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THEORETICAL MECHANICS.
THE THEORY and WORKSHOP.
PART I. KINEMATICS.

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for the students of the specialty
185 «Oil and gas engineering and technology»

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Посібник присвячено викладу основ теоретичної механіки у розділі кінематика. Надані основні теоретичні відомості, представлено прилади розв’язання типових задач, а також варіанти завдань для самостійної роботи.

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C O N T E N T S

INTRODUCTION.

Mechanics may be defined as an area of science that describes and develops the conditions of equilibrium or of the motion of the material bodies under the action of forces. Mechanics can be divided in three large parts, function of the studied object: *mechanics of the non-deformable bodies* (mechanics of the rigid bodies), *mechanics of the deformable bodies* (strength of the materials, elasticity, building analysis) and *fluid mechanics*. *Mechanics of the non-deformable bodies*, or ***theoretical mechanics***, may be divided into other three parts: **kinematics, statics** and **dynamics.**

Kinematics is the part of theoretical mechanics that deals with the motions of bodies without consideration of their masses and the forces that act on them, so kinematics studies the motion from geometrical point of view, namely the pure motion.

Statics is the part of theoretical mechanics that studies a transformation of systems of forces in other simpler systems and of the conditions of equilibrium of the bodies.

Dynamics is the part of theoretical mechanics that studies the motion of bodies considering their masses and forces that act~~s~~ on them. In all these definitions the bodies are considered rigid (non-deformable bodies). It is known that real bodies are deformable under the action of the forces. But these deformations are generally very small and they produce small effects on the equilibrium conditions and the motion. Mechanics is a science of the nature because it deals with the study of the natural phenomena. Many consider mechanics as a science joined to mathematics because it develops its theory based on mathematical proofs. On the other hand, mechanics is not an abstract science or a pure one, it is an applied science.

Theoretical mechanics studies the simplest form of the motion of material bodies, namely the **mechanical motion.** The mechanical motion is defined as the phenomenon in which a body or a part of a body modifies its position with respect to another body considered as reference system.

Theoretical mechanics uses three fundamental notions: **space, time** and **mass.** These three notions are considered independent with respect to each other. They are named fundamental notions because they may be not expressed using other simpler notions and they will form the reference frame used in the study of theoretical mechanics. The notion of **space** is associated with the notion of position. For example the position of a point P may be defined with three lengths measured on three given directions, with respect to a reference point. These three lengths are known under the name of the coordinates of the point P. The notion of space is associated also with the notion of largest of the bodies and the area of them. **The space** in theoretical mechanics is considered to be the real space where natural phenomena are produced and it is considered having the following proprieties: *infinitely large, three dimensional, continuous, homogeneous and isotropic.* The space defined in this way is the Euclidian space with three dimensions that allows to build the differential equations of motion and to obtain the differential computation.

In defining a mechanical phenomenon, generally speaking, it is not enough to use only the notion of space, namely it is not enough to define only the position and size of the bodies. Mechanical phenomena have durations and they are produced in any succession. Combined with these notions (duration and succession), theoretical mechanics considers a fundamental notion **of time** having the following proprieties: *infinitely large, one-dimensional, continuous, homogeneous and irreversible.* The time between two events is named *interval of time* and the limit among two intervals of time is named *instant.*

The notion of **mass** is used to characterize and compare the bodies in the time of mechanical events. The **mass** in theoretical mechanics is the measure of inertia of bodies in translational motion and will represent the quantity of substance of the body, constant in time of the studied phenomenon.

Besides of these fundamental notions, theoretical mechanics uses other characteristic notions, generally used in each part of the mechanics. These notions will be named as basic notions and they will be defined for each part of mechanics. In **Kinematics** the basic notions are: *the velocity* and *the acceleration.* In **Statics** we use three notions: *the force, the moment of the force about a point and the moment of the force about an axis.* In **Dynamics** we use: t*he linear momentum, the angular momentum, the kinetic energy, the work, the potential energy* and *the mechanical energy.*

Theoretical mechanics does not operate with a real object, but with a model of a real object. Through a model or a scheme we understand a representation of the body or a real phenomenon with a certain degree of approximation. But the approximation must be made in a way so that the body or the phenomenon keeps its principal proprieties. To simplify the study of the theoretical mechanics, the material bodies are considered under the form of two models coming from the general model of the **material continuum**: **the rigid body** and **the particle**.

The rigid body, by definition, is the non-deformable material body. This body has the propriety that: the distance among two any points of the body does not change indifferent to the actions of the forces or other bodies about it. This model is accepted in theoretical mechanics because, generally, the deformations of the bodies are very small and they may be neglected without an introduction of substantial errors in the computations or in the final solutions of studied problems.

In the case when the body is very small or the dimensions are not significant in the studied problem, one can use the **particle (the material point)** model**.** The particle is in fact a geometrical point that has mass of the modelled body attached as an attribute.

The rigid bodies models may have different schemes depending on the number of dimensions. Further we are going to implement next three schemes: *material lines (bars)*, *material surfaces (plates)* and *material volumes (blocks)*. *Material lines* or *bars* are rigid bodies in which one dimension (the length) is larger than the other two (width and thickness). These kinds of bodies are reduced to a line representing the locus of the centroids of cross sections. *Material surfaces* or *plates* are bodies in which two dimensions are bigger than the third (the thickness). In this case the body is reduced to a surface representing the median surface of the plate. *Material volumes* or *blocks* are bodies in which three dimensions are comparable. Finally, another classification of the bodies is made function the distribution of the mass by the body. We shall have two kinds of bodies: *homogeneous bodies* in which the mass is uniformly distributed by the entire volume of the bodies, and *non-homogeneous bodies* in which the mass is non-uniformly distributed by the bodies volume.

Historical Review

The laws of theoretical mechanics are formulated through the fruitful labor of many generations of scientists. First presentation of General concepts of mechanics are in the works of Greek philosopher Aristotle (384-322 BC), who considered the solution of practical problems using the lever. The first scientific justification of mechanics appears in the work of the geometer of Syracuse and the mechanics of Archimedes (287-212 BC). He made an attempt to describe axiomatic mechanics (statics), gave a number of scientific generalizations pertaining to the doctrine of equilibrium, center of gravity and hydrostatics (Archimedes ' principle).

The rapid development of mechanics begun with the Renaissance. Outstanding scientists of this era have developed methods of static and laid the foundations of dynamics. The most significant contribution to mechanics was made by Leonardo da Vinci (1452-1519) who studied the trajectory of the body that has been thrown at an angle to the horizon, the movement of the body of an aircraft and the phenomenon of friction and introduced the concept of moment of a force about a point; Steven Simon (1548-1620) gave an axiomatic construction of statics on the basis of the postulates of Archimedes, introduced the concept of the power triangle and proved the theorem of three forces; Nicolaus Copernicus (1473-1543) discovered the heliocentric system of the world; Galileo Galilei (1564-1642) established the basic laws of free fall of bodies, he introduced the notion of non-uniform motion and acceleration of a particle, first formulated the law of inertia, the principle of relativity of classical mechanics and investigated the action of forces on bodies that are moving; Johannes Kepler (1571-1630) discovered the laws of planetary motion; Rene Descartes (1596-1650) closer to his contemporaries approached to correct the formulation of the law of inertia, first introduced the concept of momentum of a material point and explored the question of the addition of an arbitrary number of movement points; Christian Huygens (1629-1695) developed a theory of oscillations of a physical pendulum, determined the center of its oscillation, proved theorems on centrifugal force, experimentally determined the acceleration of gravity, studied the impact of two bodies; Robert Hooke (1635-1703) discovered the law of proportionality between force applied to an elastic body, and strain (Hooke's law), which is the main ratio at the present calculation of dynamics and strength of structures and buildings, and anticipated the law of gravitation of Newton; P. Varignon (1654-1722) - established in its final form the concept of moment of force, conditions of equilibrium of a system of convergent and parallel forces, proved a theorem about the moment of the resultant.

One of the first places in the development of mechanics belongs to Gottfried Leibniz (1646-1716), who developed and applied to problems in mechanics differential and integral calculus, introduced the concept of kinetic energy and came very close to creating the calculus of variations. The establishment of the basic laws of dynamics was completed by the great English mathematician Isaac Newton (1643-1727). In his famous essay "Mathematical foundations of natural philosophy" (1687) he formulated the basic concepts of classical mechanics, axioms and some fundamental theorems of celestial mechanics and law of universal gravitation.

The period of development of mechanics after Newton is largely associated with the name of L. Euler (1707-1783), who most of his life worked at the St. Petersburg Academy of Sciences. L. Euler fully completed the process of mathematical description of particle mechanics, was the founder of solid mechanics and formulated the laws of dynamics for a continuous environment.

Further development of mechanics was associated with the study of motion of a system of material points. The development of this direction was initiated by works of L. D'alembert (1717-1783), who formulated the principle by which formal problems of the dynamics was reduced to problems of statics (D'alembert principle) and L. Lagrange (1736-1813). In his outstanding essay "Analytical mechanics", he formulated the most general principle of statics - the principle of possible displacements, found a general pattern of the general equation of dynamics, and brought differential equations of motion of a mechanical system to a generalized form (the Lagrange equations of first and second kind).

In the future works of prominent mathematicians and engineers P. L. Mopar-Tu (1698-1759), P. S. Laplace (1749-1827), K. F. Gauss (1777-1855), S. Poisson (1781-1840) Hamilton (1805-1865), Jacobi (1804-1851), M. V. Ostrogradskii (1801-1861) completed the mathematical description of mechanics of material points and rigid bodies, that were developed specific to the concepts of analytical mechanics (generalized coordinates, generalized velocity, generalized force) and described mathematical methods of solution of various tasks.

The subsequent development of mechanics is characterized by in-depth study of a number of its sections and the appearance of new. One should note the work of S. M. Kovalevskaya (1850-1891) on the theory of rotation of a heavy rigid body around a fixed point that became the starting point for the applied theory of gyroscopes. A significant contribution to the development of mechanics of non-holonomic systems, which has numerous applications in cybernetics, automated control theory, wave dynamics, was made by D. Gibbs (1839-1903), S. A. Chaplygin (1863-1945) and other scientists. Stability theory of equilibrium and motion, which was closely connected with the problem of accurate instrumentation, created and developed by the works of E. Routh (1831-1907), M. Zhukovsky (1847-1921), A. M. Lyapunov (1857-1918), H. Poincare (1854-1912). The most significant results in the theory of gyroscopes, that are basis of navigation devices, were obtained by L. Foucault (1819-1868), A. M. Krylov (1863-1945), V. Bulgakov (1901-1952) and other mechanics.

1. KINEMATICS OF A PARTICLE

1.1. Introduction to Kinematics

Kinematics is that part of the Theoretical Mechanics that deals with the study of the *mechanical motion* without consideration of the forces and masses of the bodies in motion, namely studies the geometry of the motion. We remind that through mechanical motion we understand the changing of the position of bodies (or parts of bodies) with respect to other bodies considered as reference systems.

The reference system may be fixed or in motion. If the motion of the bodies is performed with respect to a fixed reference system (or that can be considered fixed system) we shall say that the motion is *absolute motion*, but if the motion of the bodies is performed with respect to moving reference system then the motion is called *relative motion*.

In the kinematics one generally has to solve two problems: to determine the position of the particle (or of the body) in each instant of the motion and to know how the particle (or the body) moves. To define the position of the particle we can use the *vector way* (used in theoretical demonstrations generally) and the *scalar way* used in problems, which is considered in certain reference systems. To define how the motion is made we shall introduce two vector notions: *velocity* and *acceleration*.

We should also note that the elements of the absolute motion may be expressed with respect to a moving reference system and the elements of the relative motion with respect to a fixed reference system. We will consider the next reference systems used in the theoretical mechanics: *Cartesian reference system* that will be generally considered fixed one (with three fixed points) and the *Frenet's reference system* that is in motion together with the body (entirely in motion).

In mechanics we deal with three-dimensional Euclidean space in which all dimensions are measured by the methods of Euclidean geometry. The unit of length, by which distance is measured, is the *meter*.

Any motion in space takes place with *time*. Time in mechanics is considered as universal, i.e., as passing simultaneously in all our frames of reference. The unit of time is one second. Time is a continuously varying quantity. In problems of kinematics, time t is taken as an independent variable (the argument). All other variables (distance, velocity, etc.) are regarded as changing with time, i.e., as functions of time t. Time is measured from some initial instant ($t = 0$).

The principles of kinematics are based on the axioms of geometry.

To describe the motion or the law of motion, of a given body (particle) kinematically means to specify the position of that body (particle) relative to a given reference system for any moment of time. One of the main problems of kinematics is that of describing the motion of particles or bodies in terms of mathematical expressions. Hence, we shall commence the investigation of the motion of any object with determining the ways of describing that motion.

The principal problem of kinematics is that of determining all the kinematic characteristics of the motion of a body as a whole or of any of its particles (trajectory, velocity, acceleration, etc.) when the law of motion for the given body is known. In order to solve this problem, we must know either the equations of motion for the given body or for another body kinematically associated with it.

We will consider the section of kinematics in the following structure: *kinematics of the absolute motion of the particle, kinematics of the rigid body and kinematics of the complex motion of the particle.*

1.2. Vector and Scalar Method of Defining of Particle's Position. Trajectory (Path) of a Particle.

At the beginning we consider the solution to the first problem of kinematics with respect to a particle, i.e. definition of the position (location) of a particle in space. In order to determine the location of a particle we assume a *reference system* (*frame of reference*), which will be fixed relative to the origin body. The reference system consists of three parts: origin (reference point); coordinate system; a system of time reading.

If the coordinates of all the points of a body remain constant within a given frame of reference, the body is said to be at *rest* relative to that reference system. If, on the other hand, the coordinates of any points of the body change with time, the body is said to be in motion relative to the given frame of reference. To study the motion of any object we have to introduce at least one reference system.

There are three methods of a particle motion describing: the *vector* method, the c*oordinate* method, the *natural* method. The last two methods are scalar ones. In the first case the *position vector* is used, that in absolute motion is represented with respect to a fixed point.

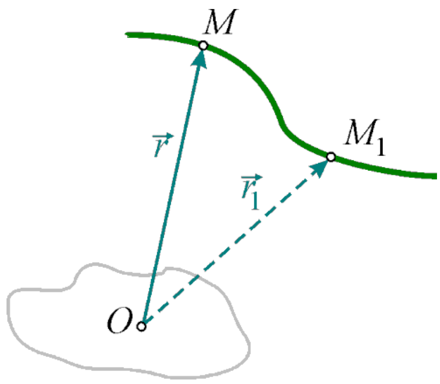


Fig.1.1 Vector Method

Let a particle *M* is moving relative to any *reference point O*. The position of the particle at any instant can be specified by a vector $\vec{r}$ drawn from the origin *O* to the particle *M* (Fig.1.1). When the particle moves, the position vector $\vec{r}$ changes with time both in magnitude and direction. Thus, position vector $\vec{r}$ is a variable vector (a vector function) depending on the argument *t*:

$$\vec{r} = \vec{r}(t) \tag{1.1}$$

Therefore, to describe the motion of a particle by vector method, one needs to set twice differentiable vector function in the form (1.1). In other words, the formula (1.1) is the law of motion of a particle in a vector form.

The continuous curve delineated by a particle moving with respect to a given reference system is called the ***trajectory (path)*** of that particle. If the trajectory is a straight line, the motion is said to be *rectilinear,* if the trajectory is a curve, the motion is *curvilinear*. In other words, trajectory (path) is the geometrical place of sequential positions of a moving particle. We can also say that the trajectory of a particle is the locus of the ends of the position vector.

If we want to express the position of the particle in scalar way with respect to a reference system, for example the Cartesian reference system, we can use three coordinates (three scalar position parameters). These coordinates are functions of time also having the same conditions as the position vector:

The position of a particle with respect to a given reference system *Oxyz* can be specified by its Cartesian coordinates x, y, z (Fig. 1.2).

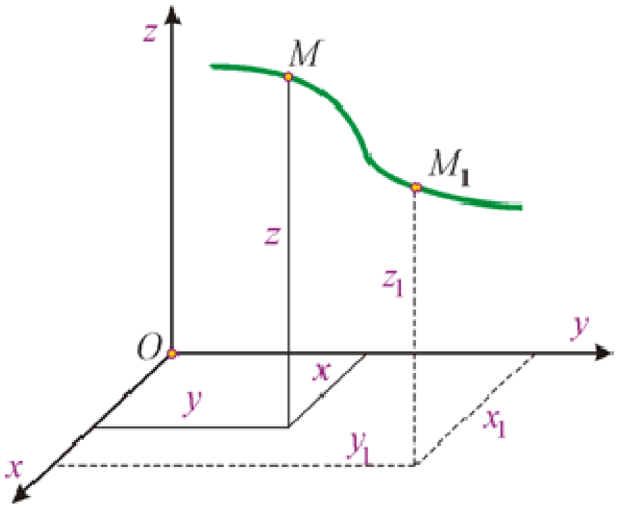


Fig.1.2. Cartesian Method

When motion takes place, the three coordinates will change with time. If we want to know the equation of particle motion, i.e., its location in space at any instant, we must know its coordinates in any moment of functions of time for the three Cartesian coordinates of the particle:

$$\begin{cases} x = f_1(t) \\ y = f_2(t) \\ z = f_3(t) \end{cases} \tag{1.2}$$

Thus, describing of the motion of a particle in the Cartesian method means to set twice differentiable scalar functions of the form (1.2).

Equations (1.2) are the equations of the particle's trajectory in parametric form, where the time t is the parameter. By eliminating time t from the equations of motion we can obtain the equation of the trajectory in the usual form, i.e., in the form of a relation between the particles coordinates.

The relation between the coordinate and vector methods of describing motion can easily be established by introducing *unit vectors i, j, k* directed along the axes *x, y, z* respectively (see Fig. 1.3). We can present the position vector $\vec{r}$ as:

$$\vec{r} = r_x\vec{i} + r_y\vec{j} + r_z\vec{k}\,, \tag{1.3}$$

where *i, j,k are the unit vectors of axes x,y,z.* As the projections of position vector $\vec{r}$ on the coordinate axes are equal to the coordinates of the particle *M*, i.e. $r_x = x$, $r_y = y$, $r_z = z$, we obtain the relationship between the coordinate and vector methods:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}\,. \tag{1.4}$$

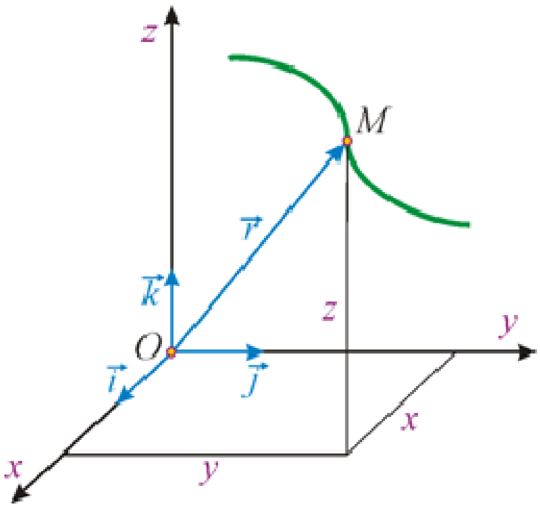


Fig.1.3. Relation between Vector and Cartesian Methods

Let consider another scalar method of describing of the motion of a particle, called *natural* method of describe motion. Natural method can be realized only if the particle trajectory is given. This method assumes that motion is completely described if position of particle on its trajectory is given as function of the time. Let the curve on the slide be the trajectory of the moving particle M (Fig. 1.4).

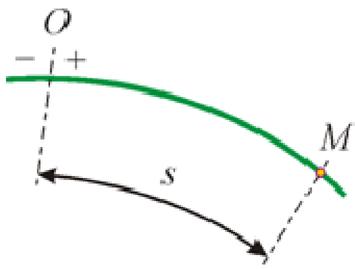


Fig.1.4. Natural Method

Take any fixed point O on the trajectory as the origin of the reference system; now, taking the trajectory as an curvilinear-coordinate axis, assume the positive and negative directions, as it is done with rectangular axes.

The position of the particle M on the trajectory is now specified by a single curvilinear coordinate s of the particle. Coordinate s is equal to the distance of moving point M from some initial point O on the trajectory (from O to M) measured along the arc of the trajectory and taken with the appropriate sign. The displacement of particle M carries it through positions M_1, M_2, ..., i.e., the distance s changes with time. In order to know the position of M on the trajectory at any instant, we must know the relation:

$$s = f(t). \tag{1.5}$$

Thus, in order to use natural method of describing of the motion of a particle if the twice differentiable scalar functions of the form (1.5) is set. In other words, the formulas (1.5) are the law of motion of a particle in a Natural form.

Thus, in order to describe the motion of a particle by the natural method, a problem must state:

1) The *trajectory* of the particle;

2) The *reference point O* on the trajectory (the position of the particle on the trajectory at the moment $t = 0$);

3) The *positive* and *negative* directions of curvilinear coordinate reading;

4) The equation of the particle's motion along the *trajectory* in the form (1.5).

Let's note that s in Eq. (1.5) denotes the *position* of the moving particle, but this coordinate does not indicate the *path* (*distance*) traversed by it.

For example, if the particle travels from O to M_1 and then reverses its motion to point M, its coordinate at that moment is $S = \overset{\frown}{OM}$, but the distance it traveled is $\overset{\frown}{OM_1} + \overset{\frown}{M_1M}$, i.e., not s.

The natural method of describing motion offers a very clear picture, but a particle's trajectory may not be known, that is why the coordinate method is employed more frequently.

Next, we consider the connection that we can establish between the natural coordinate s and Cartesian coordinates. Infinitesimal segment of a trajectory of length ds is connected with differentials dx, dy, dz by the formula:

$$ds^2 = dx^2 + dy^2 + dz^2 .$$

Following we obtain:

$$ds = \pm\sqrt{dx^2 + dy^2 + dz^2} .$$

$$ds = \pm\sqrt{dx^2 + dy^2 + dz^2}\,\frac{dt}{dt} = \pm\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt .$$

Finally, we obtain:

$$s = \pm\int_0^t \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,dt \qquad (1.6)$$

1.3. The Velocity of a Particle

One of the basic kinematic characteristics of motion of a particle is a vector quantity called *velocity*. Velocity characterizes a change in time of intensity and direction of motion of a particle in space.

First we introduce the concept of average velocity of a particle in a given time interval. Let's assume a moving particle M occupies at the moment of time t a position M defined by the position vector $\vec{r}$. At the moment of time t_1 ($t_1 = t + \Delta t$, where Δt *is the increment of time*), the particle occupies a position M_1 defined by the *vector - position* $\vec{r}_1 = \vec{r} + \Delta\vec{r}$ (Fig.1.5).

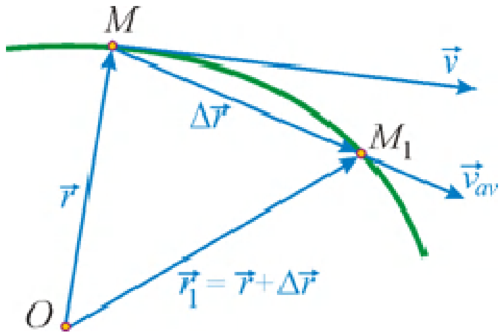


Fig.1.5. Concept of the Velocity

The displacement during the time interval $\Delta t = t_1 - t$ is defined by a vector $\overrightarrow{MM_1}$ which we shall call the *displacement* vector of the particle. From triangle OMM_1 we have $\vec{r} + \overrightarrow{MM_1} = \vec{r}_1$ whence the displacement vector of the particle may be defined as:

$$\overrightarrow{MM_1} = \vec{r}_1 - \vec{r} = \Delta\vec{r} \quad . \tag{1.7}$$

The ratio of the displacement vector of a particle to the corresponding time interval defines a vector quantity called the *average* (both magnitude and direction) velocity of a particle during the given time interval Δt :

$$\vec{v}_{av} = \frac{\overrightarrow{MM_1}}{\Delta t} = \frac{\Delta\vec{r}}{\Delta t}. \tag{1.8}$$

Vector of the average velocity $\vec{v}_{av}$ has the same direction with displacement vector $\overrightarrow{MM_1}$, i.e., along the chord MM_1, in the direction of the motion of the particle in the case of curvilinear motion, and along the trajectory itself in the case of rectilinear motion.

To obtain a characteristic of motion independent of the choice of the time interval Δt, the concept of *instantaneous* velocity of a particle is introduced.

The instantaneous velocity of a particle at any time *t* is defined as the vector quantity $\vec{v}$ towards which the average velocity $\vec{v}_{av}$ tends when the time interval Δt (increment of time) tends to zero:

$$\vec{v} = \lim_{\Delta t \to 0} \vec{v}_{av} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}$$

The limit of the ratio $\Delta \vec{r} / \Delta t$ as $\Delta t \to 0$ is the first derivative of the vector *r* with respect to *t* and is denoted, like the derivative of a scalar function, by the symbol dr / dt. Finally we obtain the vector of instantaneous velocity as:

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} \quad , \tag{1.9}$$

where the dot over the letter is a symbol of differentiation with respect to time.

Thus, *the vector of instantaneous velocity of a particle is equal to the first derivative of the position vector* $\vec{r}$ *of the particle with respect to time*. As the limiting direction of the secant MM_1 is a tangent, the vector of instantaneous velocity is *tangent* to the trajectory of the particle in the direction of motion. Eq. (1.9) also shows that the velocity vector $\vec{v}$ is equal to the ratio of the infinitesimal displacement $d\vec{r}$ of the particle tangent to its trajectory to the corresponding time interval *dt*.

In rectilinear motion the velocity vector $\vec{v}$ is always directed along the straight line in which the particle moves and can change only in magnitude; in curvilinear motion the direction of the velocity vector $\vec{v}$ changes continuously. The dimension of velocity is *displacement/time*, and the customary units are m/s or km/h.

In order to make numerical calculation of particle's motion parameters we have to use a coordinate representation of the motion. First we use fixed rectangular coordinate system *Oxyz*, origin of which coincides with the point *O*. Unite vectors of axes are $\vec{i}$, $\vec{j}$, $\vec{k}$. So, we can present the position vector $\vec{r}(t)$ accordingly (1.3) and (1.4). The velocity vector of a particle is equal to the first derivative of the position- vector $\vec{r}$ of the particle with respect to time accordingly (1.9). Let us calculate time-derivative of the position vector taking into account expression (1.4):

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx(t)}{dt}\vec{i} + \frac{dy(t)}{dt}\vec{j} + \frac{dz(t)}{dt}\vec{k} \, . \tag{1.10}$$

Here we have to take into account that the reference system *Oxyz* is fixed and because of that, unite vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ are constants vectors. From the other side velocity vector $\vec{v}$ could be presented using its projection on the axes *Oxyz*:

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k} \tag{1.11}$$

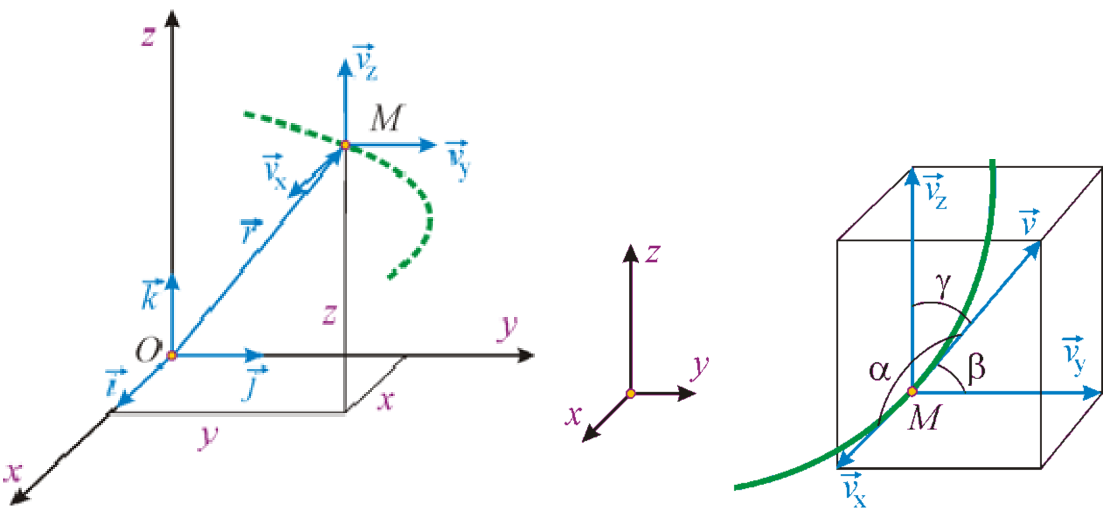


Fig. 1.6. Velocity vector in Cartesian reference system

Comparing of two formulas (1.10) and (1.11) we come to the conclusion that, *the projections of the velocity* $\vec{v}$ on the coordinate axes are equal to the first derivatives of the corresponding coordinates of the particle with respect to time:

$$\begin{cases} v_x = \dfrac{dx}{dt} = \dot{x}; \\ v_y = \dfrac{dy}{dt} = \dot{y}; \\ z_x = \dfrac{dz}{dt} = \dot{z}. \end{cases} \tag{1.12}$$

If we know the projections of the velocity $\vec{v}$, we can find the *magnitude of the velocity* $\vec{v}$:

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}; \tag{1.13}$$

To determine the *direction of the velocity* we have to calculate cosines of angles α, β, γ between the vector of velocity $\vec{v}$ and unite vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ (i.e., the angles α, β, γ which vector of velocity $\vec{v}$ makes with the coordinate axes):

$$\begin{cases} \cos\alpha = \dfrac{v_x}{v} = \dfrac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}; \\ \cos\beta = \dfrac{v_y}{v} = \dfrac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}; \\ \cos\gamma = \dfrac{v_z}{v} = \dfrac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}. \end{cases} \tag{1.14}$$

In the natural method of describing motion, velocity vector $\vec{v}$ is determined from its projections on a set of coordinate axes M*τnb* (Fig. 1.7).

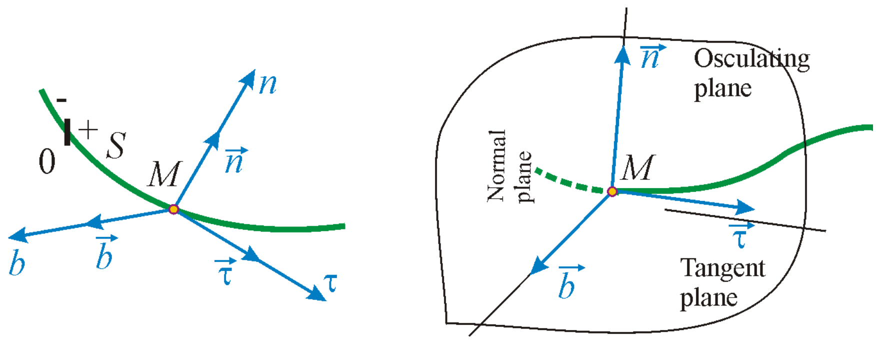


Fig.1.7. The natural coordinate system

The natural coordinate system (Frenet's coordinate system) M*τnb* is a rectangular system. Its origin coincides with the moving point *M* during the whole time of motion of the particle. Axis *Mτ* is directed along the tangent to the trajectory in the direction of the positive displacements. Correspondent unit vector is $\vec{\tau}$. Axis *Mn* is directed along the normal in the *osculating* plane towards the concavity of the trajectory. Correspondent unit vector $\vec{n}$ is directed from point *M* to the center of curvature. Axis *Mb* is directed perpendicular to the *Mτ* and *Mn* to form a right-hand set. Correspondent unit vector is $\vec{b}$.The axis *Mτ* is called *tangent* axis. The normal *Mn*, which lies in the osculating plane (or in the plane of the curve itself if the curve is two-dimensional), is called the *principal normal (*or simply a *nor-*

mal), and the normal *Mb* perpendicular to it is called the *binormal*. A plane normal to the vector $\vec{\tau}$ and intersecting tangent in the point *M* is a *normal* plane. The plane containing tangent axis $M\tau$ and binormal axis *Mb* is called a *tangent* plane (Fig. 1.7). Note that the unit vector $\vec{b}$ of the binormal axis is defined as the cross product of the unit tangent vector $\vec{\tau}$ and unit normal vector $\vec{n}$ ($\vec{b} = \vec{\tau} \times \vec{n}$).

Next, let's consider the process of determining of the velocity of a particle in the framework of the natural way of describing movement. Let us know the trajectory of the motion of the particle, and the law of change of curvilinear coordinate along the trajectory is given in the general form (1.5).

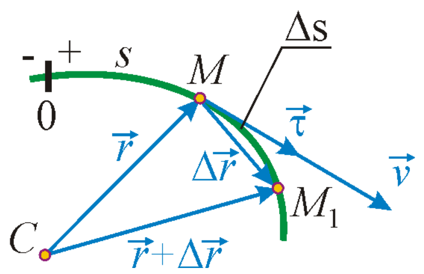


Fig.1.8. The determination of the velocity within the natural way to describe motion

If in a time interval $\Delta t = t_1 - t$ a particle moves from position *M* to position M_1, the displacement along the arc of the trajectory being $\Delta s = s_1 - s$ (Fig. 1.8), the numerical value of the average velocity will be:

$$v_{av} = \frac{\Delta s}{\Delta t} \ . \tag{1.15}$$

Applying the limit, we obtain the algebraic (or numerical) value of the instantaneous velocity for a given time *t*:

$$v = \lim_{\Delta t \to 0} v_{av} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} \tag{1.16}$$

The vector of instantaneous velocity of a particle is equal to the first derivative of the position vector $\vec{r}$ of the particle with respect to time:

$\vec{v}=\frac{d\vec{r}}{dt}$, where the infinitesimal increment of the position- vector $\vec{r}$ can be expressed as $d\vec{r} \approx ds\cdot\vec{\tau}$, where ds – is the infinitesimal displacement of the particle ($\lim\limits_{\Delta t\to 0}\frac{\Delta r}{\Delta s}=\frac{dr}{ds}=1$). Then, we obtain next formula:

$$\vec{v}=\frac{ds}{dt}\cdot\vec{\tau}\ . \tag{1.17}$$

On the other hand, the velocity vector is tangent to the trajectory at every point. Therefore, velocity vector $\vec{v}$ can be expressed as:

$$\vec{v}=v_{\tau}\vec{\tau}\,, \tag{1.18}$$

where v_{τ} is the scalar value of velocity – projection of the velocity on the tangent axis of the natural coordinate system. Comparison of two formulas (1.17) and (1.18) gives us next formula of the *algebraic value* of velocity of a particle:

$$v_{\tau}=v=\frac{ds}{dt}\ . \tag{1.19}$$

The sign of a algebraic value of velocity v is the same as that of first derivative of the curvilinear (arc) coordinate s (v=$\frac{ds}{dt}=\dot{s}$), it can be defined in the following way: if $v=\dot{s}>0$, the velocity vector $\vec{v}$ is in the positive direction of curvilinear coordinate s, if $v=\dot{s}<0$, the velocity vector $\vec{v}$ is in the negative direction of s (Fig. 1.9).

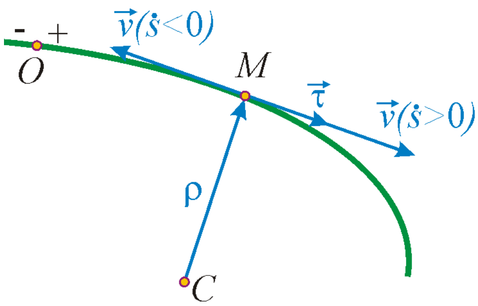


Fig. 1.9. Direction of the velocity

1.4. The Acceleration of a Particle.

Acceleration of a particle should be defined as vector measure of velocity alternation. Acceleration characterizes the time rate of change of velocity vector in magnitude and direction.

Let us assume a moving particle that occupies a position M and has a velocity $\vec{v}$ at a given time t, and let the same particle at time t_1 occupy a position M_1 and have a velocity $\vec{v}_1$ (Fig. 1.10). The increase of the velocity in the time interval $\Delta t = t_1 - t$ is $\Delta\vec{v} = \vec{v}_1 - \vec{v}$. Let us construct vector $\Delta\vec{v}$ from point M and construct the parallelogram with $\vec{v}_1$ as its diagonal and $\vec{v}$ as one of its sides. It is evident that the other side will represent vector $\Delta\vec{v}$. Increment of the velocity is $\Delta\vec{v} = \vec{v}_1 - \vec{v} = \vec{v}(t + \Delta t) - \vec{v}(t)$

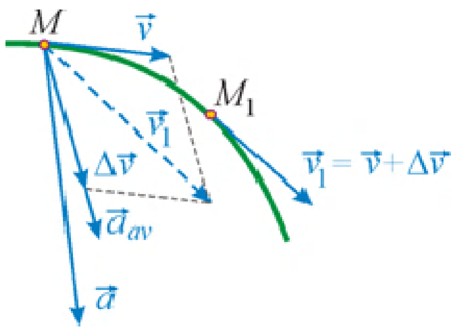


Fig.1.10. Concept of acceleration

Note that the vector $\Delta\vec{v}$ is always directed towards the inside of the trajectory. The ratio of the velocity increment vector $\Delta\vec{v}$ to the corresponding time interval Δt defines the vector of *average* acceleration of a particle in the given time interval:

$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} . \qquad (1.20)$$

Obviously, the vector of average acceleration has the same direction as velocity increment vector $\Delta\vec{v}$, i.e., towards the concavity of the trajectory.

The *instantaneous* acceleration of a particle at a given time t is defined as the vector quantity $\vec{a}$ towards which the average acceleration $\vec{a}_{av}$ tends when the time interval Δt tends to zero: (Acceleration $\vec{a}$ at the moment t – instantaneous acceleration – is defined as limit of average acceleration $\vec{a}_{av}$ when increment of time interval Δt tends to zero): $\vec{a} = \lim\limits_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$, or, taking into account Eq. (1.9):

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \dot{\vec{v}} = \ddot{\vec{r}}. \tag{1.21}$$

Thus, the vector of instantaneous acceleration of a particle is equal to the first derivative of the velocity vector or the second derivative of the position vector $\vec{r}$ of the particle with respect to time. The dimension of acceleration is *displacement/ (time)*^{2}, and the commonly used unit is m/s^2.

In rectilinear motion vector $\vec{a}$ is directed along the straight line in which the particle is moving. If the trajectory is a plane curve, the acceleration vector $\vec{a}$, just like the vector of average acceleration $\vec{a}_{av}$, lies in the plane of the curve and is directed towards the concavity of the curve. If the trajectory is a curve in space, the vector $\vec{a}_{av}$ is directed towards its concavity, in a plane through the tangent to the trajectory at point M and a line parallel to the tangent through the neighboring point M_1 (see Fig. 1.10). In the limit, when point M_1 tends to M, this plane coincides with the so-called osculating plane. Hence, in the general case, the acceleration vector $\vec{a}$ lies in the osculating plane and is directed towards the inside of the curve. Of all the planes passing through point M, the osculating plane has the greatest contact with the curve. Every point of a three-dimensional curve (e.g., a helix) has its own osculating plane. The osculating plane of a plane curve is coincident with the plane of the curve and is common for all its points.

In order to make numerical calculation of particle's acceleration we have to use a coordinate presentation of the motion. First we use fixed rectangular coordinate system $Oxyz$, origin of which coincides with the point O. Unite vectors of axes are $\vec{i}$, $\vec{j}$, $\vec{k}$. So, we can present the position vector $\vec{r}(t)$ according to (1.3) and (1.4). The acceleration vector of a particle is equal to the second derivative of the position vector $\vec{r}$ of the particle with respect to time according to (1.21). Let us calculate the second time-derivative of the position vector using the expression (1.4):

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x(t)}{dt^2}\vec{i} + \frac{d^2y(t)}{dt^2}\vec{j} + \frac{d^2z(t)}{dt^2}\vec{k}. \qquad (1.22)$$

On the other hand the acceleration vector $\vec{a}$ could be presented using its projection on the axes *Oxyz*:

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \qquad (1.23)$$

Comparing two formulas (1.22) and (1.23) we come to the conclusion that, the projections of the acceleration $\vec{a}$ on the coordinate axes are equal to the first derivatives of the projections of the velocities, or the second derivatives of the corresponding coordinates, of the particle with respect to time:

$$\begin{cases} a_x = \dfrac{dv_x}{dt} = \dfrac{d^2x}{dt^2}; \\ a_y = \dfrac{dv_y}{dt} = \dfrac{d^2y}{dt^2}; \\ a_x = \dfrac{dv_z}{dt} = \dfrac{d^2z}{dt^2}. \end{cases} \qquad (1.24)$$

The magnitude of the acceleration is defined by the next equation:

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}; \qquad (1.25)$$

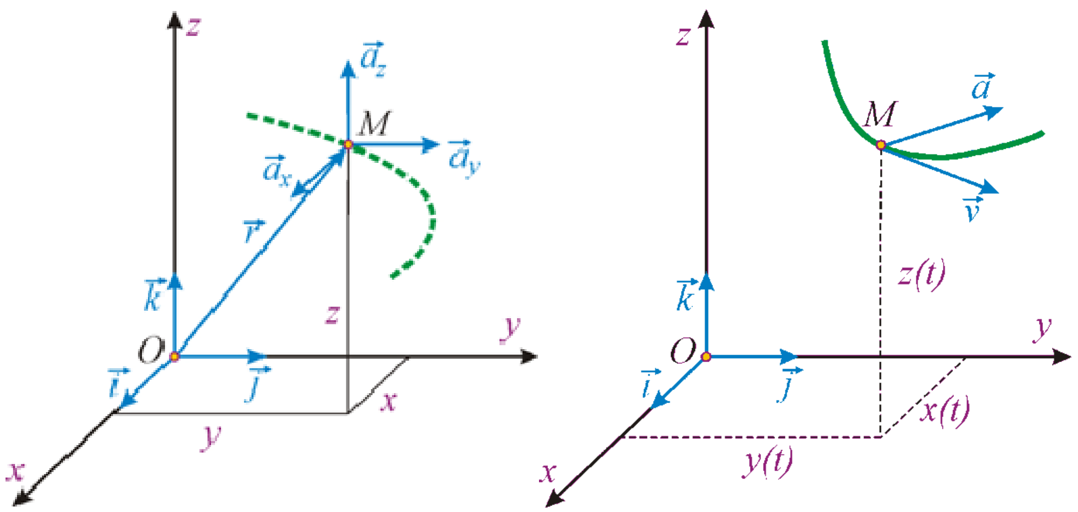


Fig. 1.11. Acceleration vector in Cartesian reference system

The direction of the acceleration is defined by the cosines of angles α_1, β_1, γ_1 between the vector of acceleration $\vec{a}$ and unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$:

$$\begin{cases} \cos\alpha_1 = \dfrac{a_x}{a} = \dfrac{\ddot{x}}{\sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}}; \\ \cos\beta_1 = \dfrac{a_y}{a} = \dfrac{\ddot{y}}{\sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}}; \\ \cos\gamma_1 = \dfrac{a_z}{a} = \dfrac{\ddot{z}}{\sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}}. \end{cases} \tag{1.26}$$

where α_1, β_1 and γ_1 are the angles between the acceleration vector and the coordinate axes.

Previously it was shown that the acceleration vector of a particle lies in the osculating plane, i.e., plane $M\tau n$, hence the projection of vector $\vec{a}$ on the binormal is zero ($a_b = 0$). Let us calculate the projections of $\vec{a}$ on the other two axes. The vector of acceleration $\vec{a}$ of a particle is equal to the first derivative of the velocity vector of the particle with respect to time. Through expression (1.17), we obtain the next formulas:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\cdot\vec{\tau}\right) = \frac{d^2s}{dt^2}\cdot\vec{\tau} + \frac{ds}{dt}\cdot\frac{d\vec{\tau}}{dt},\ \vec{a} = \ddot{s}\cdot\vec{\tau} + v\cdot\frac{d\vec{\tau}}{dt}. \tag{1.27}$$

Where $\dfrac{d\vec{\tau}}{dt} = \lim\limits_{\Delta t\to 0}\dfrac{\Delta\vec{\tau}}{\Delta t} = \lim\limits_{\Delta t\to 0}\left(\dfrac{\Delta\vec{\tau}}{\Delta t}\dfrac{\Delta s}{\Delta s}\right) = \lim\limits_{\Delta t\to 0}\dfrac{\Delta s}{\Delta t}\lim\limits_{\Delta s\to 0}\dfrac{\Delta\vec{\tau}}{\Delta s} = v\dfrac{d\vec{\tau}}{ds} = vk\vec{n}$, where k is the *curvature* of the trajectory at point M. As the curvature is the inverse of the *radius of curvature* ρ at M, we have: $k = \dfrac{1}{\rho}$. Finally we obtain:

$$\vec{a} = \ddot{s}\cdot\vec{\tau} + \frac{v^2}{\rho}\cdot\vec{n}, \tag{1.28}$$

On the other hand, vector of acceleration $\vec{a}$ could be presented using its projection on the axes $M\tau nb$:

$$\vec{a} = a_\tau \vec{\tau} + a_n \vec{n} + a_b \vec{b}\,. \tag{1.29}$$

Comparison of two formulas (1.28) and (1.29) gives us:

$$\left\{ a_\tau = \frac{dv}{dt} = \ddot{s}(t), a_n = \frac{v^2}{\rho}, a_b = 0. \right. \tag{1.30}$$

So, we have proved that the projection of the acceleration of a particle on the tangent to the trajectory is equal to the first derivative of the magnitude of velocity v, or the second derivative of the curvilinear coordinate s, with respect to time; the projection of the acceleration on the principal normal is equal to the second power of the velocity divided by the radius of curvature of the trajectory at the given point of the trajectory; the projection of the acceleration on the binormal is zero ($a_b = 0$).

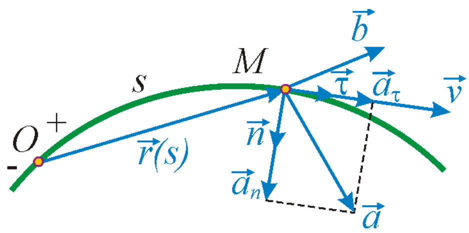


Fig. 1.12. The acceleration in natural reference system

Vectors $\vec{a}_\tau$ and $\vec{a}_n$, i.e., the normal and tangential components of the acceleration, directed along the tangent $M\tau$ and the principal normal Mn, respectively (Fig.1.12).

The first component in eq. (1.29) lies on the tangent to the trajectory. It is called tangential acceleration. It characterizes the change of the velocity magnitude. The second component in eq. (1.29) lies in the osculating plane at 90° angle to the trajectory tangent and is directed in the center of curvature. It is called normal acceleration and it characterizes the change of the velocity direction.

The component $\vec{a}_n$ is always directed along the inward normal, as $a_n > 0$, while the component $\vec{a}_\tau$ can be directed either in the positive or in the negative direction of the axis $M\tau$, depending on the sign of the projection a_τ

The acceleration vector $\vec{a}$ is the diagonal of a parallelogram constructed with the components $\vec{a}_\tau$ and $\vec{a}_n$ as its sides:

$$\vec{a} = \vec{a}_\tau + \vec{a}_n \ . \tag{1.31}$$

As the components $\vec{a}_\tau$ and $\vec{a}_n$ are mutually perpendicular vectors, the magnitude of vector $\vec{a}$ and its angle μ to the normal *Mn* are given by the equations:

$$\vec{a} = \sqrt{a_\tau^2 + a_n^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{\rho}\right)^2}\ , \ \tan\mu = \frac{|a_\tau|}{a_n}\ . \tag{1.32}$$

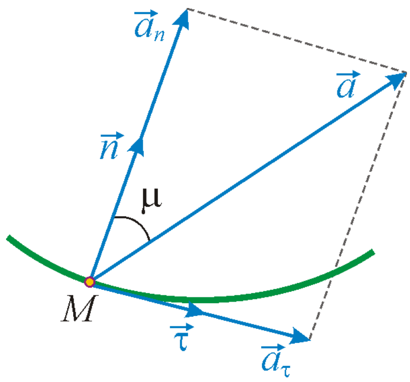


Fig. 1.13. Tangent, normal and full acceleration of a particle.

The magnitude and direction of tangent acceleration determine the character of motion of a particle. If the tangential acceleration is zero, then the motion is called uniform motion, while the velocity of the particle does not change in magnitude (determined by the value in the initial time v_0 – initial velocity), and arc (natural) coordinate depends on time linearly:

$$s = s_0 + v_0 t\ , \tag{1.33}$$

where s_0 – the initial value of the arc coordinates (initial displacement of the particle along the trajectory).

If the magnitude of tangent acceleration is not changed during movement, the motion is called uniformly variable motion. The velocity varies in time according to a linear law, and arc coordinate varies with time according to the square law:

$$v = v_0 + a_\tau t; \quad s = s_0 + v_0 t + \frac{a_\tau t^2}{2}. \qquad (1.34)$$

If the direction of the vector of tangent acceleration and vector of velocity are the same ($\vec{v} \uparrow\uparrow \vec{a}_\tau$ or $\dot{s} \cdot \ddot{s} > 0$), the motion is accelerated one, otherwise ($\vec{v} \uparrow\downarrow \vec{a}_\tau$ or $\dot{s} \cdot \ddot{s} < 0$) the motion is decelerated one.

1.5. Problems and Solutions.

Example 1. Particle's motion in the vertical plane (Fig. 1.14) described by the equations: $x = 300\ t$, m; $y = 400\ t - 5t^2$ m, where t is the time, s

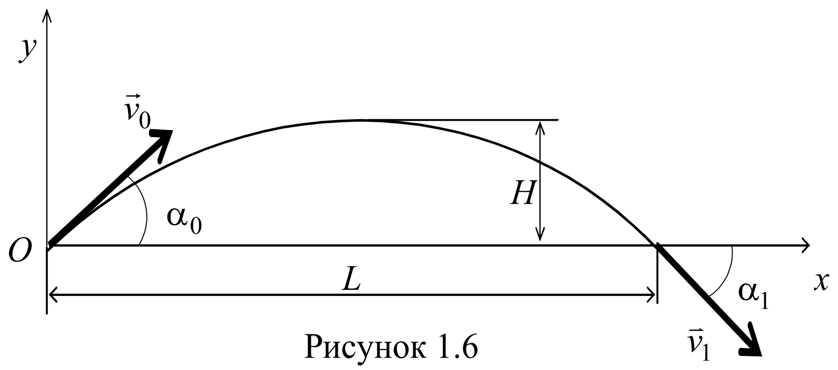


Fig. 1.14. To the first example

It is necessary to define:

– The trajectory of the particle;

– The velocity and the acceleration of a particle in the initial and final instant of time;

– The maximum height of the particle lift above the horizon H and the maximum flight range L;

– The radius of curvature of the trajectory at its start, end and highest point.

<u>Solution</u>

Find the equation of the trajectory, eliminating from the equation of motion $y = 400\ t - 5t^2$ (m) time t. First, from the equation $x = 300\ t$, we define $t = \dfrac{x}{300}$, and then obtain the trajectory equation in the following form:

$$y = \frac{4}{3}x - \frac{1}{18000}x^2.$$

The trajectory of a particle in the coordinate x and y in the vertical plane is a parabola.

Calculate the projection of the velocity and acceleration of a particle on the coordinate axis:

$$v_x = \dot{x} = 300\ \text{m/s};\ v_y = \dot{y} = 400 - 10t\ \text{m/s};\ a_x = \ddot{x} = 0;\ a_y = \ddot{y} = -10\ \text{m/s}^2.$$

Define their values in the initial instant of the time: $t = 0$:

$$v_0 = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{400^2 + 300^2} = 500\ \text{m/s};$$

$$a_0 = \sqrt{\ddot{x}^2 + \ddot{y}^2} = \sqrt{(-10)^2} = 10\ \text{m/s}^2.$$

The maximum height of the particle lift above the horizon can be determined by examining the extremum of a function $y(t)$ in variable t. This means that from the point of view of kinematics, the projection of the particle's velocity on the y-axis at a given instant of time must be equal to zero.

Then $\dot{y} = 400 - 10\tau_1 = 0$, where τ_1 – the time point of the maximum height, $\tau_1 = 40$ s. Inserting this time value into the expression for y, we obtain the $y_{\max} = H = y(40) = 8$ km.

The maximum range of the particle's flight defined from the conditions that at the time of the particle's drop the function $y(t)$ takes a value of zero $y(\tau_2) = 400\tau_2 - 5\tau_2^2 = 0$, where τ_2 – flight time of a particle. The root of this quadratic equation corresponding to the drop of a point to the ground, $\tau_2 = 80$ s, where the range $x_{\max} = x(80) = 24$ km.

Now, knowing the time of particles flight, it is possible to determine its velocity and acceleration at the end of the flight. Substituting the time τ_2 into the expression for the velocity projection of a particle on the y-axis, we get the $v_{1y} = -400$ m/s. Projections of the velocity and acceleration on the x-axis is not time-dependent and constant throughout the flight. Thus, the particle moves with a constant acceleration, which equal to 10 m/s^2 and directed vertically downward, and its velocity at the end of the flight is equal at the velocity magnitude at the beginning $|\vec{v}_1| = |\vec{v}_0| = 500$ m/s and make x axis is equal angles $|\alpha_1| = |\alpha_0|$.

To determine the radius of curvature of the particle, turn to the kinematic characteristics of motion in the natural reference system.

First find the tangential acceleration according to the formula

$$a_\tau = \frac{\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}}{|\vec{v}|},$$

and then calculate it for the initial time

$$a_{0\tau} = \frac{\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}}{|\vec{v}|} = \frac{300 \cdot 0 + 400 \cdot (-10)}{500} = -8 \text{ sm/s}^2$$

and for the final time

$$a_{1\tau} = \frac{300 \cdot 0 + (-400) \cdot (-10)}{500} = 8 \text{ m/s}^2.$$

Now we can calculate the normal acceleration according to the formula $a_n = \sqrt{|\vec{a}|^2 - |\vec{a}_\tau|^2}$, and then $a_{0n} = a_{1n} = \sqrt{10^2 - 8^2} = 6$ m/s^2. Since the radius of curvature of the trajectory included in the formula $a_n = \dfrac{v^2}{\rho}$, then

$$\rho = \frac{v^2}{a_n} = \frac{500^2}{6} = 41.667 \text{ km}.$$

The radii of curvature of the trajectory at the beginning and at the end of the flight the same. At the highest point of the trajectory

$$a_\tau = \frac{300 \cdot 0 + 0 \cdot (-10)}{500} = 0; \;\; a_n = 10 \text{ m/s}^2; \;\; |\vec{v}| = 300 \text{ m/s}; \;\; \rho = \frac{300^2}{10} = 9 \text{ km}.$$

As can be seen from the above example, the equations of motion contain all the necessary to study the characteristics of its motion at any point in time.

Example 2. The particle moves in the plane according to the law:

$$\begin{aligned} x &= 6\cos\left(\pi t^2\right) + 2; \\ y &= -3\sin\left(\pi t^2\right) + 3. \end{aligned} \tag{1.35}$$

It is necessary to define:

– The trajectory of the particle;

– The velocity and the acceleration of a particle at time $t_1 = 0{,}5$ ($x(t)$ and $y(t)$ is marked in centimeters t – seconds);

– The radius of curvature of the trajectory at the given moment of time.

Solution

The equations of motion (1.35) can be regarded as parametric equations of the particle's trajectory. In order to obtain the equation of the trajectory in coordinate form it is necessary to exclude time t from equations (1.35). Since part of the arguments are trigonometric functions, we can do the following: leave trigonometric functions on the right-hand sides of equations only, then square both equations and add them. After that we obtain:

$$\frac{(x-2)^2}{36} + \frac{(y-3)^2}{9} = 1, \tag{1.36}$$

i.e., the trajectory in this case is the ellipse with semiaxes 6 and 3. The ellipse is shown in Fig. 1.15.

In the given time $t_1 = 0{,}5\,\text{s}$ moving particle has the coordinates $x = 6{,}24\,\text{cm}$, $y = 0{,}88\,\text{cm}$, which are determined by substituting t in the original equations of motion (1.35). To check the correctness obtained equation of the trajectory (1.36) can be substituted in this equation x (or y) and get him y (or x).

The velocity vectors and the full acceleration of the particle defined in

Cartesian coordinates as follows:

$$\vec{v} = v_x\vec{i} + v_y\vec{j}; \quad \vec{a} = a_x\vec{i} + a_y\vec{j},$$

where $\vec{i}$, $\vec{j}$ – unit vectors of the axis Ox and Oy; v_x, v_y, a_x, a_y – projections of the velocity and the full acceleration of a particle on a coordinate axis. We find them by taking the time derivative of the equations of motion (1.35):

$$v_x = \dot{x} = -12\pi t\sin\left(\pi t^2\right);$$
$$v_y = \dot{y} = -6\pi t\cos\left(\pi t^2\right);$$
$$a_x = \dot{v}_x = \ddot{x} = -24\pi^2 t^2\cos\left(\pi t^2\right) - 12\pi\sin\left(\pi t^2\right);$$
$$a_y = \dot{v}_y = \ddot{y} = 12\pi^2 t^2\sin\left(\pi t^2\right) - 6\pi\cos\left(\pi t^2\right).$$

After inserting $t_1 = 0{,}5\,\text{s}$, we obtain:

$$v_x = -13{,}33\ \text{cm/s}; \quad v_y = -6{,}67\ \text{cm/s};$$
$$a_x = -68{,}53\ \text{cm/s}^2; \quad v_y = 7{,}61\ \text{cm/s}^2.$$

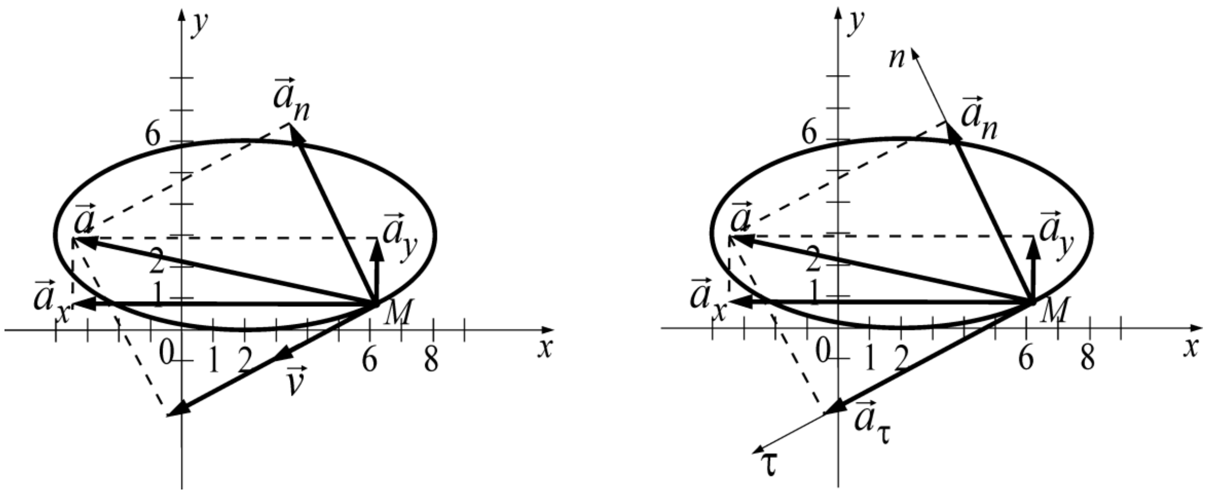


Fig. 1.15. To the second example

After the finding projections we can determine the velocity magnitude:

$$v = \sqrt{v_x^2 + v_y^2} \approx 14{,}9\ \text{cm/s};$$

Magnitude of the full acceleration:

$$a = \sqrt{a_x^2 + a_y^2} \approx -68{,}95\ \text{cm/s}^2.$$

Now we turn to the determination of the kinematic characteristics of motion in natural coordinate system. The tangent acceleration can be considered as the projection of the vector of full acceleration to the velocity vector, so its algebraic value is defined as follows:

$$a_\tau = \frac{v_x a_x + v_y a_y}{v} \approx 57{,}89\ \text{cm/s}^2.$$

The "+" sign indicates that the motion is accelerated one, i.e., the direction $\vec{v}$ and $\vec{a}_\tau$ will be the same. The magnitude of the particle's normal acceleration: $a_n = \dfrac{v^2}{\rho}$, where ρ – the radius of curvature of the trajectory at a given instant of time, if radius of curvature is unknown, then a_n can be determined by the formula:

$$a_n = \sqrt{a^2 - a_\tau^2} \approx 37{,}45\ \text{cm/s}^2.$$

The radius of curvature of the trajectory is determined from the expression:

$$\rho = \frac{v^2}{a_n} \approx 5{,}93\ \text{cm}.$$

Figure 1.15 shows the position of the particle at a given moment of time. The velocity vector is built in the scale of the projections, and the line of action of this vector must coincide with the tangent to the trajectory. The vector of full acceleration is built based from values of its components, and then is decomposed into tangent and normal acceleration.

Example 3. The particle moves in the plane according to the law:

$$x = t;$$
$$y = t^2 - 4. \tag{1.37}$$

It is necessary to define:

– The trajectory of the particle;

– The velocity and the acceleration of a particle at time $t_1 = 1$ ($x(t)$ and $y(t)$ is marked in centimeters t – seconds);

– The radius of curvature of the trajectory at the given moment of time.

<u>Solution</u>

The equations of motion (1.37) can be regarded as parametric equations of the particle's trajectory. In order to obtain the equation of the trajectory in coordinate form it is necessary to exclude time t from equations (1.37). In this case it is necessary to express the time t of the first equation and substitute into the second equation. After that we obtain:

$$y = x^2 - 4, \tag{1.38}$$

i.e., the trajectory in this case is a parabola, the graph of which is shown in Fig. 1.17 a. The solid line shows the actual trajectory of a particle which starts from the position that corresponds to the initial time: $t_0 = 0$, $x(0) = 0$, $y(0) = -4\,\text{cm}$. In a given time $t_1 = 1\,\text{s}$ particle that moves, has coordinates $x = 1\,\text{cm}$, $y = -3\,\text{cm}$ which are determined by substitution t_1 in the original equations of motion (1.13). To check the correctness of obtaining the equation of the trajectory of (1.14) can be substituted in this equation x (or y) and obtain y (or x).

The velocity and acceleration vectors of the particle defined in Cartesian coordinates as follows:

$$\vec{v} = v_x\vec{i} + v_y\vec{j}; \quad \vec{a} = a_x\vec{i} + a_y\vec{j},$$

where $\vec{i}, \vec{j}$ – unit vectors of the axis Ox and Oy; v_x, v_y, a_x, a_y – projections of the velocity and the full acceleration of a particle on a coordinate axes. We find them by time differentiating the equations of motion (1.37):

$$v_x = \dot{x} = 1; \quad v_y = \dot{y} = 2t;$$
$$a_x = \dot{v}_x = \ddot{x} = 0; \quad a_y = \dot{v}_y = \ddot{y} = 2.$$

After substituting $t_1 = 1\,\text{s}$, we obtain:

$$v_x = 1\,\text{cm/s}; \quad v_y = 2\,\text{cm/s};$$
$$a_x = 0; \quad a_y = 2\,\text{cm/s}^2.$$

Next, we determine magnitude of the velocity:

$$v = \sqrt{v_x^2 + v_y^2} \approx 2{,}24\,\text{cm/s};$$

Magnitude of the acceleration:

$$a = \sqrt{a_x^2 + a_y^2} = 2\,\text{cm/s}^2$$

We now turn to the determination of the kinematic characteristics of motion in natural coordinate system. The tangent acceleration can be considered as the projection of the vector of full acceleration to the velocity vector, so its algebraic value is defined as follows:

$$a_\tau = \frac{v_x a_x + v_y a_y}{v} \approx 1{,}79\,\text{cm/s}^2$$

The "+" sign indicates that the motion is accelerated one, i.e., the direction $\vec{v}$ and $\vec{a}_\tau$ will be coincide.

The magnitude of the normal acceleration of the particle is defined by the expression:

$$a_n = \sqrt{a^2 - a_\tau^2} \approx 0{,}89\,\text{cm/s}^2.$$

The radius of curvature of the trajectory is determined from the formula:

$$\rho = \frac{v^2}{a_n} \approx 5{,}61\text{cm}.$$

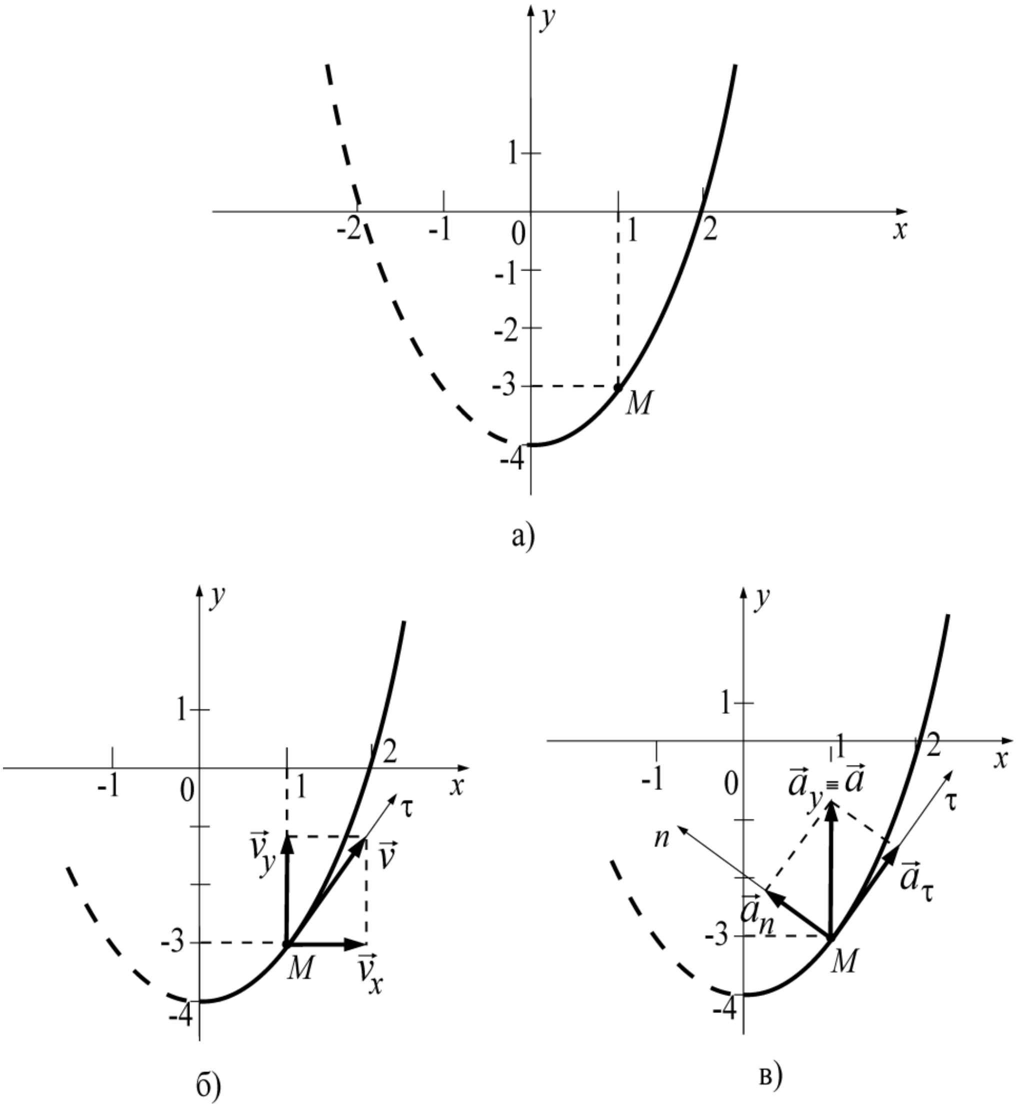


Fig. 1.16. To the third example

Fig. 1.16 shows the position of the particle in the given instant of time. The velocity vector is built based from values of its projections, and the line of action of this vector must coincide with the tangent to the trajectory (Fig. 1.16 b). The vector of full acceleration is based from values of its components, and then is decomposed into tangent and normal acceleration (Fig.1.16 c).

1.6. Tasks for solving.

Let us consider the motion of a particle in the same plane. The table 1.1 contains the laws of motion in Cartesian form. It is necessary to determine the trajectory of a particle, construct its graph, and for a given instant of time to determine the velocity and acceleration of a particle and draw its vectors on the trajectory.

Table 1.1. Lows of particle's motion.

N variant	Law of motion $x = f_1(t)$, cm	Law of motion $y = f_2(t)$, cm	Time t_1, s
1	$4\cos^2\left(\frac{\pi t}{3}\right)+2$	$4\sin^2\left(\frac{\pi t}{3}\right)$	1
2	$-\cos\left(\frac{\pi t}{3}\right)+3$	$\sin\left(\frac{pt^2}{3}\right)-1$	1
3	$2\sin\left(\frac{\pi t}{3}\right)$	$-3\cos\left(\frac{\pi t}{3}\right)+4$	1
4	$3t^2$	$-4t$	0,5
5	$3t^2-t+1$	$5t^2-\frac{5t}{3}-2$	1
6	$7\sin\left(\frac{\pi t}{6}\right)+3$	$2-7\cos\left(\frac{\pi t}{6}\right)$	1
7	$-4\cos\left(\frac{\pi t}{3}\right)$	$-2\sin\left(\frac{\pi t}{3}\right)-3$	1
8	$-4t^2$	$-3t$	0,5
9	$5\sin^2\left(\frac{\pi t}{6}\right)$	$-5\cos^2\left(\frac{\pi t}{6}\right)-3$	1
10	$5\cos\left(\frac{\pi t}{3}\right)$	$-5\sin\left(\frac{\pi t}{3}\right)$	1

11	$4\cos\left(\frac{\pi t}{3}\right)$	$-3\sin\left(\frac{\pi t}{3}\right)$	1
12	$3t$	$4t^2$	0,5
13	$7\sin^2\left(\frac{\pi t}{6}\right)-5$	$-7\cos^2\left(\frac{\pi t}{6}\right)$	1
14	$1+3\cos\left(\frac{\pi t^2}{3}\right)$	$3\sin\left(\frac{\pi t^2}{3}\right)+3$	1
15	$5t^2-4$	$3t$	1
16	$3t-6t^2$	$\frac{3t}{2}-3t^2$	1
17	$6\sin\left(\frac{\pi t}{6}\right)-2$	$6\cos\left(\frac{\pi t}{6}\right)+3$	1
18	$7t^2$	$5t$	0,25
19	$3-3t^2+t$	$4-5t^2+\frac{5t}{3}$	1
20	$-4\cos\left(\frac{\pi t}{3}\right)-1$	$-4\sin\left(\frac{\pi t}{3}\right)$	1
21	$6t$	$2t^2-4$	1
22	$8\cos^2\left(\frac{\pi t}{6}\right)+2$	$-8\sin^2\left(\frac{\pi t}{6}\right)-7$	1
23	$-3-9\sin\left(\frac{\pi t}{6}\right)$	$-9\cos\left(\frac{\pi t}{6}\right)+5$	1
24	$4t^2+1$	$-3t$	1
25	$3t-6t^2$	$\frac{3t}{2}-3t^2$	1
26	$6\sin\left(\frac{\pi t}{6}\right)-2$	$6\cos\left(\frac{\pi t}{6}\right)+3$	2
27	$7t^2$	$5t$	0,25

28	$3-3t^2+t$	$4-5t^2+\dfrac{5t}{3}$	0,5
29	$-4\cos\left(\dfrac{\pi t}{3}\right)-1$	$-4\sin\left(\dfrac{\pi t}{3}\right)$	2
30	$6t$	$2t^2-4$	2
31	$8\cos^2\left(\dfrac{\pi t}{6}\right)+2$	$-8\sin^2\left(\dfrac{\pi t}{6}\right)-7$	2
32	$-3-9\sin\left(\dfrac{\pi t}{6}\right)$	$-9\cos\left(\dfrac{\pi t}{6}\right)+5$	2

Questions

1. What is studying the mechanics?
2. What objects has the job of theoretical mechanics?
3. What part of the theoretical mechanics is divided?
4. What is a particle, a rigid body, a mechanical system?
5. Call some scientists who have developed theoretical mechanics.
6. What are the main problems of kinematics?
7. What are the main kinematic characteristics of the motion of a particle?
8. Formulate the methods of the defining of the motion of a particle.
9. What is the position vector of a particle?
10. How to define the motion of a particle in the Cartesian reference system?
11. How to set the particle motion when using natural method of description?
12. What is the Frenet's reference system?
13. What is the trajectory of a particle?
14. How to define the trajectory of a particle under Cartesian method of description of the motion of the particle?
15. What characterizes the velocity of the particle?
16. How to define the velocity under vector method of description of the motion of the particle?
17. How to define the velocity under Cartesian method of description of the motion of the particle?
18. How to define the velocity under natural method of description of the motion of the particle?
19. Where is the velocity vector of the particle relative to the trajectory?
20. What characterizes the acceleration of the particle?
21. How to define the acceleration under vector method of description of the motion of the particle?
22. How to define the acceleration under Cartesian method of description of the motion of the particle?

2. KINEMATICS OF A RIGID BODY. THE SIMPLEST MOTIONS OF A RIGID BODY.

2.1. Types of the Motion of a Rigid Body. Determination of the Position of a Rigid Body in Space.

In this section, we discuss issues related to the kinematics of rigid bodies. The main difference between the consideration of the kinematics of rigid bodies and consideration of the kinematics of particles is that under motion of the rigid body its size and shape are taken into account, and its rotation should be analyzed.

There are two approaches to the analysis of the kinematics of the rigid body. In accordance with the first approach it is possible to consider the motion of a body always as arbitrary motion and use the general formulas to determine the kinematic characteristics. In accordance with the second approach it is possible to consider certain features of body movement in advance (e.g., no rotations or displacements) and to consider some particular case of motion of a rigid body using specific formulas to determine the kinematic characteristics.

The second approach visibly simplifies the examination of certain cases of rigid body motions. It should also be noted that the elements in various technical devices make very specific types of motion (e.g., rotation around a fixed axis or fixed point) and for their analysis, a general approach is unnecessary. Also, it should be noted that the arbitrary motion of a rigid body can be consistently viewed as a combination of particular motions. It is possible to distinguish four particular cases of motion of a rigid body.

Particular cases of motion of a rigid body include:

– *translational motion*;

– *rotational motion (rotation)around a fixed axis*;

– *rotational motion(rotation)around a fixed point*;

– *plane-parallel (planar) motion.*

The translational motion of the rigid body is characterized by the lack of rotations during the motion (for example: piston pump, the slider in the mechanism to convert the movement). Rotational motion about a fixed axis is characterized by a fixed straight line that is associated with the body (example: rotor of turbine, wheel gear and belt gear, the crank in the mechanism for conversion of motions). Rotational motion around a fixed point (spherical motion) is characterized by the fact that during motion the same point of the body remains fixed (example: rotating radar antennas). Plane-parallel (planar) motion is characterized by the fact that all points of the body move in the planes, that are parallel to each other (example: wheel rolling on a straight-line path, motion of the connecting rod in the mechanism for conversion of motions).

The main tasks of the kinematics of the rigid body include:

– determining at any given time the position and orientation of the body in space as a whole;

– determining at any given time the position of an arbitrary point of the body;

– determining of kinematic characteristics of the motion of the body as a whole;

– determining of kinematic characteristics of motion of an arbitrary point of the body.

In order to determine the position of an arbitrary point of the body and its kinematic characteristics (trajectory, velocity and acceleration) it is possible to use the approaches discussed in the first section, it is necessary to have the law of motion of a point defined in the stationary frame of reference. On the other hand, there are approaches to determine the velocity and acceleration of arbitrary point of the body depending on the specific type of movement that a rigid body performs. We will discuss these approaches later.

Let us now consider the questions that are related to the definition of the position and orientation of an arbitrary rigid body in space. It can be argued that we know how the body is located in space, if we know where each point of it is located and vice versa. A rigid body can be represented as a set of an infinite number of points. To determine the position of the point in space we have to use, for example, three Cartesian coordinates that can be changed during the movement. So it might seem that to describe the motion of an arbitrary rigid body it is necessary to consider an infinite number of equations of motion. In fact, it is not true. By definition, a rigid body does not change its size and shape in the process of motion, so the distance between any two points of the body remains unchanged during the motion. We will show that if we know how any two points of the body move (i.e., how its Cartesian coordinates change in time), then we will always be able to determine how any other point of the body moves.

Let us consider three arbitrary points *A, B, C* of the free rigid body, which are not positioned on the same straight line. To describe the motion of each of these points, for example, we need to consider three Cartesian coordinates: $x_i(t), y_i(t), z_i(t), \quad i = 1,2,3$. Since the distance between the selected points do not change during the motion, then the following relations can be written:

$$\begin{cases} (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = AB^2 = \mathrm{const}_1, \\ (x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2 = BC^2 = \mathrm{const}_2, \\ (x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 = AC^2 = \mathrm{const}_3. \end{cases} \tag{2.1}$$

The system of equations (2.1) contains three algebraic equations with respect to the nine Cartesian coordinates of the selected points. It means that only six coordinates are independent with respect to each other. Thus, knowing how two arbitrary points of the free rigid body move, we can always determine how any other point of the body moves, i.e. we can find three Cartesian coordinates of the system (2.1). Knowing how any point of a rigid body moves, we can claim that we know how the whole body moves. Thus, in order to determine the position of a free solid body in space, we need to know how two of its points move, i.e. we must have six functions of time for the *six* Cartesian coordinates of two points of the body.

In general, it can be argued that to describe the position and motion of a free rigid body in space it is required to have six independent parameters, called *independent generalized coordinates*. The number of independent parameters that determine a body position in space is called the *Number of Degrees of Freedom* (DOF) of a rigid body. We can say that a free rigid body in space has 6 DOF, under the plane-parallel or spherical motion a body has 3 DOF, under rotational motion around a fixed axis a body has 1 DOF, etc.

Thus, to describe the motion of free rigid body, and for the analysis of its kinematics, we must have laws of change in time for the six independent generalized coordinates. Formally, as shown above, it can be Cartesian coordinates for two arbitrary points of the body. Then the kinematic equations of motion will have the form:

$$\begin{cases} x_A(t) = f_1(t); \\ y_A(t) = f_2(t); \\ z_A(t) = f_3(t); \\ x_B(t) = f_4(t); \\ y_B(t) = f_5(t); \\ z_B(t) = f_6(t), \end{cases} \tag{2.2}$$

where *A* and *B* – two arbitrary points of the body, $f_i(t), \quad i = 1...6$ – twice differentiable function of time. Note system equations in the form (2.2) is used very rarely to describe the motion of a rigid body, because, functions describing the rotation of a rigid body are not explicitly presented here. The functions defining the rotation of a rigid body are needed to determine its orientation in space.

There are different methods to write the kinematic equations of motion of a free rigid body which explicitly contain functions describing its rotation. *Euler's method* is considered historically as the first method for describing of the motion of a rigid body, according to which the system of kinematic equations of motion of a free rigid body has the form:

$$\begin{cases} x_O(t) = f_1(t); \\ y_O(t) = f_2(t); \\ z_O(t) = f_3(t); \\ \psi(t) = f_4(t); \\ \theta(t) = f_5(t); \\ \varphi(t) = f_6(t), \end{cases} \tag{2.3}$$

where *O* – the so-called pole (point of the body, which is chosen arbitrarily), ψ – the angle of precession, θ – the angle of nutation, φ – the spin angle, $f_i(t), \quad i = 1...6$ – twice differentiable function of time.

According to the Euler's method each new position of the body can be obtained from the previous position through four consecutive steps:

1) Movement of the body in space (without rotation) to the new location defined by the new coordinates of the pole;

2) Rotation of the body to the angle of precession;

3) Rotation of the body to the angle of nutation;

4) Rotation of the body to the angle of rotation.

Angular velocity and angular acceleration are the kinematic features of the whole body. *Angular velocity* $\vec{\omega}$ is a vector that characterizes the rate of change of the rotation angles of the body. The magnitude of the angular velocity is defined as the first derivatives of functions of angles of body's rotation and in the general case may be written as:

$$\omega = f\left(\dot{\psi}, \dot{\theta}, \dot{\varphi}\right). \tag{2.4}$$

Specific functions for the expression (2.4) can be obtained by considering a particular case of motion of a rigid body, and that will be done next.

Angular acceleration $\vec{\varepsilon}$ is a vector that characterizes the rate and direction of change of the angular velocity, in general, the relationship between angular acceleration and angular velocity is set as follows:

$$\vec{\varepsilon} = \dot{\vec{\omega}}. \tag{2.5}$$

If we present of the angular velocity vector and angular acceleration in a Cartesian coordinate system: $\vec{\omega} = \left(\omega_x, \omega_y, \omega_z\right)$, $\vec{\varepsilon} = \left(\varepsilon_x, \varepsilon_y, \varepsilon_z\right)$, then it is possible to consider the following ratios for its projections:

$$\varepsilon_x = \dot{\omega}_x,\ \varepsilon_y = \dot{\omega}_y,\ \varepsilon_z = \dot{\omega}_z. \tag{2.6}$$

Magnitudes of angular velocity and angular acceleration in this case will be determined as follows:

$$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2},\ \varepsilon = \sqrt{\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2}. \tag{2.7}$$

It should be understood that the angular velocity and angular acceleration do not depend on what point of the body we are considering for further kinematic analysis. These kinematic characteristics are determined for the whole body.

As noted above, the motion of a rigid body can almost always be represented as a set of simple motions, but there are two types of body motion that cannot be decomposed into simpler types. Such motions are called the simplest motions of a rigid body. These include translational motion and rotational motion around a fixed axis.

2.2. Translational Motion of a Rigid Body

Translational motion (*translation*) is such motion, when every straight line between any two points of the rigid body moves remaining parallel to its original direction during the motion. (Translation of a rigid body is such a motion in which any straight line through the body remains continually parallel to itself).

Under translation the trajectories of points of rigid body are equidistant curves. Translation should not be confused with rectilinear motion of a particle. Under translation the particles of a body may move on any curved trajectories. For example: when a motor car moves on a rectilinear horizontal road, the motion of its body is translation one, since every point of the body moves on a straight-line trajectory, otherwise, when a motor car moves on a curvilinear road, the motion of its body is not of translation, any straight line through the body of car does not remain parallel to itself.

The properties of translational motion are defined by the following theorem. Under translational motion, all points of a body move along *similar trajectories* (that will coincide if superimposed) and have at any instant the *same velocity* and *acceleration*.

To prove this theorem, let us consider a rigid body under translational motion with respect to a reference system *Oxyz*. We take two arbitrary points A and B on the body whose positions at time t are specified by position vector $\vec{r}_A$ and position vector $\vec{r}_B$ (Fig. 2.1). Next, we draw the vector $\overrightarrow{AB}$ joining the two points.

If is seen that the position vector $\vec{r}_B$ equals to the sum the vector - position $\vec{r}_A$ and vector $\overrightarrow{AB}$:

$$\vec{r}_B = \vec{r}_A + \overrightarrow{AB} \tag{2.8}$$

Because straight line AB moves remaining parallel to itself and point A and B are points of a rigid body and distance between them does not change during the motion we can state that vector $\overrightarrow{AB}$ is constant vector: $\overrightarrow{AB} = \text{const}$.

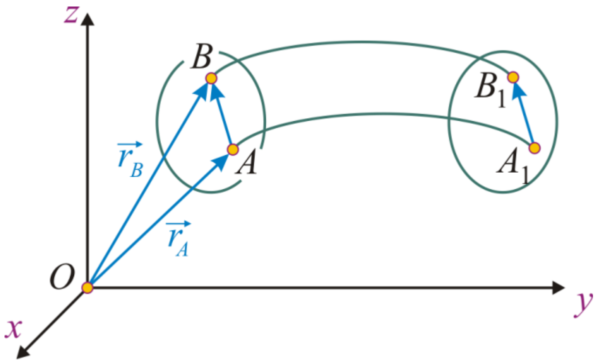


Fig.2.1. To translation motion of a body

As it followed from Eq. (2.8) (and from the fig. 2.1) the path of point B can be obtained by a parallel displacement of all the points of the path of particle A through a constant vector $\overrightarrow{AB}$. Hence, the trajectories of points A and B are identical curves (that will coincide if superimposed).

To determine the velocities of points A and B, we differentiate both parts of Eq.2.3 with respect to time. We obtain:

$$\vec{v}_B = \frac{d\vec{r}_B}{dt} = \frac{d\vec{r}_A}{dt} + \frac{d\,\overrightarrow{AB}}{dt} = \left|\overrightarrow{AB} = \text{const}\right| = \frac{d\vec{r}_A}{dt} = \vec{v}_A.$$

But the derivative of the constant vector $\overrightarrow{AB}$ is zero while the derivatives of vectors $\vec{r}_A$ and $\vec{r}_B$ with respect to time give the velocities of points A and B.

Thus we find that at any instant the velocities of points A and B are equal in magnitude and direction:

$$\vec{v}_B = \vec{v}_A \tag{2.9}$$

To determine the velocities of points A and B, we differentiate both parts of Eq.2.9 with respect to time. We obtain:

$$\vec{a}_B = \frac{d\vec{v}_B}{dt} = \frac{d\vec{v}_A}{dt} = \vec{a}_A$$

Hence, at any instant the accelerations of points A and B are equal in magnitude and direction:

$$\vec{a}_B = \vec{a}_A \tag{2.10}$$

As points A and B are arbitrary, it follows that the trajectories and the velocities and accelerations of all the points of a body at any instant are the same, which proves the theorem.

As follows from the theorem the translational motion of a rigid body is fully described by the motion of any point belonging to it. Thus, the analysis of translational motion of a rigid body is reduced to the methods of particle kinematics examined before.

The common velocity $\vec{v}$ of all the points of a body in translational motion is called the *velocity of translation*, and the common acceleration $\vec{a}$ is called the *acceleration of translation*. Vectors velocity $\vec{v}$ and acceleration $\vec{a}$ can, obviously, be shown as applied at any point of the body.

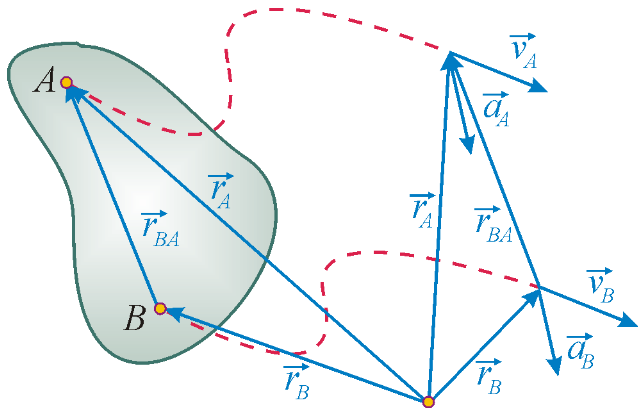


Fig.2.2. Velocities and accelerations under translation

2.3. Rotational Motion of a Rigid Body around a Fixed Axis.

2.2.1. Angle of Rotation of the Body. Angular Velocity. Angular Acceleration. *Rotational motion* of a rigid body around a *fixed axis* (*rotation*) is such motion of a rigid body that one line of the body (or of an extension of the body) remains fixed during motion. The fixed line is called the *axis of rotation*. In rotational motion all points of the body which lie on the axis of rotation are motionless, while all the other points of the body describe *circular trajectories* the planes of which are perpendicular to the axis of rotation and the centers of which lie on it. Note that a body may rotate about an axis without any point of it belonging to that axis, e.g., the rotation of a wheel on an axle, so the rotation axis can be an imaginary line

To determine the position of a rotating body, let us draw two planes through the axis of rotation z: plane *F*, which is fixed, and plane *G* through the rotating body and rotating with it (Fig. 2.3). The position of the body at any instant will be fully specified by the angle φ between the two planes, taken with the appropriate sign. Angle φ is called the *angle of rotation* of the body.

Angle φ is regarded as positive if from the positive direction of the axis of rotation *z* we can see rotation of the moving plane as counter clockwise. Angle φ is always measured in *radians* (radian is a dimensionless unit).

The position of a body at any instant is completely specified if we know the angle φ as a function of time *t*:

$$\varphi = f(t) \tag{2.11}$$

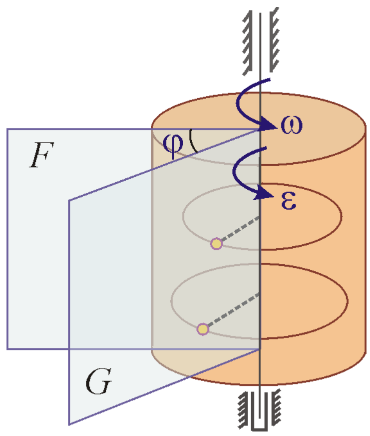


Fig.2.3. The Concept of the Angle of Rotation of the Body

Equation (2.11) can be called a law of rotational motion of a rigid body about a fixed axis.

The principal kinematic characteristics of the rotation of a rigid body are its *angular velocity* ω and *angular acceleration* ε. In accordance with the definition given earlier, the angular velocity in this case can be considered as a quantity that characterizes the rate of change of the angle of rotation of a body around a fixed axis.

If in an interval of time $\Delta t = t_1 - t$ a body turns through an angle $\Delta\varphi = \varphi_1 - \varphi$, the *average angular velocity of the body* in the given time interval is equal to the ratio of the increment of angle of rotation to the corresponding increment of time Δt:

$$\omega_{av} = \frac{\Delta\varphi}{\Delta t}.$$

The angular velocity of a body at a given time t is the value towards which the average angular velocity ω_{av} tends when the time interval Δt tends to zero:

$$\omega = \lim_{\Delta t \to 0} \omega_{av} = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} \qquad \text{or} \qquad \omega = \frac{d\varphi}{dt} \tag{2.12}$$

Thus, the angular velocity of a body at a given time is equal in magnitude to the first derivative of the angle of rotation with respect to time. Equation (2.12) also shows that the value of ω is equal to the ratio of the infinitesimal angle of rotation $d\varphi$ to the corresponding time interval dt.

The sign of ω specifies the direction of the rotation. It will is agreed that $\omega > 0$ when the rotation is counter clockwise, and $\omega < 0$ when the rotation is clockwise. The dimension of angular velocity, if the time is measured in seconds, is

$$[\omega] = \frac{\text{radian}}{\text{sec}}.$$

Angular acceleration characterizes the time rate of change of the angular velocity of a rotating body. If in a time interval $\Delta t = t_1 - t$ the change of angular velocity of a body is $\Delta\omega = \omega_1 - \omega$, then the *average angular acceleration* of the body in that interval of time is equal to the ratio of the increment of angular velocity to the corresponding increment of time Δt:

$$\varepsilon_{av} = \frac{\Delta\omega}{\Delta t}$$

The angular acceleration at a given time t is the value towards which average angular acceleration ε_{av} tends when the time interval Δt tends to zero. Thus,

$$\varepsilon = \lim_{\Delta t \to 0} \varepsilon_{av} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt},$$

or, taking into account Eq. (6) we obtain:

$$\varepsilon = \frac{d\omega}{dt} = \frac{d^2\varphi}{dt^2}. \tag{2.13}$$

Thus, the angular acceleration of a body at a given time is equal in magnitude to the first derivative of the angular velocity, or the second derivative of the angular displacement, of the body with respect to time. The dimension of angular acceleration is

$$[\varepsilon] = \frac{\text{radian}}{\sec^2}.$$

If the angular velocity ω increases in magnitude, then the rotation is *accelerated*, if it decreases, then the rotation is *decelerated* (Fig.2.4). It can be understood that the rotation is accelerated when the angular velocity ω and the angular acceleration ε are of the same sign and decelerated when they are of different signs.

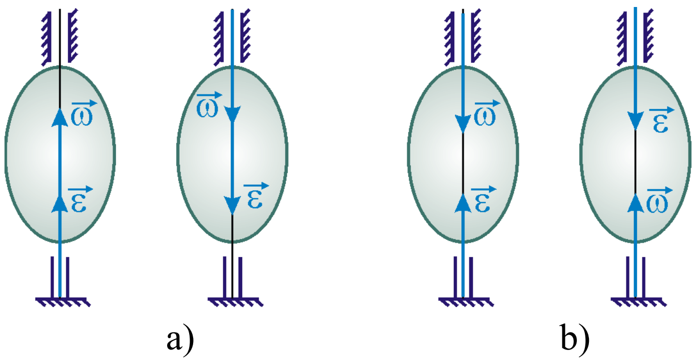


Fig.2.4 Accelerated (a) and Decelerated (b) rotation

The angular velocity of a rotating body can be denoted by a vector $\vec{\omega}$. It is directed along the axis of rotation of the body in the direction from which the rotation is seen as counter clockwise (see Fig. 2.5). The angular velocity vector in this case is defined as:

$$\vec{\omega} = \omega_z \vec{k} = \frac{d\varphi}{dt}\vec{k}, \tag{2.14}$$

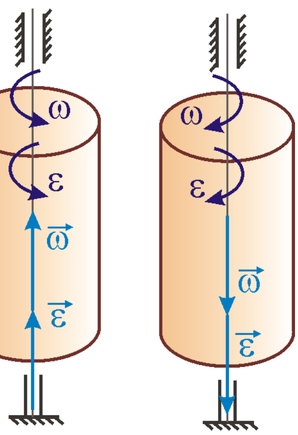
where ω_z – the projection of the vector of the angular velocity on the axis of rotation, $\vec{k}$ – the unit vector that lies on the axis of rotation.

By analogy with angular velocity, the angular acceleration of a body can be denoted by a vector $\vec{\varepsilon}$ along the axis of rotation:

$$\vec{\varepsilon} = \varepsilon_z \vec{k} = \frac{d^2\varphi}{dt^2}\vec{k} = \frac{d\omega}{dt}\vec{k}, \tag{2.15}$$

where ε_z – the projection of the vector of the angular velocity on the axis of rotation.

The direction of the vector of the angular acceleration $\vec{\varepsilon}$ coincides with that of the vector of the angular velocity $\vec{\omega}$ when the rotation is accelerated (Fig. 2.5a), and is of opposite sign when the rotation is decelerated (Fig. 2.5b).



a) b)

Fig.2.5. The Vectors of the Angular Velocity and Angular Acceleration

If the angular velocity of a rotating body does not change (ω = const), the rotation is said to be *uniform rotation*. Let us develop the equation of uniform rotation. We have from Eq. (2.12): $d\varphi = \omega dt$. Hence, assuming that at the initial moment (t = 0) angle φ equals φ_0 and by integrating the left-hand side of the equation from φ_0 to φ and the right-hand side from 0 to t, we obtain the equation of uniform rotation:

$$\varphi = \varphi_0 + \omega t, \tag{2.16}$$

where $\boldsymbol{\varphi_0}$ is the initial angle of rotation.

In engineering, the angular velocity of uniform rotation is often expressed as the number of revolutions per minute:

$$[n] = \frac{\text{revolutions}}{\text{min}}.$$

Let us establish the relation between n (rpm) and ω (rps) ($[\omega] = \dfrac{\text{radian}}{\text{sec}}$). A complete revolution turns a body through an angle of 2π radian and n revolutions take it through an angle $2\pi n$. If the duration of this rotation is t = 1 min = 60 s, then we obtain:

$$\omega = \frac{2\pi n}{60} \approx 0{,}1n.$$

If the angular acceleration of a body does not change during the rotation (ε = const), the rotation is said to be *uniformly variable rotation*. Let us develop the equation of uniformly variable rotation assuming that at the initial instant (t = 0) angle φ equals φ_0 and that the angular velocity $\omega = \omega_0$ (where ω_0 is the initial angular velocity).

From Eq. (2.13) we have $d\omega = \varepsilon dt$. Integrating the left-hand side over the interval from ω_0 to ω and the right-hand side from 0 to t, we obtain:

$$\omega = \omega_0 + \varepsilon t. \tag{2.17}$$

Let us write previous expression in the form:

$$\frac{d\varphi}{dt} = \omega_0 + \varepsilon t \qquad \text{or} \quad d\varphi = \omega_0 dt + \varepsilon t\, dt.$$

Integrating again, we obtain the equation of uniformly variable rotation:

$$\varphi = \varphi_0 + \omega_0 t + \varepsilon \frac{t^2}{2}. \qquad (2.18)$$

If ω and ε have the same sign, then the rotation is *uniformly accelerated*, if they have opposite signs, then it is *uniformly decelerated rotation*.

Velocities of the Points of a Rotating Body. Having established in the previous sections the characteristics of the motion of bodies as a whole, let us now investigate the motion of the individual points of a body. Now we have to determine the *linear velocity* $\vec{v}$ of an arbitrary point of a rotating body.

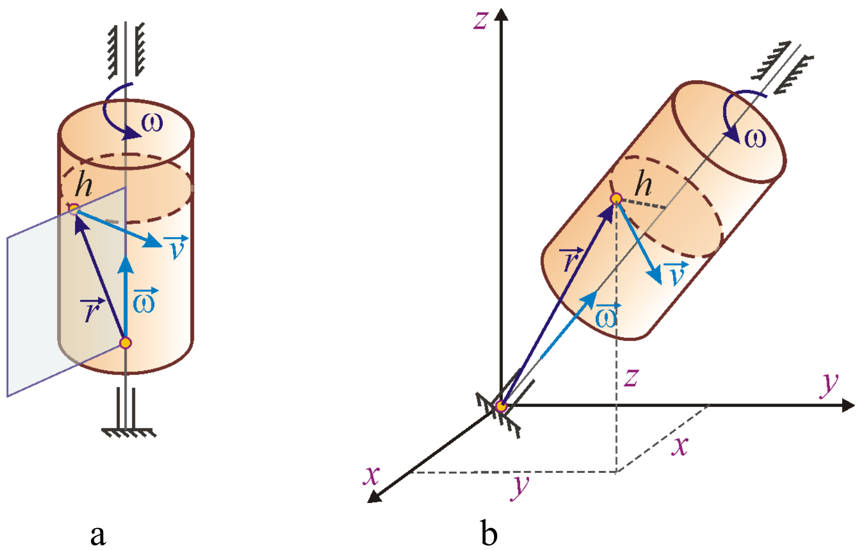


Fig.2.6. The Concept of the Velocity of a Point of the Rotating Body

Let's consider a point M of a rigid body at a distance h from the axis of rotation z (Fig. 2.6). When the body rotates, point M follows circular trajectory of radius h in a plane perpendicular to the axis of rotation with its centre C on that axis.

The distribution of velocities of points of a rotating body was first established by Euler. According to this law the vector of linear velocity $\vec{v}_M$ of a point M of a rotating rigid body is equal to the cross product of the vector of angular velocity $\vec{\omega}$ of that body and the position vector $\vec{r}_M$ of this point:

$$\vec{v}_M = \vec{\omega} \times \vec{r}_M . \qquad (2.19)$$

This expression is called Euler's formula. According to the rules of cross products, vector of velocity of the point both the perpendicular and the vector of angular velocity of the body and the position vector of the point ($\vec{v} \perp \vec{\omega}$ and $\vec{v} \perp \vec{r}$). Thus, the vector of velocity of the point lies in the plane of its trajectory on the tangent to the trajectory and directed in the direction of rotation.

The magnitude of the velocity of the point is determined by the following formula:

$$v_M = \omega \cdot r_M \cdot \sin \alpha .$$

Here α is the angle between the vectors of angular velocity and position vector of the point. Taking into account that $r_M \cdot \sin \alpha = h$ (h is the shortest distance between the point and the axis of rotation), finally we obtain:

$$v_M = \omega \cdot h . \qquad (2.20)$$

Thus, the magnitude of the velocity of a point belonging to a rotating body is equal to the product of the angular velocity of that body and the shortest distance between the point and the axis of rotation.

As the value of angular velocity ω at any given instant is the same for all points of the body, it follows from Eq. (2.20) that the linear velocity of any point of a rotating body is proportional to its distance from the axis of rotation.

The magnitude of the velocity of a point of the rotating body may be determined in another way:

$$\bar{v} = \bar{\omega} \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = (\omega_y z - \omega_z y)\bar{i} + (\omega_z x - \omega_x z)\bar{j} + (\omega_x y - \omega_y x)\bar{k} , \qquad (2.21)$$

where $\omega_x, \omega_y, \omega_z$ – the projection of the vector of the angular velocity on the Cartesian axis, x, y, z – the projection of the position vector of a point on the axis of the Cartesian coordinate, $\vec{i}, \vec{j}, \vec{k}$ – unit vectors of the coordinate axes. On the other hand velocity vector $\vec{v}$ could be presented using its projection on the axes *Oxyz*:

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

Comparing this formula with equation (2.21) we obtain expressions for the projections and the magnitude of the velocity of the point:

$$\begin{aligned} v_x &= \omega_y z - \omega_z y; \\ v_x &= \omega_z x - \omega_x z; \quad v = \sqrt{v_x^2 + v_y^2 + v_z^2}. \\ v_z &= \omega_x y - \omega_y z; \end{aligned} \tag{2.22}$$

Accelerations of the Points of a Rotating Body. Linear acceleration of the point of the rotating body can be found with the help of general formula connecting linear acceleration and linear velocity: $\vec{a} = \dfrac{d\vec{v}}{dt}$.

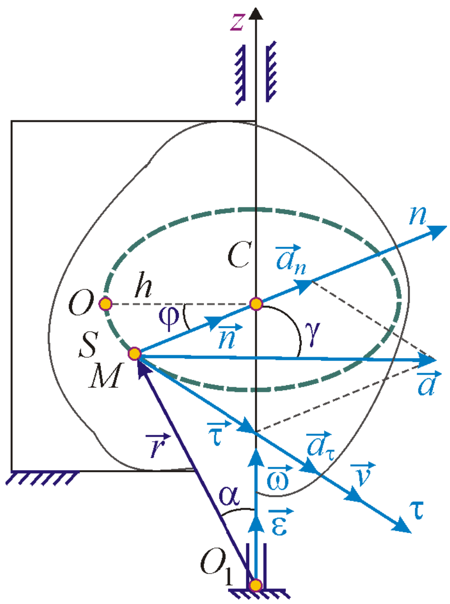


Fig.2.7. The Concept of the Acceleration of a Point of the Rotating Body

Using the Euler's formula for the velocity we easily obtain vector of linear acceleration $\vec{a}$:

$$\vec{a} = \frac{d(\vec{\omega} \times \vec{r})}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}.$$

Taking into account that the first time derivative of the angular velocity is the angular acceleration ($\frac{d\vec{\omega}}{dt} = \vec{\varepsilon}$) and the first derivative of the position vector is the velocity of a point ($\frac{d\vec{r}}{dt} = \vec{v}$), we finally obtain:

$$\vec{a} = \vec{\varepsilon} \times \vec{r} + \vec{\omega} \times \vec{v}. \qquad (2.23)$$

The first term of equation (2.23) is called *rotational acceleration*:

$$\vec{a}^{rt} = \vec{\varepsilon} \times \vec{r}. \qquad (2.24)$$

The second term of formula (2.23) is called *centripetal acceleration*:

$$\vec{a}^{cp} = \vec{\omega} \times \vec{v} = \vec{\omega} \times (\vec{\omega} \times \vec{r}). \qquad (2.25)$$

The rotational acceleration $\vec{a}^{rt}$ is tangent to the trajectory (in the direction of the rotation if it is an accelerated rotation and in the reverse direction if it is a decelerated rotation). The centripetal acceleration $\vec{a}^{cp}$ is always directed along the radius h towards the axis of rotation (Fig. 2.7). The full acceleration belongs to the plane normal to the axis of rotation (Fig.2.7). The full acceleration of a point of a rotating body is equal to the sum of the rotational and centripetal accelerations which are perpendicular to each other:

$$\vec{a} = \vec{a}^{rt} + \vec{a}^{cp}, \quad \vec{a}^{rt} \perp \vec{a}^{cp}. \qquad (2.26)$$

Magnitude of the rotational acceleration can be calculated by the formula:

$$a^{rt} = \varepsilon \cdot r \cdot \sin\alpha = \varepsilon \cdot h. \qquad (2.27)$$

Magnitude of the centripetal acceleration can be calculated using the expression:

$$a^{cp} = \omega \cdot v \cdot \sin 90° = \omega \cdot (\omega \cdot h) = \omega^2 \cdot h . \qquad (2.28)$$

Magnitude of the full acceleration can be calculated by the formula:

$$a = \sqrt{(a^{rt})^2 + (a^{cp})^2} = \sqrt{\varepsilon^2 \cdot h^2 + \omega^4 \cdot h^2} = h \cdot \sqrt{\varepsilon^2 + \omega^4} . \qquad (2.29)$$

The inclination of the vector of full acceleration to the radius of the circle described by the point is specified by the angle γ, given by the equation:

$$\mathrm{tg}\gamma = \frac{|a^{rt}|}{a^{cp}} = \frac{|\varepsilon|}{\omega^2} . \qquad (2.30)$$

Since at any given instant ε and ω are each the same for all the points of the body, it follows from formulas (2.29) and (2.30) that the accelerations of all the points of a rotating rigid body are proportional to their distances from the axis of rotation and make the same angle γ with the radiuses of the circles described by them (Fig.2.7).

Equations (2.19) – (2.29) make it possible to determine the velocity and acceleration of any point of a body if the equation of rotation of the body and the distance of the given point from the axis of rotation are known. With these formulas, knowing the motion of any single point of a body, it is possible to determine the motion of any other point and the kinematics characteristics of the motion of the body as a whole.

2.4. The Conversion of the Simplest Motions of a Rigid Body

Various mechanisms often carry out the conversion (transformation) of the simplest motions of a rigid body: translational to rotational, rotational to translational and rotational motion from one element of the mechanism to another.

In Fig.2.8 it is shown the system consisting of two bodies: the load 1 and the wheel 2. The load performs translational motion; the wheel performs rotational motion around a fixed axis. Consider the definition of the angular velocity of the wheel 2, by the given velocity of the load 1 suspended on a rope reeled up on a wheel (Fig. 2.8). The velocity of all points of the rope on which the load is suspended is the same (assuming the rope is inextensible), and equal to the velocity of the load. It is also true for the point that touches the point belonging to the wheel, that makes rotational movement, and this allows to determine the angular velocity of the wheel.

In this case we assume that the positive direction of movement of the load corresponds to positive rotation of the wheel. The algebraic value of the angular velocity of the wheel:

$$\omega_2 = \frac{v_1}{R_2}. \tag{2.31}$$

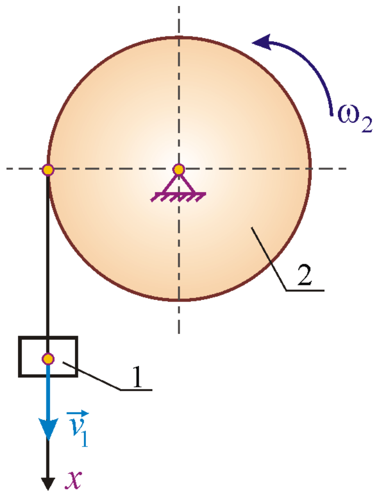


Fig.2.8. The Conversion Between Rotational and Translational Motion

Similarly, we can define the velocity of the load when angular velocity of the wheel is given:

$$v_{1x} = \omega_{2z} R_2. \tag{2.32}$$

When the rotational motions are transmitted from one body to another explicitly, either the toothed/friction gearing (Fig. 2.9), or the chain/belt transmission is used (Fig. 2.10).

In the case of the gear transmission wheels 1 and 2 have a common point, so the velocities of points on their outer diameters are the same, i.e.:

$$\omega_1 R_1 = \omega_2 R_2. \tag{2.20}$$

When defining the algebraic value of the angular velocity of the 2nd wheel one should take into account that the external gearing (see Fig. 2.3) changes the direction of rotation to reversed:

$$\omega_2 = \omega_1 \frac{R_1}{R_2}, \tag{2.21}$$

while the internal gearing (see Fig. 2.4) does not:

$$\omega_{2z} = \omega_{1z} \frac{R_1}{R_2}. \tag{2.22}$$

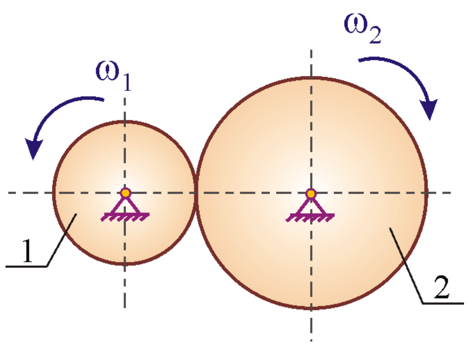


Fig.2.9.

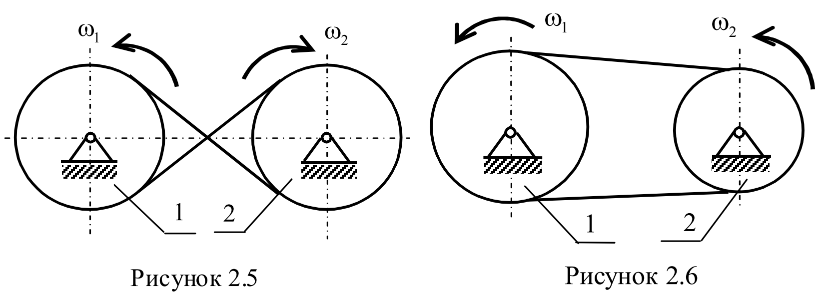


Fig.2.10

Corresponding points on the pulleys of the belt transmission (meaning the point where the belt, which is considered to be inextensible, is coming off one of the pulley and is reeled on the other) will also have the same magnitudes of velocities. The direction of rotation can either be changed to the opposite during the conversion of motion (see Fig. 2.10 a) or stay the same (see Fig. 2.10 b). The angular velocity thus determined respectively by formulas (2.16) and (2.17).

2.5. Problems and Solutions.

Example 1. Wheel 1 (Fig. 2.11) rotates around a fixed axis according to the law $\varphi = 2t^2+4$, and actuates a lifting mechanism of the load 4. The mechanism consists of two multi-level wheels 2 and 3 connected by a belt and rotating around a fixed axis.

It is necessary to determine the velocity and acceleration of the load 4 at time $t =$ 3 s, if $R_1 = 40$ cm, $R_2 = 15$ cm, $R_3 = 25$ cm, $r_2 = 10$ cm, $r_3 = 20$ cm.

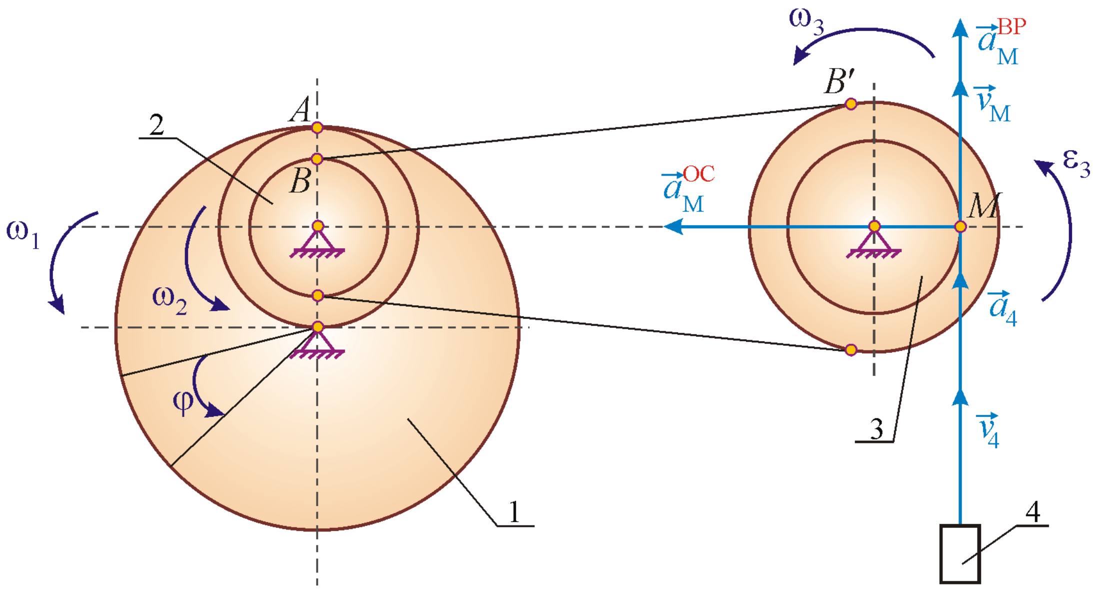


Fig.2.11. For the First Example

Solution

Determine the angular velocity of gear wheel $\omega_1 = \dot{\varphi} = 4t, \mathrm{s}^{-1}$. The point of contact of the wheels 1 and 2 is at A and it is their common point. Its velocity $v_A = \omega_1 R_1 = \omega_2 R_2$, where $\omega_2 = \omega_1 \dfrac{R_1}{R_2}$. Further, equating the speed of points B and B' belonging to the wheels 2, 3, we get the ratio between angular velocities of these wheels $\omega_2 r_2 = \omega_3 R_3$, where $\omega_3 = \omega_2 \dfrac{r_2}{R_3} = \omega_1 \dfrac{R_1}{R_2} \cdot \dfrac{r_2}{R_3}$.

Inputting ω_1 and the radii of the wheels in the expression for the angular velocity, one can determine:

$$\omega_3 = 4t\frac{40}{15}\cdot\frac{10}{25} = 4.267t\,.$$

So if $v_4 = \omega_3 r_3$, then let us calculate the velocity of the point M and the load 4, with $t = 3$ s:

$$v_M = v_4 = 4.267\cdot 3\cdot 20 = 256\ \text{cm/s}.$$

The acceleration of the load 4 is equal to the rotational acceleration of the point M, i.e.:

$$a_4 = a_M^{\text{rt}} = \varepsilon_3 r_3 = \dot{\omega}_3 r_3 = 4.267\cdot 20 = 85.34\ \text{cm/s}^2\,.$$

Example 2. Let us consider the example of mechanism, kinematic diagram of which is shown in Fig. 2.12, where the driving element is the load.

Given: the law of change of vertical coordinate of the load $x(t) = 30 + 10t^2$, cm; the radii of the wheels $R_1 = R_3 = 10$ cm, $R_2 = 30$ cm, $r_2 = 20$ cm. It is necessary to determine the velocity and acceleration of point M for time $t_1 = 1$ s.

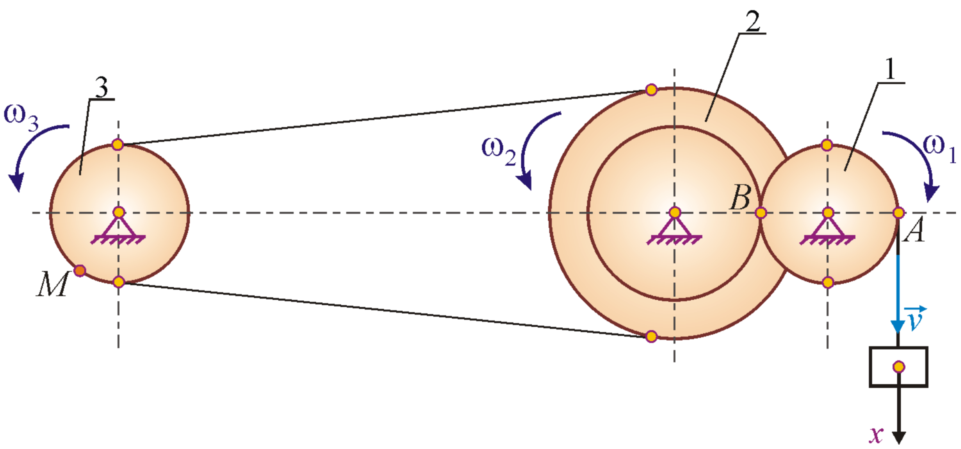


Fig.2.12. For the Second Example

Solution

Denote and show in Fig. 4.3 the points of mechanism A, B, D_1, D_2, through which the motion is converted from one link (driving) to another (driven).

We start the solution of the problem with determining the velocity of the load. Since the load performs translational motion, it can be considered as a particle, the motion of which is specified by the coordinate method, and that moves only along the one axis x. In order to determine the projection of the velocity of the load on this axis, we take the time derivative from the law of motion of the load and this derivative completely determines its velocity: $v = v_x = \dot{x} = 20t$ cm/s, under t_1 = 1 s, v= 20 cm/s.

Since the sign of the projection of the velocity of the load on the axis x is positive, then the velocity vector is directed downwards, i.e. in the direction of the positive axis.

The velocity of all points of the rope on which the load hangs are understood to be similar (the rope is considered to be inextensible), the velocity of the point from the wheel 1 that contacts the rope is equal to the velocity of the load. Since this point belongs to the body that is making the rotational motion around a fixed axis, it allows us to determine its angular velocity. The direction of the angular velocity of the wheel 1 corresponds to the direction of the velocity of point A. Let us write the algebraic value of the angular velocity of wheel 1:

$$\omega_1 = \frac{v}{R_1} = \frac{20t}{10} = 2t \ \text{s}^{-1}, \text{ under } t_1 = 1\text{s}, \quad \omega_1 = 2 \ \text{s}^{-1}.$$

Wheels 1 and 2 are connected via geared connection and have a common point B (see Fig.2.12). Therefore, the velocity of points of the wheels on their rims, are the same. When writing algebraic value of the angular velocity of the wheel 2 we will consider that the external gearing changes the direction of rotation reversed:

$$\omega_2 = -\omega_1 R_1 / r_2 = \text{s}^{-1}, \text{ under } t_1 = 1 \text{ s}, \ \omega_2 = 1 \ \text{s}^{-1}.$$

The velocities of points D_1 and D_2, located on the pulleys of the belt transmission are the same. Here, however, the direction of rotation does not change, so

$$\omega_3 = \omega_2 R_2 / R_3 = 3t \ \text{s}^{-1}, \text{ under } t_1 = 1 \text{ s}, \quad \omega_3 = 3 \ \text{s}^{-1}.$$

Let us now define the velocity of a point M of the wheel 3 at time t_1 = 1 s. The value of the velocity is the product of the magnitude of the angular velocity and the distance from the point M to the axis of rotation, which is equal to the radius R_3: $v_M = \omega_3 R_3 = 30$ cm/s. Direction of the velocity vector is shown to be perpendicular to the radius connecting the point with the axis of rotation, in accordance with the direction of rotation (Fig. 2.13).

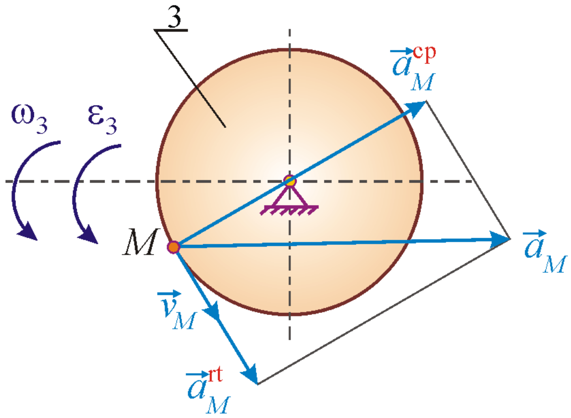


Fig.2.13. Velocity and Accelerations of Point A

To find the acceleration of the point M it is necessary to know the angular acceleration of the wheel 3. Algebraic value of the angular acceleration can be defined as the time derivative from the algebraic values of the angular velocity $\varepsilon_3 = \dot{\omega}_3 = 3$ s^{-2}. Algebraic values of angular velocity and angular acceleration have the same sign, hence, the rotational motion is accelerated motion.

The acceleration of point M can be defined as the geometric sum of the vectors of rotational and centripetal accelerations modules that are calculated by the formulae: $a_M^{\text{rt}} = \varepsilon_3 R_3 = 30$ cm/s^2; $a_M^{\text{cp}} = \omega_3^2 R_3 = 90$ cm/s^2, and the full acceleration of the point M:

$$a_M = \sqrt{(a_M^{\text{cp}})^2 + (a_M^{\text{rt}})^2} \approx 94{,}87\,\text{cm/s}^2$$

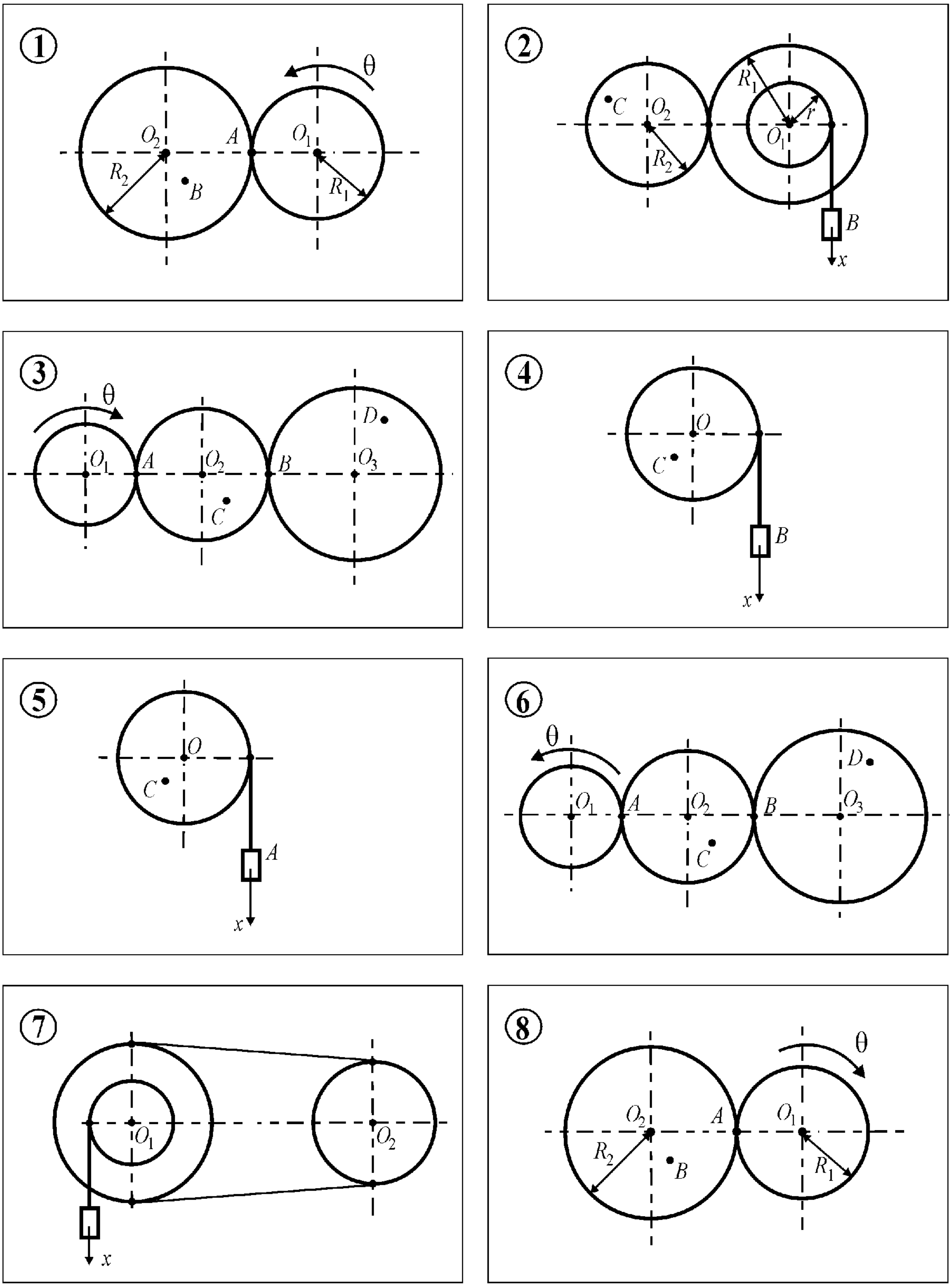
The vectors of the accelerations are shown in Fig. 4.4. The motion of the wheel 3 is accelerated; therefore, the rotational acceleration of the point M is directed in the same direction as its velocity. Centripetal acceleration is always directed to the axis of rotation.

2.6. Tasks for solving.

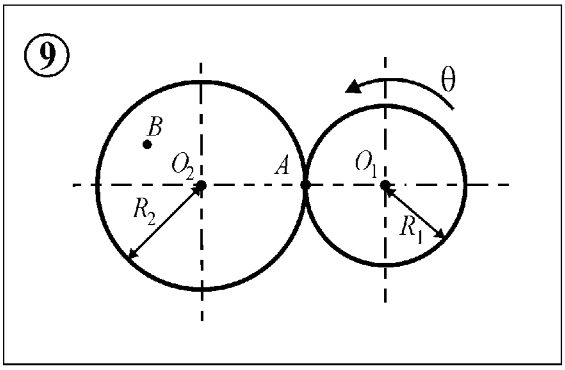
Diagrams of mechanical systems, elements of which make translational and rotational movement are shown in the table 2.1. The wheels rotate about a fixed axis that pass through their centers. The time dependence of the rotation angle of one of the wheels or the law of movement of the load is set in the variants of 2.2. The geometrical sizes are specified in the variants of 2.3.

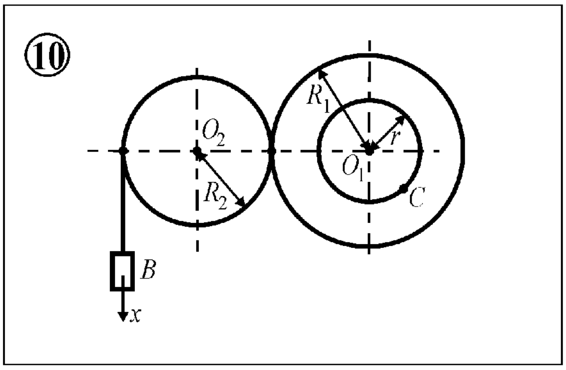
It is necessary to determine the angular velocity and angular acceleration of the wheels, and the velocity and acceleration of points indicated on the figures.

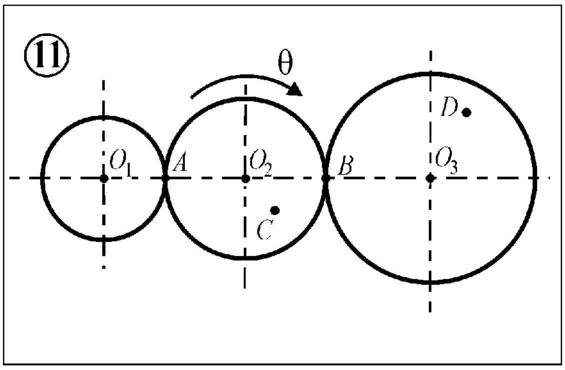
Table 2.1. Diagrams of mechanical systems.

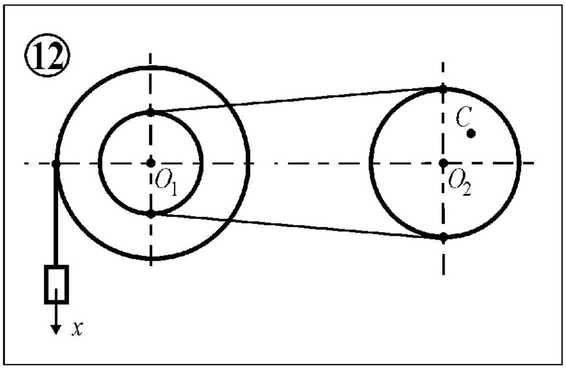


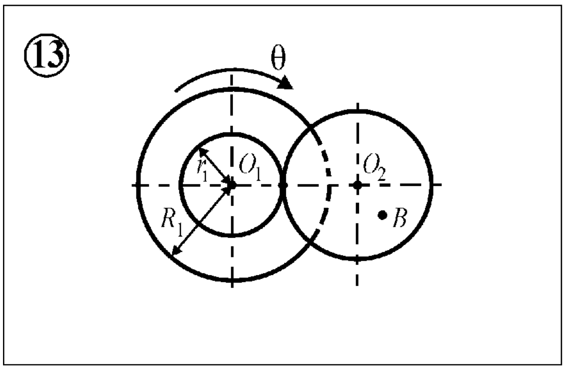
Continuation of the table 2.1.

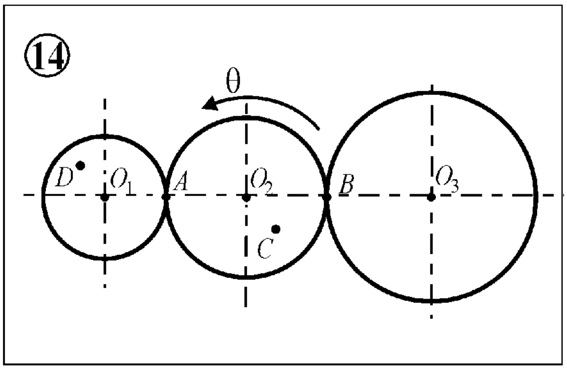


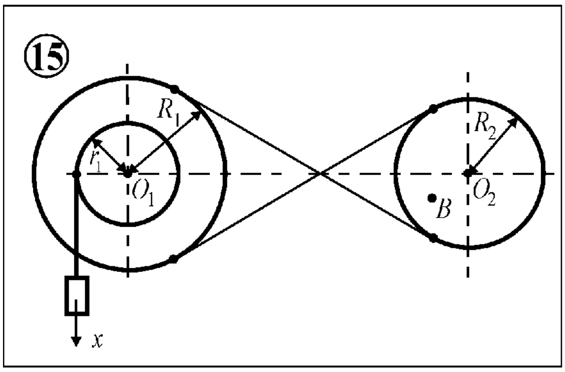


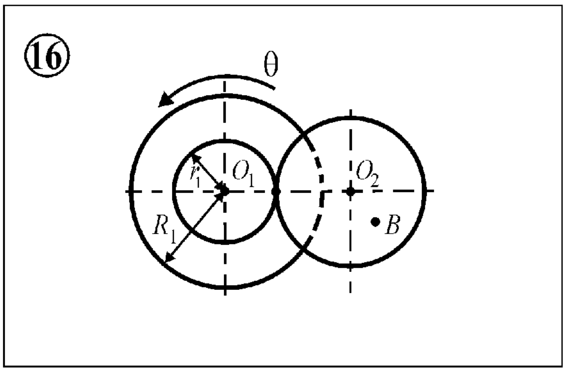












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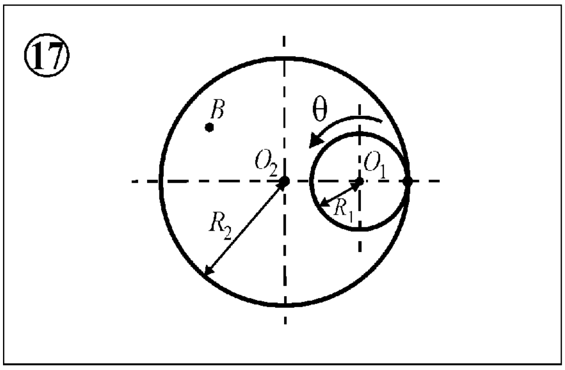

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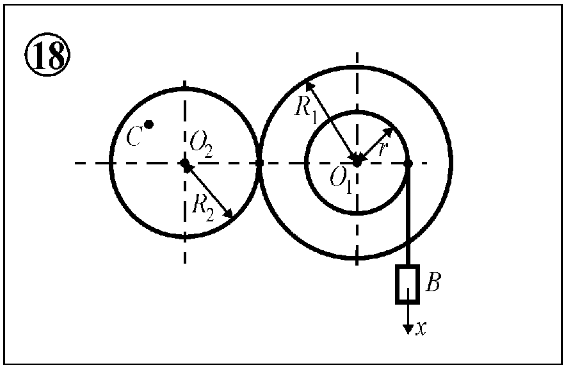

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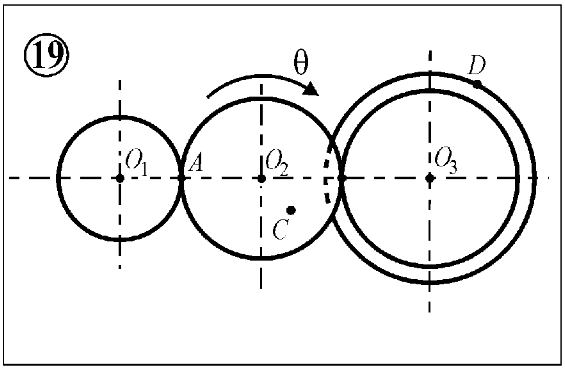

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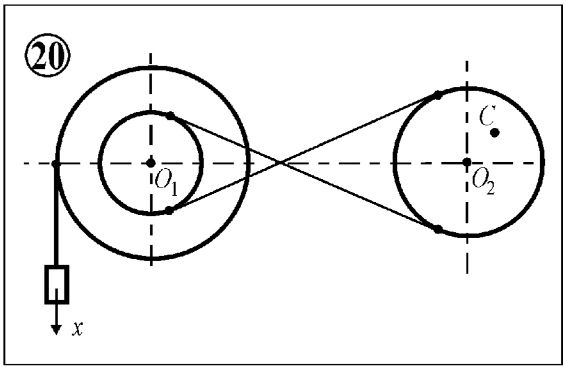

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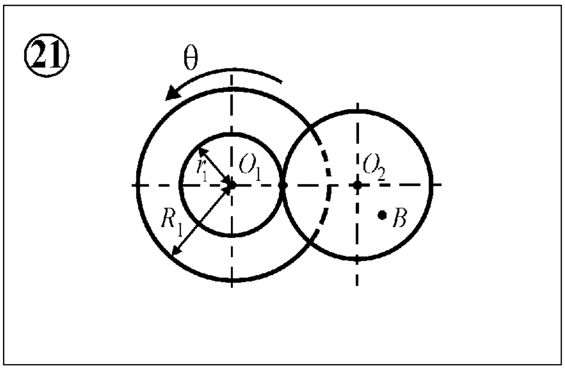

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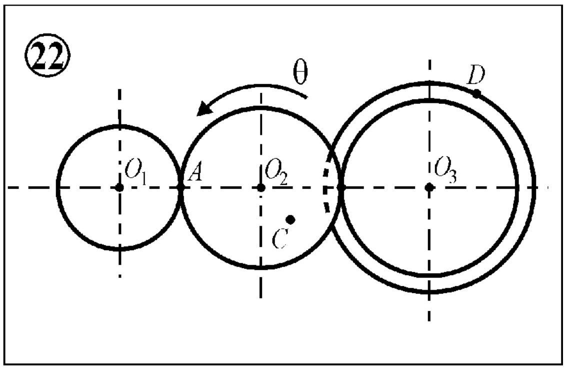

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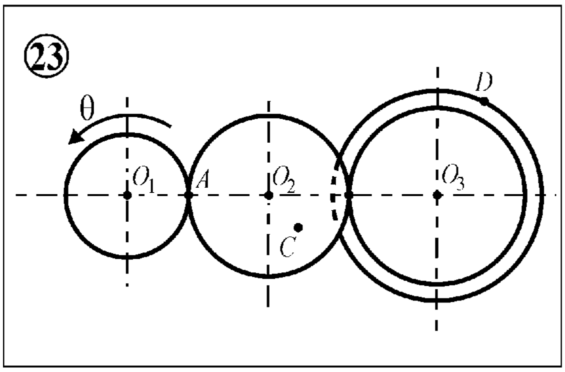

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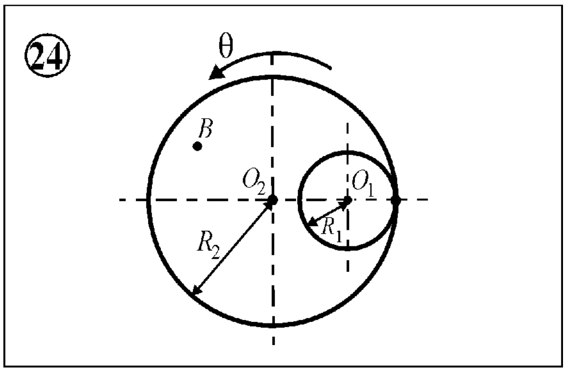
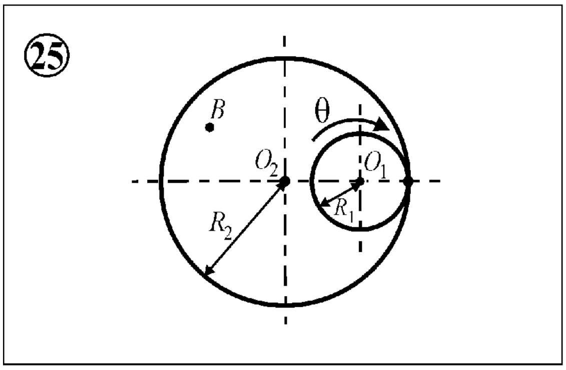
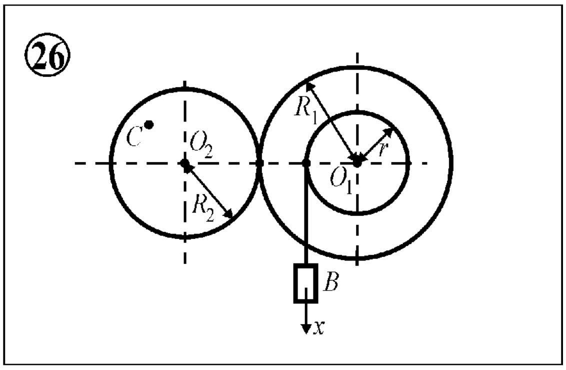
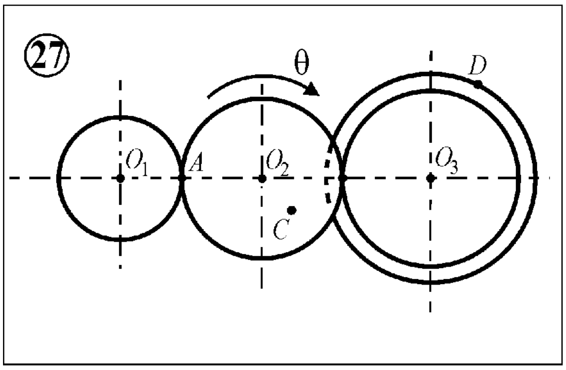
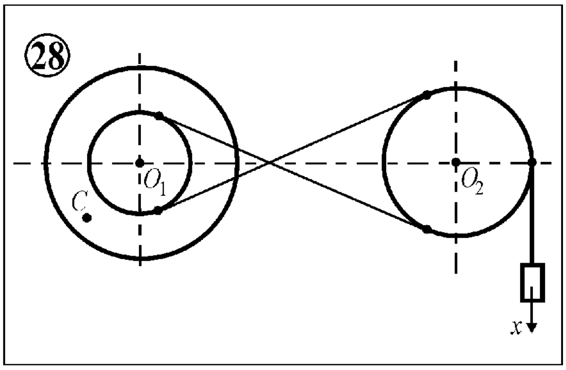


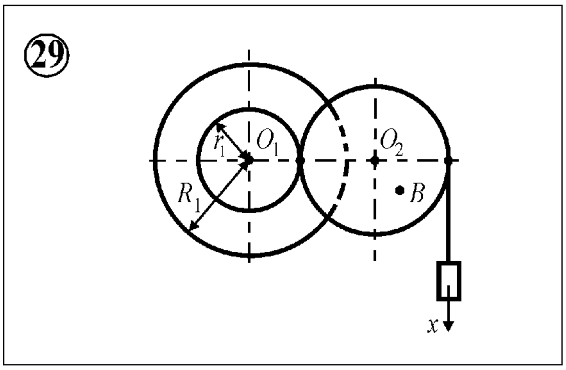
Continuation of the table 2.1.

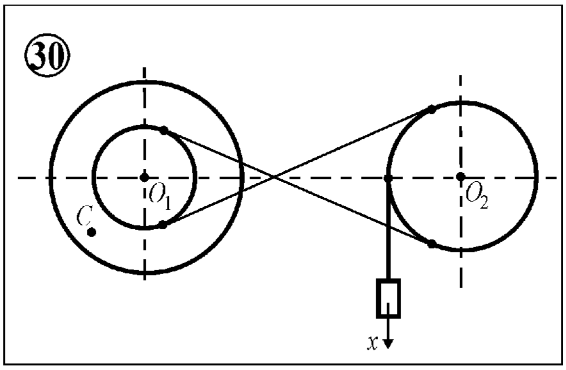


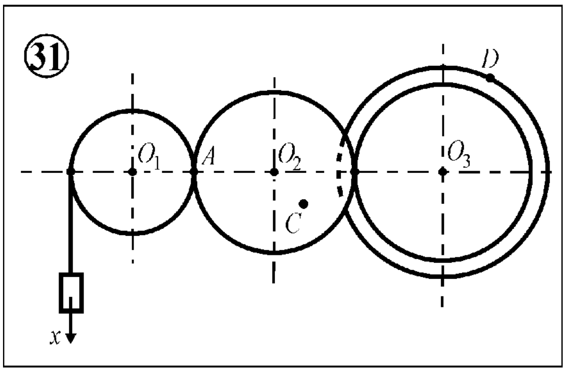












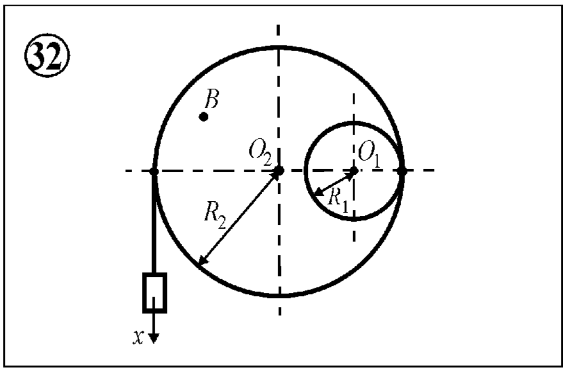


Table 2.1. Conditions of motion.

Variant N	Low of motion	Time, s
1	$\theta(t)=2t^2+3t-8$	0,5
2	$x(t)=t^2+2t-2$	1
3	$\theta(t)=t^2+2t-8$	1,5
4	$x(t)=1{,}5t^2+2t-4$	1
5	$x(t)=2t^2-2t-2$	1
6	$\theta(t)=4t^2+2t-1$	0,5
7	$x(t)=3t^2+t-1$	0,5
8	$\theta(t)=5t^2+2t-2$	0,5
9	$\theta(t)=t^2+1{,}5t-3$	1
10	$x(t)=t^2+t-1$	1,5
11	$\theta(t)=0{,}5t^2+t-1$	1
12	$x(t)=t^2+0{,}5t-3$	1,5
13	$\theta(t)=0{,}5t^2+3t-7$	1
14	$\theta(t)=1{,}5t^2+2t-6$	0,5
15	$x(t)=4t^2+t-1$	0,5
16	$\theta(t)=2t^2+t-5$	0,5
17	$\theta(t)=t^2+4t-4$	1
18	$x(t)=3t^2+3t-3$	0,5
19	$\theta(t)=5t^2+4t-3$	0,5
20	$x(t)=3t^2+3t-3$	0,5
21	$\theta(t)=2{,}5t^2+3t-2$	0,5
22	$\theta(t)=1{,}5t^2+4t-1$	0,5
23	$\theta(t)=2{,}5t^2+2t-2$	1
24	$\theta(t)=0{,}5t^2+4t-3$	1,5
25	$\theta(t)=t^2+5t-4$	1
26	$x(t)=1{,}5t^2+6t-5$	1,5
27	$\theta(t)=2t^2+6t-6$	0,5
28	$x(t)=t^2-2t-2$	1
29	$x(t)=1{,}5t^2-2t+2$	1,5
30	$x(t)=0{,}5t^2-2{,}5t-2$	1
31	$x(t)=4t^2+2{,}5t-3$	0,5
32	$\theta(t)=t^2-1{,}5t-9$	1

Table 2.2. Geometrical sizes.

Variant N	R_1, cm	r_1, cm	R_2, cm	R_3, cm
1	20	–	35	–
2	30	15	20	–
3	10	–	15	20
4	50	–	–	–
5	60	–	–	–
6	5	–	10	15
7	5	15	10	–
8	10	–	20	–
9	25	–	30	–
10	30	15	20	–
11	10	–	20	30
12	25	10	20	–
13	40	25	30	–
14	5	–	15	25
15	45	20	35	–
16	50	30	40	–
17	15	–	60	–
18	20	5	10	–
19	60	55	40	20
20	40	20	30	–
21	60	40	45	–
22	70	60	50	40
23	65	55	45	35
24	15	–	55	–
25	10	–	35	–
26	50	20	40	–
27	90	85	70	60
28	15	5	10	–
29	20	5	10	–
30	25	10	15	–
31	60	50	40	30
32	5	–	35	–

Questions

1. What is studying the mechanics?
2. What objects has the job of theoretical mechanics?
3. What part of the theoretical mechanics is divided?
4. What is a particle, a rigid body, a mechanical system?
5. Call some scientists who have developed theoretical mechanics.
6. What are the main problems of kinematics?
7. What are the main kinematic characteristics of the motion of a particle?
8. Formulate the methods of the defining of the motion of a particle.
9. What is the position vector of a particle?
10. How to define the motion of a particle in the Cartesian reference system?
11. How to set the particle motion when using natural method of description?
12. What is the Frenet's reference system?
13. What is the trajectory of a particle?
14. How to define the trajectory of a particle under Cartesian method of description of the motion of the particle?
15. What characterizes the velocity of the particle?
16. How to define the velocity under vector method of description of the motion of the particle?
17. How to define the velocity under Cartesian method of description of the motion of the particle?
18. How to define the velocity under natural method of description of the motion of the particle?
19. Where is the velocity vector of the particle relative to the trajectory?
20. What characterizes the acceleration of the particle?
21. How to define the acceleration under vector method of description of the motion of the particle?
22. How to define the acceleration under Cartesian method of description of the motion of the particle?

3. PLANE-PARALLEL MOTION A RIGID BODY.

3.1. Theoretical material.

Plane-parallel motion of bodies is one of the most common in machinery and technology. Such motion is carried out by rolling bodies (wheels, rollers, cylinders); separate details of mechanisms intended for the conversion of the rotating motion of one body to the translational one; planetary gears, etc.

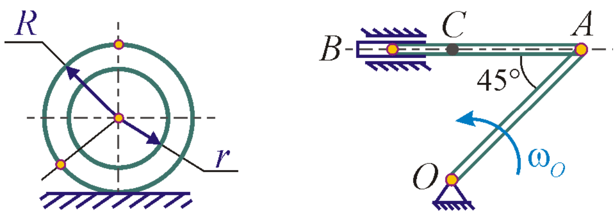


Figure 3.1 – Examples of the plane-parallel motion

Plane-parallel or plane motion is a motion of a body, when all the points of the body move in planes parallel to a certain fixed plane, as shown in Fig. 3.2 This fixed plane is called the *principle plane*.

To describe the plane motion of a body it is enough to describe the motion of the projection of a body on the principle plane. This projection is called a plane figure (see Figure 3.2).

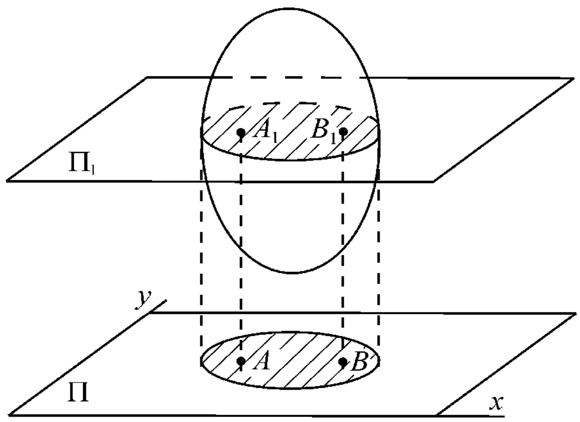


Figure 3.2 – Model of plane-parallel motion

The motion of a flat figure can be considered as the result of the formation of translational motion and rotation around one of the points, which are called a pole. As a pole, it is customary to choose a point of the body, the kinematic characteristics of which are known. In fig. 3.4 pole is the point A.

Equations of plane-parallel (flat) motion have the following form:

$$X_A = X_A(t), \quad Y_A = Y_A(t); \quad \varphi = \varphi(t), \tag{3.1}$$

where X_A, Y_A – coordinates of the pole in a fixed coordinate system; φ – angle of rotation around the pole. The position of any other point of the body that does not coincide with the pole can be determined in two ways:

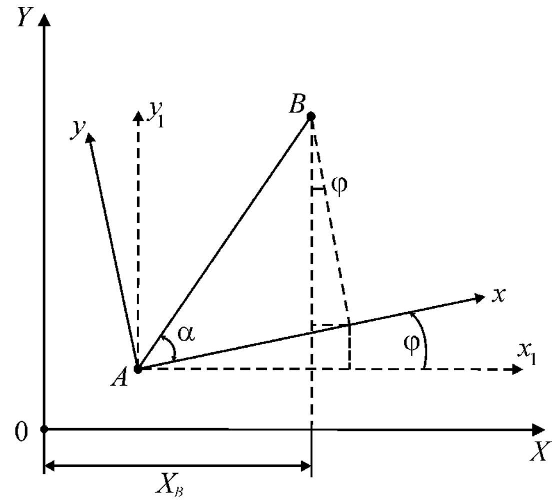


Figure 3.3 – Determination of the coordinates of an arbitrary point of the body in a plane-parallel motion

– On the one hand, if a segment AB is given then the law of the point B motion becomes the following:

$$\begin{aligned} X_B &= X_A(t) + AB \cdot \cos(\varphi(t) + \alpha); \\ Y_B &= Y_A(t) + AB \cdot \sin(\varphi(t) + \alpha), \end{aligned} \tag{3.2}$$

where X_B, Y_B - coordinates of a point B in a fixed coordinate system; α - the angle forming the segment AB from the axis x of the moving coordinate system. Since the moving coordinate system is related to the points of the solid body, the angle α under the motion of the body remains constant (see

Fig. 3.4)

– on the other hand, one can use the communication formulas when rotation of one coordinate system relative to another:

$$X_B = X_A(t) + x_B \cdot \cos\varphi(t) - y \cdot \sin\varphi(t);$$
$$Y_B = Y_A(t) + x_B \cdot \sin\varphi(t) + y \cdot \cos\varphi(t), \qquad (3.3)$$

where x_B, y_B – the coordinates of the point B in the moving coordinate system, which remain unchanged in the process of motion. According to the equations of motion of point (3.2) or (3.3), the velocity and acceleration of a point, can be determined by formulas (1.11) – (1.14) and (1.23) – (1.26).

Also, the kinematic characteristics of the point of the body in a plane-parallel motion can be determined by the *vector method*. According to this method, the velocity of any point B of a body is the vector geometrical sum of the velocity of any other point A taken as the pole and the velocity of rotation of point B about the pole (See Fig. 3.4). In this case, the velocity vector is defined as follows:

$$\vec{v}_B = \vec{v}_A + \vec{v}_{BA} = \vec{v}_A + \vec{\omega} \times \overline{AB}, \quad \text{or}$$
$$\vec{v}_B = (v_{Ax} - \omega(Y_B - Y_A))\vec{i} + (v_{Ay} - \omega(X_B - X_A))\vec{j}, \qquad (3.4)$$

where v_{Ax}, v_{Ay} – projections of the velocity vector of the pole on the axis of the fixed coordinate system; $\vec{\omega}, \omega$ – vector and magnitude of angular velocity of a body.

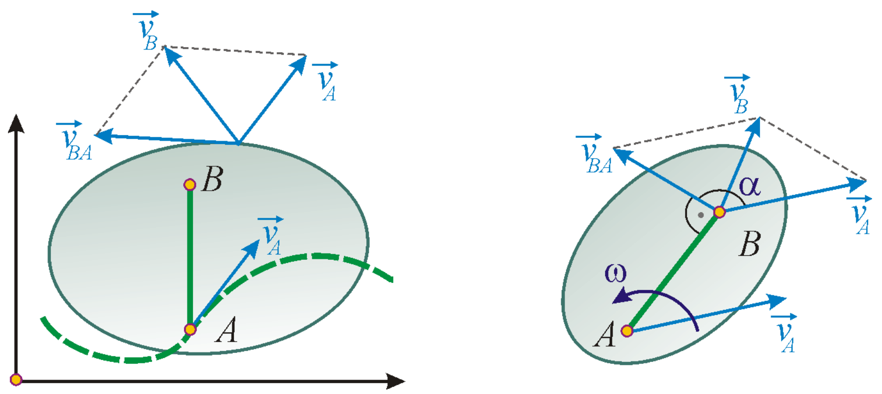


Figure 3.4 – Vector method of determination of the velocity of an arbitrary

point of the body in a plane-parallel motion

The velocity magnitudes of two arbitrary points of a body under the plane-parallel motion conditions of are governed by the next theorem: t*he projections of the velocities of two points of a rigid body on the straight line joining those points are equal* (See Fig. 3.5):

$$v_B \cos\alpha = v_B \cos\beta. \tag{3.5}$$

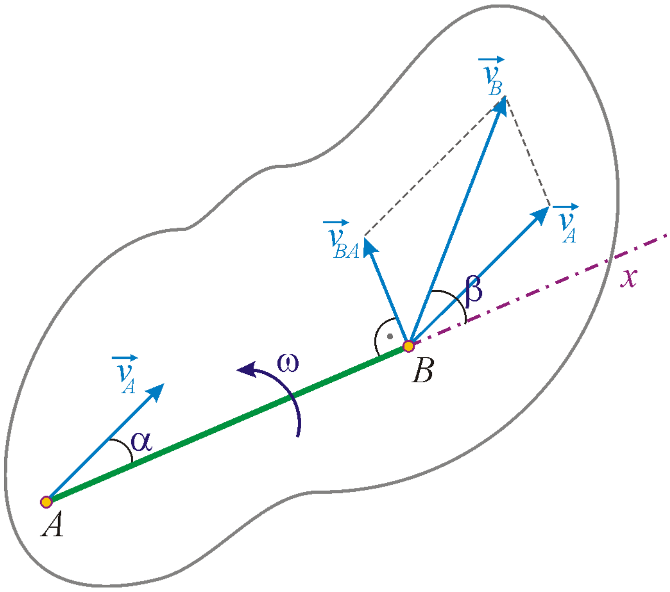


Figure 3.5 – Correlation between velocities of two arbitrary points of the body in a plane-parallel motion

The acceleration of any point *B* of a body is composed of the acceleration of any other point *A* taken for the pole and the acceleration of the point *B* in its rotation together with the body about that pole:

$$\vec{a}_B = \vec{a}_A + \vec{a}_{BA} = \vec{a}_A + \vec{a}_{BA}^{\tau} + \vec{a}_{BA}^{n} = \vec{a}_A + \vec{\varepsilon} \times \overrightarrow{AB} + \vec{\omega} \times \vec{\omega} \times \overrightarrow{AB},$$

or

$$\vec{a}_B = \left(a_{Ax} - \varepsilon(Y_B - Y_A) - \omega^2(Y_B - Y_A)\right)\vec{i} + \\ + \left(a_{Ay} - \varepsilon(X_B - X_A) - \omega^2(Y_B - Y_A)\right)\vec{j}, \tag{3.6}$$

where $a_{BA}^{\tau}, a_{BA}^{n}$ – rotational and centripetal components of $\vec{a}_{BA}$; a_{Ax}, a_{Ay} – projections of the vector of acceleration of the pole on the axis of the fixed coordinate system; $\vec{\varepsilon}, \varepsilon$ - vector and magnitude of angular acceleration of a body.

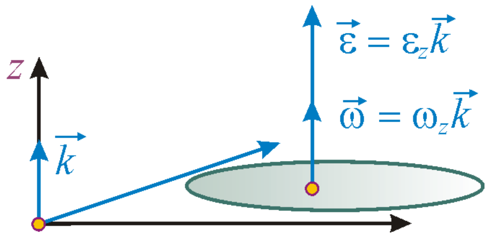


Figure 3.6 – Vectors of angular velocity and angular acceleration respectively to the principle plane

It should be noted that the vectors of the angular velocity $\vec{\omega} = \dot{\varphi}\vec{k}$ of the body and its angular acceleration $\vec{\varepsilon} = \dot{\omega}\vec{k} = \ddot{\varphi}\vec{k}$ in a plane-parallel motion are always perpendicular to the principle plane (See Fig. 3.6). From here you can get comfortable formulas for determining the angular velocity and angular acceleration of the body at its plane-parallel motion at known speeds and accelerations of its two points:

$$\omega = \frac{v_{Ax} - v_{Bx}}{Y_B - Y_A} = \frac{v_{Ay} - v_{By}}{X_B - X_A};$$
$$\varepsilon = \frac{a_{Ax} - a_{Bx} - \omega^2 (X_B - X_A)}{Y_B - Y_A} = \frac{a_{Ay} - a_{By} - \omega^2 (Y_B - Y_A)}{X_B - X_A}. \qquad (3.6)$$

If we analyze the theory of a plane-parallel motion of a solid, then we can formulate next conclusion: *at any moment of the motion of a flat figure along the main plane there is a point whose velocity is zero. This point is called the instantaneous center of zero velocity (accepted reduction - ICZV).*

The ICZV is at the intersection of perpendiculars to the velocity vectors of the points (Fig. 3.7). If we now the vector of the velocity of one point (for example *A*), then we can determine the direction of instantaneous rotation and the magnitude of the angular velocity:

$$\omega = \frac{v_A}{PA}. \tag{3.7}$$

After that we may define magnitude and direction of the velocity of any other point, for example:

$$v_B = \omega \cdot PB; \;\; v_C = \omega \cdot CB.$$

Note, that the velocity vector of each point is perpendicular to the segment between point and ICZV.

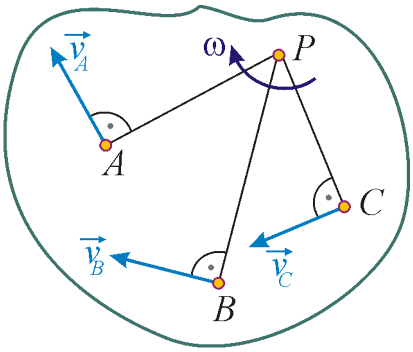


Figure 3.7 – The general case of determining the position of the ICZV

Typical cases of the definition of ICZV include the following:

1) When the wheel is rolling without sliding on a fixed surface (Fig. 3.8). At every moment of motion the ICZV is a point of contact of the body with a stationary surface. The angular velocity of the wheel is determined from the ratio of the wheel center velocity to its radius (R):

$$\omega = \frac{v_A}{PO} = \frac{v_A}{R};$$

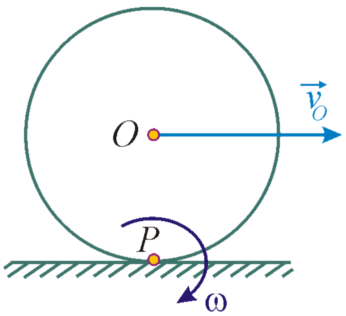


Figure 3.8 - Determination of the position of the ICZV, instantaneous angular velocity of the rolling wheel

2) When we know directions and magnitudes of velocity of two points

of a body, and velocity vectors are parallel with each other and perpendicular to the line connecting the points, than the ICZV is located at the intersection of the line connecting the points, and the line connecting the ends of the velocity vectors (Fig. 3.9 a). With the opposite direction of the velocity vectors (Fig. 3.9 b), the ICZV is located between the points whose velocities are known.

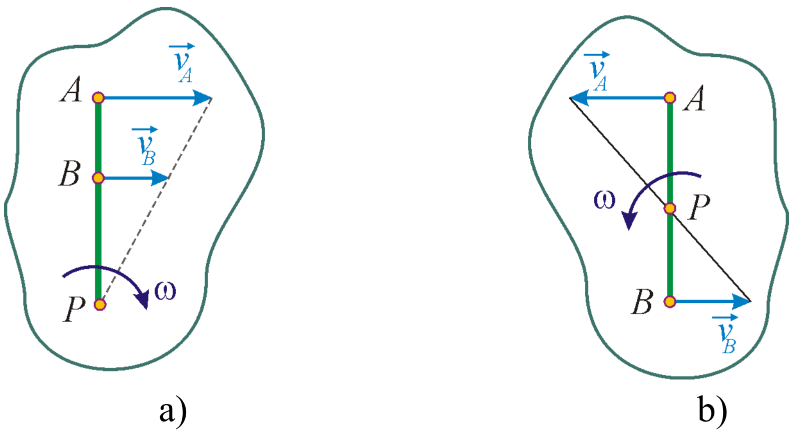


Figure 3.9 – Determination of the position of the ICZV, if the vectors of velocities of two points of a body are perpendicular to the line connecting them: a) the vectors are directed to one and the same side; b) the vectors are directed to the opposite sides

3) When the velocity vectors of the points are parallel to each other and not perpendicular to the segment connecting them (Fig. 3.10), than this is so called *instantaneous translation* and the angular velocity of the body is zero, and the velocities of all its points are equal to each other. It should also be added that the equality of velocities is observed only at the instant of motion of the body, but at the same time, the acceleration of the points of the body are different.

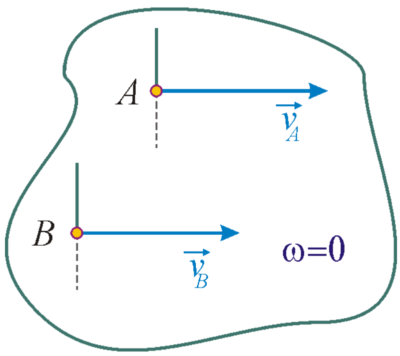


Figure 3.10 – The case of the instantaneous translation

3.2. Problems and Solutions

Example 1. The wheel with the radius $r = 40$см is rolling on a fixed horizontal plane. The velocity of wheel center is $v_A = 60\,\text{cm/sec}$, its acceleration is – $a_A = 35\,\text{cm/sec}^2$. The motion is decelerated. It is necessary to determine the velocities and accelerations of points B and C (Fig. 3.11 a), if $AC = 15\text{cm}$.

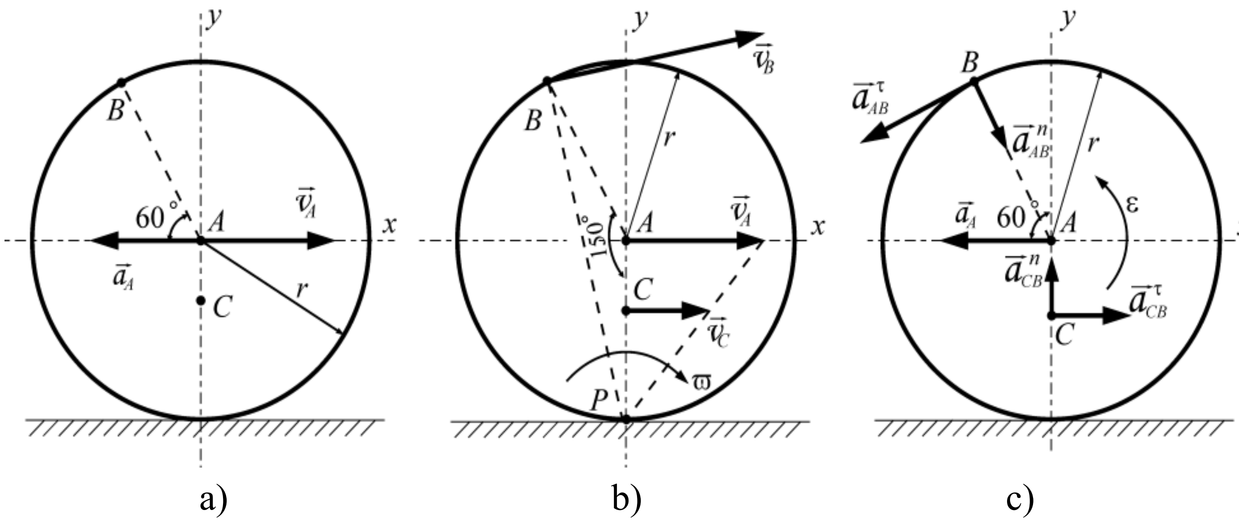


Figure 3.10 – To the first example

To determine the velocities, we use the notion of instant center of velocities (ICV). In this case, the ICV is located at the point of contact of the moving wheel and the fixed plane - at the point P (Fig.1.3б). The vectors of the velocities of the points A, B and C are perpendicular to the segments connecting these points with the ICV. According to the equations for a planar motion, the angular velocity of the wheel is determined as follows:

$$\omega = \frac{v_A}{AP} = \frac{v_A}{r} = 1{,}5\ \text{sec}^{-1}.$$

The magnitudes of velocities of the points B and C are determined by the following formulas:

$$v_B = \omega \cdot BP, \quad v_C = \omega \cdot CP,$$

where we find the corresponding distances in the following way:

$$CP = AP - AC = r - AC = 25 \text{ см},$$
$$\mathrm{BP} = \sqrt{\mathrm{AB}^2 + AP^2 - 2 \cdot AB \cdot AP \cdot \cos\angle BAP} =$$
$$= \sqrt{r^2 + r^2 - 2 \cdot r \cdot r \cdot \cos 150^\circ} = 77{,}3 \text{ cm}.$$

Finally we obtain:

$$v_B = 115{,}91 \,\text{cm/sec}, \quad v_C = 37{,}5 \,\text{cm/sec}.$$

Due to the fact that the trajectory of point A (center of the wheel) is a horizontal line, then the vector of full acceleration of point A lies on this line, and because of the fact that the motion is decelerated, it is directed towards the opposite direction to the vector of the velocity of point A (see Fig. 1.3a). The angular acceleration of the wheel can be defined as follows:

$$\varepsilon = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{v_A}{r}\right) = \frac{\dot{v}_A}{r} = \frac{a_{A\tau}}{r} = \frac{a_A}{r} = 0{,}875 \text{ sec}^{-2}.$$

To find the accelerations of the points B and C we use theorem on the acceleration of the points in a planar motion, and choosing point *A* as a pole. Thus, for point *B* we get:

$$\vec{a}_B = \vec{a}_A + \vec{a}_{BA}^{\tau} + \vec{a}_{BA}^{n}, \tag{3.8}$$

here $\vec{a}_{BA}^{\tau}$ and $\vec{a}_{BA}^{n}$ – vectors of rotational and centripetal acceleration of point B in its rotational motion around the pole (point A). Vector $\vec{a}_{BA}^{n}$ always directed from point B to point A. Vector $\vec{a}_{BA}^{\tau}$ is aligned perpendicular to $\vec{a}_{BA}^{n}$ in the direction, that is determined by the angular acceleration. (see. Fig.1.3c). The magnitudes of these vectors are defined by:

$$a_{BA}^{n} = \omega^2 \cdot AB = 90 \,\text{cm/sec}^2, \quad a_{BA}^{n} = \varepsilon \cdot AB = 35 \,\text{cm/sec}^2.$$

The magnitude of the vector $\vec{a}_B$ can be determined by projecting the vector equation (3.8) on the Cartesian coordinate axes:

$$a_{Bx} = -a_A + a_{BA}^n \cdot \sin\left(30^\circ\right) - a_{BA}^\tau \cdot \cos\left(30^\circ\right) = -20,31\ \text{cm/sec}^2;$$
$$a_{By} = -a_{BA}^n \cdot \cos\left(30^\circ\right) - a_{BA}^\tau \cdot \sin\left(60^\circ\right) = -95,44\ \text{cm/sec}^2;$$
$$a_B = \sqrt{a_{Bx}^2 + a_{By}^2} \approx 97,58\,\text{cm/sec}^2.$$

Then, in a similar way, we determine the acceleration of the point C:

$$\vec{a}_C = \vec{a}_A + \vec{a}_{CA}^\tau + \vec{a}_{CA}^n;$$
$$a_{CA}^\tau = \varepsilon \cdot AC = 13{,}125\,\text{cm/sec}^2;$$
$$a_{CA}^n = \omega^2 \cdot AC = 33,75\,\text{cm/sec}^2;$$
$$a_{Cx} = -a_A^n \cdot \cos\left(30^\circ\right) - a_A^\tau \cdot \cos\left(60^\circ\right) + a_{CA}^n \approx 136,21\,\text{cm/sec}^2;$$
$$a_{Cy} = -a_A^n \cdot \sin\left(30^\circ\right) - a_A^\tau \cdot \sin\left(60^\circ\right) + a_{CA}^\tau \approx -99{,}14\,\text{cm/sec}^2;$$
$$a_C = \sqrt{\left(a_{Cx}\right)^2 + \left(a_{Cy}\right)^2} \approx 168,47\,\text{cm/sec}^2.$$

Example 2. The diagram of the crank mechanism is shown on Fig. 3.11 at an arbitrary position. The mechanism consists of a crank *OA*, which rotates about a fixed axis located on point *O*, the connecting rod *AB* and the slide *B*, which can move only horizontally. The rotation of the crank occurs according to the law of the angle measured between the crank and the horizontal axis:

$$\varphi = t^2, \tag{3.9}$$

It is necessary to determine the velocity and acceleration of points *A*, *B*, *C* and the angular velocity and angular acceleration of rod *AB* for the instant of time when the rotation angle of the crank is $\varphi_* = \dfrac{\pi}{6}$. The elements of the mechanism have the following dimensions: $OA = 30\,\text{см}$, $AB = 60\,\text{см}$, $AC = 30\,\text{см}$.

The parts of the mechanism carry out the following types of motion: crank *OA* – rotational motion, the connecting rod *AB* – planar motion, the slider B – translational motion.

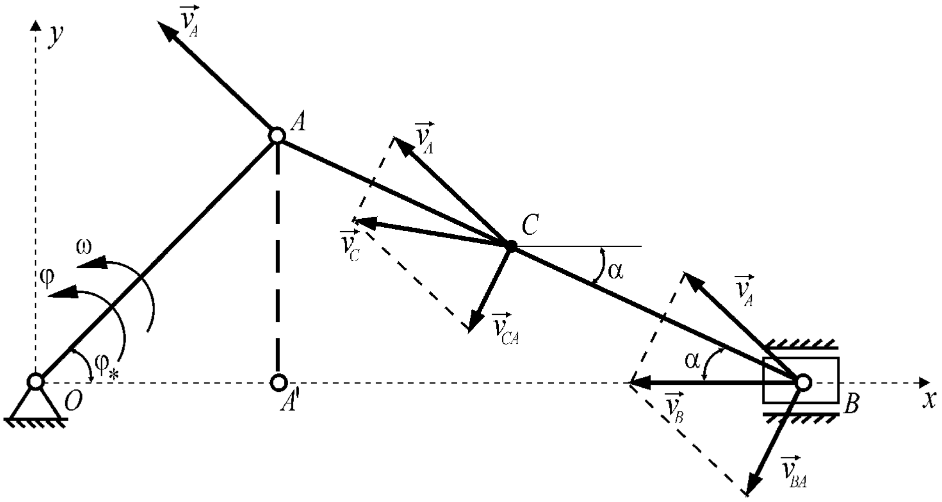


Figure 3.11 – The diagram of the crank mechanism and the distribution of the velocities

First, we define a time instant corresponding to the specified angle of rotation of the crank. It is necessary to solve the algebraic equation $\varphi = \varphi_*$, $t^2 = \frac{\pi}{6}$, $t_* \approx 0,72$ c. To determine the kinematic characteristics of the points it is necessary to know the angular velocity and angular acceleration of the crank OA:

$$\omega = \dot{\varphi} = 2t\Big|_{t=t_*} = 1,44\ \text{c}^{-1};$$
$$\varepsilon = \dot{\omega}_{OA} = 2\ \text{c}^{-2}.$$

Next, determine the velocity of point A:

$$v_A = \omega_{OA} \cdot OA = 43,2\ \frac{\text{см}}{\text{c}}.$$

Vector $\vec{v}_A$ is directed perpendicular to the crank OA in accordance with the direction of the angular velocity ω_{OA} (see Fig. 3.11). Since the slider can only move along the horizontal rails, then its velocity is directed horizontally to the side, to satisfy the theorem on the equality of the projection of the velocity on the segment *AB*. The connecting rod *AB* has plane-parallel motion, therefore, to determine the velocity of its points we use the formulas (3.4), if point *A* is selected as a pole:

$$\vec{v}_B = \vec{v}_A + \vec{v}_{BA};$$
$$v_{BA} = \omega_{AB} \cdot AB. \tag{3.10}$$

The value of the angular velocity of the rod *AB* is unknown, but we know that the vector of the velocity $\vec{v}_{BA}$ is perpendicular to the segment *AB*, as the result that we can design the vector equality of (3.10) on the axis of a Cartesian coordinate system and obtain the system of two algebraic equations, where the unknowns will be v_B and v_{BA}:

$$-v_B = -v_A \cdot \sin\varphi_* - v_{BA} \sin\alpha;$$
$$0 = v_A \cdot \cos\varphi_* - v_{BA} \cos\alpha. \tag{3.11}$$

To solve this system it is necessary to determine the trigonometric functions of the angle α, which should consider of the ratio in right triangles $OA'A$ and $BA'A$:

$$AA' = OA \cdot \sin\varphi_*; \quad \sin\alpha = \frac{AA'}{AB} = 0{,}25;$$
$$\cos\alpha = \sqrt{1 - \sin^2\alpha} = 0{,}968.$$

Finally, after solving the system (3.11), we obtain:

$$v_B \approx 31{,}26\,\frac{\text{см}}{\text{с}}, \quad v_{BA} \approx 38{,}65\,\frac{\text{см}}{\text{с}}, \quad \omega_{AB} = \frac{v_{BA}}{AB} = 0.64\ \text{c}^{-1}$$

Then in the same way we define the velocity of a particle on rod *AB*:

$$\vec{v}_C = \vec{v}_A + \vec{v}_{CA}; \quad \vec{v}_{CA} \perp AB; \quad v_{CA} = \omega_{AB} \cdot AC \approx 19{,}2\,\text{cm/sec};$$

$$v_{Cx} = -v_A \cdot \sin\varphi_* - v_{CA} \cdot \sin\alpha \approx -26{,}4\,\text{cm/sec};$$

$$v_{Cy} = v_A \cdot \cos\varphi_* - v_{CA} \cdot \cos\alpha \approx 18{,}82\,\text{cm/sec};$$

$$v_C = \sqrt{\left(v_{Cx}\right)^2 + \left(v_{Cy}\right)^2} \approx 32{,}42\,\text{cm/sec}.$$

It should be noted that the velocities of points of the body, that carried out planar movement can also be defined using the concept of instantaneous center of zero velocities.

Now we proceed to the definition of accelerations. Vector of acceleration of the point A can be defined as:

$$\vec{a}_A = \vec{a}_A^{\,\text{об}} + \vec{a}_A^{\,\text{до}},$$

where $\vec{a}_A^{\,\text{об}}$ $\vec{a}_A^{\,об}$ and $\vec{a}_A^{\,\text{до}}$ – vectors of rotational and centripetal acceleration of the point A, that are directed as shown in Fig. 3.12. Their magnitudes and the magnitude of full acceleration of point A is equal to:

$$a_A^{\text{rt}} = \varepsilon \cdot OA = 60\ \text{cm/sec}^2;\ a_A^{\text{cp}} = \omega^2 \cdot OA = 62{,}21\ \text{cm/sec}^2;$$

$$a_A = \sqrt{\left(a_A^{\text{rt}}\right)^2 + \left(a_A^{\text{cp}}\right)^2} = 86{,}43\ \text{cm/sec}^2.$$

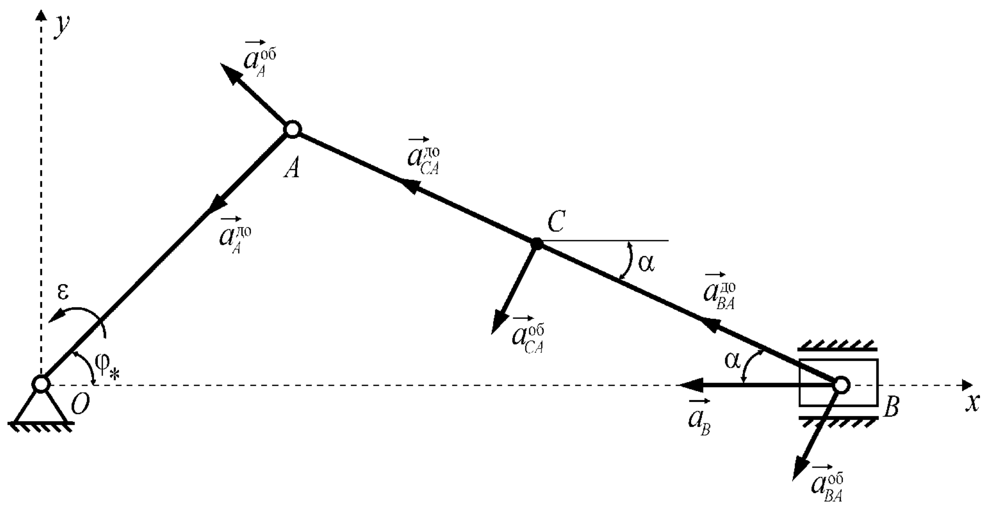


Figure 3.12 – Distribution of accelerations

To determine the accelerations of points B and C we use the theorem on accelerations of points in a planar motion, with point A chosen as the pole. Thus, we get for point B:

$$\vec{a}_B = \vec{a}_A + \vec{a}_{BA}^{\text{rt}} + \vec{a}_{BA}^{\text{cp}} = \vec{a}_A^{\text{rt}} + \vec{a}_A^{\text{cp}} + \vec{a}_{BA}^{\text{rt}} + \vec{a}_{BA}^{\text{cp}}, \tag{3.12}$$

where $\vec{a}_{BA}^{\text{об}}$ and $\vec{a}_{BA}^{\text{до}}$ – vectors of rotational and centripetal acceleration of the point B in its rotational motion around the pole (point A). Vector of centripetal acceleration of point B is directed from point B to point A, and its magnitude is equal to:

$$a_{BA}^{\text{cp}} = \omega_{AB}^2 \cdot AB = 24{,}58 \ \text{cm/sec}^2$$

For vector $\vec{a}_B$ it is known that its acting line locates on the horizontal axis along the rails of the slider. For vector $\vec{a}_{BA}^{\text{об}}$ it is known that its acting line is perpendicular to the vector $\vec{a}_{BA}^{\text{до}}$. We define the direction of these vectors randomly along these lines (see Fig. 3.12). Magnitudes of unknown accelerations we can determine using the projection of vector equality (3.12) on the axis of a Cartesian coordinate system:

$$-a_B = -a_A^{\text{rt}} \cdot \sin\varphi_* - a_A^{\text{cp}} \cdot \cos\varphi_* - a_{BA}^{\text{rt}} \cdot \sin\alpha - a_{BA}^{\text{cp}} \cdot \cos\alpha;$$
$$0 = a_A^{\text{rt}} \cdot \cos\varphi_* - a_A^{\text{cp}} \cdot \sin\varphi_* - a_{BA}^{\text{rt}} \cdot \cos\alpha + a_{BA}^{\text{cp}} \cdot \sin\alpha;$$
$$a_B \approx 114{,}64 \ \text{cm/sec}^2; \ a_{BA}^{\text{rt}} \approx 27{,}89 \ \text{cm/sec}^2.$$

The sign of the algebraic values of these accelerations is positive, i.e., their directions have been chosen correctly.

The angular acceleration of the rod AB can be determined using expression for magnitude of the rotational acceleration of the point in its motion relative to point A:

$$\varepsilon_{AB} = \frac{a_{BA}^{\text{rt}}}{AB} \approx 0{,}47 \ \text{sec}^{-2}$$

Then in the same way we define the acceleration of a point C of the rod AB:

$$\vec{a}_C = \vec{a}_A + \vec{a}_{CA}^{\text{rt}} + \vec{a}_{CA}^{\text{cp}} = \vec{a}_A^{\text{rt}} + \vec{a}_A^{\text{cp}} + \vec{a}_{CA}^{\text{rt}} + \vec{a}_{CA}^{\text{cp}};$$

$$a_{CA}^{\text{rt}} = \varepsilon_{AB} \cdot AC \approx 14{,}1\ \text{cm}/\text{sec}^2;$$

$$a_{CA}^{\text{cp}} = \omega_{AB}^2 \cdot AC \approx 12{,}29\ \ \text{cm}/\text{sec}^2;$$

$$a_{Cx} = -a_A^{\text{rt}} \cdot \sin\varphi_* + a_A^{\text{cp}} \cdot \cos\varphi_* - a_{CA}^{\text{rt}} \cdot \sin\alpha +$$
$$+a_{CA}^{\text{cp}} \cdot \cos\alpha \approx -99{,}31\ \text{cm}/\text{sec}^2;$$

$$a_{Cy} = a_A^{\text{rt}} \cdot \cos\varphi_* - a_A^{\text{cp}} \cdot \sin\varphi_* - a_{CA}^{\text{rt}} \cdot \cos\alpha +$$
$$+a_{CA}^{\text{cp}} \cdot \sin\alpha \approx 10{,}28\ \text{cm}/\text{sec}^2;$$

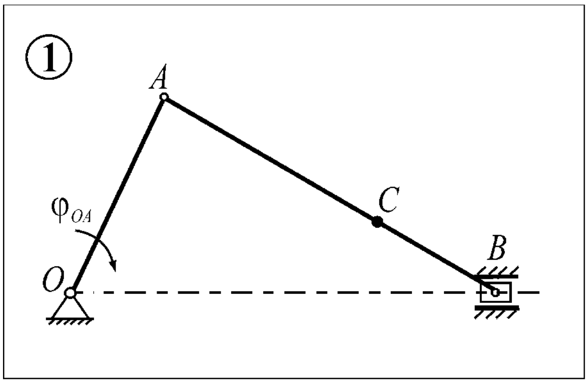
$$a_C = \sqrt{\left(a_{Cx}\right)^2 + \left(a_{Cy}\right)^2} \approx 99{,}84\ \ \text{cm}/\text{sec}^2.$$

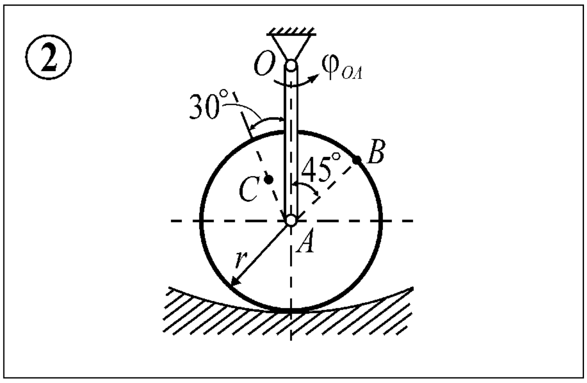
3.3. Tasks for solving.

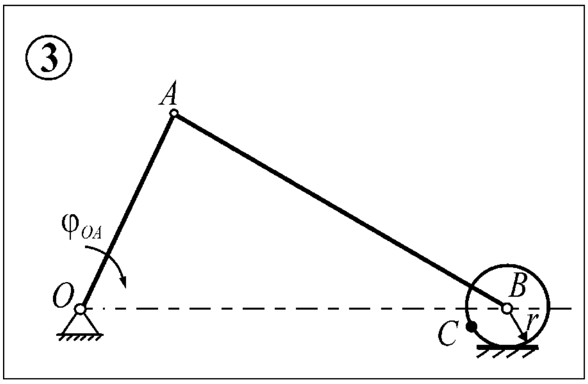
Diagrams of mechanical systems, elements of which make plane movement are shown in the table 3.1. The time dependence of the rotation angle of the cranc is set in the variants of 3.2. The geometrical sizes are specified in the variants of 3.2.

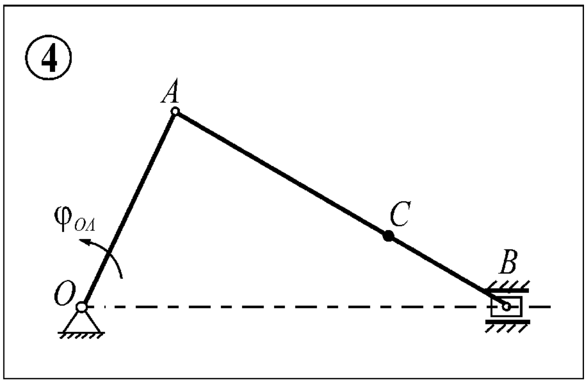
It is necessary to determine the kinematic characteristics of points and bodies in the system. The initial data are the geometric dimensions of the system elements and the law of change in the time of the angle of rotation of the body of the OA (the arrow indicates the positive direction of rotation angle change).

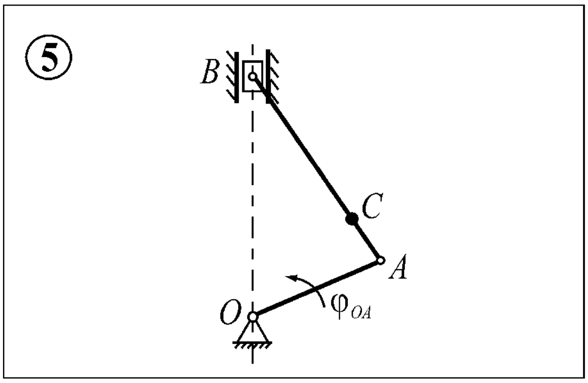
Table 3.1. Diagrams of mechanical systems.

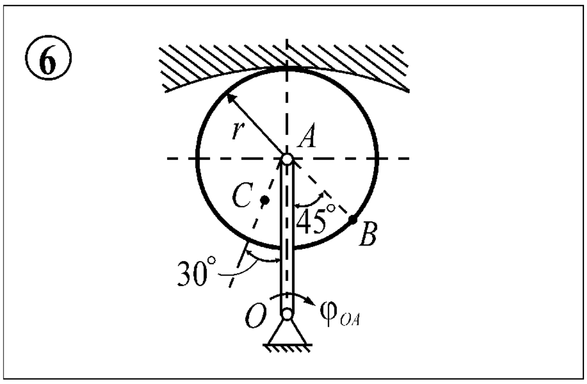


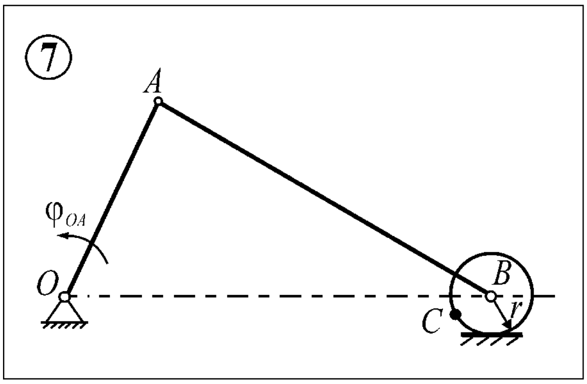


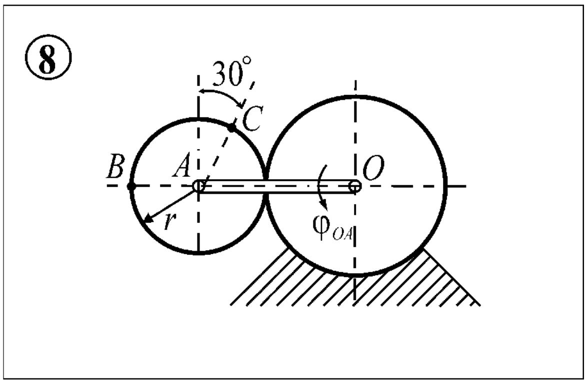












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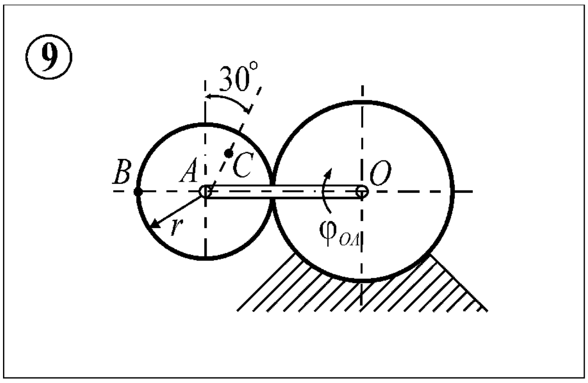

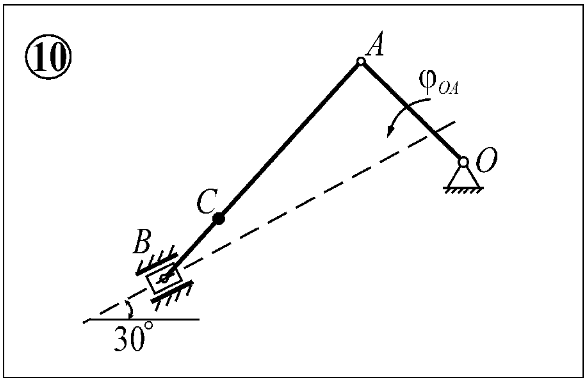

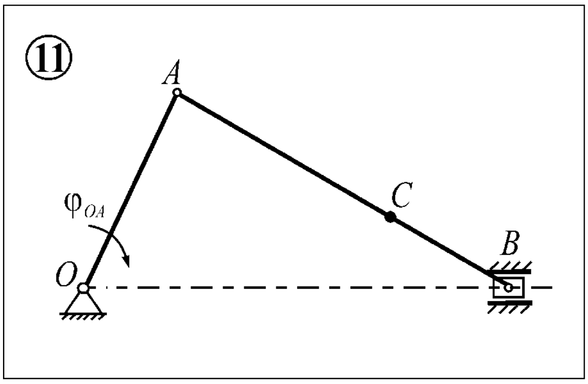

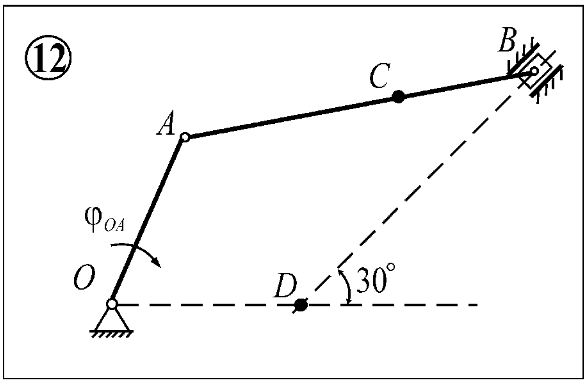

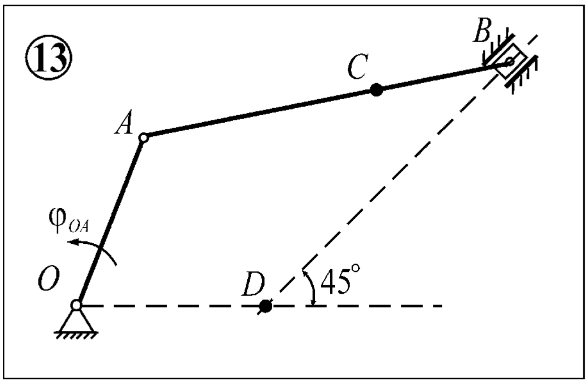

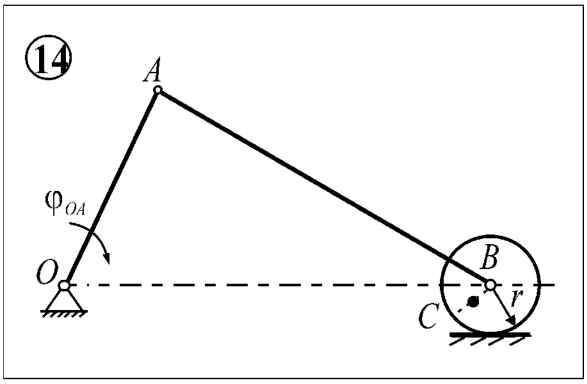

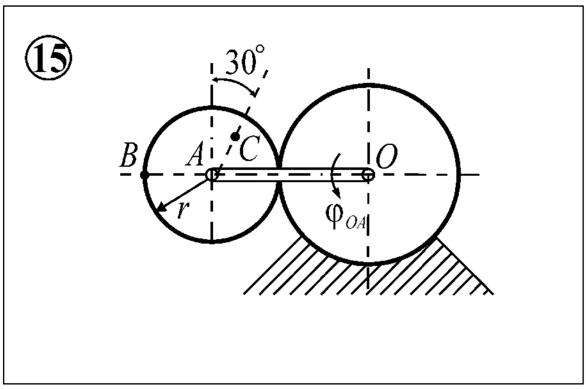

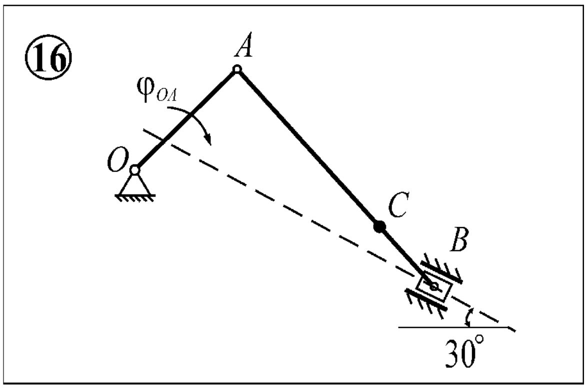

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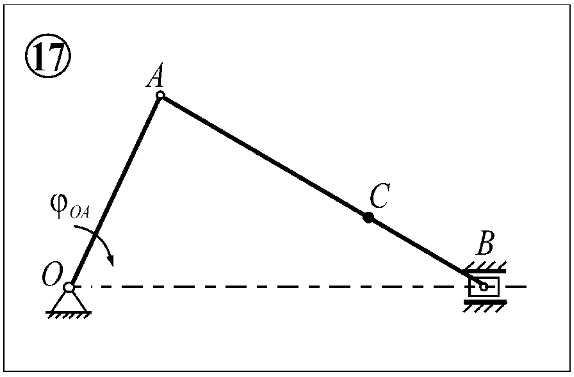

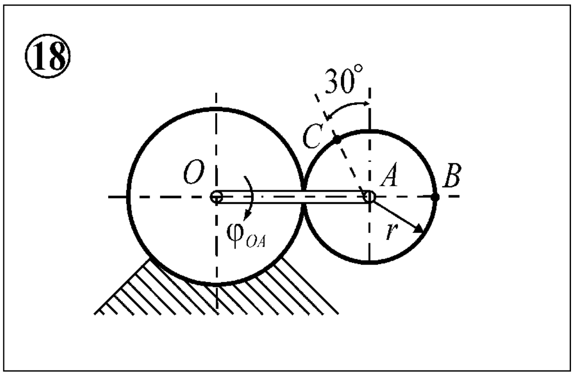

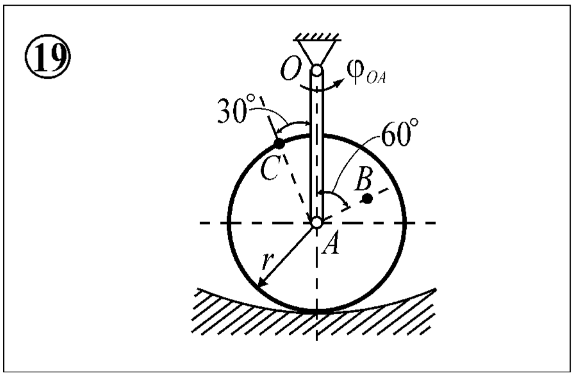

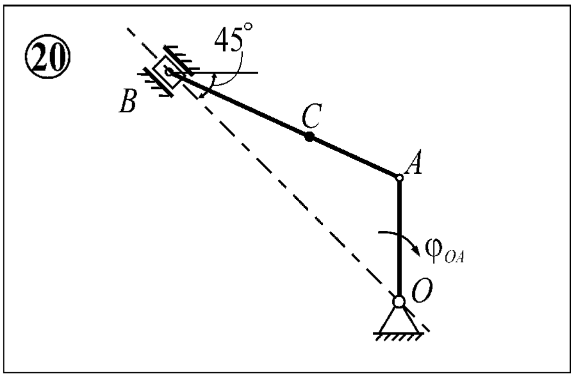

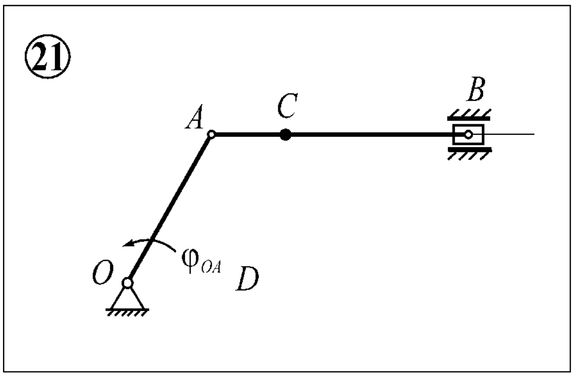

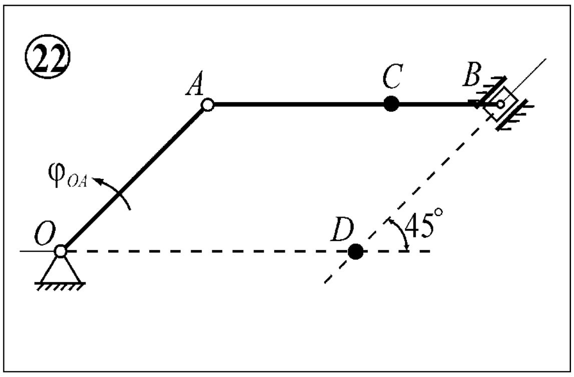

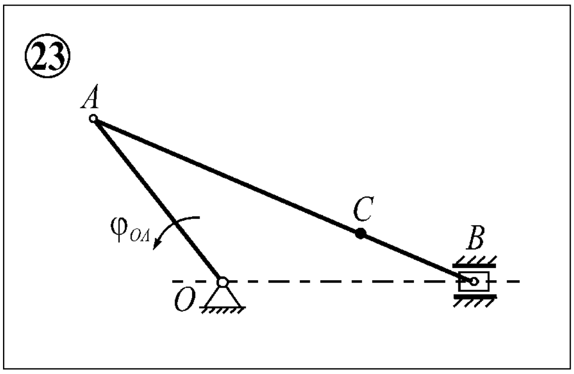

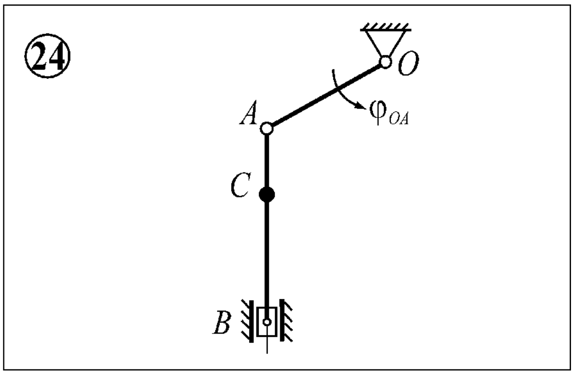

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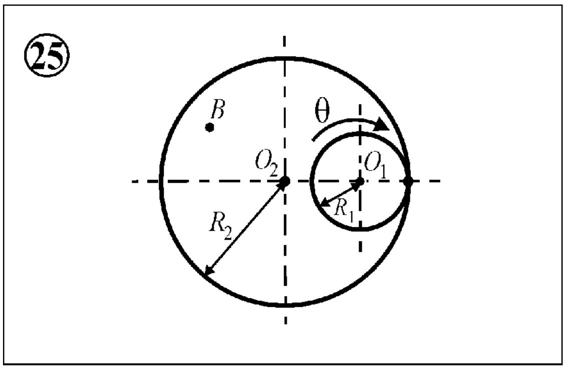
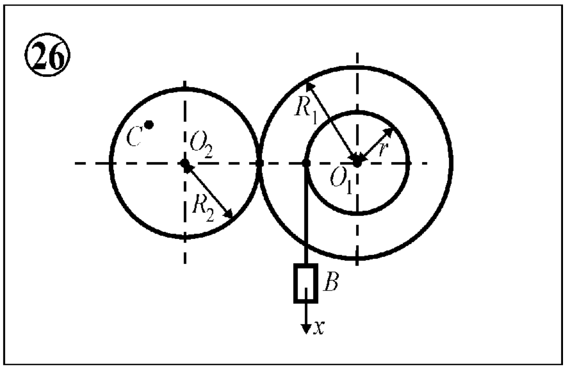
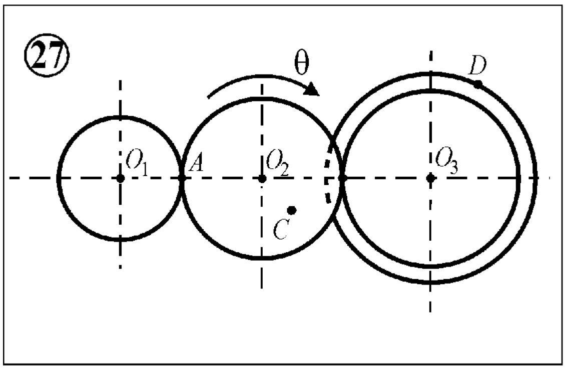
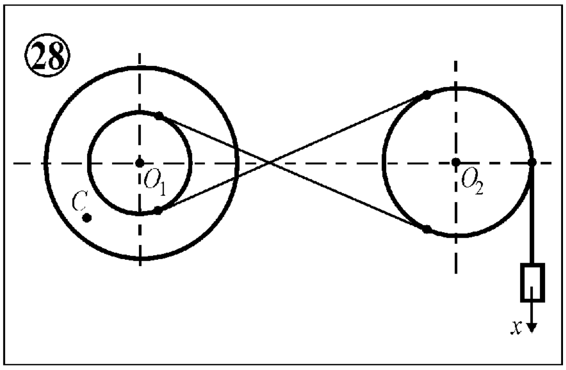
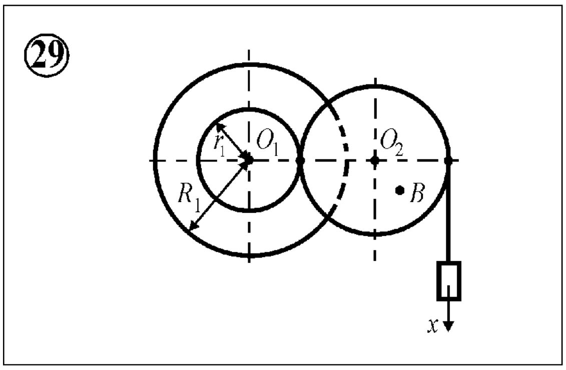
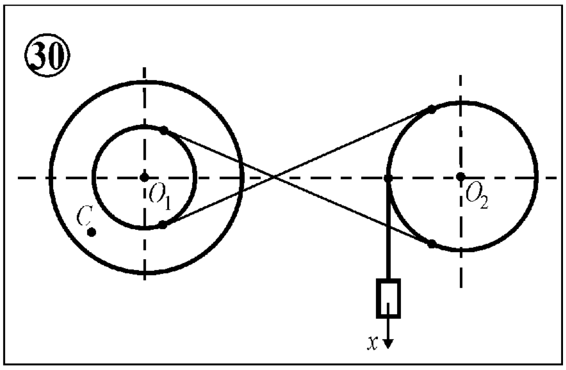


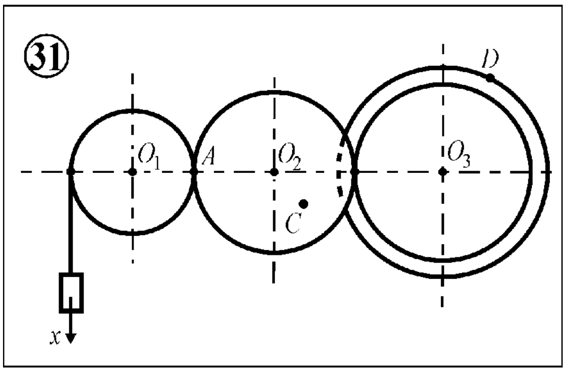












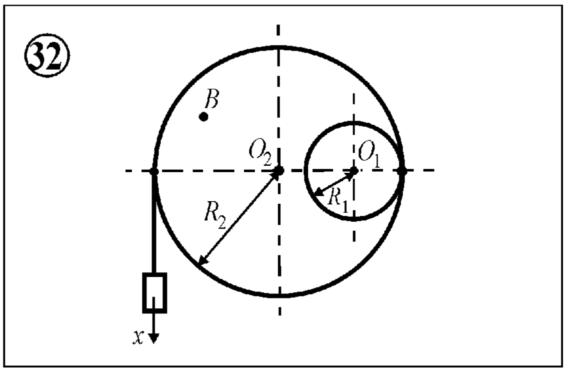


Table 3.2. Conditions of motion.

Variant N	Low of motion φ_{OA}, rad	φ_*, rad
1	$3t - t^2$	$\pi/6$
2	$2\cos(2t)$	$\pi/4$
3	$3t + 2t^2$	$\pi/3$
4	$3t - 0{,}5t^2$	$\pi/6$
5	$0{,}5\pi\sin^2(3t)$	$\pi/2$
6	$2\cos(1{,}5t)$	$2\pi/3$
7	$t + t^2$	π
8	$-\pi\sin^2(3t)$	$-\pi/6$
9	$\pi\sin^2(3t)$	$\pi/4$
10	$3t - t^2$	2π
11	$10t - 2t^2$	$-\pi/6$
12	$t + 0{,}3t^2$	$\pi/2$
13	$3t + 4t^2$	2π
14	$2t + 3t^2$	$\pi/6$
15	$-0{,}5\pi\sin^2(2t)$	$-\pi/6$
16	$-2t - t^2$	-2π
17	$-3t + t^2$	π
18	$-2\pi/3\sin^2(2t)$	$-\pi/6$
19	$4\cos(2t)$	$\pi/4$
20	$0{,}5\pi\sin^2(t)$	$\pi/4$
21	$13t - 10t^2$	π
22	$4t + 5t^2$	$\pi/4$
23	$3t + 2{,}5t^2$	$\pi/3$
24	$3t + 1{,}5t^2$	2π

Table 3.3. Geometrical sizes.

Variant N	OA, cm	AB, cm	AC, cm	BC, cm	OD, cm	r, cm
1	10	30	20	–	–	–
2	12	–	3	–	–	6
3	14	32	–	–	–	10
4	16	34	25	–	–	–
5	18	36	10	–	–	–
6	20	–	5	–	–	8
7	22	38	–	–	–	8
8	24	–	–	–	–	8
9	26	–	5	–	–	10
10	24	40	30	–	–	–
11	22	42	35	–	–	–
12	20	40	30	–	20	–
13	18	38	25	–	18	–
14	16	36	–	4	–	8
15	14	–	4	–	–	6
16	12	34	28	–	–	–
17	10	32	26	–	–	–
18	12	–	–	–	–	4
19	14	4	–	–	–	6
20	16	30	15	–	–	–
21	18	32	12	–	–	–
22	20	34	28	–	34	–
23	22	36	30	–	–	–
24	24	38	14	–	–	–

Questions

1) What is the plane-parallel motion?

2) Give the examples of technical devices that perform plane-parallel motion?

3) Give the examples of nature objects that perform plane-parallel motion?

4) How many Cartesian coordinates we need to describe the plane-parallel motion?

5) What are the kinematic equations of a plane-parallel motion?

6) How many non-zero angles of body rotation are there in the case of plane-parallel motion?

7) How many parts are there in the crank mechanism?

8) What is the crank?

9) What is the connecting rod?

10) What is the slider?

11) What type of motion has the crank?

12) What type of motion has the connecting rod?

13) What type of motion has the slider?

14) How we can define the velocity of a point under the condition of the plane-parallel motion?

15) How we can define the acceleration of a point under the condition of the plane-parallel motion?

16) Where is the vector of point velocity under the condition of the plane-parallel motion?

17) Where is the vector of point under the condition of the plane-parallel motion?

18) What direction has the vector of the angular velocity under the condition of the plane-parallel motion?

19) What direction has the vector of the angular acceleration under the condition of the plane-parallel motion?

20) What is the instantaneous center of zero velocity in a plane-parallel motion?

21) Under what conditions a solid body carries a plane-parallel motion?

22) Which formula does the vector of velocity of a point of a body, which moves plane-parallel, is calculated?

23) By what formula the vector of acceleration of a point of a body, which moves flat-parallel, is calculated?

24) Which formula can determine the value of the angular velocity of the body in a plane-parallel motion?

25) Which formula can determine the values of angular acceleration of the body in a plane-parallel motion?

4. COMPLEX MOTION OF A PARTICLE.

4.1. Theoretical material.

In the part 1 we have considered the motion of a particle with respect to one fixed frame of reference. But in solving problems of mechanics it is often more expedient (and sometimes necessary) to consider the motion of a particle (or body) simultaneously with respect to two frames of reference, one of which is assumed to be fixed and the other moving in some specified way with reference to the first. The performed motion by the particle in this case is called ***complex*** **(***ccompound* or *resultant*) motion.

The method of resolving of a motion into simpler motions by introducing a supplementary moving frame of reference is widely employed in kinematic calculations.

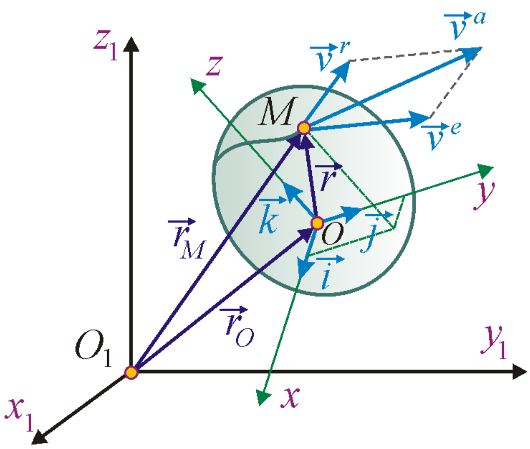


Figure 4.1 – Conception of the complex motion

Let's consider the complex motion of a particle M moving with respect to a frame of reference $Oxyz$ which is in turn moving with relation to another frame of reference $O_1x_1y_1z_1$, which we assume to be fixed.

The motion performed by the particle M with respect to the mobile coordinate system is called *relative motion* (this is the motion which is seen by an observer moving together with the mobile axes $Oxyz$).

The trajectory AB described by the particle in relative motion is called the *relative trajectory*. The velocity of the motion of particle M relative to the axes $Oxyz$ (i.e., along the relative trajectory) is called the *relative velocity* (denoted by the symbol $\vec{v}^r$), and the particle acceleration in that motion is the *relative acceleration* (denoted $\vec{a}^r$). It follows from the definition that in computing $\vec{v}^r$ and $\vec{a}^r$ axes *Oxyz* can be assumed as fixed axes.

The motion of the particle *M* performed by the moving frame of reference *Oxyz*, together with all the points of space fixed with respect to it, relative to the fixed system $O_1x_1y_1z_1$ is, for the particle *M*, the *transport motion*.

The velocity of the point fixed in the moving axes *Oxyz* with which the particle *M* coincides at a given instant is called the *transport velocity* of the particle *M* at that instant (denoted by $\vec{v}^e$), and the acceleration of that point is called the *transport acceleration* of the particle *M* (denoted by $\vec{a}^e$).

The motion of the particle *M* with respect to the fixed frame of reference $O_1x_1y_1z_1$ is called the *absolute* or *resultant motion*. The trajectory *CD* described in this motion is called the *absolute trajectory*, the velocity is the *absolute velocity* (denoted $\vec{v}^a$), and the acceleration, the *absolute acceleration* (denoted $\vec{a}^a$).

There are two ways of defining of the kinematics characteristics of a particle under condition of complex motion: coordinate way and vector way. According to the coordinate way we have to define the low of particle motion in the fixed reference frame and after that we have to use formulas (1.11) – (1.14) and (1.23) – (1.26). The second way (vector way) is based on two theorems. The first theorem defines the absolute velocity of a particle, the second theorem defines the absolute acceleration of a particle.

To define the absolute velocity of a particle under the condition of complex motion we have to use the following theorem of the composition of velocities: *The absolute velocity of a particle in complex motion is equal to the vector sum of the transport (transfer or bulk) and the relative velocities*:

$$\vec{v}^a = \vec{v}^r + \vec{v}^e . \tag{4.1}$$

The conception of geometrical interpretation of this theorem is shown in Fig. 4.2.

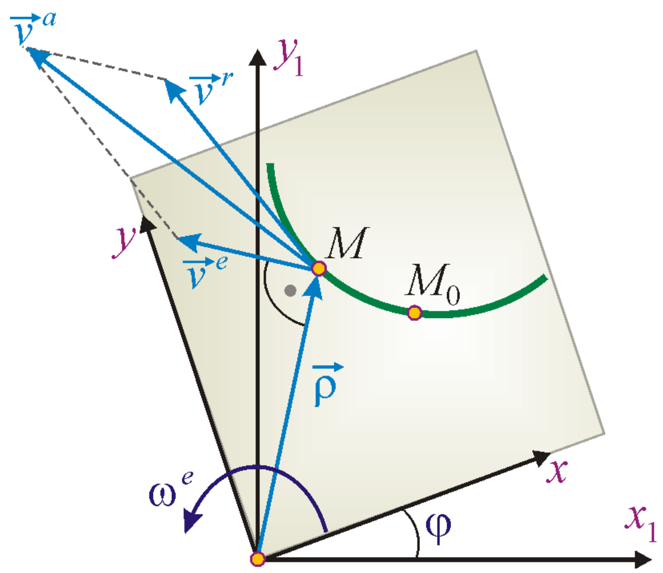


Figure 4.2 – The composition of the velocities

To define the absolute acceleration of a particle under the condition of complex motion we have to use the following theorem of the composition of velocities (so called the Coriolis theorem): *the absolute acceleration of a particle is equal to the geometrical sum of three accelerations: the relative acceleration, which characterizes the time rate of change of the relative velocity in the relative motion, the transport acceleration, which characterizes the time rate of change of the transport velocity in the transport motion, and the Coriolis acceleration, which characterizes the time rate of change of the relative velocity in the transport motion and of the transport velocity in the relative motion*:

$$\vec{a}^{a} = \vec{a}^{r} + \vec{a}^{e} + \vec{a}^{Cor}, \tag{4.2}$$

where $\vec{a}^{Cor}$ – the Coriolis acceleration. The Coriolis acceleration is calculated as:

$$\vec{a}^{Cor} = 2\left(\vec{\omega}^{e} \times \vec{v}^{r}\right), \tag{4.3}$$

where $\vec{\omega}^{e}$ – the angular velocity of the motion of transport.

Thus, the Coriolis acceleration of a particle is equal to the double cross-product of the angular velocity of the motion of transport and the relative velocity of the particle.

If the angle between the vectors $\vec{\omega}^e$ and $\vec{v}^r$ is α, then the magnitude of the Coriolis acceleration is defined by the next formula:

$$a^{Cor} = 2\left|\omega^e v^r\right|\sin\alpha \,. \tag{4.4}$$

The vector Coriolis acceleration $\vec{a}^{Cor}$ has the same sense with the vector $\vec{\omega}^e \times \vec{v}^r$, i.e., normal to the plane through vectors $\vec{\omega}^e$ and $\vec{v}^r$ in the direction from which a counter-clockwise rotation would be seen to carry vector $\vec{\omega}^e$ into vector $\vec{v}^r$ through the smaller angle (Fig. 4.3).

It can also be seen from that the direction of vector $\vec{a}^{Cor}$ can be obtained by projecting vector $\vec{v}^r$ on plane *P*, which is normal to $\vec{\omega}^e$, and turning the projection $\vec{v}^r_P$ on $90°$ in the direction of the rotation of transport. If the relative trajectory is a plane curve moving in its plane, then angle α = 90° and the magnitude of Coriolis acceleration is:

$$a^{cor} = 2\left|\omega^e \cdot v^r\right|. \tag{4.5}$$

It can be seen that in this case the direction of the Coriolis acceleration can be obtained by turning the vector of the relative velocity on 90° in the direction of the rotation of transport (i.e., clockwise or counterclockwise depending on the sense of the rotation).

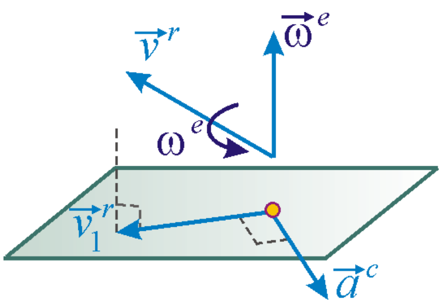


Figure 4.3 – The location of the Coriolis acceleration

From formula (4.4) we can see that the Coriolis acceleration is zero when:

1) $\omega^e = 0$, i.e., if the motion of transport is translational or if the angu-

lar velocity of the rotation of transport becomes zero at a given instant;

2) $v^r = 0$, i.e., if there is no relative motion or if the relative velocity becomes zero at a given instant;

3) Angle $\alpha = 0$ or $\alpha = 180°$, i.e., if the relative motion is parallel to the axis of the rotation of transport or if vector $\vec{v}^r$ is parallel to that axis at a given instant.

Finally, in general case, when the transport motion is non-translational and the relative motion is curvilinear, then the vector of absolute acceleration is defined (geometrical interpretation is given on Fig. 4.4):

$$\vec{a}^a = \vec{a}_\tau^r + \vec{a}_n^r + \vec{a}_\tau^e + \vec{a}_n^e + \vec{a}^{Cor} . \tag{4.6}$$

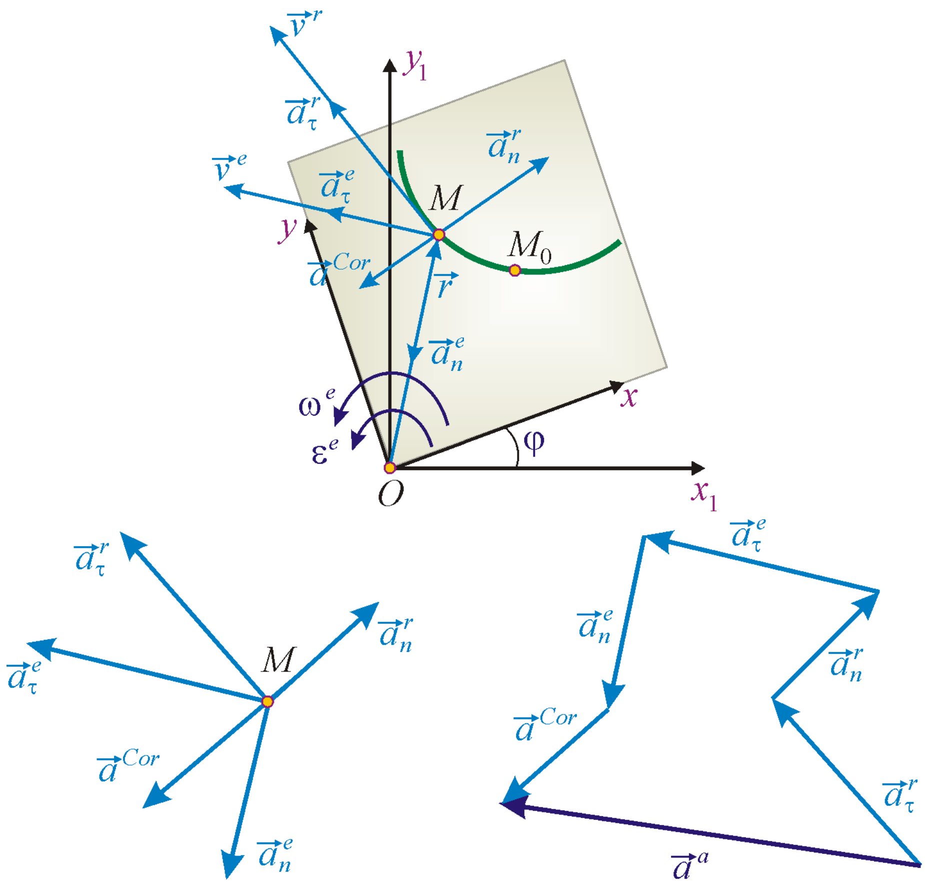


Figure 4.4 – Geometrical characteristics under the general case of complex motion

4.2. Problems and Solutions

Example 1. Let us consider a square plate which rotates around the axis passing through the fixed point O_1 perpendicularly to the plane of the plate (Fig. 3.16 a).

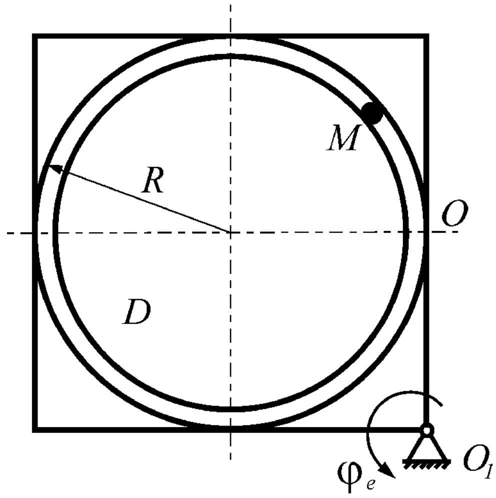


Figure 4.4 – The diagram of complex motion

The movement of plates occurs under the law:

$$\varphi_e = 2t - t^3 \text{(rad)}, \tag{4.7}$$

here the positive direction of the reference angle φ_e indicated by the arc arrow (see Fig. 4.4). The particle M moves along a circle with a radius of $R = 30\,\text{cm}$, which is inscribed in a square. The movement of the particle takes place according to the law:

$$OM = s_r = 20\pi \cdot \sin\frac{\pi t}{6} \text{ (cm)}, \tag{4.8}$$

the positive direction of the reference coordinate S_r is taken from point O to particle M.

It is necessary to determine the absolute velocity and absolute acceleration of particle M at time $t_1 = 1\,\text{sec}$.

The movement of the particle M consider as complex, considering the movement along the circumference relative rotation of the plate – portable. We first define the position of a point M at a given point in time:

$$s_r(t_1) = OM(t_1) = 10\pi \text{ (cm)},$$

then the angle OO_2M is equal to the value:

$$\angle OO_2M = \frac{OM}{R} = \frac{\pi}{3} = 60^0 .$$

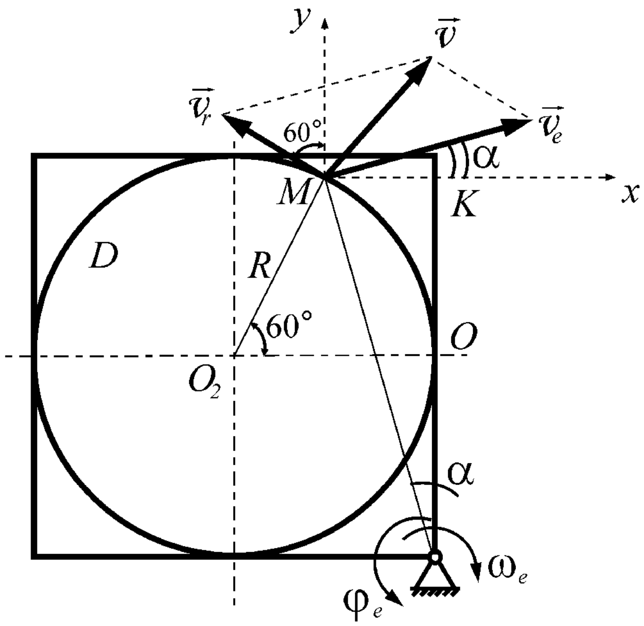


Figure 4.5 – The distribution of the velocities

Draw a particle M in this position (Fig. 4.5). Distance O_1M is the radius of a circle described around the axis of rotation is the point of the plate (or plates), which coincides at a given instant of time the mass particle M, Then we find some auxiliary geometrical quantities:

$$MK = R - R\cos 60^\circ = 15\,\text{см},\;\; O_1K = O_1O + OK =$$
$$= R + R\sin 60^\circ = 55{,}98\,\text{см};$$
$$O_1M = \sqrt{(O_1K)^2 + (MK)^2} = 57{,}96\,\text{см};$$
$$\cos\alpha = \frac{O_1K}{O_1M} = 0{,}966; \quad \sin\varepsilon = \frac{MK}{O_1M} = 0{,}259.$$

To determine the absolute velocity of the point M we will use the theorem on addition of speeds in complex motion:

$$\vec{v} = \vec{v}_r + \vec{v}_e,$$

where $\vec{v}$ – the vector of the absolute velocity of the point M, $\vec{v}_r$ – the vector of the relative velocity of point m, $\vec{v}_r$ – the vector quantity of the transport velocity of the point M. the Algebraic value of the relative speed is equal to

$$\tilde{v}_r = \frac{ds_r}{dt} = \frac{10}{3}\pi^2 \cos\left(\frac{\pi t}{6}\right),$$
$$t_1 = 1\,\text{с} \Rightarrow \tilde{v}_r = 28{,}49\,\text{см/с}, \quad v_r = |\tilde{v}_r|.$$

Here v_r is the magnitude of the relative velocity. The algebraic value of the relative speed is positive, that is, its vector pointing in the direction of increasing coordinates s_r (see Fig. 3.16 b). Next, find the magnitude load speed and find out the direction of the vector:

$$v_e = \omega_e \cdot O_1M,\, \omega_e = |\tilde{\omega}_e|,$$
$$\tilde{\omega}_e = \frac{d\varphi_e}{dt} = 2 - 3t^2,$$
$$t_1 = 1\,\text{с} \Rightarrow \tilde{\omega}_e = -1\,\text{с}^{-1},\, v_e = 57{,}96\;\text{см}/c.$$

Here $\tilde{\omega}_e$ i ω_e is the algebraic value of the magnitude of the angular velocity of the plate. Algebraic value of the angular velocity is negative, that is, the plate rotates in the direction which is opposite to the direction of the reference angle φ_e, which is indicated with the corresponding arc of the arrow ω_e (see Fig. 3.16 b). Vector portable speed point M and is perpendicular to the segment O_1M and aimed in the direction of rotation of the plate. The vector of the absolute velocity is shown in accordance with the rules of vector addition (see Fig. 4.5), we define its magnitude by decomposing the vectors along the axes of a Cartesian coordinate system:

$$v = (v_x, v_y),\ v_r = (v_{rx}, v_{ry}),\ v_e = (v_{ex}, v_{ey}),$$

$$v_x = v_{rx} + v_{ex} = -v_r \cdot \cos 30^\circ + v_e \cdot \cos\alpha \approx 31{,}31 \text{см/с},$$

$$v_y = v_{ry} + v_{ey} = -v_r \cdot \sin 30^\circ + v_e \cdot \sin\alpha \approx 29{,}25 \text{см/с},$$

$$v = \sqrt{v_x^2 + v_y^2} \approx 42{,}84 \text{ см/с}.$$

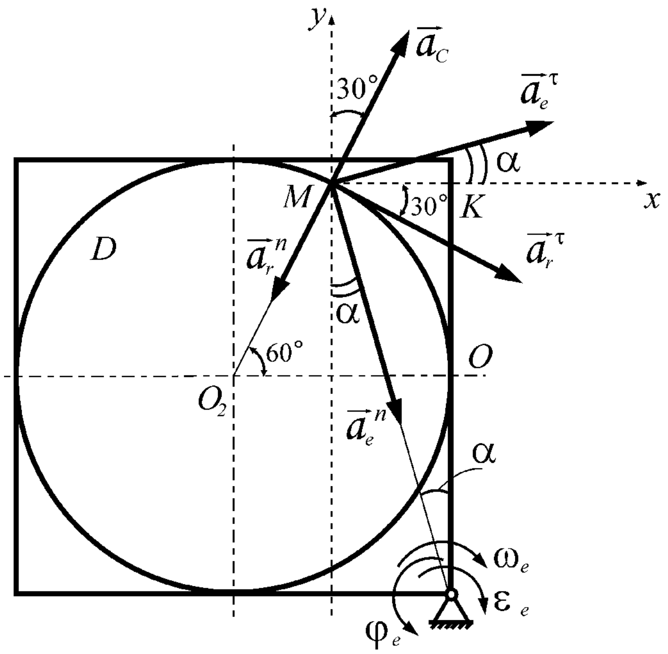


Figure 4.6 – The distribution of the accelerations

To determine the absolute acceleration of the point M we will use theorem on composition of accelerations in complex motion (Coriolis theorem):

$$\vec{a} = \vec{a}_r + \vec{a}_e + \vec{a}_c,$$

where $\vec{a}$ – the vector of the absolute acceleration, $\vec{a}_r$ – the vector of the relative acceleration, $\vec{a}_e$ – the vector of the portable acceleration, $\vec{a}_c$ – the vector of the Coriolis acceleration of a particle *M*. Due to the fact that relative and figurative movements of the point M are not straightforward, the formula for acceleration must be written in expanded form:

$$\vec{a} = \vec{a}_r^{\,n} + \vec{a}_r^{\,\tau} + \vec{a}_e^{\,\text{cp}} + \vec{a}_e^{\,\text{rt}} + \vec{a}_c ,$$

where $\vec{a}_r^{\,n}$ – the vector of the relative normal acceleration, $\vec{a}_r^{\,\tau}$ – the vector of the relative tangent acceleration, $\vec{a}_e^{\,\text{до}}$ – the vector of the portable centripetal acceleration, $\vec{a}_e^{\,\text{об}}$ – the vector of the portable rotational acceleration of the point M. The vector of the relative normal acceleration is always directed from the point M to the center of curvature of the relative trajectory – to the point O_2 (see Fig. 3.16), and its magnitude is equal to the value:

$$a_r^n = \frac{v_r^2}{R} \approx 27{,}06 \text{ см/с}^2 .$$

Then we find the algebraic value of the relative tangential acceleration $\tilde{a}_r$ and its magnitude a_r:

$$\tilde{a}_r = \frac{d\tilde{v}_r}{dt} = -\frac{5}{9}\pi^3 \sin\frac{\pi t}{6},$$

$$t_1 = 1\,\text{с} \Rightarrow \tilde{a}_r = -8{,}61\,\text{см/с}^2 , \quad a_r = |\tilde{a}_r|.$$

The algebraic value of the relative tangential acceleration is negative, so the vector of the relative tangential acceleration is directed, unlike the vector of the relative velocity in the direction of decreasing relative coordinate s_r (see Fig. 4.5).

Vector of the centripetal acceleration is always directed towards the center of curvature of the portable trajectory of point O_1 (see Fig. 3.19 in), and its magnitude is equal to the value:

$$a_e^{\text{cp}} = \omega_e^2 \cdot O_1M \approx 57{,}96\text{cm}/sec^2 .$$

Magnitude of the transport rotational acceleration a_e^{rt} defined as follows:

$$a_e^{\text{об}} = \varepsilon_e \cdot O_1 M, \qquad \varepsilon_e = \left|\tilde{\varepsilon}_e\right|,$$

$$\tilde{\varepsilon}_e = \frac{d\omega_e}{dt} = -6t,$$

$$t_1 = 1\,\text{с} \Rightarrow \tilde{\varepsilon}_e = -6\,\text{с}^{-2},\ a_e^{\text{об}} = 347,73\ \text{см}/\text{с}^2.$$

Here $\tilde{\varepsilon}_e$ and ε_e is the algebraic value of the portable magnitude of the angular acceleration of the plate. Algebraic value of the angular acceleration is negative, thus, the vectors $\vec{\omega}_e$ and $\vec{\varepsilon}_e$ are directed in the same direction, and the portable motion of the point M is accelerated. Portable vector rotational acceleration is directed in the same way as the vector load speed (see Fig. 3.16). The vector of Coriolis acceleration and the magnitude is determined as follows:

$$\vec{a}_c = 2 \cdot [\vec{\omega}_e, \vec{v}_r]; \qquad a_c = 2 \cdot \omega_e \cdot v_r \cdot \sin(\vec{\omega}_e, \vec{v}_r).$$

Due to the fact that the relative movement of the point occurs in the plane perpendicular to the axis of rotation of the portable, than $\sin(\vec{\omega}_e, \vec{v}_r) = \sin 90^0 = 1$. According to previous formula we get that $a_c = 56{,}98\,\text{cm}/sec^2$. The magnitude of the absolute acceleration of the particle *M*, we define by projecting vector equations on the axis of a Cartesian coordinate system:

$$a_x = a_{rx}^n + a_{rx}^\tau + a_{ex}^{\text{до}} + a_{ex}^{\text{об}} + a_{cx} = -a_r^n \cdot \cos 60^\circ +$$
$$+ a_r^\tau \cdot \cos 30^\circ + a_e^{\text{до}} \cdot \sin\alpha + a_e^{\text{об}} \cos\alpha +$$
$$+ a_c \cdot \cos 60^\circ \approx 358{,}39\ \text{см}/\text{с}^2;$$
$$a_y = a_{ry}^n + a_{ry}^\tau + a_{ey}^{\text{до}} + a_{ey}^{\text{об}} + a_{cy} = -a_r^n \cdot \sin 60^\circ -$$
$$- a_r^\tau \cdot \sin 30^\circ - a_e^{\text{до}} \cdot \cos\alpha + a_e^{\text{об}} \sin\alpha +$$
$$+ a_c \cdot \sin 60^\circ \approx 64{,}24\ \text{см}/\text{с}^2;$$
$$a = \sqrt{a_x^2 + a_y^2} \approx 364{,}1\,\text{см}/\text{с}^2.$$

Thus, using the vector relations of the theory of complex motion of a particle, we have defined absolute velocity and absolute acceleration of a particle moving along the plate, which is in turn rotates around a fixed axis.

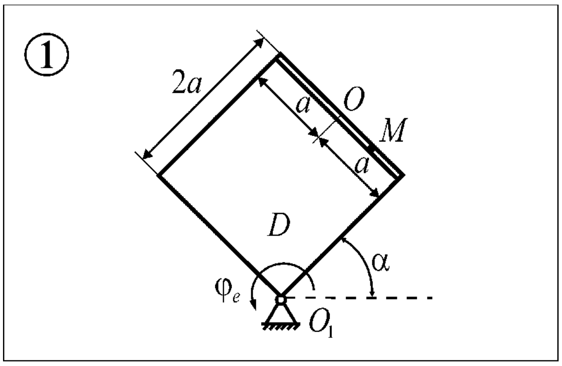
4.3. Tasks for solving.

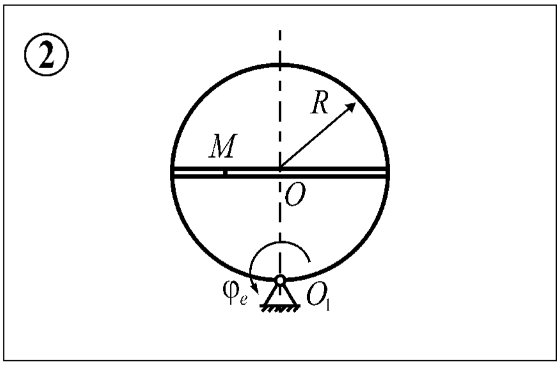
It is necessary to determine the kinematic characteristics of the particle *M* using the vector relations of the kinematics of the complex motion for a given instant.

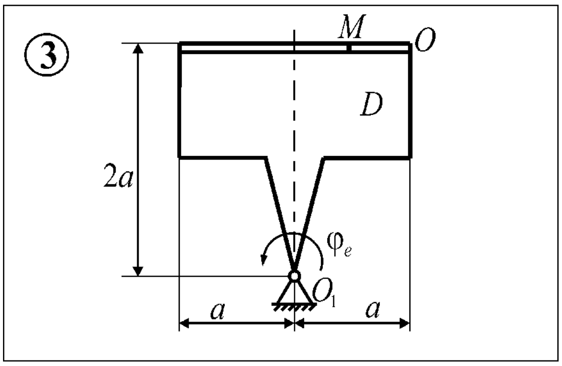
In tabl. 4.1 – 4.3, there are variants of diagrams and initial data are proposed, according to which it is necessary to determine the kinematic characteristics of the particle *M* in the complex motion along with the body *D*. At the initial moment of time, the body *D* (plate) from the resting state starts the rotational motion around the axis passing through point O_1 is perpendicular to the plane of the picture, while the particle *M* begins to move relative to the body in a previously known trajectory. The initial data are the geometric dimensions of the plate, the law of change in the time of its angle of rotation, and the law of time variation of the natural coordinate particle *M* along the relative trajectory.

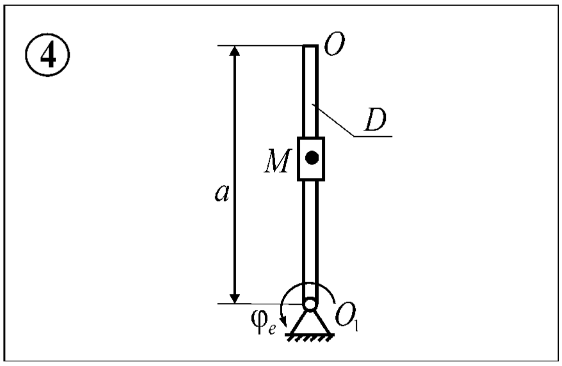
Note. In variants 1, 8 and 31 - body *D* is shown in the initial position, in all other variants - in the arbitrary. Point *O* is the beginning of the countdown along the relative trajectory, particle *M* in all variants is shown shifted in a positive direction along the relative trajectory.

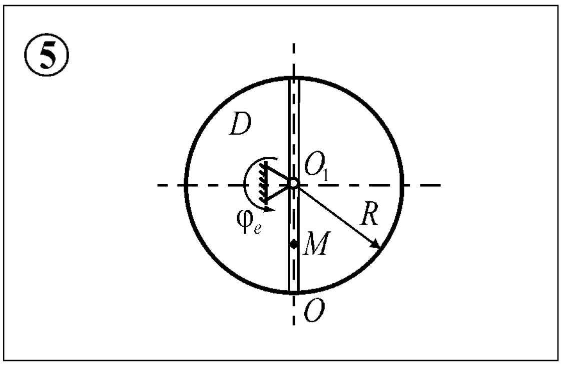
Table 4.1. Diagrams of mechanical systems.

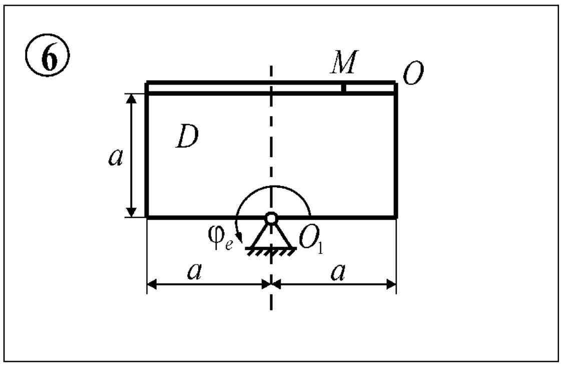


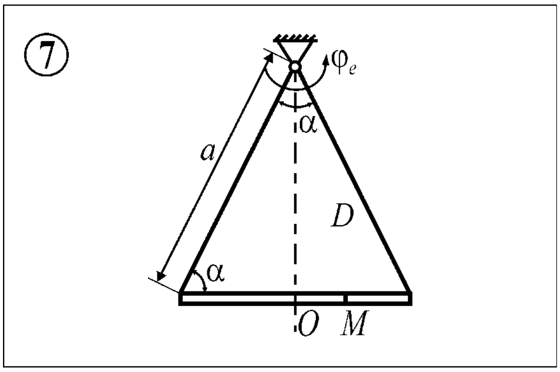


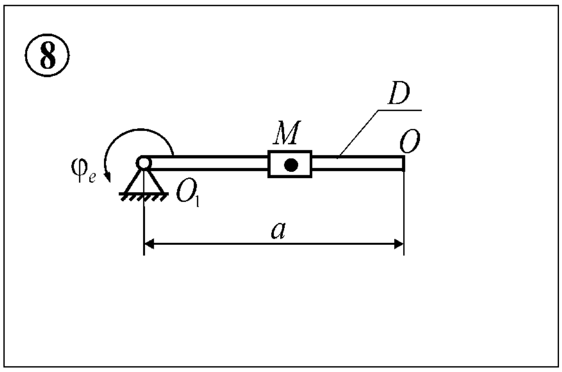












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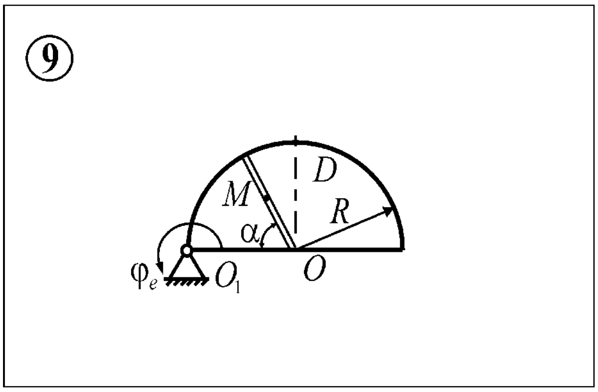

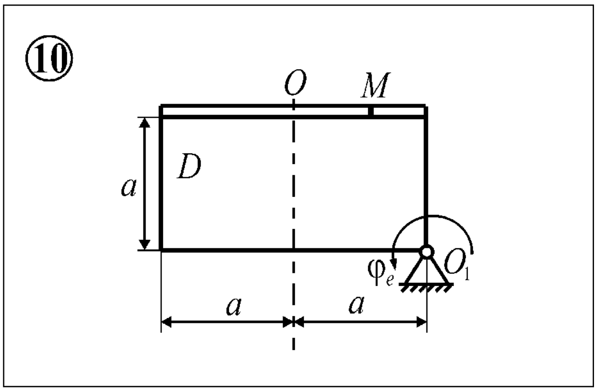

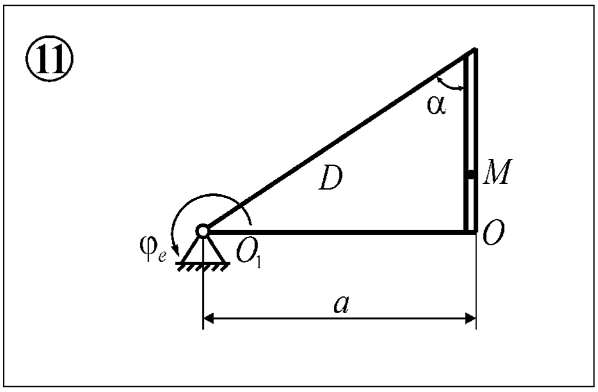

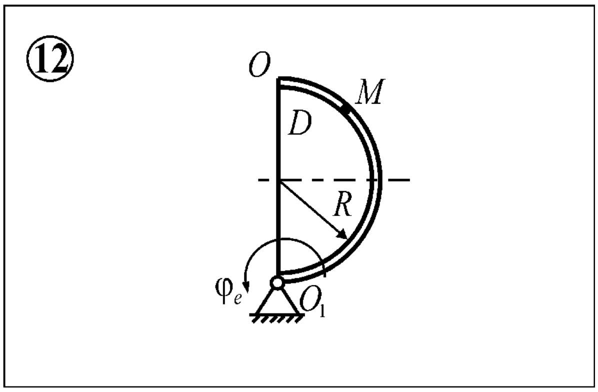

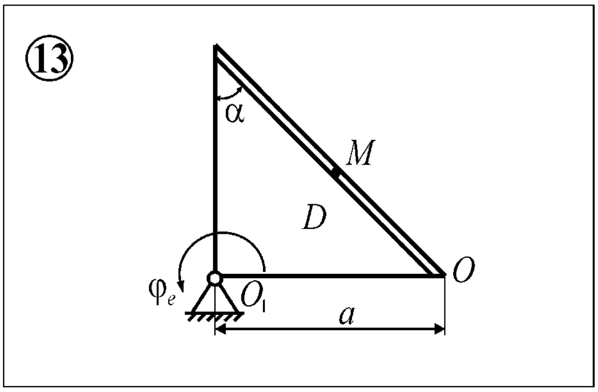

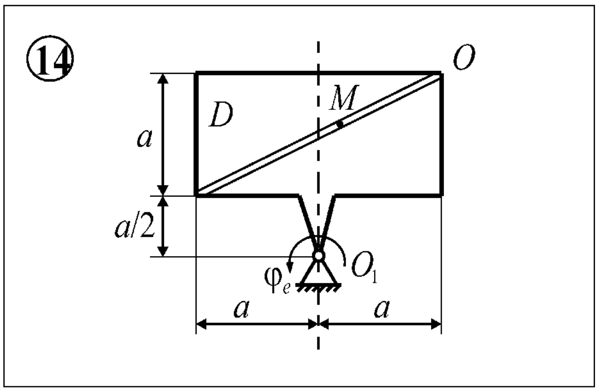

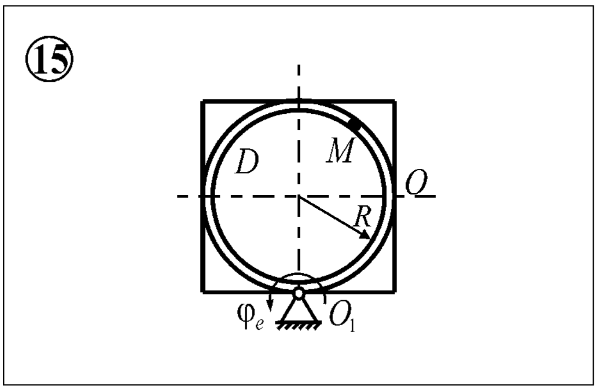

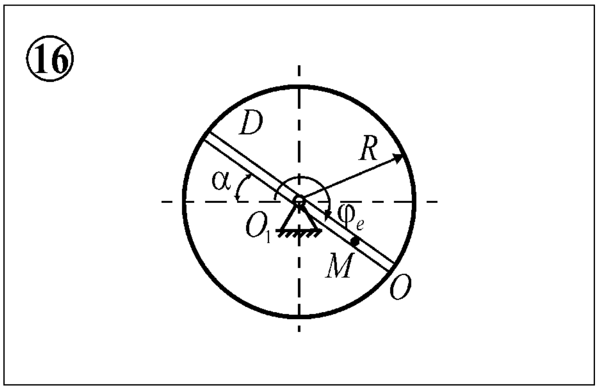

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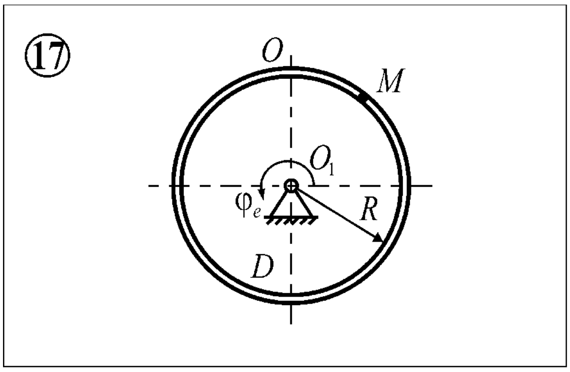

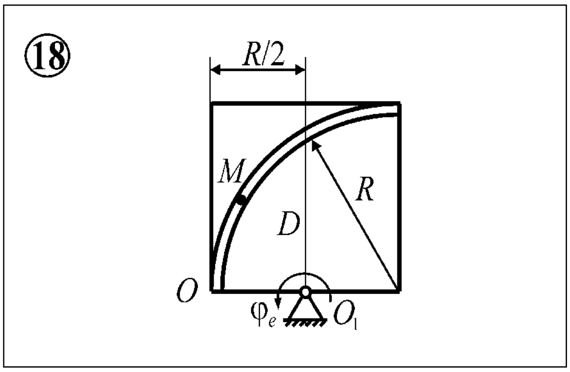

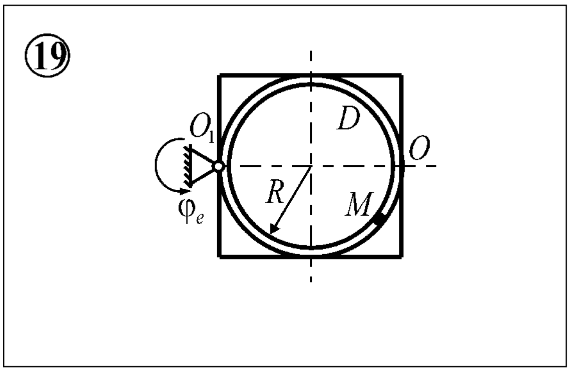

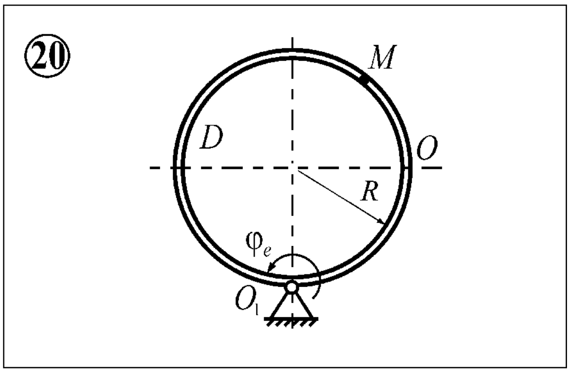

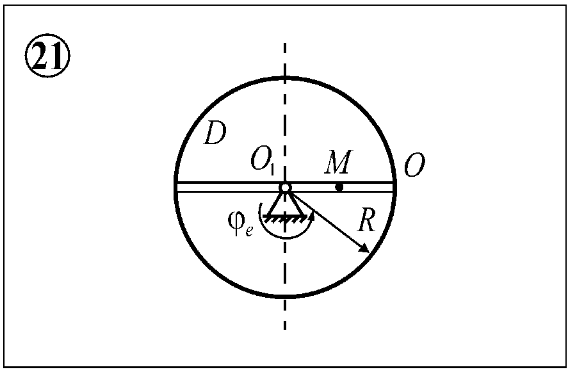

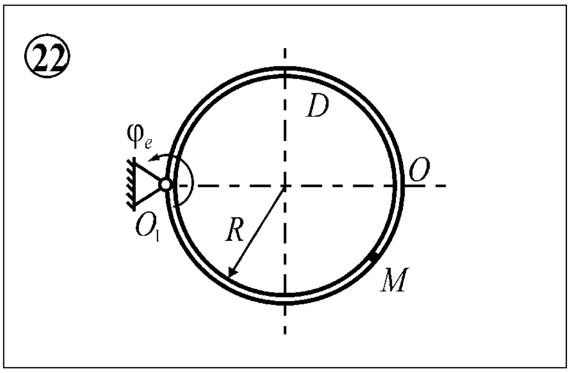

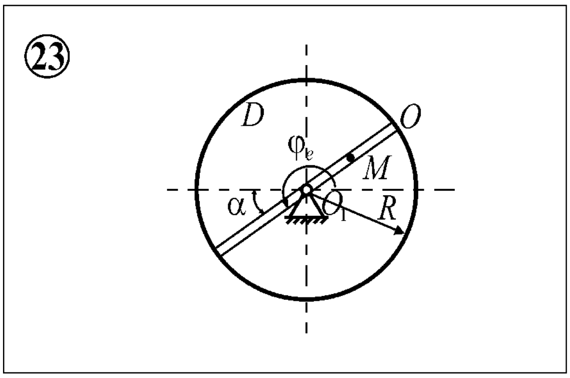

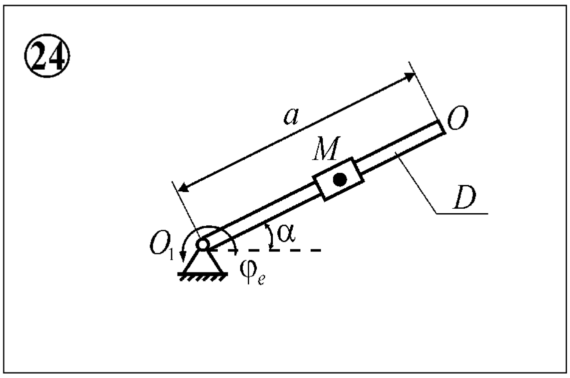

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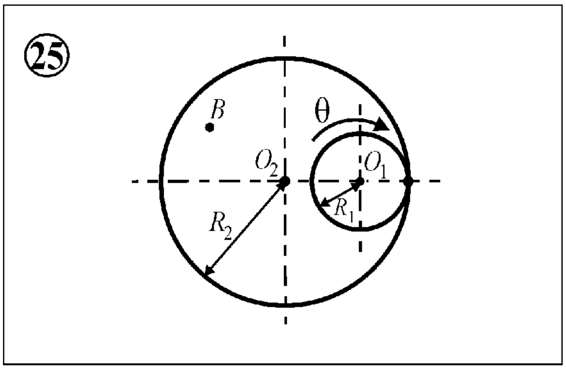

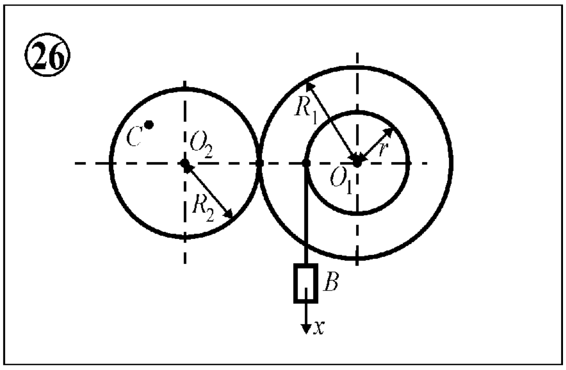

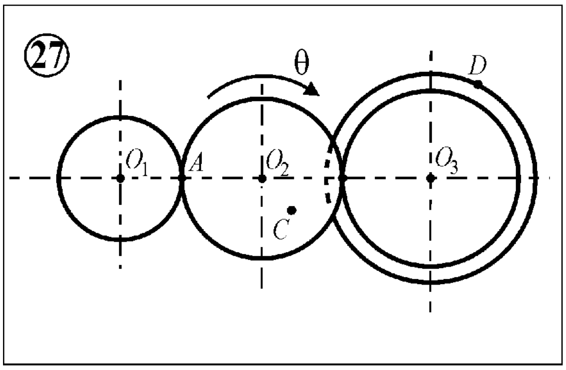

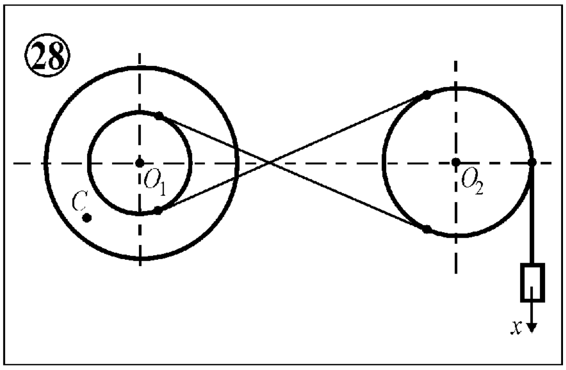

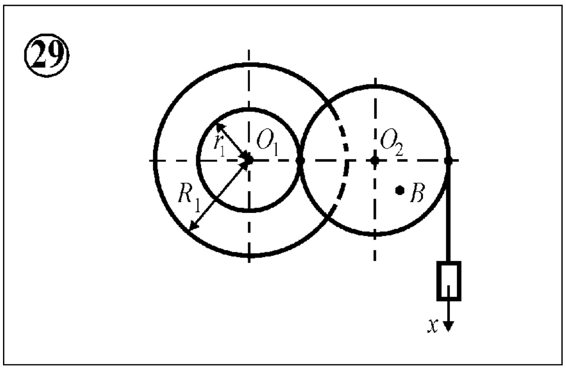

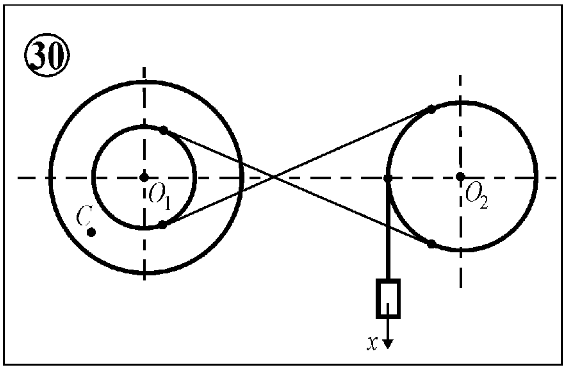

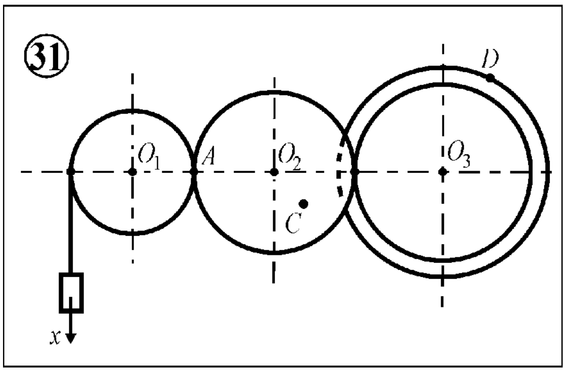

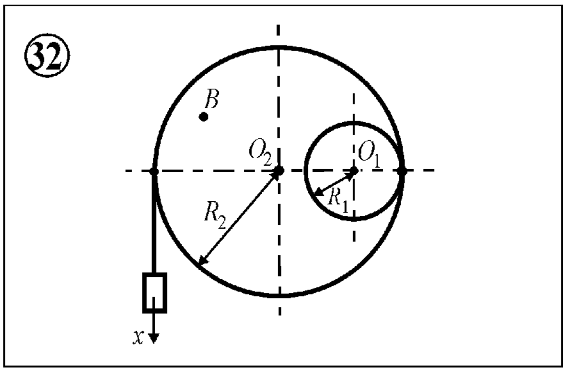

Table 4.2. Conditions of motion and geometrical sizes.

Variant N	Low of motion φ_e, rad	Low of motion s_r, cm	Time t_*, sec
1	$0{,}3t + 0{,}5t^2$	$10\sin(2t)$	3,25
2	$0{,}1t + t^2$	$15\sin(3t)$	2,46
3	$t - 0{,}2t^2$	$28\sin^2(2t)$	3,64
4	$t + 0{,}3t^2$	$16\sin^2(t)$	3,2
5	$t + 0{,}4t^2$	$72\sin^2(t)$	2,91
6	$0{,}2t + 0{,}3t^2$	$40\sin^2(t)$	4,25
7	$0{,}2t + 0{,}9t^2$	$11\cos(t)$	2,53
8	$0{,}1t - 0{,}5t^2$	$26\sin^2(0{,}8t)$	3,45
9	$t + 0{,}2t^2$	$40\sin^2(0{,}3t)$	3,63
10	$0{,}3t + 0{,}2t^2$	$22\sin(t)$	4,91
11	$t^3 + 0{,}6t^2$	$10\sin^2(t)$	2,51
12	$3t + 0{,}1t^2$	$70\sin^2(2{,}3t)$	1,97
13	$t + t^2$	$16\sin^2(1{,}8t)$	3,6
14	$0{,}3t + 0{,}4t^2$	$22\sin^2(0{,}2t)$	3,61
15	$0{,}4t + 0{,}2t^2$	$70\cos(2{,}5t)$	1,1
16	$2t + t^2$	$28\sin^2(0{,}3t)$	1,3
17	$0{,}3t + 0{,}2t^2$	$100\cos(t)$	1,6
18	$4t + 5t^2$	$40\sin^2(0{,}4t)$	1,8
19	$3t + 0{,}2t^2$	$45\sin(2{,}5t)$	2
20	$0{,}3t + 0{,}7t^2$	$50\sin(1{,}5t)$	0,8
21	$-4t^3 - 2t^2$	$90\sin^2(3{,}6t)$	1
22	$0{,}1t + 0{,}9t^2$	$95\sin(0{,}7t)$	1,1
23	$t + 0{,}5t^2$	$34\sin^2(1{,}8t)$	1,5
24	$0{,}4t + t^2$	$38\sin^2(1{,}4t)$	1,8

Table 4.3. Geometrical sizes.

Variant N	a, cm	R, cm	α, rad
1	10	–	π/4
2	–	15	–
3	14	–	–
4	16	–	–
5	–	36	–
6	20	–	–
7	22	–	π/3
8	26	–	–
9	–	40	π/3
10	22	–	–
11	18	–	π/3
12	–	36	–
13	14	–	π/4
14	10	–	–
15	–	30	–
16	–	14	π/6
17	–	32	–
18	–	34	–
19	–	36	–
20	–	38	–
21	–	45	–
22	–	35	–
23	–	17	π/6
24	38	–	π/6

Questions.

1) What is the complex motion of a particle?

2) How many reference frames we use for describing of the complex motion of a particle?

3) What is the relative motion of a particle?

4) What is the transport motion of a particle?

5) What is the absolute motion of a particle?

6) What is the relative velocity under the complex motion of a particle?

7) What is the transport velocity under the complex motion of a particle?

8) What is the absolute velocity under the complex motion of a particle?

9) What is the relative acceleration under the complex motion of a particle?

10) What is the transport acceleration under the complex motion of a particle?

11) What is the absolute acceleration under the complex motion of a particle?

12) What is the sense of the Coriolis acceleration of a particle under the complex motion?

13) Give the examples of technical devices that include the complex motion of a particle?

14) Give the examples of the complex motion of a particle in a nature?

15) Where is the vector of Coriolis acceleration if the particle has plane complex motion?

16) Which formula determines the vector of absolute velocity in the case of a complex particle movement?

17) What is the formula for determining the absolute acceleration vector in the case of a complex particle movement?

18) Which formula determines the vector of the acceleration of the Coriolis in the case of a complex particle movement?

19) What is the geometrical interpretation of the absolute acceleration

under the complex motion?

20) What is the geometrical interpretation of the absolute velocity under the complex motion?

21) How we can determine the magnitude of the absolute velocity?

22) How we can determine the magnitude of the absolute acceleration?

23) How we can determine the magnitude of the Coriolis acceleration?

24) If the transport angular velocity is equaled zero, then Coriolis acceleration is equaled?

25) If the vectors of the transport angular velocity and relative velocity is parallel, then Coriolis acceleration is equaled?

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Навчальне видання

АНІЩЕНКО Галина Оттівна
ЛАВІНСЬКИЙ Денис Володимирович

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Part I. Kinematics.

Навчальний посібник

(англійською мовою)

для студентів-бакалаврів, які навчаються за спеціальністю
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