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THE METHOD OF *R*-FUNCTIONS IN THE SOLUTION OF ELASTIC PROBLEMS ON THE BASIS OF REISSNER'S MIXED VARIATIONAL PRINCIPLE

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A method is presented for solving boundary-value elastic problems on the basis of the variational–structural method of *R*-functions and Reissner's mixed variational principle. A mathematical formulation is given to problems on the deformation of elastic bodies under mixed boundary conditions and bodies interacting with smooth rigid dies. Solutions satisfying all the boundary conditions are proposed. For undetermined components of these solutions, the resolving equations are derived and their properties are studied. A posteriori estimation of numerical solutions is made. As examples, solutions are found to a problem on the stress–strain state of a short cylinder and to a contact problem on a cylinder interacting with a smooth die. A numerical method of solving such problems is analyzed for convergence, and the accuracy of the solutions is estimated.

Popular variational methods used to solve elastic problems by minimizing the Lagrange functional possess a significant shortcoming—the stresses are determined less accurately than the displacements. To improve the accuracy of the solution, it is expedient to determine the parameters of the stress–strain states independently, which can be done with the help of mixed variational principles [1]. In the present study, a mathematical formulation is given to elastic problems on the basis of Reissner's principle, a solution technique and a technique for estimation of numerical solutions are proposed, and numerical results for elastic solids of revolution are presented.

1. Mathematical Formulation of the Problem. Let us consider in arbitrary curvilinear coordinates α^i $(i=\overline{1,3})$ an elastic body occupying a volume V bounded by a surface $S = S_t \cup S_u \cup S_c$. Here S_t is that fraction of the surface on which external distributed loads of intensity $\vec{t}$ act, S_u is that fraction of the surface on which conditions limiting the displacements of points of the body are specified, and S_c is that fraction of the surface on which mixed conditions are specified, as, for example, in the case where the body interacts with a smooth rigid die that occupies a domain determined by the inequality $\Psi(\alpha^1, \alpha^2, \alpha^3) \le 0$. The contact interaction of bodies is also discussed in [9, 10]. When the contact region S_c is known, the components of the stress–strain state of the body are determined from the stationarity condition for the Reissner functional

$$\iiint_V \left\{ \delta\sigma^{ij}\left[\frac{1}{2}(\nabla_i u_j + \nabla_j u_i) - d_{ijkl}\sigma^{kl}\right] - \delta u_j (\nabla_i \sigma^{ij} + f^j)\right\} dv$$

$$+\iint_{S_t} \delta u_j (\sigma^{ij} n_i - t^i) ds - \iint_{S_u} n_i \delta\sigma^{ij} (u^j - u_j^*) ds - \iint_{S_c} \delta\sigma^{ij} n_i n_j (u_m n^m - u_n^*) ds$$

$$+\iint_{S_c} \delta u_j (\sigma^{ij} n_i - \sigma^{km} n_k n_m n^j) ds = 0, \tag{1.1}$$

where u_n^* is the indentation made by forcing the die into the elastic body normally to its undeformed surface or the distance between the body and the die.

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Let us show that for the equality $\sigma^{ij} n_i - \sigma^{km} n_k n_m n^j = 0$ to hold, it is necessary and sufficient that the conditions $\sigma^{ij} n_i \tau_j = 0$ and $\sigma^{ij} n_i b_j = 0$ be satisfied. They express the equality to zero (at points of the contact surface) of the load components acting along the tangent $\vec{\tau} = \tau_i \vec{e}^{\,i}$ and binormal $\vec{b} = b_i \vec{e}^{\,i}$ to the body surface with the external normal $\vec{n} = n_i \vec{e}^{\,i}$. If the contact region is unknown beforehand, it can be determined by supplementing the variational equation (1.1) with the nonpositivity condition for the contact stresses,

$$\sigma^{ij} n_i n_j \le 0 \quad \text{for all} \quad \alpha^k \in S_c, \tag{1.2}$$

and a condition whereby points of the elastic body cannot penetrate into the rigid die,

$$u_i n^i < u_n^* \quad \text{for all} \quad \alpha^k \in S_t. \tag{1.3}$$

For the rigid die, the quantity u_n^* depends on its shape. To establish this dependence, we assume that upon contact interaction the displacement components of points of the elastic body's surface must satisfy the condition $\Psi(\alpha^i + u^i) \ge 0$. If the die is of arbitrary shape, this inequality is usually linearized by expanding it into a Maclaurin series with respect to the displacement components, which are assumed small,

$$\Psi(\alpha^k) + g^{ij} u_i \nabla_j \Psi(\alpha^k) \ge 0, \tag{1.4}$$

where g^{ij} are the components of the metric (fundamental) tensor.

Let us represent the displacement vector in terms of the projections onto the normal (u_n), tangent (u_τ), and binormal (u_b) to the undeformed surface of the body:

$$u_i = u_n n_i + u_\tau \tau_i + u_b b_i.$$

Then, in view of $n^i \nabla_i \Psi(\alpha^k) < 0$, from (1.4) we obtain

$$u_n \le -\frac{\Psi(\alpha^k)}{n^i \nabla_i \Psi(\alpha^k)} - u_\tau \frac{\tau^j \nabla_j \Psi(\alpha^k)}{n^i \nabla_i \Psi(\alpha^k)} - u_b \frac{b^j \nabla_j \Psi(\alpha^k)}{n^i \nabla_i \Psi(\alpha^k)}. \tag{1.5}$$

It is obvious that (1.5) holds when the normals to the surfaces of the body and the die are nonorthogonal. From (1.5), it follows that under the assumptions adopted, the normal displacement of a point of the body under the die is dependent on the die shape and is determined by the tangent and binormal displacements. The shapes of the contacting surfaces frequently turn out to be such that the last two terms in (1.5) may be neglected compared with the first one. In this case, from (1.5) we obtain an expression for the normal indentation the die produces,

$$u_n^* = -\frac{\Psi(\alpha^k)}{n^i \nabla_i \Psi(\alpha^k)}. \tag{1.6}$$

2. Solution Technique. The Reissner functional is minimax. The absence of an extremum at the stationarity point complicates considerably the solution of the problem, since the convergence of the direct methods has been proved for extreme functionals. It is possible to formulate a sufficient condition of convergence of the Ritz method for the Reissner functional. To this end, it is sufficient, as shown in [6], that the Riesz representations of the sought-for solutions satisfy all the main and natural boundary conditions of the variational equation (1.1). This can be made possible by using the variational–structural method known in the theory of *R*-functions [4]. According to this method, the solution of problem (1.1) is represented as structures identically satisfying the boundary conditions of the problem for an arbitrary choice of the undetermined components included in these structures. The undetermined components are determined from the stationarity condition for the functional.

The existing applications of the variational–structural method for the solution of equilibrium problems for deformable bodies are mainly based on the Lagrange variational principle. When Reissner's mixed principle is used, by approximating independently the stresses and displacements, we can simplify the structures of the solutions to boundary-value problems in arbitrary curvilinear coordinates. To construct such structures, it is necessary to write analytically the equations of the boundary regions of the body's surface and the expressions for the components of the orthonormalized vectors of the normal n^i, tangent τ^i, and binormal b^i to the surface of the elastic body. This can be done by applying the mathematical apparatus of the theory of *R*-functions, as shown, for example, in [4].

The structures of the solutions for the displacement and stress components determined from Eq. (1.1) and satisfying all the natural conditions are obtained in the form [5]

$$u_i = w_c(A_1\tau_i + A_2 b_i + u_n^* n_i) + w_u u_i^* + w_{cu} B_i, \tag{2.1}$$

$$\sigma^{ij} = \omega_t[t^i n^j + t^j n^i - g^{ij}(t^k n_k)] - \omega_c C_1 n^i n^j + C_2 \tau^i \tau^j + C_3 b^i b^j + C_4(\tau^i b^j + \tau^j b^i) + \omega_{ct} D^{ij}, \tag{2.2}$$

where $A_1, A_2, B_i\ (i = \overline{1,3}), C_1, C_2, C_3, C_4$, and $D^{ij}\ (i, j = \overline{1,3}, D^{ij} = D^{ji})$ are the undetermined components of the structures, and

$$w_c = \begin{cases} 1 & \text{for all} \quad \alpha^k \in S_c \\ 0 & \text{for all} \quad \alpha^k \in S_u \end{cases}, \qquad w_u = \begin{cases} 1 & \text{for all} \quad \alpha^k \in S_u \\ 0 & \text{for all} \quad \alpha^k \in S_c \end{cases},$$

$$w_{cu} = 0 \quad \text{for all} \quad \alpha^k \in S_c \cup S_u,$$

$$\omega_t = \begin{cases} 1 & \text{for all} \quad \alpha^k \in S_t \\ 0 & \text{for all} \quad \alpha^k \in S_c \end{cases}, \qquad \omega_u = \begin{cases} 1 & \text{for all} \quad \alpha^k \in S_c \\ 0 & \text{for all} \quad \alpha^k \in S_t \end{cases},$$

$$\omega_{ct} = 0 \quad \text{for all} \quad \alpha^k \in S_c \cup S_t$$

are the structure components carrying geometrical information.

In the further numerical studies, we represent the undetermined components of structures (2.1) and (2.2) as

$$A_1 = \sum_{j=1}^{H_1} A_{1j}\phi_{1j}, \qquad A_2 = \sum_{j=1}^{H_2} A_{2j}\phi_{2j}, \qquad B_i = \sum_{j=1}^{w_1} B_{ij}\vartheta_{ij}, \tag{2.3}$$

$$C_1 = \sum_{m=1}^{N_1} C_{1m}\chi_{1m}, \qquad C_2 = \sum_{m=1}^{N_2} C_{2m}\chi_{2m}, \qquad C_3 = \sum_{m=1}^{N_3} C_{3m}\chi_{3m},$$

$$C_4 = \sum_{m=1}^{N_4} C_{4m}\chi_{4m}, \qquad D^{ij} = \sum_{m=1}^{N_{ij}} D_m^{ij}\chi_m^{ij}, \tag{2.4}$$

where $A_{1j}\ (j = \overline{1, H_1}), A_{2j}\ (j = \overline{1, H_2}), A_{ij}\ (j = \overline{1, W_i}), C_{km}, k = \overline{1,4},\ m = \overline{1, N_k},\ D_m^{ij}$, and $m = \overline{1, N_{ij}}$ are the approximation coefficients of the undetermined components and $\phi_{1j}\ (j = \overline{1, H_1}),\ \phi_{2j}\ (j = \overline{1, H_2}), \vartheta_{ij}\ (j = \overline{1, W_i}),\ \chi_{km},\ k = \overline{1,4},\ m = \overline{1, N_k},\ \chi_m^{ij}$, and $m = \overline{1, N_{ij}}$ are coordinate functions, which are complete and linearly independent.

The resolving equations are derived by substituting structures (2.1) and (2.2) into Eq. (1.1). From this equality, equating the coefficients of variations to zero, we obtain a system of linear algebraic equations for the unknown approximation coefficients in (2.3) and (2.4). This system has the following block-matrix form [5]:

$$\begin{bmatrix} [R_{11}] & [R_{12}] \\ [R_{21}] & 0 \end{bmatrix} \begin{Bmatrix} q_\sigma \\ q_u \end{Bmatrix} = -\begin{Bmatrix} p_{1\sigma} + p_{2\sigma} \\ p_u \end{Bmatrix}, \tag{2.5}$$

where q_u and q_σ are vectors made up of the coefficients appearing in (2.3) and (2.4), respectively. The coefficients of the matrix of system (2.5) are usually calculated numerically, for example, by the Gaussian quadrature formulas.

The matrix of system (2.5) possesses the following properties. First, the quadratic form $q_\sigma^T [R_{11}] q_\sigma$ coincides with the additional work taken with the opposite sign, and, consequently, the matrix $[R_{11}]$ is negative definite and symmetric. Second, the matrix of system (2.5) is also symmetric. Indeed, using the obvious equality

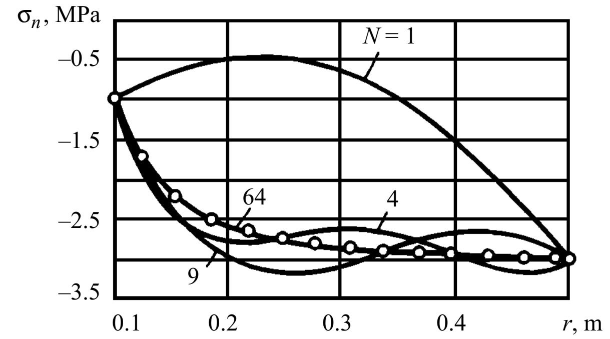

Fig. 1

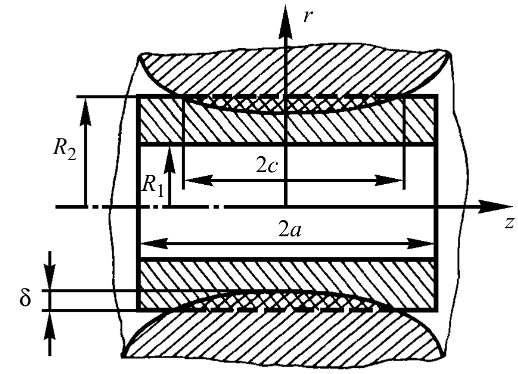

Fig. 2

$$\frac{1}{2}\iiint\limits_V \sigma^{ij}(\nabla_i u_j + \nabla_j u_i)dv = -\iiint\limits_V u_j(\nabla_i \sigma^{ij})dv,$$

which is valid for the parameters of the stress–strain state of the body under homogeneous boundary conditions, it is easy to prove that $[R_{12}]^T = [R_{21}]$.

Note that when the contact region is unknown beforehand, its boundary can be determined, as, for example, in [2]. To this end, it is necessary to eliminate successively from the set of contact points those points on the body surface at which condition (1.2) is not satisfied and to include in this set those points at which condition (1.3) is not satisfied. The process should be continued until both of these conditions are satisfied.

3. Estimation of Solutions. The solutions that the variational–structural method produces will be approximate if a finite number of coordinate functions are retained in expansions (2.3) and (2.4). In numerical solutions, the number of coordinate functions is determined by the required accuracy of the solutions. This is especially important for problems with unknown contact regions, since an insufficiently accurate solution of a problem at a step of an iterative process may lead to its divergence.

To estimate the accuracy of approximate solutions, we may use various energy norms calculated on these solutions. As such norms, the values of the Lagrange, Castigliano, and Reissner functionals were used. Following [7], it is possible to show that if to minimize the Lagrange functional

$$\inf_{u\in K_u}\left\{\frac{1}{2}\iiint\limits_V c^{ijkl}\varepsilon_{ij}\varepsilon_{kl}dv - \iiint\limits_V u_i f^i dv - \iint\limits_{S_u} u_i t^i ds\right\}, \tag{3.1}$$

we introduce a perturbation in the form

$$\Phi = \frac{1}{2} - \iiint\limits_V c_{ijkl}(\varepsilon_{ij} + p_{ij})(\varepsilon_{kl} + p_{kl})dv - \iiint\limits_V u_i f^i dv - \iint\limits_{S_u} u_i t^i ds, \tag{3.2}$$

where $\varepsilon_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i)$, then maximization of the Castigliano functional is a problem dual to problem (3.1) with respect to this perturbation, and the Lagrangian of the minimization problem (3.1) coincides with the Reissner functional with respect to perturbation (3.2). Thus, by comparing the values of the Lagrange, Castigliano, and Reissner functionals, calculated on some approximate solution (2.1), (2.2), we can estimate the accuracy of this solution.

4. Numerical Examples. Let us solve some axisymmetric elastic problems by the method proposed above.

First, we will consider a finite cylinder with free ends under external and internal pressures uniform along the length. Figure 1 shows how the radial stresses (solid lines) converge to an analytical solution (points) presented by Lurie in [3]. Table 1 summarizes the values (in J) of the Castigliano (Φ_C), Reissner (Φ_R), and Lagrange (Φ_L) functionals depending on the number of coordinate functions (N) and shows that the integral estimates converge. It is seen that the integral estimates corresponding to

TABLE 1

N	1	9	64	—
Φ_C	–2.028	–2.357	–2.359	–2.359
Φ_R	–2.028	–2.357	–2.359	–2.359
Φ_L	–1.446	–2.347	–2.359	–2.359

the analytical solution (last column) are equal. Thus, the proposed solution technique and integral estimate turn out to be very efficient in numerical studies.

Further, we will consider a cylinder of length $2a = 0.4$ m, internal radius $R_1 = 0.1$ m, and external radius $R_2 = 0.2$ m compressed by a smooth rigid die (Fig. 2). Let the equation of the die in cylindrical coordinates (r, z) have the form

$$r - R_2 + \delta(1 - 2z^2 / a^2) = 0, \tag{4.1}$$

where $\delta = 0.001$ mm is the indentation depth (Fig. 2).

Let us formulate conditions on the contact surfaces. To this end, we will write the vectors $\vec{n} = \{1, 0\}$ and $\vec{\tau} = \{0, 1\}$ of the normal and tangent to the undeformed surface of the elastic body and the gradient to the die surface at points of the contact surface of the elastic body, $\operatorname{grad} \Psi|_{r=R_2} = \{1, -4\delta z / a^2\}$. From these expressions, it is obvious that the normals to the surfaces of the cylinder and the die are not orthogonal. Performing simple transformations and applying formula (1.4), we obtain linearized contact conditions ($u_n = u_r$ and $u_\tau = u_z$),

$$u_r - 4\delta z u_z / a^2 = -\delta(1 - 2z^2 / a^2), \qquad -a \le z \le a. \tag{4.2}$$

Let us rearrange this expression:

$$u_r / \delta - 4\delta z u_z / a^2 = -(1 - 2z^2 / a^2). \tag{4.3}$$

For the adopted conditions $u_r / \delta \le 1$ and $u_z / a \ll 1$, the second term on the left-hand side of Eq. (4.3) will be very small. Therefore, it can be neglected. For the normal indentation, we obtain the following equality:

$$u_r^* = -\delta(1 - 2z^2 / a^2).$$

On this basis, for the points of the external radius of the cylinder under the die, the impenetrability conditions and the conditions of negativity of the normal contact stresses (1.2) and (1.3) must be satisfied. For a smooth die ($\sigma_{zr} = 0$), these conditions can be written in the form

$$\sigma_{rr} \le 0, \qquad -c \le z \le c, \tag{4.4}$$

$$u_r < u_r^*, \qquad -c > z > c, \tag{4.5}$$

where $2c$ is the extent of the contact region to be determined by the method of successive iterations.

As the initial approximation of the contact region, we assume $c = 2a$. Figure 3 illustrates the iterative determination of the contact boundary. From Fig. 3, it is seen that conditions (4.4) for the normal contact stresses are satisfied exactly at the fourth iteration. Note that condition (4.5) whereby the points of the ring cannot penetrate into the die is satisfied at each iteration. Since both conditions (4.4) and (4.5) are satisfied at the fourth iteration, the contact problem with one-sided constraints is considered solved. From the calculation data, it is found that $c = 0.543a$.

To assess the reliability of the solutions obtained from Eq. (1.1) at each iteration, we will take advantage of the method of a posteriori integral estimation of approximate solutions (Section 3) employing the fact that the stationary values of the

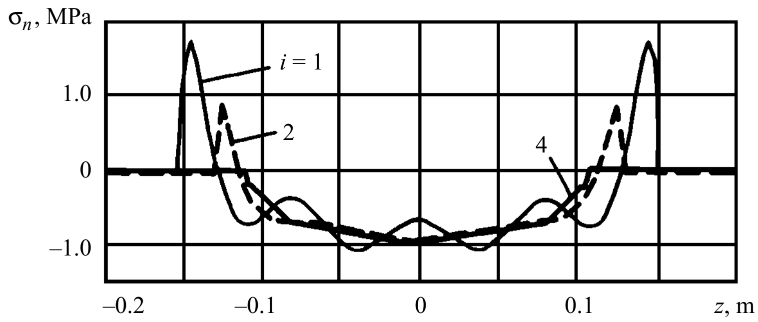

Fig. 3

TABLE 2

i	0	1	3	4
Φ_C	0.0176	0.0141	0.0133	0.0135
Φ_R	0.0176	0.0141	0.0132	0.0132
Φ_L	0.0176	0.0143	0.0134	0.0133

TABLE 3

i	0	1	2	3	4	5
P	128.7	179.2	190.5	192.3	195.2	190.5

Castigliano, Reissner, and Lagrange functionals are equal. From Table 2, it is seen that solutions of linear problems of the form (1.1) are obtained with a high accuracy at each iteration step i.

The integral estimates of approximate solutions presented in Table 2 provide information on the accuracy of the solutions at each step of the iterative process. Together with inequalities (4.4) and (4.5), they give a final estimate of the accuracy of the solution to the initial nonlinear problem. An indirect estimate of the solution accuracy can be made from the accuracy of determination of the contact area. To this end, we may use the maximality condition for the contact pressure forcing a rigid smooth die into an elastic body to a prescribed depth. This condition was formulated in [8] as follows: if $P(S_c)$ is the force required for the die to come into contact with the region S_c, then for a fixed indentation depth the inequality imposed on the components of the stress–strain state will hold if and only if S_c is selected so as to maximize P. In the problem in question, the pressing contact force P is determined by the formula

$$P = -2\pi R_2 \int_{-a}^{a} \sigma_{rr}(R_2, z)dz. \tag{4.6}$$

To estimate how accurately the contact area is determined, we set $c = 0.48a$ to be smaller than the value that has been calculated, $c = 0.543a$. The stress–strain analysis made for all iterations and under the adopted condition ($i = 5$) shows that the pressing contact force (P, kN) given in Table 3 has a maximum for the fourth-iteration solution. Thus, the calculated contact boundary differs from the exact boundary by no greater than 11.6%.

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