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A New Proof of the Existence of Suitable Weak Solutions and Other Remarks for the Navier-Stokes Equations

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Abstract

We prove that the limits of the semi-discrete and the discrete semi-implicit Euler schemes for the 3D Navier-Stokes equations supplemented with Dirichlet boundary conditions are suitable in the sense of Scheffer [1]. This provides a new proof of the existence of suitable weak solutions, first established by Caffarelli, Kohn and Nirenberg [2]. Our results are similar to the main result in [3]. We also present some additional remarks and open questions on suitable solutions.

Keywords

Navier-Stokes Equations, Regularity, Caffarelli-Kohn-Nirenberg Estimates, Semi-Implicit Euler Approximation Schemes

1. Introduction

The main objective of this paper is to provide a new proof of the existence of suitable weak solutions to the Navier-Stokes equations. Specifically, we show that the semi-discrete and the completely discrete semi-implicit Euler schemes lead to families of approximate solutions that converge to a weak solution that is suitable in the sense of Caffarelli, Kohn and Nirenberg [2].

We will be concerned with the 3D Navier-Stokes equations completed with initial and Dirichlet boundary conditions in bounded domains $\Omega\times(0,T)$ (as usual, Ω is the spatial domain, a regular, bounded and connected open set in $\mathbb{R}^3$ "filled" by the fluid particles; $(0,T)$ is the time observation interval).

The key concept of suitable weak solution was introduced in [2]. In few words, this is a weak solution satisfying a local energy inequality. Due to its definition, it is expected that suitable solutions are regular (and unique). However, up to now, this is unknown. The best we can prove is that the set of singular points of a suitable solution is small, in the sense that has Hausdorff dimension ≤1. As shown below, in order to improve this result, we would need an estimate that we do not have at hand at present. A similar analysis can be performed in the context of the Boussinesq system; see [4].

The estimate in [2] of the Hausdorff dimension relies on some technical results asserting that adequate criteria, applied to suitable solutions in a given space-time region, imply the regularity of the points in a subregion. During the last years, several authors have tried to improve or weaken these criteria and some achievements have been obtained:

- In Seregin [5], a family of sufficient conditions that contains the Caffarelli-Kohn-Nirenberg condition as a particular case is introduced. Its formulation is given in terms of functionals invariant with respect to scale transformations.
- In Vasseur [6], an interesting criterion appears: if we normalize the solution and the sum of the associated kinetic and viscous energies and the L^p norm of the pressure is small enough, we get regularity. The proof of this assertion is inspired by a method of De Giorgi designed to prove the regularity of elliptic equations, see [7].
- In Wolf [8], the author provides a notion of local pressure. It permits to estimate the integrals involving the pressure in terms of the velocity and, again, deduce regularity. The method is interesting and can be adapted to get partial regularity results for other systems, such as the equations of quasi-Newtonian or Boussinesq (heat conducting) fluids.
- Finally, in Choe, Wolf and Yang [9], an improved version of the Caffarelli-Kohn-Nirenberg criterion is furnished, using ideas from [5].

Note that, in a recent paper, Buckmaster and Vicol [10] have proved that, for a very weak class of distributional solutions in spatially periodic domains, non-uniqueness occurs.

The techniques employed in this paper can be applied to many other approximation schemes that lead to energy inequalities, as those in [11] [12] [13] [14]. More precisely, we first use the well known energy estimates, together with appropriate interpolation results and recall that the approximate solutions converge to a weak solution (u, p). Then, we analyze the role of the pressure p; this reduces in fact to a detailed study of the behavior of the time derivate of the velocity field. This way, we are able to take (lower) limits in the local energy identities satisfied by the approximate solutions and deduce that (u, p) is suitable.

Our results can be compared to other previous proofs of existence: the one in the Appendix in [2] (based on the construction of a family of time delayed linear approximations), the main result in Da Veiga [15] (relying on regularization with vanishing fourth-order terms), the main result in Guermond [3] (where Faedo-Galerkin techniques are employed) and, also, the results by Berselli and

Spirito [16] [17], where the Voigt approximations and the artificial compressibility method are shown to converge.

We think that our results can be useful at least from two points of view. First, a new (relatively simple) constructive argument is used to prove the existence of suitable solutions. Then, there is some practical interest: by inspection of the behavior of the computed approximations in a prescribed region, we may try to deduce if the related points are regular. In other words, checking whether or not the Caffarelli-Kohn-Nirenberg criteria are satisfied on the computed numerical solutions can serve to identify or discard singular points. Based on this idea, we will present in a forthcoming paper several numerical experiments for which interesting conclusions can be obtained.

The plan of the paper is the following:

- In Section 2, we review the main results in the papers [2] and [18]. In particular, we explain why suitable solutions are relevant in the context of the regularity problem.
- In Section 3, we recall the Euler approximation schemes and we establish the convergence to a suitable solution of the Navier-Stokes equations.
- Finally, Section 4 is devoted to some additional comments and open questions.

2. Background: The Basic Results by Caffarelli, Kohn and Nirenberg

In the sequel, we denote by $|\cdot|$ and $(\cdot,\cdot)$ the usual L^2 norm and scalar product, respectively. The symbol C will be used to denote a generic positive constant.

2.1. The Main Properties of Suitable Solutions

In this section, we will recall the main contributions of Caffarelli, Kohn and Nirenberg, see [2]. In this reference, the best results known to date in relation to the regularity of the Navier-Stokes equations are established.

Let $\Omega \subset \mathbb{R}^3$ be a nonempty, regular, bounded and connected open set and assume that $T > 0$. Let us set $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$. We will consider local and global solutions to the Navier-Stokes equations in three dimensions

$$\begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla p = f(x,t), \\ \nabla \cdot u = 0, \end{cases} \tag{1}$$

where $f = (f^1, f^2, f^3)$ verifies $f \in L^q(\Omega \times (0,T))^3$ with $q \ge 2$ and $\nabla \cdot f = 0$.

At a local level, we will consider solutions in sets of the form $G \times (a,b)$, where $G \subset \Omega$ is open and connected and $(a,b) \subset (0,T)$:

Definition 2.1 Let the open set $D := G \times (a,b)$ be given. It will be said that the couple (u,p) is a **weak solution** to the Navier-Stokes equations, Equation (1) in D if the following holds:

- $u \in L^2\left(0,T;H^1(G)^3\right) \cap L^\infty\left(0,T;L^2(G)^3\right)$ and $p \in L^{5/3}\left(G \times (a,b)\right)$.
- u and p satisfy the Navier-Stokes equations in Equation (1) in the distributional sense in D.

On the other hand, for the definition of a global solution, it will be convenient to use the spaces H and V, with

$$H := \left\{ v \in L^2(\Omega)^3 : \nabla \cdot v = 0 \text{ in } \Omega, v \cdot n = 0 \text{ on } \partial\Omega \right\},$$

$$V := \left\{ v \in H_0^1(\Omega)^3 : \nabla \cdot v = 0 \text{ in } \Omega \right\}.$$

Let us assume that $u_0 \in H$ and let us consider the initial-boundary value problem

$$\begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla p = f(x,t), (x,t) \in Q, \\ \nabla \cdot u = 0, (x,t) \in Q, \\ u(x,t) = 0, (x,t) \in \Sigma, \\ u(x,0) = u_0(x), x \in \Omega. \end{cases} \tag{2}$$

Definition 2.2 It will be said that the couple (u, p) is a (global) **weak solution** to Equation (2) if the following holds:

- $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and $p \in L^{5/3}(Q)$.
- u and p satisfy the Navier-Stokes equations in Equation (2) in the distributional sense in Q
- $u(x,0) = u_0(x)$ a.e. in Ω.

It is known that any couple (u, p) satisfying the previous first and second points also verifies

$$u \in L^{10/3}(Q)^3 \cap C_w^0([0,T];H), u_t \in L^{4/3}(0,T;V').$$

In particular, u can be viewed as a well-defined H-valued function and the third assertion in Definition 2.2 has a sense as an equality in H.

In order to understand the role and relevance of the terms in the estimates that follow, it is convenient to associate a dimension to each variable in Equation (2). Note that, if the pair (u, p) is a weak solution to the Navier-Stokes equations in $D = G \times (a,b)$, then for each $\lambda > 0$ the functions

$$u_\lambda(x,t) := \lambda u\left(\lambda x, \lambda^2 t\right) \text{ and } p_\lambda(x,t) := \lambda^2 p\left(\lambda x, \lambda^2 t\right)$$

solve a similar problem in $D_\lambda := \lambda^{-1} G \times \left(\lambda^{-2} a, \lambda^{-2} b\right)$, with force $f_\lambda := \lambda^3 f\left(\lambda x, \lambda^2 t\right)$. Thus, for any integer k, we say that a variable or a linear differential operator is of dimension k if it is non-dimensionalized when it is multiplied by λ^{-k}, where λ is a characteristic length. We can affirm that

- x_i has dimension 1 and t is of dimension 2,
- u^i has dimension −1 and p has dimension −2,
- f has dimension −3,
- ∂_i has dimension −1 and ∂_t has dimension 2,

so that all the terms of the motion equation in Equation (2) have dimension −3.

The analysis of the existence of a weak solution to Equation (2) can be found

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for instance in [19] and [20]. Now, we will speak of the regularity problem, that is, the possible regularity properties of the weak solution. To this purpose, let us consider the following definition:

Definition 2.3 Let (u,p) be a weak solution to Equation (1) in $D = G\times(a,b)$ and let $(x_0,t_0)\in D$ be given. It will be said that (x_0,t_0) is a **singular point** if u is not L^∞ in any neighborhood of (x_0,t_0), that is, there are no r and C such that $|u(x,t)|\le C$ for (x,t) a.e. in $B((x_0,t_0);r)$. The remaining points, those where u is locally bounded, will be called **regular points**.

According to a result by Serrin [21], it is known that, if (u,p) is a weak solution to Equation (2) and $(x_0,t_0)\in D$ is a regular point for u, then u coincides a.e. with a C^∞ function in a neighborhood of (x_0,t_0). This gives an idea of how interesting can be to get a description of the set S of singular points.

In fact, Serrin proved that, in order to have u of class C^∞ near (x_0,t_0), one just needs an estimate of the kind L^r in time and L^s in space, with sufficiently large r and s. Note that, in [22], it is shown that a weaker condition is sufficient for C^∞ regularity. Note also that, in accordance with the results in [23], if one component of the velocity field is essentially bounded in a region, there is no singular point in a subregion.

The first papers devoted to describe S are due to Scheffer [1] [24] [25]. There, some estimates of the size of the set were given in terms of appropriate Hausdorff measures. Actually, the main result in [1] is the following:

Theorem 2.4 Assume that $f\equiv 0$. There exists a weak solution to Equation (2) whose associated singular set S satisfies:

$$\mathcal{H}^{5/3}(S)<+\infty \quad \text{and} \quad \mathcal{H}^1\left(S\cap\left(\Omega\times\{t\}\right)\right)<+\infty \quad \text{uniformly in } t.$$

Here, $\mathcal{H}^k$ denotes the usual Hausdorff k-dimensional measure in $\mathbb{R}^4$.

This result was improved by Caffarelli, Kohn and Nirenberg in [2] in several directions. There, the authors used a particular class of weak solution, denoted **suitable weak solution** or simply suitable solution, according to the following definition:

Definition 2.5 Let $D = G\times(a,b)$ be a cylinder in $\mathbb{R}^3\times\mathbb{R}$. It is said that (u,p) is a **suitable weak solution** to the Navier-Stokes equations in D if it satisfies points 1 and 2 of Definition 2.1 and, furthermore, the following generalized energy inequality property: for any $\phi\in C_0^\infty(D)$ with $\phi\ge 0$,

$$2\iint_D|\nabla u|^2\phi\le\iint_D\left(|u|^2(\phi_t+\Delta\phi)+\left(|u|^2+2p\right)u\cdot\nabla\phi+2(u\cdot f)\phi\right).$$

Then, the authors of [2] introduced the so called "parabolic" Hausdorff measure $\mathcal{P}^1$, as follows:

- First, for any small $\delta>0$ and any $X\subset\mathbb{R}^4$, they set

$$\mathcal{P}^1_\delta(X):=\inf\left\{\sum_{i\ge1}r_i : X\subset\bigcup_{i\ge1}Q_{r_i},\, r_i<\delta\right\}.$$

Here, each Q_{r_i} is a parabolic cylinder, that is, a set of the form

$$Q_r := \left\{ (x,t) \in \mathbb{R}^4 : |x-\overline{x}| \le r, |t-\overline{t}| \ge r^2 \right\}$$

for some $(\overline{x},\overline{t}) \in Q$.

- Then, for any $X \subset \mathbb{R}^4$, they set

$$\mathcal{P}^1(X) := \lim_{\delta \to 0^+} \mathcal{P}^1_\delta(X)$$

With the help of $\mathcal{P}^1$, a local partial regularity result can be established for any suitable solution:

Theorem 2.6 Let (u,p) be a suitable solution to Equation (1) in *D*. Then the associated singular set satisfies $\mathcal{P}^1(S) = 0$.

This result improves Theorem 2.4 in several aspects: first, it has local character; then, it allows a rather general force term *f*; finally, it gives a better estimate of the Hausdorff dimension of *S*, since one has

$$\mathcal{H}^1(X) \le C\mathcal{P}^1(X), \forall X \subset \mathbb{R}^4$$

for some $C > 0$.

In the sequel, for any (x,t) and $r > 0$, we will denote by $Q_r(x,t)$ the following parabolic cylinder:

$$Q_r(x,t) := \left\{ (y,\tau) : |y-x| < r, t-r^2 < \tau < t \right\}.$$

For the proof of Theorem 2.6, we need two results. The first one is the following:

Proposition 2.7 Suppose that (u,p) is a suitable weak solution to Equation (1) in $Q_1 := Q_1(0,0)$ and $f \in L^q(Q_1)^3$ with $q > 5/2$. There exist $\epsilon_1 > 0$, $C_1 > 0$ and $\epsilon_2 = \epsilon_2(q) > 0$ such that, if

$$\iint_{Q_1} \left(|u|^3 + |u||p| \right) + \int_{-1}^{0} \left(\int_{|x|<1} |p| \, dx \right)^{5/4} dt \le \epsilon_1 \tag{3a}$$

and

$$\iint_{Q_1} |f|^q \le \epsilon_2 \tag{3b}$$

then

$$|u(x,t)| \le C_1 \quad a.e. \text{ in } Q_{1/2} := Q_{1/2}(0,0). \tag{3c}$$

In particular, $(0,0)$ is a regular point.

Proposition 2.7 shows that the sizes of the data have an influence on the regularity of suitable solutions. Now, if we introduce

$$M(r) := \frac{1}{r^2} \iint_{Q_r} \left(|u|^3 + |u||p| \right) + r^{-13/4} \int_{t-r^2}^{t} \left(\int_{|y-x|<r} |p| \, dy \right)^{5/4} d\tau \tag{4}$$

and

$$F_q(r) := r^{3q-5} \iint_{Q_r} |f|^q, \tag{5}$$

taking into account the dimensions of these quantities, we can easily deduce the following:

Corollary 2.8 Suppose that (u,p) is a suitable solution to Equation (1) in the cylinder $Q_r(x,t)$ and $f \in L^q(Q_r(x,t))^3$, with $q > 5/2$. Then, if $M(r) \le \epsilon_1$ and $F_q(r) \le \epsilon_2$, one has

$$|u| \le C_1 r^{-1} \text{ a.e. in } Q_{r/2}(x,t)$$

and, consequently, every point in $Q_{r/2}(x,t)$ is regular.

Let us now set

$$Q_r^*(x,t) := \left\{ (y,\tau) : |y-x| < r, t - \frac{7}{8}r^2 < \tau < t + \frac{1}{8}r^2 \right\}.$$

The second fundamental result used in the proof of Theorem 2.6 is the following:

Proposition 2.9 Let (u,p) be a suitable solution to Equation (1) in a neighborhood of (x,t). There exists $\epsilon_3 > 0$ such that, if

$$\limsup_{r\to 0} \frac{1}{r} \iint_{Q_r^*(x,t)} |\nabla u|^2 \le \epsilon_3,$$

then (x,t) is a regular point.

For the proofs of Propositions 2.7 and 2.9, Caffarelli, Kohn and Nirenberg used the generalized energy inequality with well chosen test functions ϕ; a simpler proof is given in [18]. Then, Theorem 2.6 is deduced from these results by contradiction using a covering lemma and the usual energy estimates.

2.2. Sketch of the Proofs of Theorems 2.4 and 2.6

Theorem 2.6 is a consequence of Proposition 2.9. The argument is explained below.

Consider first the proof of the fact that S has Hausdorff dimension less than or equal to 5/3, that is, Theorem 2.4. Using Corollary 2.8 and a covering lemma, we can easily see that, for each $\delta > 0$, S can be covered by a family of parabolic cylinders $\{Q_{r_i}^*(x_i,t_i)\}$ such that $r_i < \delta$, the $Q_{r_i/5}^*(x_i,t_i)$ are mutually disjoint and

$$\frac{1}{r^2} \iint_{Q_{r_i/5}^*(x_i,t_i)} \left(|u|^3 + |u||p| \right) + r^{-13/4} \int_{t_i - 7r_i^2/8}^{t_i + r_i^2/8} \left(\int_{|x-x_i|<r_i} |p| \, \mathrm{d}y \right)^{5/4} \mathrm{d}\tau > C\epsilon_1 \tag{6}$$

for all i. Using Hölder's inequality, we deduce that

$$r_i^{-5/3} \iint_{Q_{r_i/5}^*(x_i,t_i)} \left(|u|^{10/3} + |p|^{5/3} \right) \ge C(\epsilon_1)$$

and, therefore,

$$\sum r_i^{5/3} \le \iint_{\cup Q_{r_i/5}^*(x_i,t_i)} \left(|u|^{10/3} + |p|^{5/3} \right) \le C.$$

Taking $\delta \to 0$, we find that $\mathcal{P}^{5/3}(S) = 0$, whence we see in particular that the Hausdorff dimension of S is at most 5/3.

To show that $\mathcal{P}^1(S) = 0$ by a similar method, instead of the integral of $\left(|u|^{10/3} + |p|^{5/3} \right)$, we need a global quantity of dimension 1. This is furnished by Proposition 2.9. Indeed, this result allows to replace Equation (6) by

$$\frac{1}{r_i} \iint_{Q_{r_i/5}^*(x_i,t_i)} |\nabla u|^2 > C(\epsilon_3) \tag{7}$$

and, this way, we are led to the estimate

$$\sum r_i \le C \iint_{\cup Q^*_{r_i/5}(x_i,t_i)} |\nabla u|^2,$$

whence we conclude that $\mathcal{P}^1(S) = 0$.

It is natural to ask if we can get a better estimate of the dimension of S. In other words, can we find $k < 1$ such that $\mathcal{P}^k(S) = 0$? Unfortunately, this question has not been answered up to now. Actually, the answer does not seem simple and is related to the possibility of demonstrating an additional estimate of the (suitable) weak solutions of order less than 1.

It is important to note that the assumption $f \in L^q(Q)^3$ with $q > 5/2$ is mainly needed to prove Proposition 2.7. On the other hand, note that, in Theorem 2.6, Caffarelli, Kohn and Nirenberg chose to estimate the measure $\mathcal{P}^1$ of the set S, instead of the standard measure $\mathcal{H}^1$. Both definitions are special cases of a construction made by Carathéodory that is detailed in [26].

The argument used by Caffarelli, Kohn and Nirenberg is valid for any suitable solution. In the Appendix of [2], they prove the existence of such a solution. Thus, the following holds:

Theorem 2.10 Suppose that $u_0 \in V$, $f \in L^q(Q)^3$ with $q > 5/2$ and $\nabla \cdot f = 0$ in Q. Then, there exists at least one suitable weak solution (u, p) to the Navier-Stokes equations in Q satisfying

$u(t) \to u_0$ weakly in H as $t \to 0$.

In addition, one has:

$$\begin{aligned} &\int_{\Omega\times\{t\}} |u|^2 \phi + 2\int_0^t \int_\Omega |\nabla u|^2 \phi \\ &\le \int_\Omega |u_0|^2 \phi(x,0) + \int_0^t \int_\Omega \left(|u|^2 (\phi_t + \Delta\phi) + \left(|u|^2 + 2p\right) u \cdot \nabla\phi + 2(u \cdot f)\phi \right) \end{aligned} \quad (8)$$

for all functions $\phi \in \mathcal{D}(\Omega \times [0,T])$ with $\phi \ge 0$ and $\phi = 0$ near $\partial\Omega \times (0,T)$.

2.3. On the Existence of Suitable Weak Solutions

The existence of a suitable weak solution to Equation (2) is established in [2] by introducing a family of linear approximated problems and checking that the generalized energy inequalities are satisfied in the limit. A second proof is given in [3], using Faedo-Galerkin approximations. In both cases, the main difficult point is passing to the limit in the term $pu \cdot \nabla\phi$ in the right-hand side of the inequality. This requires nontrivial estimates on the pressure. In particular, Guermond [3] is able to achieve by reproducing for the discrete pressure some a priori estimates similar to the estimates of Sohr and Von Wahl in [27].

3. Some Convergence Results

3.1. The Convergence of the Semi-Approximate Problems

In this section, we will give a new proof of Theorem 2.10. To do this, we will apply the semi-implicit Euler scheme to produce a family of approximations to the Navier-Stokes problem Equation (2). We will see that, at least for a subsequence, we have convergence to a suitable weak solution.

The scheme is the following. We take N large enough (the number of time steps) and we define the time step size $\tau := T/N$, the instants $t^m := m\tau$ and the approximations

$$f^m := \frac{1}{\tau}\int_{t^{m-1}}^{t^m} f(x,t)\,\mathrm{d}t,\quad u^m \approx u(\cdot,t^m) \text{ and } p^m \approx p(\cdot,t^m), \text{ with } u^0 = u_0 \text{ and}$$

$$\begin{cases} \dfrac{u^{m+1}-u^m}{\tau} + (u^m\cdot\nabla)u^{m+1} - \Delta u^{m+1} + \nabla p^{m+1} = f^{m+1}, x\in\Omega \\ \nabla\cdot u^{m+1} = 0, x\in\Omega; \int_\Omega p^{m+1}\mathrm{d}x = 0, \\ u^{m+1} = 0, x\in\partial\Omega, \end{cases} \tag{9}$$

for $m = 0,1,\cdots,N-1$.

First of all, let us check that the u^m are well defined:

Lemma 3.1 The Euler scheme in Equation (9) is well defined. In other words, for every $m \ge 0$, there exists a unique solution (u^{m+1}, p^{m+1}) to Equation (9).

The proof is immediate by induction. We only need to note that for each m Equation (9) is a Dirichlet problem for a linear PDE system that can be written in the form

Find $u\in V$ such that $1/\tau(u,v) + ((w\cdot\nabla)u,v) + (\nabla u,\nabla v) = (g,v)\ \forall v\in V$, Now, let us see that the u^m are uniformly bounded in the L^2 norm. We have:

$$\left(\frac{u^{m+1}-u^m}{\tau}, u^{m+1}\right) + \left((u^m\cdot\nabla)u^{m+1}, u^{m+1}\right) + \left(\nabla u^{m+1}, \nabla u^{m+1}\right) = \left(f^{m+1}, u^{m+1}\right), \tag{10}$$

which can be rewritten in the form

$$\frac{1}{2}\left(\left|u^{m+1}\right|^2 - \left|u^m\right|^2\right) + \frac{1}{2}\left|u^{m+1}-u^m\right|^2 + \tau\left|\nabla u^{m+1}\right|^2 = \tau\left(f^{m+1}, u^{m+1}\right). \tag{11}$$

Using the Cauchy-Schwarz and Young inequalities, we easily get that

$$\frac{1}{2}\left(\left|u^{m+1}\right|^2 - \left|u^m\right|^2\right) + \tau\left|\nabla u^{m+1}\right|^2 \le C\left|f^{m+1}\right|^2 + \frac{\tau}{2}\left|\nabla u^{m+1}\right|^2. \tag{12}$$

Hence,

$$\left|u^{n+1}\right|^2 + \tau\sum_{m=0}^{n}\left|\nabla u^{m+1}\right|^2 \le C\left|f^{m+1}\right|^2 + \left|u^0\right|^2 \le C \tag{13}$$

for all n and, certainly, u^m is uniformly bounded in H.

Using this Euler scheme, we can construct the approximate solutions of the Navier-Stokes system. More precisely, let us introduce the functions u_N and u_N^* as follows:

- $u_N : [0,T] \mapsto V$ is the unique continuous piecewise linear function satisfying
$$u_N(t^m) = u^m \text{ for } m = 0,1,\cdots,N.$$
- $u_N^* : [0,T] \mapsto V$ is the piecewise constant function characterized by
$$u_N^*(t) = u^{m+1} \text{ in } (t^m, t^{m+1}] \text{ for } m = 0,1,\cdots,N-1.$$

In a similar way, we can introduce the approximate pressures p_N^* and forces f_N^* (again piecewise constant). The following holds:

Lemma 3.2 For any N and almost every $t \in (0,T)$, one has

$$\begin{cases} u_{N,t} + \left(u_N^*(t-\tau)\cdot\nabla\right)u_N^* - \Delta u_N^* + \nabla p_N^* = f_N^*, \\ \nabla\cdot u_N^* = 0. \end{cases} \tag{14}$$

We can now present the main result of this section. It is related to the convergence of u_N and u_N^* towards a suitable weak solution to the Navier-Stokes equation:

Theorem 3.3 After eventual extraction of a subsequence, the functions u_N^* converge weakly in $L^2(0,T;V)$, weakly-* in $L^\infty(0,T;H)$, strongly in $L^2(Q)^3$ and a.e. in Q towards a suitable weak solution to Equation (2) as $N \to +\infty$.

For the proof of Theorem 3.3, it will be convenient to recall the following well known lemma (for instance, see the proof in [19]):

Lemma 3.4 Let u be a function satisfying $u \in L^2(0,T;V)$ and $u_t \in L^2(0,T;V')$. Then, u is a.e. equal to a continuous function from $[0,T]$ into H. In addition, the function $t \mapsto |u(t)|^2$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|^2 = 2\langle u_t(t), u(t)\rangle \text{ a.e. in } (0,T),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product in $V'\times V$.

Proof of Theorem 3.3:

Let us first try to find the spaces where the functions u_N^*, u_N and p_N^* are uniformly bounded. This is classical and very well known, but we will give the details for completeness.

Consider the Equation (11). Let us fix N and n with $0 \le n \le N-1$ and let us carry out summation in m, from 0 to n. The following is obtained:

$$\frac{1}{2}\left|u^{n+1}\right|^2 + \frac{1}{2}\sum_{m=0}^{n}\left|u^{m+1}-u^m\right|^2 + \tau\sum_{m=0}^{n}\left|\nabla u^{m+1}\right|^2$$
$$= \tau\sum_{m=0}^{n}\left(f^{m+1}, u^{m+1}\right) + \frac{1}{2}\left|u^0\right|^2.$$

Obviously, this can also be written in the form

$$\frac{1}{2}\left|u_N^*(t)\right|^2 + \frac{1}{2}\sum_{m=0}^{n}\left|u^{m+1}-u^m\right|^2 + \sum_{m=0}^{n}\int_{t^m}^{t^{m+1}}\left|\nabla u_N^*(t)\right|^2 \mathrm{d}t$$
$$= \sum_{m=0}^{n}\int_{t^m}^{t^{m+1}}\left(f_N^*(t), u_N^*(t)\right)\mathrm{d}t + \frac{1}{2}\left|u^0\right|^2 \quad \forall t \in \left(t^m, t^{m+1}\right].$$

Therefore,

$$\frac{1}{2}\left|u_N^*(t)\right|^2 + \int_0^{t^{n+1}}\left|\nabla u_N^*(t)\right|^2 \mathrm{d}t \le \int_0^{t^{n+1}}\left(f_N^*(t), u_N^*(t)\right)\mathrm{d}t + \frac{1}{2}\left|u^0\right|^2 \tag{15}$$

and, from the Cauchy-Schwarz and Young inequalities, we easily see that

$$\left|u_N^*(t)\right|^2 + \int_0^T\left|\nabla u_N^*(t)\right|^2 \mathrm{d}t \le \left|u^0\right|^2 + C\left\|f_N^*\right\|_{L^2(Q)}^2 \quad \forall t \in [0,T].$$

This means that

$$u_N^* \text{ is uniformly bounded in } L^2(0,T;V) \text{ and } L^\infty(0,T;H). \tag{16}$$

On the other hand, it can also be deduced from Equation (9) that

$$\int_0^T \left|u_N(t)-u_N^*(t)\right|^2 \mathrm{d}t \le \tau \sum_{m=0}^{N-1}\left|u^{m+1}-u^m\right|^2 \le C\tau,$$

whence $\left\|u_N^* - u_N\right\|_{L^2(Q)}^2 \le C\tau$.

To estimate u_N, we use its definition and the fact that, for any $t\in\left(t^m,t^{m+1}\right)$,

$$\left|u_N(t)\right| \le \left|u^m\right| + \left|u^{m+1}\right| \quad \text{and} \quad \left|\nabla u_N(t)\right| \le \left|\nabla u^m\right| + \left|\nabla u^{m+1}\right|.$$

Accordingly, we also have that

$$u_N \text{ is uniformly bounded in } L^2(0,T;V) \text{ and } L^\infty(0,T;H). \tag{17}$$

Now, from classical interpolation results, we deduce that u_N^* and u_N is uniformly bounded in

$$L^r\left(0,T;L^{\frac{6r}{3r-4}}(\Omega)^3\right) \quad \forall r\in[2,+\infty]. \tag{18}$$

It is well known that the estimates of Equation (17) and Equation (18) allow us to prove that the u_N belong to and are uniformly bounded in the Sobolev spaces of fractional order $H^\gamma(0,T;H)$ for $0<\gamma<1/4$; see for example [19]. Therefore, as a consequence of Aubin-Lions' Theorem, the u_N belong to a compact set of $L^2(Q)$.

As a consequence, at least for a subsequence (again indexed by *N*), we must have:

$$\begin{cases} u_N \to u \text{ weakly in } L^2(0,T;V) \text{ and weakly-}* \text{ in } L^\infty(0,T;H), \\ u_N \to u \text{ strongly in } L^2(Q)^3 \text{ and a.e. in } Q. \end{cases} \tag{19}$$

This is enough to pass to the limit in Equation (14) and deduce that u is a weak solution of Equation (2). Note that it can also be assumed that

$$u_N \to u \quad \text{strongly in } L^r\left(0,T;L^q(\Omega)^3\right) \text{ for all } 2<r<+\infty,\ 1\le q<6r/(3r-4). \tag{20}$$

To show that u is suitable, we have to give new estimates. To this purpose, we will use some regularity results that, as those in [3], play the role of the Sohr and Wahl's estimates in [27].

For $0<s<1$, the space $H^s(\Omega):=\left[H^1(\Omega),L^2(\Omega)\right]_s$ can be defined by the method of real interpolation between $H^1(\Omega)$ and $L^2(\Omega)$, *i.e.* the so-called K-method of Lions and Peetre [28]; see also [29] and [30]. We will denote by H_0^s the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$. For any $s<0$, the space $H^{-s}(\Omega)$ and the corresponding norm $\|\cdot\|_{H^{-s}}$ are defined by duality and, in particular,

$$\|v\|_{H^{-s}} := \sup_{w\in\mathcal{D}(\Omega)\setminus\{0\}} \frac{(v,w)}{w_{H^s}} \quad \forall v\in L^2(\Omega).$$

We will look for a uniform estimate of $u_{N,t}$ in a space of the form $L^a\left(0,T;H^{-\sigma}(\Omega)^3\right)$. This way, by applying De-Rham's Lemma (see [31]), we will get a bound of p_N^* in $L^a\left(0,T;H^{1-\sigma}(\Omega)^3\right)$ and we will be able to take limits in the generalized energy inequality.

Note that, for all *m*, one has $u^m = w^m + z^m$, where the w^m and the z^m are respectively given by

$$\begin{cases} \frac{1}{\tau}\left(w^{m+1} - w^m\right) + Aw^{m+1} = 0, \\ w^0 = u^0 \end{cases} \tag{21}$$

and

$$\begin{cases} \frac{1}{\tau}\left(z^{m+1} - z^m\right) + Az^{m+1} = F^{m+1}, \\ z^0 = 0, \end{cases} \tag{22}$$

where $F^{m+1} = f^{m+1} - \left(u^m \cdot \nabla\right)u^{m+1}$ and A is the Stokes operator. Recall that $A : D(A) \subset H \mapsto H$, with

$$D(A) = H^2(\Omega)^3 \cap V, Av = P(-\Delta v) \;\; \forall v \in D(A)$$

(here, $P : L^2(\Omega)^3 \mapsto H$ is the orthogonal projector). Also, recall that there exists an orthogonal basis of V formed by eigenfunctions ξ_j,

$$A\xi_j = \lambda_j \xi_j, \xi_j \in V, \left|\xi_j\right| = 1, \lambda_j \nearrow \infty$$

and

$$D\left(A^r\right) = \left\{ v \in H : \sum_{j \geq 1} \lambda_j^{2r} \left|\left(v, \xi_j\right)\right|^2 < +\infty \right\}$$

for all $r \geq 0$.

In the sequel, we will consider the functions w_N, w_N^*, z_N and z_N^*, whose definitions can be obtained from the z^m and the w^m in a way similar to u_N and u_N^*.

First, note that

$$w^m = \left(Id. + \tau A\right)^{-m} u_0 \quad \forall m = 0, 1, \cdots, N,$$

whence

$$\begin{aligned} \left\|w_{N,t}\right\|_{L^2(Q)}^2 &= \tau \sum_{m=0}^{N-1} \left| A\left(Id. + \tau A\right)^{-(m+1)} u_0 \right|^2 \\ &= \tau \sum_{m=0}^{N-1} \sum_{j \geq 1} \frac{\lambda_j^2}{\left(1 + \tau\lambda_j\right)^{2(m+1)}} \left|\left(u_0, \xi_j\right)\right|^2 \\ &= \tau \sum_{j \geq 1} \left(\sum_{m=0}^{N-1} \frac{1}{\left(1 + \tau\lambda_j\right)^{2(m+1)}} \right) \lambda_j^2 \left|\left(u_0, \xi_j\right)\right|^2 \\ &\leq \sum_{j \geq 1} \frac{\lambda_j^2}{\left(1 + \tau\lambda_j\right)^{2(m+1)}} \left|\left(u_0, \xi_j\right)\right|^2 = \left|u_0\right|^2. \end{aligned} \tag{23}$$

Therefore,

$$w_{N,t} \text{ and } Aw_N^* \text{ are uniformly bounded in } L^2(Q)^3. \tag{24}$$

Let us now see what can be said of $z_{N,t}$ and Az_N^*. For all $m \geq 1$, we have

$$z^m = \sum_{l=1}^{m} (Id. + \tau A)^{-(m+1-l)} F^l$$

Let $s, \sigma \in (0,1)$ be such that with $\sigma > s$. Then

$$\left\| Az^{m+1} \right\|_{H^{-\sigma}} \le \tau \sum_{l=1}^{m+1} \left\| A(Id. + \tau A)^{-(m+2-l)} \right\|_{\mathcal{L}(H^{-s};H^{-\sigma})} \left\| F^l \right\|_{H^{-s}} = \tau \sum_{l=1}^{m} a_{m-l} b_l,$$

where the a_n and the b_l are given by

$$a_n = \left\| A(Id. + \tau A)^{-(m+2-l)} \right\|_{\mathcal{L}(H^{-s};H^{-\sigma})}, b_l = \left\| F^l \right\|_{H^{-s}}.$$

We will apply the following result, that must be viewed as a discrete version of the well known Young inequality for convolution products:

Lemma 3.5 Let us assume that $k \ge 1$, $a \in l^p$ and $b \in l^q$. Then, if $r \in [1, +\infty]$ and

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

one has

$$\left(\sum_{n=1}^{k} \left| \sum_{l=1}^{n} a_{n-l} b_l \right|^r \right)^{1/r} \le \left(\sum_{n=1}^{k} |a_n|^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{k} |b_n|^q \right)^{1/q} \tag{25}$$

for all $k \ge 1$.

The proof of this result can be found in [32]. Using Lemma 2.5 with $r = a$, $p = 1$ and $q = a$, we find that

$$\left(\tau \sum_{n=1}^{N} \left\| Az^{n+1} \right\|_{H^{-\sigma}}^a \right)^{1/a} \le \left(\tau^{1+a} \sum_{n=1}^{N} \left| \sum_{l=1}^{n} a_{n-l} b_l \right|^a \right)^{\frac{1}{a}} \le \left(\sum_{n=1}^{N} a_n \right) \left(\sum_{n=1}^{N} b_n^a \right)^{1/a}.$$

From the estimates in Equation (16) already obtained for u_N^*, it is immediate that, for any $a \in [1,2]$, F_N^* is uniformly bounded in $L^a\left(0,T;L^{3a/(4a-2)}(\Omega)^3\right)$ and, consequently, also in $L^a\left(0,T;H^{-(5a-4)/(2a)}(\Omega)^3\right)$. Thus, choosing a in $[1,2]$ and taking $s = (5a-4)/(2a)$, we get:

$$\left\| F_N^* \right\|_{L^a(0,T;H^{-s})} = \left(\tau \sum_{m=1}^{N} b_m^a \right)^{1/a} \le C(a).$$

On the other hand, for any smooth *z*, one has

$$\left\| A(Id. + \tau A)^{-(n+2)} z \right\|_{H^{-\sigma}}^2 = \sum_{j \ge 1} \lambda_j^{-\sigma} \frac{\lambda_j^2}{(1 + \tau \lambda_j)^{2(n+2)}} \left| z, \xi_j \right|^2 \le \left[\sup_j \frac{\lambda_j^{2(1-\epsilon)}}{(1 + \tau \lambda_j)^{2(n+2)}} \right] \left\| z \right\|_{H^{-s}}^2,$$

where $\epsilon = (\sigma - s)/2$. Therefore, recalling the definition of the a_n, we deduce that

$$a_n \le \frac{C(\epsilon)}{(n\tau)^{1-\epsilon}}, \quad \tau \sum_{n=1}^{m} a_n \le C(\epsilon) \int_0^T \frac{\mathrm{d}s}{s^{1-\epsilon}} \le C(\epsilon)$$

and, finally,

$$\left\| Az_N^* \right\|_{L^a(0,T;H^{-\sigma})} \le C \left(\sum_{m=1}^{N} \tau a_m \right) \left(\tau \sum_{m=1}^{N} \left\| F^m \right\|_{H^{-s}}^a \right)^{\frac{1}{a}} \le \left\| CF_N^* \right\|_{L^a(0,T;H^{-s})} \tag{26}$$

Note that this estimate is valid for all $a \in [1,2]$, with $s = (5a-4)/(2a)$ and $\sigma \in (0,1)$, $\sigma > s$.

Obviously, the same estimate is valid for $\|z_{N,t}\|_{L^a(0,T;H^{-\sigma})}$. This proves that

$$
\begin{aligned}
& z_{N,t} \text{ and } Az_N^* \text{ are uniformly bounded in } L^a\left(0,T;H^{-\sigma}(\Omega)^3\right) \\
& \forall a \in [0,1], \forall \sigma > s = (5a-4)/(2a).
\end{aligned} \tag{27}
$$

In view of Equation (24) and Equation (27) and recalling that $u_N^* = w_N^* + z_N^*$, it follows that Au_N^* and $u_{N,t}^*$ are also uniformly bounded in $L^a\left(0,T;H^{-\sigma}(\Omega)^3\right)$.

Now, from De-Rham's Lemma [31], we see that p_N^* is uniformly bounded in $L^a\left(0,T;H^{1-\sigma}(\Omega)\right)$, which is continuously embedded in $L^a\left(0,T;L^{6/(1+2\sigma)}(\Omega)\right)$. In particular, for $a = 5/3$, we have $s = 13/10$ and we can take σ as close as desired to s, which gives $6/(1+2\sigma)$ as close as desired to 5/3.

As a consequence of these estimates, after extracting a new sequence (if this is needed), we see that

$$
p_N^* \text{ converges weakly in } L^{5/3}\left(0,T;L^\beta(\Omega)\right) \quad \forall \beta < 5/3. \tag{28}
$$

Let us check that the local energy inequality holds for u and p.

If we multiply the Equation (14) by the function $u_N^*\phi$, where $\phi \in C_0^\infty(\Omega \times [0,T])$ is non-negative and we integrate in space, we have:

$$
\begin{aligned}
& \int_\Omega u_{N,t} \cdot u_N^*\phi + \int_\Omega \left(u_N^*(t-\tau)\cdot\nabla\right)\left|u_N^*\right|^2 \phi + \int_\Omega \left(-\Delta u_N^*\right)\cdot u_N^*\phi + \int_\Omega \nabla p_N^* \cdot u_N^*\phi \\
& = \int_\Omega f_N^* \cdot u_N^*\phi.
\end{aligned} \tag{29}
$$

If $t \in [0,T]$, there exists n such that $t \in \left(t^n, t^{n+1}\right]$ and then, using Lemma 3.4, one has:

$$
\begin{aligned}
\int_\Omega u_{N,t} \cdot u_N^*\phi &= \int_\Omega u_{N,t} \cdot u_N\phi + \int_\Omega u_{N,t} \cdot \left(u_N^* - u_N\right)\phi \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_N|^2 \phi - \frac{1}{2}\int_\Omega |u_N|^2 \phi_t + \int_\Omega u_{N,t}\cdot\left(u_N^* - u_N\right)\phi.
\end{aligned}
$$

Note moreover that $\int_\Omega u_{N,t} \cdot \left(u_N^* - u_N\right)\phi \ge 0$, because $u_N^* - u_N$ is by definition equal to $\left(t^{n+1} - t\right)u_{N,t}$.

- Also,

$$
\int_\Omega \left(u_N^*(t-\tau)\cdot\nabla\right)\left|u_N^*\right|^2 \phi = -\frac{1}{2}\int_\Omega u_N^*(t-\tau)\left|u_N^*\right|^2 \cdot \nabla\phi,
$$

- On the other hand,

$$
\int_\Omega \left(-\Delta u_N^*\right)\cdot u_N^*\phi = \int_\Omega \left|\nabla u_N^*\right|^2 \phi - \frac{1}{2}\int_\Omega \left|u_N^*\right|^2 \Delta\phi.
$$

- Finally,

$$
\int_\Omega \nabla p_N^* \cdot u_N^*\phi = -\int_\Omega p_N^* \cdot u_N^* \nabla\phi.
$$

Consequently, we see from Equation (29) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_N|^2\phi + \int_\Omega |\nabla u_N^*|^2\phi \le \frac{1}{2}\int_\Omega |u_0|^2\phi + \iint_{\Omega\times(0,t)} |u_N|^2\phi_t$$
$$+\iint_{\Omega\times(0,t)}\left[\left(\frac{1}{2}u_N^*(t-\tau)|u_N^*|^2 + p_N^* u_N^*\right)\cdot\nabla\phi + \frac{1}{2}|u_N^*|^2\Delta\phi + f_N^*\cdot u_N^*\phi\right]$$

If we integrate in time, we find that

$$\int_\Omega |u_N|^2\phi + 2\iint_{\Omega\times(0,t)} |\nabla u_N^*|^2\phi \le \int_\Omega |u_0|^2\phi + \iint_{\Omega\times(0,t)} |u_N|^2\phi_t$$
$$+\iint_{\Omega\times(0,t)}\left[\left(u_N^*(t-\tau)|u_N^*|^2 + 2p_N^* u_N^*\right)\cdot\nabla\phi + |u_N^*|^2\Delta\phi + 2f_N^*\cdot u_N^*\phi\right]$$

Thanks to the energy estimates Equation (16), we can take the lower limit in the left-hand side. On the other hand, thanks to Equation (19), Equation (20) and Equation (28), we can take limits in all the terms in the right; for example, since u_N^* converges strongly in $L^{5/2}\left(0,T;L^{19/5}(\Omega)^3\right)$ and p_N^* converges weakly in $L^{5/3}\left(0,T;L^{14/9}(\Omega)\right)$, we see that $p_N^* u_N^*\cdot\nabla\phi$ converges weakly in $L^1(Q)$ towards $pu\cdot\nabla\phi$.

The final result is that

$$2\iint_{\Omega\times(0,t)}|\nabla u|^2\phi$$
$$\le \int_\Omega |u_0|^2\phi + \iint_{\Omega\times(0,t)}\left(|u|^2(\phi_t+\Delta\phi) + \left(|u|^2+2p\right)u\cdot\nabla\phi + 2(u\cdot f)\phi\right), \tag{30}$$

as desired.

3.2. The Convergence of the Fully Discretized Problems

In this section, we will argue as in [3] and we will check that the approximate solutions obtained via the semi-implicit Euler discrete scheme, used together with an appropriate approximation in space, converge to a suitable solution to Equation (2).

As before, let us introduce N, $\tau := T/N$ and the $t^m := m\tau$. We will also consider two families of finite dimensional spaces $\{X_h\}_{h>0}$ and $\{P_h\}_{h>0}$ with the $X_h \subset H_0^1(\Omega)^3$ and the $P_h \subset L^2(\Omega)$ such that

$$\begin{cases} \inf_{v_h\in X_h}\|v-v_h\|_{H^1} \to 0 \quad \forall v\in H_0^1(\Omega)^3, \\ \inf_{q_h\in P_h}\|q-q_h\|_{L^2} \to 0 \quad \forall q\in L^2(\Omega), \end{cases} \tag{31}$$

and the (X_h, P_h) are uniformly compatible, in the sense that there exists a constant $\mu > 0$ independent of h such that the following *inf-sup* conditions are satisfied:

$$\inf_{q_h\in P_h\setminus\{0\}}\sup_{v_h\in X_h\setminus\{0\}}\frac{(\nabla q_h, v_h)}{\|v_h\|_{H^{-1-s}}\|q_h\|_{H^s}} \ge \mu. \tag{32}$$

Now, we consider the approximations

$$u_h^m = u(\cdot,t^m)\in X_h,\ p_h^m = p(\cdot,t^m)\in P_h,$$

with $u_h^0 = u_{0h}$ (the orthogonal projection of u_0 on X_h) and

$$\begin{cases} \left(\dfrac{u_h^{m+1}-u_h^m}{\tau}, v_h\right)+\left(\left(u_h^m\cdot\nabla\right)u_h^{m+1}, v_h\right)+\left(\nabla u_h^{m+1},\nabla v_h\right)+\left(\nabla p_h^{m+1}, v_h\right) \\ =\left(f_h^{m+1}, v_h\right), \forall v_h\in X_h, \\ \left(q_h,\nabla\cdot u_h^{m+1}\right)=0, \\ \left(u_h^{m+1}, p_h^{m+1}\right)\in\left(X_h, P_h\right), \end{cases} \tag{33}$$

for $m=0,1,\cdots,N-1$. The following result, which is a consequence of Equation (32), gives coherence to our scheme:

As before, the u_h^m and p_h^m serve to construct approximate solutions to the Navier-Stokes system. Thus, we define the functions $u_{N,h}, u_{N,h}^*, p_{N,h}^*$ etc. similarly to u_N, u_N^*, p_N^* etc. The main result of this section is the following:

Theorem 3.7 After eventual extraction of a subsequence, the functions $u_{N,h}^*$ converge weakly in $L^2(0,T;V)$, weakly-* in $L^\infty(0,T;H)$, strongly in $L^2(Q)^3$ and a.e. in Q towards a suitable weak solution to Equation (2) as $N\to+\infty$ and $h\to 0$.

Sketch of the proof:

Arguing as in the proof of the Theorem 3.3, it can be seen that the $u_{N,h}$ and the $u_{N,h}^*$ are uniformly bounded in $L^2(0,T;V)$ and $L^\infty(0,T;H)$ and, furthermore, $\left\|u_{N,h}-u_{N,h}^*\right\|_{L^2(Q)}^2\le C\tau$. As in [19], we can also prove that the $u_{N,h}$ are uniformly bounded in $H^\gamma(0,T;H)$ for any $\gamma\in(0,1/4)$. Consequently, at least for a subsequence (still indexed with N and h), one has:

$$\begin{cases} u_{N,h}\to u \text{ weakly in } L^2(0,T;V) \text{ and weakly-}* \text{ in } L^\infty(0,T;H), \\ u_{N,h}\to u \text{ strongly in } L^2(Q)^3 \text{ and a.e. in } Q. \end{cases} \tag{34}$$

Also,

$$u_{N,h}\to u \quad \text{strongly in } L^r\left(0,T;L^q(\Omega)^3\right) \text{ for all } 2<r<+\infty,\ 1\le q<6r/(3r-4). \tag{35}$$

As before, this is enough to pass to the limit and deduce that u is a weak solution of Equation (2).

In order to prove that u is suitable, we can argue as in Guermond [3]. Here, we need the spaces $\tilde{H}_0^s(\Omega):=\left[L^2(\Omega),H_0^1(\Omega)\right]_s$ for $s\in(0,1)$ and their dual spaces $\tilde{H}_0^{-s}(\Omega)$.

The following estimates are established in [3]:

- For any $\alpha\in[1/4,1/2]$ and any $\delta<\bar{\delta}(\alpha)=2(1+\alpha)/5$, one has
$$\left\|u_{N,h,t}\right\|_{H^{\delta-1}(0,T;\tilde{H}^{-\alpha})}+\left\|u_{N,h}^*\right\|_{H^\delta(0,T;\tilde{H}^{-\alpha})}\le C(\alpha),$$
- For any and $s\in[1/2,7/10]$ any, one also has
$$\left\|u_{N,h,t}\right\|_{H^{-r}(0,T;\tilde{H}^{-s})}+\left\|p_{N,h}^*\right\|_{H^{-r}(0,T;\tilde{H}^{1-s})}\le C(s),$$

As a consequence, it can be assumed that the $p_{N,h}^*$ converge weakly (for instance) in $H^{-r}\left(0,T;H^{3/8}(\Omega)\right)$ for all $r>7/16$ and the $u_{N,h}^*$ converge strongly in $H^{\delta}\left(0,T;\tilde{H}^{-\alpha}(\Omega)^3\right)$ for all $\alpha<3/8$ and $\delta<11/20$. This is sufficient to ensure that $p^*_{N,h}u^*_{N,h}\cdot\nabla\phi$ converges weakly in $L^1(Q)^3$ towards $pu\cdot\nabla\phi$.

Hence, arguing as in the final part of the proof of Theorem 3.3, it is not difficult to check that the limit (u,p) of the $\left(u^*_{N,h},p^*_{N,h}\right)$ is a suitable weak solution to Equation (2).

This ends the proof.

4. Some Additional Comments and Questions

4.1. The Same Results Hold for the Boussinesq System

The Boussinesq system is the following:

$$\begin{cases} u_t-\Delta u+(u\cdot\nabla)u+\nabla p=f+\theta k,(x,t)\in Q,\\ \nabla\cdot u=0,(x,t)\in Q,\\ \theta_t+u\cdot\nabla\theta-\Delta\theta=g,(x,t)\in Q,\\ u(x,t)=0,\theta(x,t)=0,(x,t)\in\Sigma,\\ u(x,0)=u_0(x),\theta(x,0)=\theta_0(x),x\in\Omega. \end{cases}\tag{36}$$

We assume here that

$$u_0\in V,\theta_0\in H^1_0(\Omega),f\in L^2\left(0,T;H^{-1}(\Omega)^3\right),k\in\mathbb{R}^3\ \text{ and }\ g\in L^2\left(0,T;L^2(\Omega)\right).\tag{37}$$

As in [4], we can speak of weak solutions to Equation (36) and, also, of suitable weak solutions to the previous equations in any set of the form $D=G\times(a,b)$, with $G\subset\mathbb{R}^3$ a connected open set.

The results in Section 3 can be extended to this framework. Thus, we can for instance consider the semi-implicit Euler scheme

$$\begin{cases} \dfrac{u^{m+1}-u^m}{\tau}+\left(u^m\cdot\nabla\right)u^{m+1}-\Delta u^{m+1}+\nabla p^{m+1}=f^{m+1}+\theta^{m+1}k,\\ \nabla\cdot u^{m+1}=0,\\ \dfrac{\theta^{m+1}-\theta^m}{\tau}+u^m\cdot\nabla\theta^{m+1}-\Delta\theta^{m+1}=g^{m+1} \end{cases}\tag{38}$$

and prove that, at least for a subsequence, the corresponding (u_N,p_N,θ_N) and $\left(u^*_N,p^*_N,\theta^*_N\right)$ converge, in an appropiate sense, to a suitable weak solution (u,p,θ).

4.2. Possible Extensions to Other Systems

It would be interesting to prove similar results to Theorems 3.3 and 3.7 for the solutions to the variable-density Navier-Stokes equations

$$\begin{cases} \rho_t+\nabla\cdot(\rho u)=0,(x,t)\in Q,\\ \rho\left(u_t+(u\cdot\nabla)u\right)-\Delta u+\nabla p=\rho f,(x,t)\in Q\\ \nabla\cdot u=0,(x,t)\in Q,\\ u(x,t)=0,(x,t)\in\Sigma,\\ u(x,0)=u_0(x),\rho(x,0)=\rho_0(x),x\in\Omega. \end{cases}\tag{39}$$

However, this is not clear at present. Note that the "reasonable" definition of a suitable weak solution should involve the following property: for any $\phi \in D(Q)$ with $\phi \geq 0$,

$$2\iint_D |\nabla u|^2 \phi \leq \iint_D \left(|u|^2 (\rho\phi_t + \Delta\phi) + \left(\rho|u|^2 + 2p\right)(u \cdot \nabla\phi) + 2\rho(u \cdot f)\phi \right).$$

But, unfortunately, the apparent lack of regularity of p makes it difficult to prove this.

4.3. Extensions to Other Approximation Schemes for the Navier-Stokes Equations

As we already said, Theorems 3.3 and 3.7 can be adapted to many other time approximation schemes. Among then, let us simply recall the following:

- Crank-Nicholson scheme:

$$\frac{u^{m+1} - u^m}{\tau} + \left(u^m \cdot \nabla\right)u^{m+1} - \Delta\left(\frac{u^{m+1} + u^m}{2}\right) + \nabla p^{m+1} = f^{m+1}, \nabla \cdot u^{m+1} = 0.$$

- Gear scheme:

$$\frac{3u^{m+1} - 4u^m + u^{m-1}}{2\tau} + \left(u^m \cdot \nabla\right)u^{m+1} - \Delta u^{m+1} + \nabla p^{m+1} = f^{m+1}, \nabla \cdot u^{m+1} = 0.$$

θ-scheme: For α and β such that $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$, we compute $\left(u^{n+\theta}, p^{n+\theta}\right)$, then $u^{n+1-\theta}$ and finally $\left(u^{n+1}, p^{n+1}\right)$ as follows:

$$\frac{u^{n+\theta} - u^n}{\theta\Delta t} - \alpha\nu\Delta u^{n+\theta} + \nabla p^{n+\theta} = f^{n+\theta} + \beta\nu\Delta u^n - \left(u^n \cdot \nabla\right)u^n, \nabla \cdot u^{n+\theta} = 0,$$

$$\frac{u^{n+1-\theta} - u^{n+\theta}}{(1-2\theta)\Delta t} - \beta\nu\Delta u^{n+1-\theta} + \left(u^{n+1-\theta} \cdot \nabla\right)u^{n+1-\theta} = f^{n+\theta} + \alpha\nu\Delta u^{n+\theta} - \nabla p^{n+\theta}.$$

$$\frac{u^{n+1} - u^{n+1-\theta}}{\theta\Delta t} - \alpha\nu\Delta u^{n+1} + \nabla p^{n+1} = f^{n+1} + \beta\nu\Delta u^{n+1-\theta} - \left(u^{n+1-\theta} \cdot \nabla\right)u^{n+1-\theta}, \nabla \cdot u^{n+1} = 0.$$

As we mentioned in Section 1, it would be interesting to establish an analog of Propositions 2.7 and 2.9 for a family of approximated solutions. This should help to detect or discard the occurrence of singular points just observing the results of appropriate numerical experiments.

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References

[1] Scheffer, V. (1977) Hausdorff the Navier-Stokes Equations. *Communications in Mathematical Physics*, **55**, 97-112. https://doi.org/10.1007/BF01626512

[2] Caffarelli, L., Kohn, R. and Nirenberg, L. (1982) Partial Regularity of Suitable Weak Solutions of the Navier-Stokes Equations. *Communications on Pure and Applied Mathematics*, **35**, 771-831. https://doi.org/10.1002/cpa.3160350604

[3] Guermond, J.-L. (2007) Faedo-Galerkin Weak Solutions of the Navier-Stokes Equations with Dirichlet Boundary Conditions Are Suitable. *Journal de Mathématiques Pures et Appliquées*, **88**, 87-106. https://doi.org/10.1016/j.matpur.2007.04.009

[4] Boling, G. and Guangwei, Y. (1996) On the Suitable Weak Solutions for the Cauchy-Problem of the Boussinesq Equations. *Nonlinear Analysis Theory, Methods and Applications*, **26**, 1367-1385.

[5] Seregin, G.A. (2007) On the Local Regularity of Suitable Weak Solutions of the Navier-Stokes Equations. *Uspekhi Matematicheskikh Nauk*, **62**, 149-168.

[6] Vasseur, A.F. (2007) A New Proof of Partial Regularity of Solutions to Navier-Stokes Equations. *Nonlinear Differential Equations and Applications NoDEA*, **14**, 753-785.

[7] De Giorgi, E. (1957) Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat., Vol. 3, 25-43.

[8] Wolf, J. (2015) On the Local Regularity of Suitable Weak Solutions to the Generalized Navier-Stokes Equations. *Annali dell' Universita di Ferrara*, **61**, 149-171. https://doi.org/10.1007/s11565-014-0203-6

[9] Choe, H.J., Wolf, J. and Yang, M. (2018) A New Local Regularity Criterion for Suitable Weak Solutions of the Navier-Stokes Equations in Terms of the Velocity Gradient. *Mathematische Annalen*, **370**, 629-647.

[10] Buckmaster, T. and Vicol, V. (2017) Nonuniqueness of Weak Solutions to the Navier-Stokes Equations.

[11] Dean, E.J. and Glowinski, R. (1993) On Some Finite Element Methods for the Numerical Simulation for Incompressible Viscous Flow. Cambridge University Press, New York, 109-150.

[12] Douglas, J. and Rachford, H.H. (1956) On the Solution of the Heat Conduction Problem in 2 and 3 Space Variables. *Transactions of the American Mathematical Society*, **82**, 421-439. https://doi.org/10.1090/S0002-9947-1956-0084194-4

[13] Glowinski, R. (2013) Numerical Methods for Fluids (Part 3). North-Holland, Vol. IX.

[14] Peaceman, D.H. and Rachford, H.H. (1955) The Numerical Solution of Parabolic and Elliptic Differential Equations. *Journal of the Society for Industrial and Applied Mathematics*, **3**, 28-41. https://doi.org/10.1137/0103003

[15] Da Veiga, H.B. (1985) On the Suitable Weak Solution to the Navier-Stokes Equations in the Whole Space. *Journal de Mathématiques Pures et Appliquées*, **64**, 77-86.

[16] Berselli, L.C. and Spirito, S. (2017) Suitable Weak Solutions to the 3D Navier-Stokes Equations Are Constructed with the Voigt Approximation. *Journal of Differential Equations*, **262**, 3285-3316.

[17] Berselli, L.C. and Spirito, S. (2018) On the Construction of Suitable Weak Solutions to the 3D Navier-Stokes Equations in a Bounded Domain by an Artificial Compressibility Method. *Communications in Contemporary Mathematics*, **20**, Article ID: 1650064.

[18] Lin, F.H. (1998) A New Proof of the Caffarelli-Kohn-Nirenberg Theorem. *Communications on Pure and Applied Mathematics*, **51**, 241-257. https://doi.org/10.1002/(SICI)1097-0312(199803)51:3<241::AID-CPA2>3.0.CO;2-A

[19] Temam, R. (1977) Navier-Stokes Equations. Theory and Numerical Analysis. Studies in Mathematics and Applications, North-Holland Publishing Co., Amsterdam, 2.

[20] Lions, P.-L. (1996) Mathematical Topics in Fluid Mechanics, 1: Incompresible Models. Clarendon Press, Oxford.

[21] Serrin, J. (1962) On the Interior Regularity of Weak Solutions of the Navier-Stokes Equations. *Archive for Rational Mechanics and Analysis*, **9**, 187-195. https://doi.org/10.1007/BF00253344

[22] Wolf, J. (2010) A New Criterion for Partial Regularity of Suitable Weak Solutions to the Navier-Stokes Equations. In: Rannacher, A.S.R., Ed., *Advances in Mathematical Fluid Mechanics*, Springer, Berlin, 613-630.

[23] Neustupa, J. and Penel, P. (1999) Regularity of a Suitable Weak Solution to the Navier-Stokes Equations as a Consequence of Regularity of One Velocity Component. In: *Applied Nonlinear Analysis*, Kluwer/Plenum, New York, 391-402.

[24] Scheffer, V. (1976) Partial Regularity of Solutions to the Navier-Stokes Equations. *Pacific Journal of Mathematics*, **66**, 535-552. https://doi.org/10.2140/pjm.1976.66.535

[25] Scheffer, V. (1980) The Navier-Stokes Equations on a Bounded Domain. *Communications in Mathematical Physics*, **73**, 1-42. https://doi.org/10.1007/BF01942692

[26] Federer, H. (1969) Geometric Measure Theory. Springer-Verlag, New York.

[27] Sohr, H. and Wahl, W. (1986) On the Regularity of the Pressure of Weak Solutions of Navier-Stokes Equations. *Archiv der Mathematik*, **46**, 428-439. https://doi.org/10.1007/BF01210782

[28] Lions, J.-L. and Peetre, J. (1964) Sur une classe d'espaces d'interpolation. *Publications mathématiques de l'IHÉS*, **19**, 5-68. https://doi.org/10.1007/BF02684796

[29] Lions, J.-L. and Magenes, E. (1968) Problèmes aux limites non homogènes et applications. Vol. 1, Dunod, Paris.

[30] Adams, R.A. and Fournier, J.J.F. (2003) Sobolev Spaces. *Journal of Pure and Applied Mathematics*, **140**, 305.

[31] Simon, J. (1990) Non-Homogeneous Viscous Incompressible Fluids: Existence of Velocity, Density and Pressure. *SIAM Journal on Mathematical Analysis*, **21**, 1093-1117. https://doi.org/10.1137/0521061

[32] Galperin, Y. (2014) Young's Convolution Inequalities for Weighted Mixed (Quasi-) Norm Spaces. *Journal of Inequalities and Special Functions*, **5**, 1-12.