e of $\mathrm{Fin}\, I$. Suppose $e = \emptyset$. Then

(i) $\sum_{\kappa=0}^{e} x(\kappa) = 0_L$, and

(ii) for every elements e, f of $\mathrm{Fin}\, I$ such that e misses f holds $\sum_{\kappa=0}^{e \cup f} x(\kappa) = \sum_{\kappa=0}^{e} x(\kappa) + \sum_{\kappa=0}^{f} x(\kappa)$.

The theorem is a consequence of (4).

Let I be a non empty set, L be an Abelian group, and x be a (the carrier of L)-valued many sorted set indexed by I. The functor $(\sum_{\alpha=0}^{\kappa} x(\alpha))_{\kappa \in \mathbb{N}}$ yielding a function from $\Omega_{\mathcal{F}(I)}$ into the carrier of L is defined by

(Def. 5) for every element j of $\mathrm{Fin}\, I$, $it(j) = \sum_{\kappa=0}^{j} x(\kappa)$.

6. Product of Family as Limit in Commutative Topological Group

Let I be a non empty set, L be a commutative semi topological group, x be a (the carrier of L)-valued many sorted set indexed by I, and J be an element of $\operatorname{Fin} I$. The functor $\operatorname{Product}(x, J)$ yielding an element of L is defined by

(Def. 6) there exists a one-to-one finite sequence p of elements of I such that $\operatorname{rng} p = J$ and $it =$ (the multiplication of L) $\odot \#_x^p$.

(35) Let us consider a set I, a semi topological group G, a function f from $\Omega_{\mathcal{F}(I)}$ into the carrier of G, a point x of G, and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x). Then $x \in \operatorname{LimF}(f)$ if and only if for every element b of B, there exists an element i of $\mathcal{F}(I)$ such that for every element j of $\mathcal{F}(I)$ such that $i \leqslant j$ holds $f(j) \in b$. The theorem is a consequence of (5).

(36) Let us consider a non empty set I, a commutative semi topological group L, a (the carrier of L)-valued many sorted set x indexed by I, an element J of $\operatorname{Fin} I$, and an element e of $\operatorname{Fin} I$. Suppose $e = \emptyset$. Then

(i) $\operatorname{Product}(x, e) = \mathbf{1}_L$, and

(ii) for every elements e, f of $\operatorname{Fin} I$ such that e misses f holds $\operatorname{Product}(x, e \cup f) = \operatorname{Product}(x, e) \cdot \operatorname{Product}(x, f)$.

The theorem is a consequence of (4).

Let I be a non empty set, L be a commutative semi topological group, and x be a (the carrier of L)-valued many sorted set indexed by I. The functor the partial product of x yielding a function from $\Omega_{\mathcal{F}(I)}$ into the carrier of L is defined by

(Def. 7) for every element j of $\operatorname{Fin} I$, $it(j) = \operatorname{Product}(x, j)$.

(37) Let us consider a non empty set I, a commutative semi topological group G, a (the carrier of G)-valued many sorted set s indexed by I, a point x of G, and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x). Then $x \in$ LimF(the partial product of s) if and only if for every element b of B, there exists an element i of $\mathcal{F}(I)$ such that for every element j of $\mathcal{F}(I)$ such that $i \leqslant j$ holds (the partial product of s)$(j) \in b$.

7. Summable Family in Commutative Topological Group

Let I be a non empty set, L be an Abelian semi additive topological group, x be a (the carrier of L)-valued many sorted set indexed by I, and J be an element of $\operatorname{Fin} I$. The functor $\sum_{\kappa=0}^{J} x(\kappa)$ yielding an element of L is defined by

(Def. 8) there exists a one-to-one finite sequence p of elements of I such that $\operatorname{rng} p = J$ and $it =$ (the addition of L) $\odot \#_x^p$.

Now we state the propositions:

(38) Let us consider a set I, a semi additive topological group G, a function f from $\Omega_{\mathcal{F}(I)}$ into the carrier of G, a point x of G, and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x). Then $x \in \operatorname{LimF}(f)$ if and only if for every element b of B, there exists an element i of $\mathcal{F}(I)$ such that for every element j of $\mathcal{F}(I)$ such that $i \leqslant j$ holds $f(j) \in b$. The theorem is a consequence of (5).

(39) Let us consider a non empty set I, an Abelian semi additive topological group L, a (the carrier of L)-valued many sorted set x indexed by I, an element J of $\operatorname{Fin} I$, and an element e of $\operatorname{Fin} I$. Suppose $e = \emptyset$. Then

(i) $\sum_{\kappa=0}^{e} x(\kappa) = 0_L$, and

(ii) for every elements e, f of $\operatorname{Fin} I$ such that e misses f holds $\sum_{\kappa=0}^{e \cup f} x(\kappa) = \sum_{\kappa=0}^{e} x(\kappa) + \sum_{\kappa=0}^{f} x(\kappa)$.

The theorem is a consequence of (4).

Let I be a non empty set, L be an Abelian semi additive topological group, and x be a (the carrier of L)-valued many sorted set indexed by I. The functor $(\sum_{\alpha=0}^{\kappa} x(\alpha))_{\kappa \in \mathbb{N}}$ yielding a function from $\Omega_{\mathcal{F}(I)}$ into the carrier of L is defined by

(Def. 9) for every element j of $\operatorname{Fin} I$, $it(j) = \sum_{\kappa=0}^{j} x(\kappa)$.

Now we state the proposition:

(40) Let us consider a non empty set I, an Abelian semi additive topological group G, a (the carrier of G)-valued many sorted set s indexed by I, a point x of G, and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x). Then $x \in \operatorname{LimF}((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}})$ if and only if for every element b of B, there exists an element i of $\mathcal{F}(I)$ such that for every element j of $\mathcal{F}(I)$ such that $i \leqslant j$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(j) \in b$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[5] Grzegorz Bancerek, Noboru Endou, and Yuji Sakai. On the characterizations of compactness. *Formalized Mathematics*, 9(**4**):733–738, 2001.

[6] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and

beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[7] Sylvie Boldo, Catherine Lelay, and Guillaume Melquiond. Formalization of real analysis: A survey of proof assistants and libraries. *Mathematical Structures in Computer Science*, pages 1–38, 2014.

[8] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[9] Nicolas Bourbaki. *Topologie générale: Chapitres 1 à 4*. Eléments de mathématique. Springer Science & Business Media, 2007.

[10] Nicolas Bourbaki. *General Topology: Chapters 1–4*. Springer Science and Business Media, 2013.

[11] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[12] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[13] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[14] Czesław Byliński. Introduction to real linear topological spaces. *Formalized Mathematics*, 13(**1**):99–107, 2005.

[15] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[16] Roland Coghetto. Convergent filter bases. *Formalized Mathematics*, 23(**3**):189–203, 2015. doi:10.1515/forma-2015-0016.

[17] Roland Coghetto. Groups – additive notation. *Formalized Mathematics*, 23(**2**):127–160, 2015. doi:10.1515/forma-2015-0013.

[18] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[19] Noboru Endou, Takashi Mitsuishi, and Yasunari Shidama. Dimension of real unitary space. *Formalized Mathematics*, 11(**1**):23–28, 2003.

[20] Noboru Endou, Yasunari Shidama, and Katsumasa Okamura. Baire's category theorem and some spaces generated from real normed space. *Formalized Mathematics*, 14(**4**): 213–219, 2006. doi:10.2478/v10037-006-0024-x.

[21] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Formalized Mathematics*, 6(**1**):117–121, 1997.

[22] Johannes Hölzl, Fabian Immler, and Brian Huffman. Type classes and filters for mathematical analysis in Isabelle/HOL. In *Interactive Theorem Proving*, pages 279–294. Springer, 2013.

[23] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.

[24] Artur Korniłowicz. The definition and basic properties of topological groups. *Formalized Mathematics*, 7(**2**):217–225, 1998.

[25] Artur Korniłowicz. Introduction to meet-continuous topological lattices. *Formalized Mathematics*, 7(**2**):279–283, 1998.

[26] Michał Muzalewski and Wojciech Skaba. From loops to Abelian multiplicative groups with zero. *Formalized Mathematics*, 1(**5**):833–840, 1990.

[27] Beata Padlewska. Locally connected spaces. *Formalized Mathematics*, 2(**1**):93–96, 1991.

[28] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[29] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.

[30] Bartłomiej Skorulski. First-countable, sequential, and Frechet spaces. *Formalized Mathematics*, 7(**1**):81–86, 1998.

[31] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1 (**2**):369–376, 1990.

[32] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1 (**1**):187–190, 1990.

[33] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *For-*

malized Mathematics, 1(**3**):569–573, 1990.
[34] Wojciech A. Trybulec. Binary operations on finite sequences. *Formalized Mathematics*, 1 (**5**):979–981, 1990.
[35] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[36] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[37] Wojciech A. Trybulec and Grzegorz Bancerek. Kuratowski – Zorn lemma. *Formalized Mathematics*, 1(**2**):387–393, 1990.
[38] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[39] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.

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