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RESEARCH PAPER

Existence of solution for some quasi-homogenous and quasi-elliptic Nonlinear Eigenvalue Problems

Fatima M. ABOUD1

[1]Department of Mathematics, College of Science, University of Diyala/Diyala, Iraq.

A B S T R A C T:

The existence of solutions for a non linear eigenvalue problems is well studied and proved for n even. In this article we will study the case of odd dimension n>1 for the family of quasi-homogeneous and quasi-elliptic operators and we will give some examples for the case n=3. We study the conditions for which we can prove the existence of non trivial solution for each case.

KEY WORDS:.Nonlinear eigenvalue problems, spectra, trace, quasi-elliptic operators, quasi-homogeneous operators.
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INTRODUCTION :

In this article we study the existence of the non-trivial solutions for a non-linear eigenvalueproblem with some quasi-homogenous and quasi-elliptic operators defined on an odd dimensional space.

In 2003, Helffer, Robert and Wang have proved the existence of eigenvalues for the non-linear eigenvalue problems for every even dimension.

For the case of odd dimension: the case n = 1 was studied by [Robert, 2004], [Christ, 1992] and [Aboud, 2009, 2010]. The case odd n > 1 was studied in [Aboud, 2009] (doctorate thesis), in which the author proved the existence of non-trivial eigenvalues for some quadratics families of operators for the cases of dimensions n = 1, 3, 5and 7 and a conjecture was given for the upper odd dimensions.

* **Corresponding Author:**
Fatima M ABOUD
E-mail fatima.aboud@sciences.uodiyala.edu.iq
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In [Aboud, 2018], the existence of the non-trivial solutions for a non-linear eigenvalue problem with a quasi-homogeneous operator defined on an odd dimensional space were proved.

2. PRELIMINARIES

In this section we give some well-known results. Let$T$be a compact operator onH, where H is a separable Hilbert space. If $r(T) \neq 0$ (spectral radius) one order the non-zero eigenvalues of T in a decreasing sequence in modulo

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots \geq |\lambda_n(T)| \geq \cdots$$

every eigenvalue be repeated following its algebraic multiplicity. For proving the existence of non-trivial eigenvalues, we use the following theorem:

Theorem 2.1 ***Theorem of Lidskii (1958)*** For every $T \in C^1(H)$ we have $Tr(T) = \sum_{j\geq 1} \lambda_j(T)$.

We get directly the following corollary:

Corollary 2.1 *If there exists* $k \geq 1$ *such that* $Tr(T^k) \neq 0$*, then* $\sigma(T) \neq \{0\}$.

Definition 2.1 *Let* X *be a Banach space, an application* $f: R^n \backslash \{0\} \to X$*, a multi-index*

$$\underline{h} \in N^n, \underline{h} = (h_1, \cdots, h_n)$$

and $d \in C$. One says that f is quasi-homogeneous of order d and of type h if

$$f(\rho^{h_1} x_1, \cdots, \rho^{h_n} x_n) = \rho^d (x_1, \cdots, x_n),$$

for each $\rho > 0$ and $(x_1, \cdots, x_n) \in R^n \backslash \{0\}$.

We give now the definition of quasi-homogeneous symbol.

Definition 2.2 *For the two multi-indices* $\underline{h} \in N^n, \underline{h} = (h_1, \cdots, h_n)and\underline{k} \in N^n, \underline{k} = (k_1, \cdots, k_n)$

we set

$$M = LCM\{h_1, \cdots, h_n, k_1, \cdots, k_n\}.$$

For, a is a C^∞ function such that

$$a: R^{2n} \backslash \{0\} \to L(H)$$

with

$$a(\rho^{h_1} x_1, \cdots, \rho^{h_n} x_n, \rho^{h_1} \xi_1, \cdots, \rho^{h_n} \xi_n) = \rho^M a(x_1, \cdots, x_n, \xi_1, \cdots, \xi_n),$$

where $L(H)$ is the space of continuous linear operators from$H$ to H, $x \in R^n$, $(x_1, \cdots, x_n)$ and $\xi \in R^n$, $(\xi_1, \cdots, \xi_n)$and

$$\varphi_{\underline{h},\underline{k}}(x, \xi) = \phi_{\underline{h},\underline{k}}(x, \xi) = (1 + \sum_{i=1}^{n} |x_i|^{2p_i} + \sum_{i=1}^{n} |\xi_i|^{2q_i})$$

where $p_i = \frac{M}{h_i}$, $q_i = \frac{M}{k_i}$, $1 \leq i \leq n$, $(x, \xi) \in R^{2n}$.

2.1 Conditions of trace class and Hilbert-Schmidt class of the symbols

In the following we give a result which help us to know if the operator is of trace class (or of Hilbert-Schmidt class) by using the formula of the associated symbol. For the proof of the following lemmas see [Rndeaux, 1984] :

Lemma 2.1 *Let*$\sigma^w(L)$ *be the Weyl symbol of L. One has L is an operator Hilbert Schmidt iff* $\sigma^w(L) \in L^2(R^{2n})$.

Lemma 2.2 *Let*$\sigma^w(L)$ *be the Weyl symbol of L. One has* $L \in C_p$*if :*

$$\sum_{0 \leq |\alpha| + |\beta| \leq k(p)} \left(\iint |D_x^\alpha D_\xi^\beta \sigma(L)|^p dx d\xi \right)^{\frac{1}{p}},$$

with $k(p)$ even integer $> 2n((\frac{2}{p}) - 1)$if$p \in [1,2]$, $k(p)$ even integer $> 3n(1 + (\frac{2}{p}))$ if $p \in]2, \infty[$.

We recall the following lemma (for the proof see [Aboud, 2009]) we prove the following lemma:

Lemma 2.3 *For an operator L the symbol* $\sigma(L)$ *quasi homogenous of order M and of type*

$(\underline{k}, \underline{\ell}) = (k_1, k_2, \ell_1, \ell_2)$such that

$$\sigma(L)(\rho^{k_1} x, \rho^{k_2} y, \rho^{\ell_1} \xi, \rho^{\ell_2} \eta) = \rho^M (x, y, \xi, \eta),$$

i.e.:$\sigma(L) \in S^M_{\underline{k},\underline{\ell}}$,

where $(x, y) \in R^{n_1} \times R^{n_2}$, $(\xi, \eta) \in R^{n_1} \times R^{n_2}$ and $n_1 + n_2 = n$. Then:

i) L is of trace class if $\sigma(L) \in S^M_{\underline{k},\underline{\ell}}$ with the condition:$M + (n_1 k_1 + n_2 k_2 + \ell_1 n_1 + \ell_2 n_2) < 0.$ (1)

ii) L is of Hilbert-Schmidt class if $\sigma(L) \in S^M_{\underline{k},\underline{\ell}}$ with the condition:

$$2M + (n_1 k_1 + n_2 k_2 + \ell_1 n_1 + \ell_2 n_2) < 0.$$

Theorem 2.2 ***(Theorem of Composition)***

One suppose that φ, ϕ are weight functions, if $A \in L^{\mu_1}_{\varphi,\phi}$ with symbol $a \in S^{\mu_1}_{\varphi,\phi}$ and $B \in L^{\mu_2}_{\varphi,\phi}$ of symbol $b \in S^{\mu_2}_{\varphi,\phi}$, then $AB \in L^{\mu_1+\mu_2}_{\varphi,\phi}$. Moreover, the symbol $a.b$ of AB admits an asymptotic expression

$$a.b \approx \sum_{\alpha,\beta} \Gamma(\alpha,\beta)\left(\partial_\xi^\alpha D_x^\beta a\right)\left(\partial_\xi^\beta D_x^\alpha b\right),$$

in the following sense:

for an integer number N one has

$$a.b - \sum_{|\alpha+\beta|<N} \Gamma(\alpha,\beta)\left(\partial_\xi^\alpha D_x^\beta a\right)\left(\partial_\xi^\beta D_x^\alpha b\right) \in S^{\mu_1+\mu_2-(N,N)},$$

where $\Gamma(\alpha,\beta) = (\frac{1}{2})^\alpha(-\frac{1}{2})^\beta \frac{1}{\alpha!\beta!}$.

3 Functional Analysis of the problem

Let H be a complex Hilbert space and let

$$L(\lambda) = H_0 + \lambda H_1 + \ldots + \lambda^{m-1}H_{m-1} + \lambda^m, \qquad (3)$$

L is a family of non bounded operators on H, where $\lambda \in C$.

In addition, H_0 is a closed operator with a dense domain$D(H_0)$. The operators $H_1, \ldots, H_{m-1}$ are defined on $D(H_0)$.

We consider the following hypothesis:

Hypothesis (H1):H_0 is a self-adjoint positive operator of the domain $D(H_0)$ in H.

Hypothesis (H2): For each integer j, $0 \le j \le k-1$, the operators $H_0^{-1}H_j$ and $H_0^{-1}H_j$ are bounded in H.

To give the third hypothesis, we give the following definition:

Definition 3.1 *LetH_1, H_2 be two complex Hilbert spaces and $T: H_1 \to H_2$ a compact operator. We denote by $(\mu_j(T))_{j\ge1}$ the decreasing sequence of eigenvalues of $(T^*T)^{\frac{1}{2}}$ where each eigenvalue repeated corresponding to its multiplicity. Let p strictly positive real number. We say that:*
$T \in C^p(H_1, H_2)$if

$$\sum_{j=1}^{\infty} \mu_j(T)^p < +\infty$$

where C^p denote the class of Schatten.

Hypotheses (H3): There exists a real $p > 0$ such that:$H_0^{\frac{-1}{m}} \in C^p(H)$.

So we have the following result:

Proposition 3.1 *Under the previous hypothesis one has:*

i) $L(\lambda)$ define a closed operator of domain $D(H_0)$.

ii) If $L(\lambda)^{-1}$ exists, then it is compact.

iii) $\forall\lambda \in C$, $L(\lambda)$ is an operator with index and $Ind(L(\lambda)) = 0$.

4 Quasi-elliptic and quasi-homogenous operators

4.1 ELLIPTIC Symbols

Definition 4.1 *Let A be an operator with the symbol$\sigma(A)(x,\xi)$ such that:*

$$\sigma(A)(x,\xi) = \sum_{j\ge0} a_{m-j}(x,\xi).$$

we say that A is elliptic if$a_m(x,\xi) \ne 0$ for $(x,\xi) \in R^n \times (R^n \backslash \{0\})$.

Now, let A be an elliptic operator of order $m > 0$, invertible and has minimum

increasing radius (i.e. the radius $arg\,\lambda = \theta$ in C is a minimum increasing radius if this radius does not contain any proper value of $a_m(x,\xi)$). The symbol of $A$is:

$$\sigma(A)(x,\xi) = \sum_{j=0}^{\infty} a_m(x,\xi),$$

where a_{m-j} is homogenous of order $m-j$.

Definition 4.2 *One says that the symbol* $a(x,\xi)$ *is poly-homogenous, if* $a(x,\xi)$ *of the following form:*

$$a(x,\xi)=\sum_{\ell\in N} a_{M-\ell}(x,\xi),$$

with $a_{M-\ell}(x,\xi)$ is homogenous of order $M-\ell$.

Definition 4.3 *One says that the symbol* $a(x,\xi)$ *is poly-quasi-homogenous, if* $a(x,\xi)$ *is of the following form:* $a(x,\xi)=\sum_{\ell\in N} a_{M-\ell}(x,\xi)$,

with $a_{M-\ell}(x,\xi)$ is quasi-homogenous of order $M-\ell$.

Definition 4.4 *One says that the symbol* $a(x,\xi)$ *is quasi-elliptic if it is poly- quasi-homogenous with the fact that its principal symbol principal does not vanish out of zero, i.e.:* $a_M(x,\xi)\neq 0, for (x,\xi)\in R^{2n}\setminus\{0\}$.

4.2 QUASI ELLIPTIC SYMBOLS

We consider the family of operators: $L_P(\lambda)=a(D_x)+(P(x)-\lambda)^2, x\in R^n$ (4)

where $P(x)=P(x_1,\cdots,x_n)$ is a positive quasi-homogenous polynomial of order M and of type $(k_1,\cdots,k_n)$ such that:

$$P(\rho^{k_1}x_1,\cdots,\rho^{k_n}x_n) =$$

$$\rho^M P_0(x_1,\cdots,x_n)+\rho^{M-\gamma}P_1(x_1,\cdots,x_n) \qquad (5)$$

where $\gamma\geq 1$ and P is quasi-elliptic such that:

$$P_0(x)\neq 0, x\in R^n\setminus\{0\},$$

and $a(D_x)=a(D_{x_1},\cdots,D_{x_n})$ is a positive pseudodifferential quasi-homogenous operator M' and of type $(\ell_1,\cdots,\ell_n)$ such that:

$$a(\rho^{\ell_1}D_{x_1},\cdots,\rho^{\ell_n}D_{x_n})=$$

$$\rho^{M'}a_0(D_{x_1},\cdots,D_{x_n})+\rho^{M'-\delta}a_1(D_{x_1},\cdots,D_{x_n}) \qquad (6)$$

where $\delta\geq 1$ and a is quasi-elliptic such that:

$$a_0(\xi)\neq 0, \xi\in R^n\setminus\{0\}.$$

We have associated to L_P the non self-adjoint matrix operator on $L^2(R^n)\times L^2(R^n)$ such that:

$$\hat{A}=\begin{pmatrix}0 & 1\\ -\hat{H}_0 & -\hat{H}_1\end{pmatrix} \qquad (7)$$

where

$$\hat{H}_0=a(D_x)+(P(x))^2=$$

$$a_0(D_x)+a_1(D_x)+(P_0(x)+P_1(x))^2$$

and

$$\hat{H}_1=-2P(x)=-2(P_0(x)+P_1(x)),$$

then the symbol of $\hat{A}$ is:

$$A=A_0+A_1=\begin{pmatrix}0 & 1\\ -H_0 & -H_1\end{pmatrix}+\begin{pmatrix}0 & 1\\ -H_0' & -H_1'\end{pmatrix}$$

where

$$\begin{aligned} H_0 &= a_0(\xi)+(P_0(x))^2,\\ H_1 &= -2(P_0(x)),\\ H_0' &= a_1(\xi)+2P_0(x)P_1(x)+P_1^2(x),\\ H_1' &= -2(P_1(x)) \end{aligned}$$

the principal symbol of $\hat{A}$ is A_0, in place of A_0 we take a matrix equivalent to A_0 and we call it also A_0 such that:

$$A_0=\begin{pmatrix}0 & \sqrt{H_0}\\ -\sqrt{H_0} & -H_1\end{pmatrix}$$

the principal symbol A_0 belongs to $S^1_{\underline{M},\underline{M'}}$, where:

$$\underline{M}=(\frac{k_1}{M},\cdots,\frac{k_n}{M}), \quad \underline{M'}=(\frac{\ell_1}{M'},\cdots,\frac{\ell_n}{M'}).$$

we have

$$\mu_{\mp}(x,\xi)=P_0(x,\xi)\mp\sqrt{a_0(x,\xi)}. \qquad (8)$$

So, we will prove the following proposition:

Proposition 4.1 *For the operator* $\hat{A}$ *in (7) with a symbol* $A\in S^1_{\underline{M},\underline{M'}}$*, such that:*

$$(\underline{M},\underline{M}') = \left((\frac{k_1}{M}, \cdots, \frac{k_n}{M}), (\frac{2\ell_1}{M'}, \cdots, \frac{2\ell_n}{M'}) \right),$$

such that there exists j, $j \geq 1$ such that:$\ell_1 + \cdots + \ell_n = M'j$.

The parametrix of $(\hat{A}_P + \lambda)^m$ have the following symbol:

$$B(x,\xi,\lambda) \approx \sum_{j\geq 0} b_{j,\lambda}(x,\xi).$$

Then for:

$$m > max\left\{\frac{k_1 + \cdots + k_n}{M} + \frac{2(\ell_1 + \cdots + \ell_n)}{M'}, n, \frac{2(\ell_1 + \cdots + \ell_n)}{M'} + M\right\}$$

the asymptotic form of this parametrix is the following :

$$Tr(\hat{A}_P + \lambda)^{-m} \approx \sum_{j\geq 0} c_{j,-m}(\lambda)$$

$$\approx c_0\lambda^{\alpha_0} + c_\delta\lambda^{\alpha_1} + \cdots ,$$

$$c_{j,-m}(\lambda) = \int_{R^n}\int_{R^n} b_{j,\lambda}(x,\xi)dxd\xi,$$

with

$$\begin{array}{lcl} c_0 & = & \int_{R^n} (P_0(x)+1))^{-m-\rho}dx \int_{R^n} \sqrt{a_0(\xi)}+1)^{-m}d\xi \\ c_1 & = & 0 \\ \vdots & & \vdots \\ c_{\delta-1} & = & 0 \end{array}$$

and

$$\begin{array}{rcl} c_{\delta,-m}(\lambda) & = & \int_{R^{2n}} b_{\delta,\lambda}(x,\xi)dxd\xi \\ & = & \lambda^{\alpha_\delta}c_\delta \end{array}$$

where

$$\begin{array}{rcl} \alpha_0 & = & -m + \frac{k_1n_1 + k_2n_2}{M'} + \frac{\ell_1n_1 + n_2\ell_2}{M''} \\ \alpha_\delta & = & \alpha_0 - \delta, \\ \delta & = & MM'.min\left\{\frac{k_1}{M} + \frac{2\ell_1}{M'}, \cdots, \frac{k_n}{M} + \frac{2\ell_n}{M'}\right\}, \\ \rho & = & \frac{2(\ell_1 + \cdots + \ell_n)}{M'}, \end{array}$$

in addition, we have $c_0 \neq 0$.

Proof : We apply the condition (1) for a general case of quasi-homogenous operators then the operator $(\hat{A} + \lambda)^{-m}$ is of trace class for m such that :

$$m \geq \frac{k_1}{M} + \cdots + \frac{k_n}{M} + \frac{2\ell_1}{M'} + \cdots + \frac{2\ell_n}{M'}, \qquad (9)$$

and for finding the asymptotic form of $Tr(\hat{A} + \lambda)^{-m}$, when $\lambda \to +\infty$, we find the asymptotic form of the symbol B of $(A + \lambda)^{-m}$. The symbol B belongs to $S^{-m}_{\underline{M},\underline{M}'}$, such that:

$$B(x,\xi,\lambda) \approx \sum_{j\geq 0} b_{j,\lambda}(x,\xi),$$

where

$$\begin{array}{rcl} b_{0,\lambda}(x,\xi) & = & (A_0(x,\xi) + \lambda)^{-m}, \\ b_{j+1,\lambda}(x,\xi) & = & -b_{0,\lambda}(x,\xi)\sum_{\Lambda} \Gamma(\alpha,\beta)\partial_\xi^\alpha D_x^\beta (A_0(x,\xi) + \lambda)^m \partial_\xi^\beta D_x^\alpha b_{l,\lambda}(x,\xi) \\ & & -b_{0,\lambda}(x,\xi)\sum_{\Lambda'} \Gamma(\alpha,\beta)\partial_\xi^\alpha D_x^\beta A_{i,m}(x,\xi,\lambda)\partial_\xi^\beta D_x^\alpha b_{l,\lambda}(x,\xi) \end{array}$$

(10)

where

$$\begin{array}{lcl} \Lambda & = & \{|(\underline{M} + \underline{M}').(\alpha+\beta)| + l = j+1, l \leq j\} \\ \Lambda' & = & \{|(\underline{M} + \underline{M}').(\alpha+\beta)| + l + i = j+1, l \leq j, 0 \leq i \leq m\} \end{array}$$

(11)

$$A_{i,m}(x,\xi,\lambda) =$$

$\sum_{s+t=i_{s\neq m,t\neq 0}} \frac{m!}{s!t!}(A_0 + \lambda)^s A_1^t, 0 \leq s,t \leq m, 0 \leq i \leq m$with

$$|(\underline{M}+\underline{M}').(\alpha+\beta)| = (\frac{k_1}{M}+\frac{2\ell_1}{M'})(\alpha_1+\beta_1)+\cdots+(\frac{k_n}{M}+\frac{2\ell_n}{M'})(\alpha_n+\beta_n).$$

Then we find the asymptotic form of the trace:

$$Tr(\hat{A}+\lambda)^{-m} \approx \sum_{j\geq 0} \int_{R^{2n}} tr(b_{j,\lambda}(x,\xi))dxd\xi,$$

we set $c_{j,-m}(\lambda) = \int_{R^{2n}} tr(b_{j,\lambda}(x,\xi))dxd\xi$

then $Tr(\hat{A}+\lambda)^{-m} \approx \sum_{j\geq 0} c_{j,-m}(\lambda)$.

We start by calculate the first term:

$$c_{0,-m} = \int_{R^{2n}} tr(b_{0,\lambda}(x,\xi))dxd\xi = \int_{R^{2n}} \Re\big(P_0(x)+i\sqrt{a_0(\xi)}+\lambda)^{-m}\big)dxd\xi,$$

to calculate the integral we do the changement of variables

$$\begin{array}{lllll} x_1 & = & \lambda^{\frac{k_1}{M}}x_{1'} & \Rightarrow & dx_1 = \lambda^{\frac{k_1}{M}}dx_{1'} \\ \vdots & & \vdots & & \vdots \\ x_n & = & \lambda^{\frac{k_n}{M}}x_{n'} & \Rightarrow & dx_n = \lambda^{\frac{k_n}{M}}dx_{n'} \\ \xi_1 & = & \lambda^{\frac{2\ell_1}{M'}}\xi_{1'} & \Rightarrow & d\xi_1 = \lambda^{\frac{2\ell_1}{M'}}d\xi_{1'} \\ \vdots & & \vdots & & \vdots \\ \xi_n & = & \lambda^{\frac{2\ell_n}{M'}}\xi_{n'} & \Rightarrow & d\xi_n = \lambda^{\frac{2\ell_n}{M'}}d\xi_{n'} \end{array}$$

(12)

we have:

$c_{0,-m}(\lambda) = \lambda^{\alpha_0}\int_{R^{2n}} \Re(P_0(x')+i\sqrt{a_0(\xi')}+1)^{-m}dx'd\xi'$,where

$$\alpha_0 = -m+\frac{k_1}{M}+\cdots+\frac{k_n}{M}+\frac{2\ell_1}{M'}+\cdots+\frac{2\ell_n}{M'},$$

where m satisfy the condition (9).We set

$$f(\alpha) = \int_{R^{2n}} (P_0(x')+\alpha\sqrt{a_0(\xi')}+1)^{-m}dx'd\xi',$$

then we do the changement of variables:

$$\begin{array}{lllll} \xi_{1'} & = & \alpha^{-\frac{2\ell_1}{M'}}\xi_{1''} & \Rightarrow & d\xi_{1'} = \alpha^{-\frac{2\ell_1}{M'}}d\xi_{1''} \\ \vdots & & \vdots & & \vdots \\ \xi_{n'} & = & \alpha^{-\frac{2\ell_n}{M'}}\xi_{n''} & \Rightarrow & d\xi_{n'} = \alpha^{-\frac{2\ell_n}{M'}}d\xi_{n''} \end{array} \quad (13)$$

we have

$$f(\alpha) = \alpha^{\rho}\int_{R^{2n}} (P_0(x')+ {}^{-m''}\sqrt{a_0(\xi''}$$

where $\rho = -(\frac{2\ell_1}{M'}+\cdots+\frac{2\ell_n}{M'})$. Then by doing an analytic extension of $f(\alpha)$ we can take $\alpha = i$then:

$$c_{0,-m}(\lambda) = \lambda^{\alpha_0}\Re(i)^{\rho}\int_{R^{2n}} (P_0(x') + {}^{-m''}\sqrt{a_0(\xi''}$$

to have that $\Re(i^{\rho}) \neq 0$ we must have that ρ is an even number, i.e. there exists j such that:

$$\ell_1+\cdots+\ell_n = M'j,$$

so

$$c_{0,-m} = (-1)^{\frac{\rho}{2}}\int_{R^{2n}} (P_0(x')+ {}^{-m''}\sqrt{a_0(\xi''}$$

for the integral

$$\int_{R^{2n}} (P_0(x')+ {}^{-m''}\sqrt{a_0(\xi''}$$

we have that the previous integral is defined and non-zero for

$$m > max\left\{\frac{k_1+\cdots+k_n}{M}+\frac{2(\ell_1+\cdots+\ell_n)}{M'}\ ,\ n\ ,\ \frac{2(\ell_1+\cdots+\ell_n)}{M'}+M\right\} \quad (14)$$

For the second term we set:

$$\kappa = min\left\{\frac{k_1}{M}+\frac{2\ell_1}{M'}\ ,\ \cdots\ ,\ \frac{k_n}{M}+\frac{2\ell_n}{M'}\right\},$$

we suppose that:$\kappa = \frac{k_1}{M}+\frac{2\ell_1}{M'}$

then the second term is $b_{\kappa MM'}$ we take in consideration the type of homogeneity (c.f. also (11)), so when $l=0$, $i=0$ and $\alpha_1+\beta_1=MM'$ we have

$$b_{\kappa MM',\lambda} = -\Gamma(MM',0)b_{0,\lambda}\partial_{\xi_1}^{MM'}(A_P+\lambda)^m D_{x_1}^{MM'}b_{0,\lambda}$$

$$-\Gamma(0,MM')b_{0,\lambda}D_{x_1}^{MM'}(A_P+\lambda)^m \partial_{\xi_1}^{MM'}b_{0,\lambda} - \Gamma(MM'-1,1) b_{0,\lambda}\partial_{\xi_1}^{MM'-1}D_{x_1}^{1}(A_P+\lambda)^m\partial_{\xi_1}^{1}D_{x_1}^{MM'-1}b_{0,\lambda} -\Gamma(MM'-2,2)b_{0,\lambda}\partial_{\xi_1}^{MM'-2}D_{x_1}^{2}(A_P+\lambda)^m \partial_{\xi_1}^{2}D_{x_1}^{MM'-2}b_{0,\lambda}\ \cdots - \Gamma(1,MM'-1) b_{0,\lambda}\partial_{\xi_1}^{1}D_{x_1}^{MM'-1}(A_P+\lambda)^m\partial_{\xi_1}^{MM'-1}D_{x_1}^{1}b_{0,\lambda}.$$

We do the changement of variables (12) we have:

$$\int_{R^{2n}} tr\left(b_{\kappa MM',\lambda}(x,\xi)\right)dxd\xi = \lambda^{\alpha_1}\int_{R^{2n}} tr(b'_{\kappa M'M}(x',\xi'))dx'd\xi'$$

where

$$\alpha_1 = -m + \frac{k_1n_1+k_2n_2}{M} + \frac{n_1\ell_1+n_2\ell_2}{M'} - \kappa MM' = \alpha_0 - \kappa MM',$$

and $b'_{\kappa MM'}(x',\xi')$ is the rest of $b_{\kappa MM',\lambda}(x,\xi)$ after the application of the changement of variables and taking λ as a factor.

We use the proposition (4.1) and the theorem of Lidskii we can proving the following theorem:

Theorem 4.1 *The operator L_P en (4) is defined on $L^2(R^n)$ where $P(x)=P(x_1,\cdots,x_n)$ is a positive quasi-homogenous polynomial of order M and of type $(k_1,\cdots,k_n)$ such that:*

$$P(\rho^{k_1}x_1,\cdots,\rho^{k_n}x_n) = \rho^M P_0(x_1,\cdots,x_n) + \rho^{M-\gamma}P_1(x_1,\cdots,x_n) \quad (15)$$

where $\gamma \geq 1$ and P is quasi-elliptic such that:

$$P_0(x) \neq 0, x \in R^n\backslash\{0\},$$

$a(D_x) = a(D_{x_1},\cdots,D_{x_n})$ is a positive pseudodifferential quasi-homogenous operator of order M' and of type $(\ell_1,\cdots,\ell_n)$ such that:

$$a(\rho^{\ell_1}D_{x_1},\cdots,\rho^{\ell_n}D_{x_n}) = \rho^{M'}a_0(D_{x_1},\cdots,D_{x_n}) + \rho^{M'-\delta}a_1(D_{x_1},\cdots,D_{x_n}) \quad (16)$$

where $\delta \geq 1$ and a is quasi-elliptic such that:

$$a_0(\xi) \neq 0, \xi \in R^n\backslash\{0\}.$$

If there exists j such that:

$$\ell_1+\cdots+\ell_n = M'j.$$

Then there exists $\lambda_0 \in C$ and $u_0 \in L^2(R^n)$, $u_0 \neq 0$ such that $L_P(\lambda_0)u_0 = 0$.

5 Quadratic Quasi-homogenous Family

We consider the following quadratic family of operators:

$$L(\lambda) = H_0 + \lambda H_1 + \lambda^2, \quad (17)$$

where H_0 and H_1 are pseudo-differential operators with the symbols H_0 and H_1 of order M and $\frac{M}{2}$, respectively and of type (k,ℓ)i.e.:

$$H_0(\rho^k x,\rho^\ell\xi) = \rho^M H_0(x,\xi),\ H_1(\rho^k x,\rho^\ell\xi) = \rho^{\frac{M}{2}}H_1(x,\xi).$$

H_0 and H_1 satisfy the hypotheses (1),(2) and (3) of the section (3), in addition we suppose that H_0 and $H_0 - H_1^2$ are positive operators.

We associate to $L(\lambda)$, the following non self-adjoint matrix operator:

$$\hat{A} = \begin{pmatrix} 0 & \widehat{H}_0^{\frac{1}{2}} \\ -\widehat{H}_0^{\frac{1}{2}} & -\widehat{H}_1 \end{pmatrix}, \quad (18)$$

with a matrix symbol A which belongs to $S^1_{\underline{k,\ell}}(L^2(R^n)\times L^2(R^n))$where

$$(\underline{k},\underline{\ell}) = (\frac{k}{M},\frac{\ell}{M}),$$

such that:

$$A = \begin{pmatrix} 0 & H_0^{\frac{1}{2}} \\ -H_0^{\frac{1}{2}} & -H_1 \end{pmatrix}, \qquad (19)$$

and $\mu_{\mp} = H_1 \mp i\sqrt{H_0 - H_1^2}$.

We find the asymptotic form of the trace of the operator $(A+\lambda)^{-m}$, for m large enough. The symbol B of $(A+\lambda)^{-m}$ belongs to $S_{\underline{k},\underline{\ell}}^{-m}(L^2(R^n)\times L^2(R^n))$ such that

$$B(x,\xi,\lambda) \approx \sum_{j\geq 0} b_{j,\lambda}(x,\xi),$$

where

$$\begin{aligned} b_{0,\lambda}(x,\xi) &= (A(x,\xi)+\lambda)^{-m}, \\ b_{j+1,\lambda}(x,\xi) &= \end{aligned}$$

$$-b_{0,\lambda}(x,\xi)\sum_{\Lambda} \Gamma(\alpha,\beta)\partial_\xi^\alpha D_x^\beta$$

$(A(x,\xi)+\lambda)^m \partial_\xi^\beta D_x^\alpha b_{l,\lambda}(x,\xi)$,

$$\Lambda = \{|(\underline{k}+\underline{\ell}).(\alpha+\beta)| + l = j+1, l \leq j\}.$$

So the trace has the following asymptotic form:

$$\begin{aligned} Tr(\hat{A}+\lambda)^{-m} &\approx \sum_{j\geq 0} \int_{R^{2n}} tr(b_{j,\lambda}(x,\xi))dxd\xi \\ &\approx \sum_{j\geq 0} c_{j,-m}(\lambda), \end{aligned}$$

setting

$$c_{j,-m}(\lambda) = \int_{R^{2n}} tr(b_{j,\lambda}(x,\xi))dxd\xi,$$

and

$$b_{j,\lambda}(x,\xi) \in S_{\underline{k},\underline{\ell}}^{-m-j}(L^2(R^n)\times L^2(R^n)).$$

Proposition 5.1 *The operator $\hat{A}$ in (18) is defined in $L^2(R^n)\times L^2(R^n)$ where H_0 and H_1 are positive pseudo-differential operators of orders M and $\frac{M}{2}$, respectively and are of type (k,ℓ) such that there exists j, $j \geq 1$ with $n(k+\ell) = Mj$.*

For m large enough and $\lambda \to +\infty$ we have

$$\begin{aligned} Tr(\hat{A}_P+\lambda)^{-m} &\approx \sum_{j\geq 0} c_{j,-m}(\lambda) \\ &\approx c_0\lambda^{\alpha_0} + c_\delta \lambda^{\alpha_0-\delta} + \cdots, \end{aligned}$$

such that

$$c_{j,-m}(\lambda) = \int_{R^{2n}} b_{j,\lambda}(x,y,\xi,\eta)dxdyd\xi d\eta,$$

where $b_{j,\lambda}$ are the terms of the symbol of the parametrix of $(\hat{A}_P+\lambda)^m$and

$$\alpha_0 = -m + \frac{2n(k+\ell)}{M},$$

$$c_0 = (-1)^{-\frac{\rho}{2}}\int_{R^{2n}} \left(H_1(x,\xi) + \sqrt{H_0(x,\xi) - H_1^2(x,\xi)} + 1\right)^{-m} dxd\xi,$$

$$\delta = M.\left\{\frac{k}{M},\frac{l}{M}\right\},$$

$$\rho = \frac{2n(k+l)}{M},$$

$$\begin{aligned} c_0 &= (-1)^{-\frac{\rho}{2}}\int_{R^{2n}} \left(H_1(x,\xi) + \sqrt{H_0(x,\xi) - H_1^2(x,\xi)} + 1\right)^{-m} dxd\xi, \\ \delta &= M.min\left\{\frac{k}{M},\frac{\ell}{M}\right\}, \\ \rho &= \frac{2n(k+\ell)}{M}, \end{aligned}$$

of plus $c_0 \neq 0$.

Proof: we have :

$$tr(b_{0,\lambda}) = \Re(H_1 + i\sqrt{H_0 - H_1^2} + \lambda)^{-m},$$

then we find the integral

$$\int_{R^{2n}} (H_1 + i\sqrt{H_0 - H_1^2} + \lambda)^{-m} dx d\xi,$$

we do the changement of variables:

$$x = \lambda^{\frac{2k}{M}} x' \Rightarrow dx = \lambda^{\frac{2kn}{M}} dx' \qquad \xi = \lambda^{\frac{2\ell}{M}} \xi' \Rightarrow d\xi = \lambda^{\frac{2\ell n}{M}} d\xi' \qquad (20)$$

we have

$$\int_{R^{2n}} tr(b_{0,\lambda}(x,\xi)) dx d\xi = \lambda^{\alpha_0} \int_{R^{2n}} \Re\left(H_1(x',\xi') + i\sqrt{H_0(x',\xi') - H_1^2(x',\xi')} + 1\right)^{-m} dx' d\xi',$$

where $\alpha_0 = -m + \frac{2n(k+\ell)}{M}$,

we set

$$c_0 = \int_{R^{2n}} \Re\left(H_1(x',\xi') + i\sqrt{H_0(x',\xi') - H_1^2(x',\xi')} + 1\right)^{-m} dx' d\xi',$$

i.e. $c_{0,-m}(\lambda) = \lambda^{\alpha_0} c_0$.

To calculate c_0 we use the same method of the previous section. we suppose that : $f(\alpha) = \int_{R^{2n}} \left(H_1(x',\xi') + \alpha\sqrt{H_0(x',\xi') - H_1^2(x',\xi')} + 1\right)^{-m} dx' d\xi'$,by doing the changement of variables :

$$x' = \left(\frac{1}{\alpha}\right)^{\frac{2k}{M}} x'' \Rightarrow dx' = \left(\frac{1}{\alpha}\right)^{\frac{2kn}{M}} dx'' \qquad \xi' = \left(\frac{1}{\alpha}\right)^{\frac{2\ell}{M}} \xi'' \Rightarrow d\xi' = \left(\frac{1}{\alpha}\right)^{\frac{2\ell n}{M}} d\xi''$$

we get

$$f(\alpha) = \left(\frac{1}{\alpha}\right)^{\rho} \int_{R^{2n}} \left(H_1(x'',\xi'') + \sqrt{H_0(x'',\xi'') - H_1^{2}(x'',\xi'')} + 1\right)^{-m} dx'' d\xi''$$

where $\rho = \frac{2n(k+\ell)}{M}$.

An analytic extension of $f(\alpha)$, allow to suppose that $\alpha = i$, and we have

$$c_0 = R(i)^{-\rho} \int_{R^{2n}} \left(H_1(x'',\xi'') + \sqrt{H_0(x'',\xi'') - H_1^{2}(x'',\xi'')} + 1\right)^{-m} dx'' d\xi''$$

then $c_0 \neq 0$, if there exists $j, j \geq 1$ such that

$$n(k+\ell) = Mj, \qquad (21)$$

For n and M even numbers we must having k and ℓ such that (21) is satisfied. For n odd and M even we must obtain that the sum $k+\ell$ be even and (21) is satisfied. For example, the case $n = 3$, $(k,\ell) = (1,3)$ and $M = 6$.

Then we have

$$c_0 = (-1)^{-\frac{\rho}{2}} \int_{R^{2n}} \left(H_1(x'',\xi'') + \sqrt{H_0(x'',\xi'') - H_1^{2}(x'',\xi'')} + 1\right)^{-m} dx'' d\xi''$$

and $c_0 \neq 0$ because

$$\left(H_1(x'',\xi'') + \sqrt{H_0(x'',\xi'') - H_1^{2}(x'',\xi'')} + 1\right) dx'' d\xi''$$

is positive.

We find the second term of the trace of $(\hat{A}+\lambda)^{-m}$ we set

$$\kappa = min\left\{\frac{k}{M}, \frac{\ell}{M}\right\},$$

then we take in consideration the type of homogeneity we have $b_{\kappa M,\lambda}$ is the second term when $l=0$ and $\alpha+\beta=M$ i.e.

$$\begin{aligned} b_{\kappa M,\lambda}(x,\xi) = & -\Gamma(M,0)b_{0,\lambda}(x,\xi)\partial_\xi^M(A_P(x,\xi)+\lambda)^m D_x^M b_{0,\lambda}(x,\xi) \\ & -\Gamma(0,M)b_{0,\lambda}(x,\xi)D_x^M(A_P(x,\xi)+\lambda)^m \partial_\xi^M b_{0,\lambda}(x,\xi) \\ & -\Gamma(M-1,1)b_{0,\lambda}(x,\xi)\partial_\xi^{M-1}D_x^1(A_P(x,\xi)+\lambda)^m \\ & \quad \partial_\xi^1 D_x^{M-1}b_{0,\lambda}(x,\xi) \\ & -\Gamma(M-2,2)b_{0,\lambda}(x,\xi)\partial_\xi^{M-2}D_x^2(A_P(x,\xi)+\lambda)^m \\ & \quad \partial_\xi^2 D_x^{M-2}b_{0,\lambda}(x,\xi) \\ & \quad \cdots \\ & -\Gamma(1,M-1)b_{0,\lambda}(x,\xi)\partial_\xi^1 D_x^{M-1}(A_P(x,\xi)+\lambda)^m \\ & \quad \partial_\xi^{M-1}D_x^1 b_{0,\lambda}(x,\xi). \end{aligned}$$

we have $b_{\kappa M,\lambda} \in S_{\underline{k},\underline{\ell}}^{-m-(\kappa M)}$ then we do the changement of variables (20) we find :

$$\int_{R^{2n}} tr(b_{\kappa M,\lambda}(x,\xi))dxd\xi = \lambda^{\alpha_1}\int_{R^{2n}} tr(b'_{\kappa M}(x',\xi'))dx'd\xi',$$

where

$$\alpha_1 = \alpha_0 - \kappa M,$$

and $b'_{\kappa M}(x',\xi')$ is the rest of $b_{\kappa M,\lambda}(x,\xi)$ after the application of the changement of variables (20) and taking λ as a factor.

We use the proposition (5.1) and the theorem of Lidskii we can proving the following theorem:

Theorem 5.1 *Let L_P be the operator in (17), which is defined on $L^2(R^n)$ with H_0, H_1 are positives pseudo-differential operators with the symbols H_0, H_1 of order M and $\frac{M}{2}$, respectively and of type (k,ℓ), if there exists j, $j \geq 1$ such that :*

$$n(k+\ell) = Mj.$$

Then there exists $\lambda_0 \in C$ and $u_0 \in L^2(R^n)$, $u_0 \neq 0$ such that $L_P(\lambda_0)u_0 = 0$.

5.1 EXAMPLE WITH $n = 3$

We consider for example the operator $L(\lambda)$ in $L^2(R^3)$ with H_0 and H_1 of order6 and 3 respectively and of type (1,3) such that:

$$\begin{aligned} H_0(x,D_x) &= |x|^6 - D_x^2, \\ H_1(x,D_x) &= |x|^3, \end{aligned}$$

Then we have the matrix operator $\hat{A}$ has the symbol $A$with :

$$\begin{aligned} H_0(x,\xi) &= |x|^6 + |\xi|^2, \\ H_1(x,\xi) &= |x|^3, \end{aligned}$$

For $m > 9$ and $\lambda \to +\infty$, the asymptotic form of the trace is the following :

$$\begin{aligned} Tr(\hat{A}+\lambda)^{-m} &\approx \sum_{j\geq 0} c_{j,-m}(\lambda), \\ &\approx \lambda^{\alpha_0}c_0 + \lambda^{\alpha_1}c_1 + \cdots, \end{aligned}$$

with

$$\begin{aligned} c_0 &= \int_{R^3} (|x|^3+1)^{-m+3}dx \int_{R^3} (|\xi|+1)^{-m}d\xi \\ c_1 &= 0 \\ \alpha_0 &= -m+4 \\ \alpha_1 &= -m+3 \end{aligned}$$

and $c_0 \neq 0$since$(|x|^3+1)^{-m+3}$ and $(|\xi|+1)^{-m}$ are positives.

Then by using the proposition (5.1) and the theorem (5.1) we can prove the existence of non trivial solutions for the operator L.

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