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RESEARCH PAPER

Hosoya and Wiener Index of Zero-Divisor Graph of Z pm q2

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A B S T R A C T:

In this work, we study zero-divisor graph of the ring Zpmq2 and give some properties of this graph. Furthermore we find Hosoya polynomial and Wiener index for this graph.

KEY WORDS: zero-divisor graph, Clique number , Hosoya polynomial, Wiener index.
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1. INTRODUCTION :

Let R be a commutative ring with identity $1\neq0$ and let Z(R) be the set of zero divisor graph of R, we denoted Γ (R) is a simple graph with vertices in $Z(R)^*=Z(R)-\{0\}$,such that u ,v $\in Z(R)^*=V(\Gamma(R))$ are adjacent if and only if uv=0. Many authors studied this concept see (Anderson and Badawi,2017; Anderson and Livindston,1999 ; Axtell , Stickles and Trampbachls,2009; Atiyah and Macdonald ,1969) .The distance between a pair of vertices (u,v)of the graph is the length of the shortest path between u,v. The diameter of a connected graph Γ, denoted diam(Γ), is the maximum distance between two vertices. The eccentricity e(v) of a vertex is the maximum distance from it to any other vertex, that is $e(v)=\max\{d(u,v)$ where $u,v\in V(\Gamma(R))\}$, the radius of Γ is rad (Γ) = min{d(x,y) : x and y are vertices of Γ } and the center of Γ is defined by $Cent(\Gamma)=\{x\in V(\Gamma) : d(x,y)= rad (G)\}$, for any $y\in V(\Gamma)$ (Buckley and Harary,1990) . $\lceil\frac{m}{2}\rceil$($\lfloor\frac{m}{2}\rfloor$ resp.) Its means that the smallest integer is not less than $\frac{m}{2}$ (the greatest integer is not greater than $\frac{m}{2}$ resp.). A complete sub-graph K_n of a graph Γ is called a clique , and ω (Γ) is the clique number of Γ, which is the greatest integer $n\geq1$such that $K_n\subseteq\Gamma$. In (Hosoya,1985) Hosoya gave the concept of Hosoya polynomials as follows: $H(\Gamma;x)=\sum_{k=0}^{diam(\Gamma)} d(\Gamma,k)x^k$,such that $d(\Gamma,k),k\geq0$, be the number of vertex pairs at distance k in connected graph Γ . The Wiener index of Γ is the sum of the distance between all pair of vertices of Γ, that is $W(\Gamma)=\sum d(u,v)$,where $u,v\in V$,and we can find this index by differentiating Hosoya polynomial with respect to x then putting x=1, see (Wiener,1947; Gutman,1993) . In (Ahmadi and Jahani-Nezhad ,2011) Ahmadi and Jahani-Nazhad studied the Wiener index of graph $\Gamma(Z_{pq})$, $\Gamma(Z_{q^2})$ where p,q are distinct prime .In (Mohammed and Authman,2018) studied the Hosoya polynomials and Wiener index of $\Gamma(Zp^mq)$,$\Gamma(Zp^m)$ and gave the properties of them

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.In this work we find Wiener index of $\Gamma(Z_{p^m q^2})$,where m≥2 and p,q are distinct prime ,and we give new properties of Hosoya polynomials of $\Gamma(Z_{p^m q^2})$.

2. Properties of $\Gamma(Zp^m q^2)$

In this section we give some properties of $\Gamma(Z_{p^m q^2})$ and find the degree of $\Gamma(Z_{p^m q^2})$, where m≥2 and p,q are distinct prime number .

Lemma 2.1: (Duane**,2006)**

Let Z_n be a ring of integers modulo n. then the number of all non-zero-divisors for $\frac{n}{k}$ are $\frac{n}{k}-1$.

Definition 2.2:

Let $R=Z_{p^m q^2}$, where m≥2 and p,q are prime, we can write $Z(R)^*=\cup_{i=1}^{m} A_i \ \cup_{j=0}^{m} B_j \ \cup_{k=0}^{m-1} C_k$, where $A_1=(P^{m-1}q^2)-\{0\}$, $A_i=(p^{m-i}q^2)-\{(p^{m-i+1}q^2)\}$, i=2,3,…,m.
$B_0=(p^m q)-\{(p^m q^2)\}$, $B_j=(p^{m-j}q)-\{(p^{m-j}q^2)\cup(p^{m-j+1}q)\}$, j=1,2,3,…,m
$C_0=(p^m)-(p^m q)$ and $C_k=(p^{m-k})-\{(p^{m-k}q)\cup(p^{m-k+1})\}$, k=1,2,3,…,m-1. Using lemma 2.1, we have $|A_1|=p-1$, $|A_i|=\left(\frac{p^m q^2}{p^{m-i}q^2}-1\right)-\left(\frac{p^m q^2}{p^{m-i+1}q^2}-1\right)=p^i-p^{i-1}$, where i=2,…,m. Similarly $|B_0|=q-1$, $|B_j|=(p^j-p^{j-1})(q-1)$,$|C_0|=q(q-1)$ and $|C_k|=(p^k-p^{k-1})(q-1)q$, for any j=1,2,3,…,m and k=1,2,3,…,m-1.

Remark 2.3:

For any $s\in Z^+$ we get,
$\sum_{i=1}^{s}|A_i|=(p-1)+(p^2-p)+(p^3-p^2)+\ldots+(p^{s-1}-p^{s-2})+(p^s-p^{s-1})=p^s-1$,
$\sum_{j=0}^{s}|B_j|=(q-1)+(p-1)(q-1)+(p^2-p)(q-1)+\ldots+(p^s-p^{s-1})(q-1)=p^s(q-1)$,
$\sum_{k=0}^{s}|C_k|=(q-1)q+(p-1)(q-1)q+\ldots+(p^s-p^{s-1})(q-1)q(q-1)q=p^s(q-1)q$,

Next, we shall give the following results

Theorem 2.4:

Let A_i , B_j , C_k be a subsets as Definition 2.2, then for each $x\in Z(Z_{p^m q^2})^*$,

$$\deg(x)_{x\in\Gamma(p^m q^2)}=\begin{cases} p^{m-i}q^2-2, & \text{if } x\in A_i \text{ and } 1\le i\le\left\lfloor\frac{m}{2}\right\rfloor. \\ p^{m-i}q^2-1, & \text{if } x\in A_i \text{ and } \left\lfloor\frac{m}{2}\right\rfloor+1\le i\le m. \\ p^{m-j}q-2, & \text{if } x\in B_j \text{ and } 0\le j\le\left\lfloor\frac{m}{2}\right\rfloor. \\ p^{m-j}q-1, & \text{if } x\in B_j \text{ and } \left\lfloor\frac{m}{2}\right\rfloor+1\le j\le m. \\ p^{m-k}-1, & \text{if } x\in C_k \text{ and } 0\le k\le m-1. \end{cases}$$

Proof:

First let $x\in A_i$ for each 1≤i≤m and let $y\in Z(Z_{p^m q^2})^*$, then there are three cases.

Case 1:

If $y\in A_j$, where j=1,2,…,m, since xy=0 (mod $p^m q^2$) if and if i+j ≤ m, then x adjacent with y if and only if j=1,… , m-i.Since $\sum_{j=1}^{m-i}|A_j|=p^{m-i}-1$, but $x\in A_j$ has loop if and only if $1\le i\le\left\lfloor\frac{m}{2}\right\rfloor$,we get the number of adjacent elements with A_j in this case $p^{m-i}-2$, if $1\le i\le\left\lfloor\frac{m}{2}\right\rfloor$ and $p^{m-i}-1$, if $\left\lfloor\frac{m}{2}\right\rfloor+1\le i\le m$

Case 2:

If $y\in B_j$, since xy=0 if and only if i+j≤m, 1≤j≤m, then x adjacent with y if and only if j=1,2,..,m-i, then the number of adjacent elements with A_i in this case $\sum_{i=0}^{m-i}|B_i|=p^{m-i}(q-1)$.

Case 3:

If $y\in C_k$, since xy=0 if and only if i+k ≤ m, 0≤k≤m-1**,** this implies that $\sum_{k=0}^{m-i}|C_k|=p^{m-i}q(q-1)$. Therefore the degree of vertex A_i where $x\in A_i$, $1\le i\le\left\lfloor\frac{m}{2}\right\rfloor$, is

$$\deg(x)_{x\in A_i}=\begin{cases} p^{m-i}q^2-2, & \text{if } 1\le i\le\left\lfloor\frac{m}{2}\right\rfloor. \\ p^{m-i}q^2-1, & \text{if} \left\lfloor\frac{m}{2}\right\rfloor+1\le i\le m. \end{cases}$$

Now, if $x\in B_j$, j=0,1,…,m and $y\in Z(Z_{p^m q^2})^*$, if $y\in C_k$, then xy≠0 ,we have two cases:

Case 4:

If $y\in A_i$, then xy=0 if and only if i+j≤m, i=1,…,m-j .So that the number of adjacent elements with B_j in this case is $\sum_{i=1}^{m-j}|A_i|=p^{m-j}-1$.

Case 5:

If $y\in B_j$, then xy=0 if and only if i+j≤m, i=1,2,…m-j, then $\sum_{i=0}^{m-j}|B_j|=p^{m-j}(q-1)$, but x has loop if and only if $0\le j\le\left\lfloor\frac{m}{2}\right\rfloor$, then the number of adjacent elements with B_j is $p^{m-j}(q-1)-1$, if $0\le j\le\left\lfloor\frac{m}{2}\right\rfloor$ and $p^{m-j}(q-1)$,if $\left\lfloor\frac{m}{2}\right\rfloor+1\le j\le m$. Hence $\deg(x)_{x\in B_j}=p^{m-j}-1+p^{m-j}(q-1)-1=p^{m-j}q-2$, if

$0\le j\le\lfloor\frac{m}{2}\rfloor$. In a similar way we can find, $deg(x)_{x\in B_j}=p^{m-j}q-1$, if $\lfloor\frac{m}{2}\rfloor+1\le j\le m$.

Finally, If $x\in C_k$, for each k=0,1,…,m-1, ,since xy=0 if and only if i+k≤m, $y\in A_i$,where i=1,…,m, so x adjacent with y if and only if k=1,..,m-k, but $\sum_{k=1}^{m-k}|C_k|=p^{m-k}-1$,if follows that $deg(x)_{x\in C_k}=p^{m-k}-1$

Theorem 2.5:

Let A_i, B_j, C_k be a subsets as Definition 2.2,where i=1,2,3,…,m, j=1,2,3,…,m and k=1,2,3,…,m-1,then $\omega(\Gamma(Z_{p^m q^2}))=p^{\lfloor\frac{m}{2}\rfloor}q-1$.

Proof:

Let N be a sub-graph of $\Gamma(Z_{p^m q^2})$, with different sets of vertices A_i and B_j, where$1\le i\le\lfloor\frac{m}{2}\rfloor$,$0\le j\le\lfloor\frac{m}{2}\rfloor$,so $\sum_{i=1}^{\lfloor\frac{m}{2}\rfloor}|A_i|=p^{\lfloor\frac{m}{2}\rfloor}-1$ and $\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}|B_j|=p^{\lfloor\frac{m}{2}\rfloor}(q-1)$. we note that any vertex in A_i is adjacent to each other vertices in B_j, and the elements of A_i are adjacent with each other also the elements of B_j are adjacent with each other. Therefore, the sub-graph N is complete.

If $x\in V(\Gamma(R))$ such that $x\notin N$, then x is non a adjacent with any element of $B_{\lfloor\frac{m}{2}\rfloor}$.So that N the greats complete sub-graph of Γ(R). Therefore $\omega(\Gamma(Z_{p^m q^2}))=\sum_{i=1}^{\lfloor\frac{m}{2}\rfloor}|A_i|+\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}|B_j|=p^{\lfloor\frac{m}{2}\rfloor}q-1$.

Theorem 2.6:

Let A_i, B_j, C_k be a subset as Definition 2.2**,** where i=1,2,3,…,m, j=0,1,2,3,…,m, k=0,1,…,m-1.Then $Cent(\Gamma(Z_{p^m q^2}))=\bigcup_{i=1}^{m-1}A_i\bigcup_{j=0}^{m-1}B_j$.

Proof:

Let $x\in\Gamma(Z_{p^m q^2})$. Then we have the following two cases:

Case1.

If min e(x)=1,then x is adjacent to all other vertices .This mean that $Z_{p^m q^2}$is a local ring or $Z_{p^m q^2}\cong F_1\times F_2$, where F_i , i∈{1,2} is a field but not local (Wang,2005) , this contradict the fact that $(Z_{p^m q^2})$ is not local ring or isomorphic with F_1 x F_2.

Case2.

If min e(x) =2 or 3 .Since diam Γ(R)≤3, then max e(x)≤3.So min e(x)=2 this mean the eccentricity of the vertices of the center of $\Gamma(Z_{p^m q^2})$ must be 2 .We observe that $x\in A_i$ or $y\in B_j$, where i=1,,2,…,m-1 ,j=0,1,…,m-1, has the eccentricity 2.

Example .1:

Let $\Gamma(Z_{p^m q^2})$,is graph such that p=3,q=5,m=3 , then $Z(Z_{675})^*=Z(Z_{3^3.5^2})^*=$ {3, 5, 6, 9, 10, 12, 15, 18, 20, 21, ... , 663, 669, 672}.

$Z(Z_{674})^*=Z(Z_{3^3.5^2})^*=\bigcup_{i=1}^{m}A_i\bigcup_{j=0}^{m}B_j\bigcup_{k=0}^{m-1}C_k$.

$A_1=(3^2.5^2)-\{0\}=\{225,450\}$.

$A_2=(3.5^2)-\{A_1\cup\{0\}\}=\{75, 150, 300, 375, 525, 600\}$.

$A_3=(5^2)-\{A_2\cup A_1\cup\{0\}\}=\{25, 50, 100, 125, 175, 200, 250, 275, 325, 350, 400, 425, 475, 500, 550, 575, 625, 650\}$.

$B_0=(3^3.5)-\{0\}=\{135, 270, 405, 540\}$.

$B_1=(3^2.5)-\{B_0\cup A_2\cup A_1\cup\{0\})=\{45, 90, 180, 315, 360, 495, 585, 630\}$.

$B_2=(3.5)-\{B_1\cup A_1\cup\{0\}\}=\{15, 30, 60, 105, 120, 165, 195, 210, 240, 255, 285, 330, 345, 390, 420, 435, 465, 480, 510, 555, 570, 615, 645, 660\}$.

$B_3=(5)-\{B_2\cup A_3\cup\{0\}\}=\{5,10,20, 35, 40, 55, 65, 70, 80, 85, 95, 110, 115, 130, 140, 145, 155, 160, 170, 185, 190,\dots 655, 665,670\}$

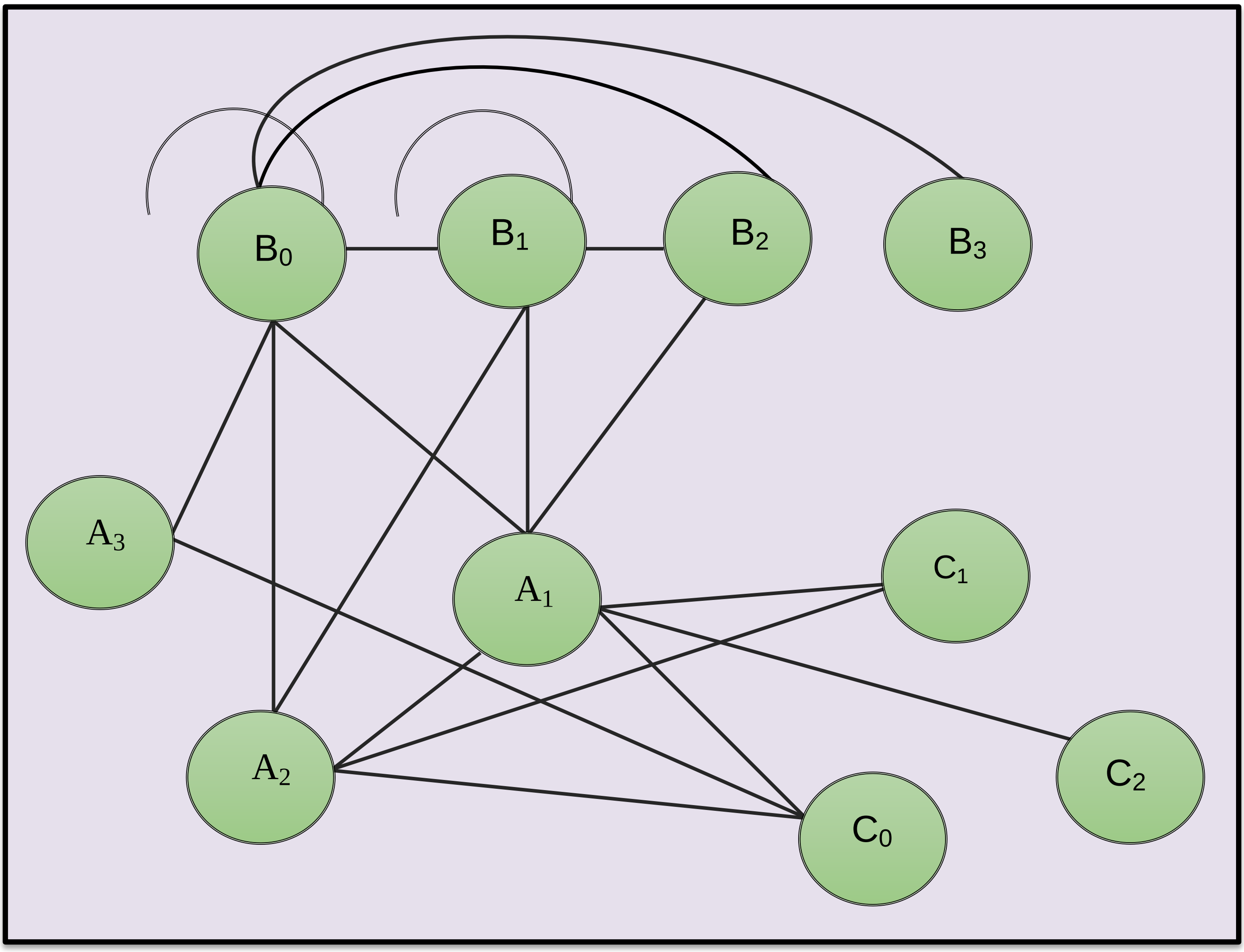
$C_0=(3^3)-\{0\}=\{27, 54, 81, 108, 162, 189, 216, 243, 297, 324,351, 378, 432, 459, 486, 513, 567, 594, 621, 648\}$.

$C_1=(3^2)-\{C_0\cup B_3\cup A_3\cup\{0\}\}=\{9, 18, 36, 63, 72, 99, 117, 126, 144, 153, 171, 198, 207, 234, 252, 261, 279, 288, 306,\dots,666\}$.

$C_2=(3)-\{C_1\cup B_3\cup A_3\cup\{0\}\}=\{3, 6, 12, 21, 24, 33, 39, 42, 48, 51, 57, 66, 69, 78, 84, 87, 93, 96, 102, 111, 114, 123, 129, 132, 138,\dots, 672\}$.

Then the

$deg(\Gamma(Z_{675}))_{x\in A_1}=224$,$deg(\Gamma(Z_{675}))_{x\in A_2}=74$, $deg(\Gamma(Z_{675}))_{x\in A_3}=24$,$deg(\Gamma(Z_{675}))_{x\in B_0}=134$, $deg(\Gamma(Z_{675}))_{x\in B_1}=44$,$deg(\Gamma(Z_{675}))_{x\in B_2}=14$, $deg(\Gamma(Z_{675}))_{x\in B_3}=4$,$deg(\Gamma(Z_{675}))_{x\in C_0}=26$, $deg(\Gamma(Z_{675}))_{x\in C_1}=8$, $deg(\Gamma(Z_{675}))_{x\in C_2}=2$, and $Cent(\Gamma(Z_{3^3 5^2}))=A_1\cup A_2\cup B_0\cup B_1\cup B_2$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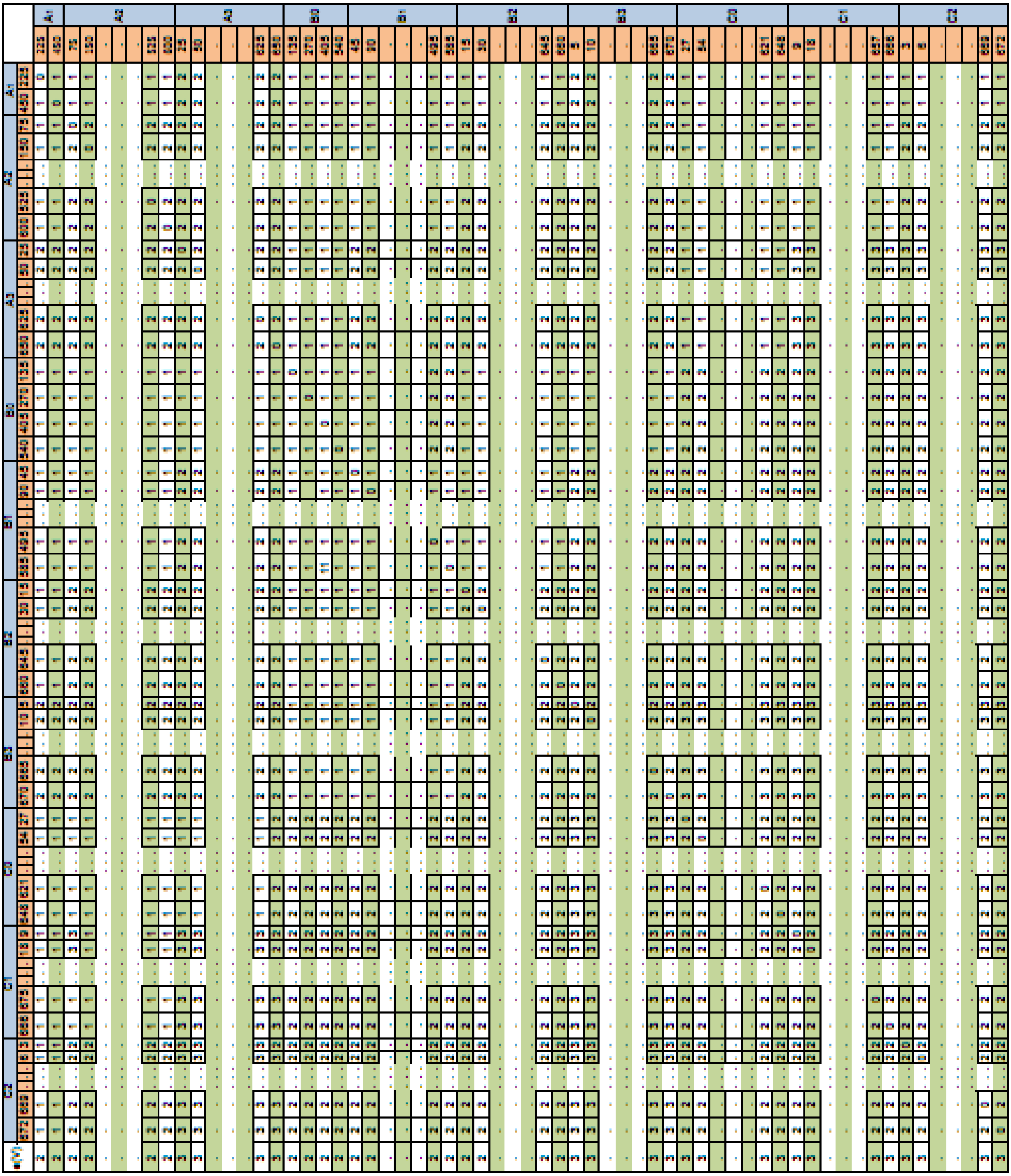

Table distances between the vertices of $\Gamma(Z_3^{3}.5^{2})$

3. Hosoya and Wiener index of zero divisor graph of $\Gamma(Zp^m q^2)$

In this section we will compute Hosoya polynomials and Wiener index of $\Gamma(Z_{p^m q^2})$.First we find the order and size of $\Gamma(Z_{p^m q^2})$**,** where m≥2 and p, q are distinct prime number.

We begin this section with the following results.

"Lemma 3.1:(Shuker and Mohammad,2013**)**

Let $R\cong R_1 \times R_2$,where R_1 and R_2 are local rings, then $|Z(R)^*|=|R_1|\times|Z(R_2)|+|R_2|\times|Z(R_1)|-|Z(R_1)||Z(R_2)|-1$".

Theorem 3.2:

Let $R \cong Z_{p^m q^2}$ then the order of R is $a_0= p^{m-1}(p^2+pq-q)-1$ and the size is

$$a_1 = \frac{1}{2}[q^2\left(\frac{1-p^m}{1-p}\right)+q\left(\frac{1-p^{m+1}}{1-p}\right)+p\left(\frac{1-p^m}{1-p}\right)-\left(\left\lfloor\frac{m}{2}\right\rfloor+6m+1\right)].$$

Proof:

Let $R\cong Z_{p^m q^2}$, since $Z_{p^m q^2}\cong Z_{p^m}\times Z_{q^2}$, and $Z(Z_{p^m})=p^{m-1}-1$, $Z(Z_{q^2})=q-1$, then by lemma 3.1

$|Z(R)|^*=|Z(Z_{p^m})|\,|Zq^2|+|Z(Z_{q^2})|\,|Z_{p^m}|-|Z(Z_{p^m})|\,|Z(Z_{q^2})|-1 = p^{m-1}(p^2+p\,q-q)-1$

Next we shall find the size a_1 of R. Since

$$2a_1=\sum_{v\in\Gamma(R)} deg(v)=\sum_{v\in A_i} deg(v)+\sum_{v\in B_j} deg(v)+\sum_{v\in C_k} deg(v).$$

where i=1,2,3,…,m, j=0,1,2,3,…,m, k=0,1,…,m.

$$2a_1=\sum_{i=1}^{\lfloor\frac{m}{2}\rfloor}(p^{m-i}q^2-2)+\sum_{i=\lfloor\frac{m}{2}\rfloor+1}^{m}(p^{m-i}q^2-1)+\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}(p^{m-j}q-2)\quad+\sum_{j=\lfloor\frac{m}{2}\rfloor+1}^{m}(p^{m-j}q-1)+\sum_{k=0}^{m-1}(p^{m-k}-1).$$

Hence

$$a_1 = \frac{1}{2}[q^2\left(\frac{1-p^m}{1-p}\right)+q\left(\frac{1-p^{m+1}}{1-p}\right)+p\left(\frac{1-p^m}{1-p}\right)-\left(\left\lfloor\frac{m}{2}\right\rfloor+6m+1\right)]$$

Theorem 3.3:

Let $\Gamma(Z_{p^m q^2})$ be a zero divisor graph, and p, q are distinct prime numbers and m>2. Then $diam\Gamma(Z_{p^m q^2})=3$

Proof:

Let $Z(Z_{p^m q^2})^*=\cup A_i\cup B_j\cup C_k$, where A_i ,B_j, C_k , i= 1 ,2,…,m j=0,1,2,…,m, k=0,1,…,m, be subsets as definition 2.2.we can find $v_1\in C_{m-1}$,$v_2\in B_m$ .Since every element in C_{m-1} is adjacent with only element in A_1 and every element in B_m is adjacent with only element in B_0, and since every element in B_0 is adjacent with only element in A_1.Then we have $d(v_1,v_2)=3$. So that $diam\Gamma(Z_{p^m q^2})=3$

The following lemma is given by (Hosoya,1985)

Lemma3.4:

Let G be a connected graph of order r, then $\sum_{i=0}^{dima(G)} d(G,i) =\frac{1}{2}\, r(r+1)$.

Theorem 3.5:

Let R= $Z_{p^m q^2}$ mod($p^m q^2$), since $diam\Gamma(Z_{p^m q^2})=3$, then $H(\Gamma(R),x)=a_0+a_1x+a_2x^2+a_3x^3$, where $a_0=p^{m-1}(q^2+pq-q)-1$, $a_1=\frac{1}{2}[q^2\left(\frac{1-p^m}{1-p}\right)+q\left(\frac{1-p^{m+1}}{1-p}\right)+p\left(\frac{1-p^m}{1-p}\right)-\left(\left\lfloor\frac{m}{2}\right\rfloor+6m+1\right)]$,$a_2=\frac{1}{2}[p^{m-1}(q^2+pq-q)-1](p^{m-1}(q^2+pq-q)-1+1)-p^{m-1}(q^2+pq-q)-\frac{1}{2}[q^2\left(\frac{1-p^m}{1-p}\right)+q\left(\frac{1-p^{m+1}}{1-p}\right)+p\left(\frac{1-p^m}{1-p}\right)-(\left\lfloor\frac{m}{2}\right\rfloor+6m+1)]-p^{m-1}(p^m-p^{m-1})(q-1)q^2$, $\left\lceil\frac{m}{2}\right\rceil<i\leq m$, $\left\lceil\frac{m}{2}\right\rceil<j\leq m$, $0\leq k\leq m-1$. $a_3= p^{m-1}(p^m-p^{m-1})(q-1)q^2$

Proof:

By Theorem (3.1), $a_0=p^{m-1}(q^2+pq-q)-1$, $a_1=\frac{1}{2}[q^2\left(\frac{1-p^m}{1-p}\right)+q\left(\frac{1-p^{m+1}}{1-p}\right)+p\left(\frac{1-p^m}{1-p}\right)-\left(\left\lfloor\frac{m}{2}\right\rfloor+6m+1\right)]$.

Now to find a_3. Let

$x,y\in Z(Z_{p^m q^2}))^*$,$Z(Z_{p^m q^2}))^*=\cup_{i=1}^{m} A_i\cup_{j=1}^{m} B_j\cup_{k=0}^{m-1} C_k$, where

$A_i=p^{m-i}q^2-\{p^{m-i+1}q^2\}$,i=1,2,…,m, $B_j=p^{m-i}q-\{(p^{m-j}q^2)\}\cup\{(p^{m-j+1}q)\}$,j=0,…,m,$C_k=p^{m-k}-\{(p^{m-k}q)\}\cup\{(p^{m-k+1})\}$, k=0 ,…,m-1.

Then we have three cases:

Case1:

1. If $x\in A_i$,and $y\in A_j$,i= 1,2,…, m,j=1,2,…,m, $x=k_1p^{m-i}q^2$,$p^{m-i+1}\nmid k_1$, $y=k_2p^{m-j}q^2$,$p^{m-j+1}\nmid k_2$, then x and y adjacent $p^{m-1}q^2$,that is $d(x,y)\leq 2$ a contradiction.
2. If $x\in A_i$,and $y\in B_j$, i=1,2,…,m, j= 0,1 ,2, … ,m.$x=k_1p^{m-i}q^2$,$p^{m-i+1}\nmid k_1$, $y=k_2\,p^{m-j}q$, $p^{m-j+1}q\nmid k_2$, then x and y adjacent p^mq, that is $d(x,y)\leq 2$ a contradiction .

3. If $x \in A_i$,and $y \in C_k$, i=1,2,…,m, k= 0,1,2, … ,m-1. $x = k_1p^{m-i}q^2$,$p^{m-i+1} \nmid k_1$, $y = k_2\, p^{m-k}$, $p^{m-k+1} \nmid k_2$. If i ≠m, then x and y adjacent $p^{m-1}q^2$, that is d(x,y)≤ 2 a contradiction . if i = m, then x adjacent with only elements in B_0 or C_0, but y non adjacent with every elements in B_0 or C_0, so that d(x,y)=3 for any k=1,….,m-1 and the number of element in this cases $|A_m| .\sum_{k=1}^{m-1}|C_k| = (p^m-p^{m-1})p^{m-1}(q-1)q$

Case2:

1. If $x \in B_i$, and $y \in B_j$, i, j=0,1,2,…,m.

$x = k_1p^{m-i}q$, $y = k_2p^{m-j}q$, where $p^{m-i+1}q \nmid k_1$, $p^{m-i}q^2 \nmid k_1$, $p^{m-j+1}q \nmid k_2$ and $p^{m-j}q^2 \nmid k_j$, since x and y are adjacent with elements p^mq , that is d(x,y)≤ 2 a contradiction.

2. If $x\in B_i$ and $y\in C_j$,i=0,1,…,m, j=0,1,…,m-1. $x=k_1p^{m-1}q$, $y=k_2p^{m-j}$, $p^{m-i+1}q\nmid k_1$ and $p^{m-i}q^2\nmid k_1$,$p^{m-j+1}\nmid k_2$ and $p^{m-j}q^2\nmid k_2$, then x and y adjacent $p^{m-1}q^2$, for any i,j=0, …m-1 that is d(x,y)≤ 2, a contradiction. If i=m, and j∈{0,…,m-1}. Then by the similar way in case 1 for i=m, we can show that d(x,y)=3, so that the number of element in this cases$|B_m|.\sum_{k=0}^{m-1}|C_k| = p^{m-1}(p^m-p^{m-1})(q-1)^2q$

Case3:

If $x \in C_i$, and $y \in C_j$,where i,j = 0,1,2,…,m-1. Then $x=k_1p^i$,$y=k_2p^j$, q or $p^{i+1} \nmid k_1$, q or $p^{j+1} \nmid k_2$, then x and y adjacent pm-1q2, that is d(x,y)≤ 2 a contradiction.

Therefore, we get $a_3= p^{m-1}(p^m-p^{m-1})(q-1)q^2$,$a_2=\frac{1}{2}[p^{m-1}(q^2+pq-q)-1](p^{m-1}(q^2+pq-q)-1+1) - p^{m-1}(q^2+pq-q) - \frac{1}{2}[q^2\left(\frac{1-p^m}{1-p}\right) + q\left(\frac{1-p^{m+1}}{1-p}\right) + p\left(\frac{1-p^m}{1-p}\right) - (\left\lfloor\frac{m}{2}\right\rfloor+6m+1)] - p^{m-1}(p^m-p^{m-1})(q-1)q^2$, $\left\lceil\frac{m}{2}\right\rceil<i\leq m$, $\left\lceil\frac{m}{2}\right\rceil<j\leq m$, 0≤k≤m-1.

Corollary 3.6:

$W(\Gamma(Z_{p^mq^2}))= \frac{1}{2}[q^2\left(\frac{1-p^m}{1-p}\right) + q\left(\frac{1-p^{m+1}}{1-p}\right) + p\left(\frac{1-p^m}{1-p}\right) - \left(\left\lfloor\frac{m}{2}\right\rfloor + 6m + 1\right)+2[\frac{1}{2}(p^{m-1}(q^2+pq-q)-1)(p^{m-1}(q^2+pq-q)-1+1)-p^{m-1}(q^2+pq-q) - \frac{1}{2}[q^2\left(\frac{1-p^m}{1-p}\right) + q\left(\frac{1-p^{m+1}}{1-p}\right) + p\left(\frac{1-p^m}{1-p}\right) - (\left\lfloor\frac{m}{2}\right\rfloor+6m+1)] - (p^{m-1}(p^m-p^{m-1})(q-1)q^2]+3[p^{m-1}(p^m-p^{m-1})(q-1)q^2]$, $\left\lceil\frac{m}{2}\right\rceil<i\leq m$, $\left\lceil\frac{m}{2}\right\rceil<j\leq m$, 0≤k≤m-1.

Proof:

From definition Wiener index, since $W(\Gamma(Z_{p^mq^2}))=\frac{d}{dx}H(\Gamma(Z_{p^mq^2});x)|_{x=1}$. Therefore, then $W(\Gamma(Z_{p^mq^2}))=0+a_1+2a_2x+3a_3x^2=$ $\frac{1}{2}[q^2\left(\frac{1-p^m}{1-p}\right) q\left(\frac{1-p^{m+1}}{1-p}\right) +p\left(\frac{1-p^m}{1-p}\right) - \left(\left\lfloor\frac{m}{2}\right\rfloor + 6m + 1\right)+2[\frac{1}{2}(p^{m-1}(q^2+pq-q)-1)(p^{m-1}(q^2+pq-q)-1+1)-p^{m-1}(q^2+pq-q) - \frac{1}{2}[q^2\left(\frac{1-p^m}{1-p}\right) +q\left(\frac{1-p^{m+1}}{1-p}\right) + p\left(\frac{1-p^m}{1-p}\right) - (\left\lfloor\frac{m}{2}\right\rfloor+6m+1)] - (p^{m-1}(p^m-p^{m-1})(q-1)q^2]x+3[p^{m-1}(p^m-p^{m-1})(q-1)q^2]\, x^2|_{x=1}$,$\left\lceil\frac{m}{2}\right\rceil<i\leq m$,$\left\lceil\frac{m}{2}\right\rceil<j\leq m$, 0≤k≤m-1.Therefore we have $W(\Gamma(Z_{p^mq^2}))= \frac{1}{2}[q^2\left(\frac{1-p^m}{1-p}\right) + q\left(\frac{1-p^{m+1}}{1-p}\right) + p\left(\frac{1-p^m}{1-p}\right) - \left(\left\lfloor\frac{m}{2}\right\rfloor + 6m + 1\right)+2[\frac{1}{2}(p^{m-1}(q^2+pq-q)-1)(p^{m-1}(q^2+pq-q)-1+1)-p^{m-1}(q^2+pq-q) - \frac{1}{2}[q^2\left(\frac{1-p^m}{1-p}\right) + q\left(\frac{1-p^{m+1}}{1-p}\right) +p\left(\frac{1-p^m}{1-p}\right) - (\left\lfloor\frac{m}{2}\right\rfloor+6m+1)] - (p^{m-1}(p^m-p^{m-1})(q-1)q^2] + 3[p^{m-1}(p^m-p^{m-1})(q-1)q^2]$,$\left\lceil\frac{m}{2}\right\rceil<i\leq m$, $\left\lceil\frac{m}{2}\right\rceil<j\leq m$, 0≤k≤m-1.

Example 2:

In Example 1,we find Hosoya and Wiener index of zero divisor graph of $\Gamma(Z_{3^3.5^2})$, $d(\Gamma(Z_{3^3.5^2},0))=314$,$d(\Gamma(Z_{3^3.5^2},1))=272$,$d(\Gamma(Z_{3^3.5^2},2))=32669$,$d(\Gamma(Z_{3^3.5^2},3))=16200$.Therefore we have$(\Gamma(Z_{3^35^2});x)=314+272x+32669x^2+16200x^3$. By definition Wiener index of zero divisor graph of $\Gamma(Z_{3^3.5^2})$. $W(\Gamma(Z_{3^3.5^2}))=272+2(32669)+3(16200)=114210$.

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