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Research Paper

Electro-magneto-hydrodynamics Flows of Burgers' Fluids in Cylindrical Domains with Time Exponential Memory

Abdul Rauf[1], Yasir Mahsud[2]

[1] Department of Computer Science and Engineering, Air University Multan
Abdali Road, Multan, 60000, Pakistan, abdul.rauf@aumc.edu.pk

[2] Abdus Salam School of Mathematical Sciences, GC University Lahore, Pakistan
68-B, New Muslim Town, Lahore, 54600, Pakistan, yasir.mahsud@sms.edu.pk

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Corresponding author: Abdul Rauf, abdul.rauf@aumc.edu.pk

Abstract. This paper investigates the axial unsteady flow of a generalized Burgers' fluid with fractional constitutive equation in a circular micro-tube, in presence of a time-dependent pressure gradient and an electric field parallel to flow direction and a magnetic field perpendicular on the flow direction. The mathematical model used in this work is based on a time-nonlocal constitutive equation for shear stress with time-fractional Caputo-Fabrizio derivatives; therefore, the histories of the velocity gradient will influence the shear stress and fluid motion. Thermal transport is considered in the hypothesis that the temperature of the cylindrical surface is constant. Analytical solutions for the fractional differential momentum equation and energy equation are obtained by employing the Laplace transform with respect to the time variable t and the finite Hankel transform with respect to the radial coordinate r. It is important to note that the analytical solutions for many particular models such as, ordinary/fractional Burgers fluids, ordinary/fractional Oldryd-B fluids, ordinary/fractional Maxwell fluids and Newtonian fluids, can be obtained from the solutions for the generalized fractional Burgers' fluid by particularizing the material coefficients and fractional parameters. By using the obtained analytical solutions and the Mathcad software, we have carried out numerical calculations in order to analyze the influence of the memory parameters and magnetic parameter on the fluid velocity and temperature. Numerical results are presented in graphical illustrations. It is found that ordinary generalized Burgers' fluids flow faster than the fractional generalized Burgers' fluids.

Keywords: Electro-magneto-hydrodynamic (EMHD) flow, Porous medium, Thermal-fluidic transports, Fractional model, Micro scale flow.

1. Introduction

In modern era, viscoelastic fluids have wide research interest due to their many useful applications. Lubricants, polymeric solutions, colloidal solutions, and artificial and natural gels are few examples of viscoelastic fluids. Flows of such fluids through porous media are important for engineering and bioengineering.

Since 1695, Leibniz and L'Hospital have initiated the differential calculus of fractional order. The theory of fractional derivatives, fractional integrals and of the fractional differential equations has intensively developed in last year's. Also, the fractional calculus became an important tool for the mathematical modeling of the complex transport phenomena such as anomalous diffusion, flows with heat and mass transfer of viscoelstic materials, etc. Interesting results about flows of viscoelastic fluids modeled by the fractional calculus are given in the references [1-6].

The space-fractional derivatives and time-fractional derivatives have the advantage of describing heredity of materials and

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processes, respectively the memory properties of materials. Mathematical models with time-fractional derivatives are more realistic for to describing many practical phenomena than the differential equations of integer order.

Combined electro-magneto-hydrodynamic (EMHD) effects in the thermo-fluidic transport through small scale channels lead to the improvement of fluidic devices performances. Chakraborty et al. [7] investigated the thermal transport in the EMHD flows in the presence of Joule heating and viscous dissipation when the wall heat flux is constant. They have studied the influence of the applied magnetic field on the fluid temperature and velocity.

Analytical solutions for the transient rotating electro-magneto-hydrodynamic flow in a parallel micro-channel have been determined by Jian et al [8]. They described analytically several particular cases of alternating current (AC) and direct current (DC) magnetic and electric fields. Their results demonstrate that the fluid velocity is strongly influenced by the applied magnetic field. It is also observed that the EMHD speed rises in the normal direction with the increase in the rotating Reynolds number and diminishes along the axis of rotation.

The analysis of heat transfer through a capillary domain in the EMHD flow of blood was carried out by Sinha and Shit [9]. They determined that the Joule heating parameter can be used to control the blood temperature. Additionally, their investigation highlighted that the magnetic field can be used as a tool to control blood flow, especially in surgical operations. Wang et al. [10] studied the convective EMHD flows of a third grade fluid between two micro-parallel plates. Analytical solutions for the fluid temperature and velocity have been obtained by employing the perturbation method. Results demonstrate that both temperature and velocity distribution are maximum for Newtonian fluids and are diminishing for non-Newtonian fluids. The Lorentz force increases with the rise in the value of magnetic parameter and it results in considerable suppression of the convection.

The flows of fluids through permeable porous media are of major significance in industry, biomechanics, geomechanics, etc. It is known that the flows in porous media can be described by the Darcy's law. The flow of a fluid with the Jeffrey's constitutive equation in a circular cylinder filled with porous medium has been investigated by Jyothi et al. [11], while Ghosh et al. [12] studied the flow in a channel of an Oldroyd-B fluid generated by a rectified sine pulses. The peristaltic transport in a porous medium through an asymmetric channel has been examined by Elshehawey et al. [13]. Their results could be useful in medical applications.

In the present paper we consider unsteady flows of a generalized Burgers fluid in a circular microtube filled with the porous medium. The mathematical model is described by a time-fractional constitutive equation for shear stress. The generalized fractional constitutive equation is based on the Caputo-Fabrizio time fractional derivative with exponential kernel. In this mathematical model, the histories of the velocity gradient influence the shear stress.

The thermal transport is considered in hypothesis of constant temperature on the cylinder surface under presence of Joule heating and viscous dissipation. The fluid motion is an axial flow generated by a time-dependent pressure gradient and an electric field in the flow direction and an applied magnetic field perpendicular on the flow direction.

Analytical solutions for the velocity and temperature fields are determined by using the Laplace transform with respect to the variable t and the finite Hankel transform of order zero with respect to the radial coordinate r.

Obviously, the solutions for the fractional generalized Burgers fluid contain solutions for particular cases (fractional/ordinary Burgers fluids, fractional/ordinary Oldroyd-B fluids, fractional/ordinary Maxwell fluids and Newtonian fluids) obtained from the general case by particularizing the material coefficients and fractional parameters.

By using the analytical solutions and the software Mathcad, the curves corresponding velocity and temperature for different values of the fractional parameters and magnetic parameter have been plotted. It is found that the fluids modeled by the fractional constitutive equation flow slower than ordinary fluids. Also, the temperature values are lower for the fractional fluids.

2. Mathematical Formulation of the Problem.

Let us consider an incompressible viscoelastic electrolyte solution with density ρ and the constant shear viscosity μ. The solution is in micro-channel of length L and radius R. The electrolyte solution has electric conductivity σ and is in the influences of the combined applied electromagnetic force and oscillating pressure-gradient. The capillary is filled with porous medium with porosity ψ. The combined effects of the the electric field $\mathbf{E} = E\mathbf{e}_z$ and the magnetic field $\mathbf{B} = B\mathbf{e}_r$ generates the electro-magneto-hydrodynamic (EMHD) force in the direction of flow, where $\mathbf{e}_z$, and $\mathbf{e}_r$ are the unit vectors along z and radial directions, respectively, of cylindrical coordinate system (r,θ,z). The flow of fluid is along axial direction (z-axis) of the capillary and no flow is assumed in the radial or azimuthal direction. In porous medium, the continuity and momentum equations for incompressible fluid are

$$\nabla.\mathbf{v} = 0, \tag{1}$$

$$\mathbf{v}.\nabla\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho}(\nabla p + \nabla.\mathcal{T} + \mathbf{b} + \mathbf{R}_d), \tag{2}$$

where ρ is density, p is pressure, $\mathcal{T}$ is stress tensor, $\mathbf{v}$ is velocity, $\mathbf{b}$ is body force and $\mathbf{R}_d$ is Darcy's resistance of porous medium. The shear stress tensor $\mathcal{T}$ for generalized Burgers' fluids is given by [14]:

$$\mathcal{T}+\lambda_1\frac{\delta\mathcal{T}}{\delta t}+\lambda_2\frac{\delta^2\mathcal{T}}{\delta t^2}=\mu\left[\mathbf{A}+\lambda_3\frac{\delta\mathbf{A}}{\delta t}+\lambda_4\frac{\delta^2\mathbf{A}}{\delta t^2}\right] \tag{3}$$

where $\mathbf{A}=\mathbf{L}+\mathbf{L}^T$, $\mathbf{L}=\nabla\mathbf{v}$ and μ is the dynamic viscosity of the fluid, and λ_1, λ_3 are the relaxation time, respectively the retardation time, and λ_2, λ_4 are the material parameters heaving the dimension $[s^2]$ and $\delta/\delta t$ denotes the upper convected derivative defined as

$$\frac{\delta^2\mathcal{T}}{\delta t^2}=\frac{\delta}{\delta t}\left(\frac{\delta\mathcal{T}}{\delta t}\right)=\frac{\delta}{\delta t}\left(\frac{d\mathcal{T}}{dt}-\mathbf{L}\mathcal{T}-\mathcal{T}\mathbf{L}^T\right). \tag{4}$$

The generalized Burgers' model given by Eq. (3) reduces at Burgers fluid for $\lambda_4=0$, Oldroyd-B fluid for $\lambda_2=\lambda_4=0$, Maxwell fluid for $\lambda_2=\lambda_3=\lambda_4=0$ and the Newtonian fluid for $\lambda_1=\lambda_2=\lambda_3=\lambda_4=0$.

For axisymmetric flow binary electrolyte solution, the Poisson-Boltzmann equation describes the electric potential $\mathfrak{E}$ of electrical double layer by [15-18]

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\mathfrak{E}(r)}{\partial r}\right)=k^2\mathfrak{E}(r),\ \ k=\left(\frac{2z_v^2e_0^2n_0}{\epsilon k_BT_a}\right)^{1/2} \tag{5}$$

with conditions

$$\mathfrak{E}(R)=\mathfrak{E}_w,\ \frac{\partial\mathfrak{E}}{\partial r}\Big|_{r=0}=0, \tag{6}$$

where $T_a,k_B,e_0,n_0,z_v,k^{-1},k$ and ϵ are the absolute temperature of the electrolytic solution, the Boltzmann constant, the electronic charge, the ionic concentration in the bulk phase, the absolute value of the valence for a electrolyte ions, the thickness of electrical double layer, the Debye-Hückel parameter, and the dielectric constant respectively. The imposed body force $\mathbf{b}$ on the fluid is a combined effect of applied magnetic and electric field effects to the system.

The EMHD body force is [20-25, 27, 30]

$$\mathbf{b}=\mathbf{J}\times\mathbf{B}+\rho_c(r)\mathbf{E}, \tag{7a}$$

where $\mathbf{J}$ is given by the Ohm's law

$$\mathbf{J}=\sigma\mathbf{E}+\sigma\mathbf{v}\times\mathbf{B}, \tag{7b}$$

and σ is the electric conductivity of the flow. For the considered problem the body force $\mathbf{b}$ takes the form

$$\mathbf{b}=(\rho_c(r)E-\sigma B^2v(r,t))\mathbf{e}_z+\sigma EB\mathbf{e}_\theta, \tag{8}$$

where $\rho_c(r)=-\epsilon k^2\mathfrak{E}(r)$, $\mathbf{E}=E\mathbf{e}_z$, $\mathbf{v}=v(r,t)\mathbf{e}_z$, $\mathbf{B}=B\mathbf{e}_r$ are the net charge density of the electrolyte solution, the axial component of applied electric field, the axial velocity of the fluid, and the constant external magnetic field in orthogonal direction to the flow respectively. The Darcy's resistance for generalized Burgers' fluid is given by the following equation [26]

$$\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\mathbf{R}_d=-\frac{\mu\psi}{k_0}\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\mathbf{v}, \tag{9}$$

where $\mathbf{R}_d=\Re\mathbf{e}_z$, k_0 is the permeability constant of the medium and ψ is the porosity of the medium. With the help of Eqs. (2) and (8) one obtains the momentum equation for incompressible flow with combined electroosmotic and pressure effects in the form

$$\rho\frac{\partial v}{\partial t}-p_0p(t)+\epsilon k^2E\,\mathfrak{E}(r)+\sigma B^2v(r,t)=\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\Re, \tag{10}$$

where $-\nabla p=p_0p(t)$ is the pressure gradient along axial direction z, $p(t)$ is continuous function and τ_{rz} is shear stress in the axial direction and is given by generalized Burger fluid model [28]

$$\tau_{rz}+\lambda_1\frac{\partial\tau_{rz}}{\partial t}+\lambda_2\frac{\partial^2\tau_{rz}}{\partial t^2}=\mu\left(\lambda_4\frac{\partial^2}{\partial t^2}+\lambda_3\frac{\partial}{\partial t}+1\right)\frac{\partial v}{\partial r}, \tag{11}$$

where λ_i ($i=1$ to 4) are the material constants. The appropriate initial and boundary conditions for velocity are

$$v(r,0)=\frac{\partial v(r,t)}{\partial t}|_{t=0}=0, \tag{12a}$$

$$\mathrm{v}(\mathrm{R},\mathrm{t})=0. \tag{12b}$$

We now introduce the following dimensionless parameters in Eqs. (5), (6), (9)-(12)

$$\begin{gathered}v^{*}=\frac{v}{v_{h}};v_{h}=\frac{-\epsilon\mathfrak{E}_{w}E}{\mu};r^{*}=\frac{r}{R};t^{*}=\frac{\nu t}{R^{2}};\mathfrak{E}^{*}=\frac{\mathfrak{E}}{\mathfrak{E}_{w}};\lambda_{1}^{*}=\frac{\lambda_{1}\nu}{R^{2}};\lambda_{3}^{*}=\frac{\lambda_{3}\nu}{R^{2}};\lambda_{2}^{*}=\frac{\lambda_{2}\nu^{2}}{R^{4}};\\ \lambda_{4}^{*}=\frac{\lambda_{4}\nu^{2}}{R^{4}};p^{*}=\frac{p}{p_{0}^{*}};p_{0}^{*}=-\frac{\epsilon\mathfrak{E}_{w}E}{p_{0}R^{2}};\tau_{rz}^{*}=\frac{\tau_{rz}}{\tau_{0}};\tau_{0}=\frac{\mu v_{h}}{R};\mathfrak{R}^{*}=\frac{\mathfrak{R}}{\mathfrak{R}_{0}};\mathfrak{R}_{0}=\frac{\mu\psi v_{h}}{k_{0}};\\ H_{a}^{2}=\frac{\sigma B^{2}R^{2}}{\mu};\frac{1}{K_{0}}=\frac{\psi R^{2}}{k_{0}};K^{2}=k^{2}R^{2},\end{gathered} \tag{13}$$

where v_h, H_a, K and K_0 are the Hemholtz-Smoluchowsky velocity, the Hartmann number, the non-dimensional electrokinetic width and the permeability parameter of the porous medium respectively. Dropping "*" notations, the governing equations in the non-dimensional form takes the form

$$\frac{\partial v}{\partial t}-p(t)-K^{2}\mathfrak{E}(r)+H_{a}^{2}v(r,t)=\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{K_{0}}\mathfrak{R}, \tag{14}$$

$$\left(\lambda_{2}\frac{\partial^{2}}{\partial t^{2}}+\lambda_{1}\frac{\partial}{\partial t}+1\right)\tau_{rz}=\left(\lambda_{4}\frac{\partial^{2}}{\partial t^{2}}+\lambda_{3}\frac{\partial}{\partial t}+1\right)\frac{\partial v}{\partial r}, \tag{15}$$

$$\left(\lambda_{2}\frac{\partial^{2}}{\partial t^{2}}+\lambda_{1}\frac{\partial}{\partial t}+1\right)\mathfrak{R}=-\left(\lambda_{4}\frac{\partial^{2}}{\partial t^{2}}+\lambda_{3}\frac{\partial}{\partial t}+1\right)v, \tag{16}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\mathfrak{E}(r)}{\partial r}\right)=K^{2}\mathfrak{E}(r), \tag{17}$$

along with initial and boundary conditions

$$\mathfrak{E}(1)=1,\ \frac{\partial\mathfrak{E}(r)}{\partial r}|_{r=0}=0, \tag{18}$$

$$v(r,0)=\frac{\partial v(r,t)}{\partial t}|_{t=0}=0, v(0,t)=0, \tag{19}$$

$$\tau_{rz}(r,0)=\frac{\partial\tau_{rz}(r,t)}{\partial t}|_{t=0}=0, \tag{20}$$

$$\mathfrak{R}(r,0)=\frac{\partial\mathfrak{R}(r,t)}{\partial t}|_{t=0}=0. \tag{21}$$

The solution of Eqs. (17), (18) takes the form,

$$\mathfrak{E}(r)=\frac{I_{0}(Kr)}{I_{0}(K)}, \tag{22}$$

where I_0 is the first kind of modified Bessel function with order zero. In order to consider a fractional model, we generalize the constitutive Eq. (15) with Caputo-Fabrizio time fractional derivative with non-singular kernel ${}^{CF}\partial_t^{\alpha}$, namely we have

$$(1+\lambda_{1}{}^{CF}\partial_{t}^{\alpha}+\lambda_{2}{}^{CF}\partial_{t}^{\alpha+1})\tau_{rz}=(1+\lambda_{3}{}^{CF}\partial_{t}^{\beta}+\lambda_{4}{}^{CF}\partial_{t}^{\beta+1})\frac{\partial v}{\partial r}, \tag{23}$$

where ${}^{CF}\partial_t^{\alpha}$ is defined as [29],

$$^{CF}\partial_t^{\alpha}(f(t)) = \frac{1}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right) f'(s)ds, 0<\alpha<1, \tag{24}$$

with the properties

$$\lim_{\alpha\to 1} {^{CF}\partial_t^{\alpha}}(f(t)) = f'(t), \tag{25}$$

$$^{CF}\partial_t^{\alpha+1}(f(t)) = {^{CF}\partial_t^{\alpha}}\{\partial_t f(t)\}, \tag{26}$$

and the Laplace transform $\mathcal{L}$ of Caputo-Fabrizio time fractional derivative $^{CF}\partial_t^{(\alpha+n)}$ is given by [29]

$$\mathcal{L}[^{CF}\partial_t^{(\alpha+n)} f(t)] = \frac{q^{n+1}\mathcal{L}(f(t)) - q^n f(0) - q^{n-1}f'(0) - \cdots - f^{n}(0)}{q+\alpha(1-q)}. \tag{27}$$

Finally, our goal is to find solutions for the differential equations (14), (16), (23) along with conditions (19)-(21). To do this, we will employ the Laplace transform with respect to the time variable t and the finite Hankel transform with respect to the radial coordinate r.

3. Solution of the Problem

In order to solve the fractionalized form of generalized Burger bio-fluid model with the combined electric and magnetic effects in porous medium we will use the Laplace transformation and zeroth order finite Hankel transform. We apply the Laplace transform to Eqs. (14), (16) and (23) and we use the initial conditions from Eqs. (19)-(21) along with the formula (27) we get

$$q\bar{v}(r,q) - P(q) - \frac{K^2\mathfrak{E}(r)}{q} + H_a^2\bar{v}(r,q) = \frac{1}{K_0}\bar{\mathfrak{R}}(r,q) + \frac{1}{r}\frac{\partial}{\partial r}(r\bar{\tau}_{rz}(r,q)), \tag{28}$$

$$\left(1+\lambda_1 q+\lambda_2 q^2\right)\bar{\mathfrak{R}}(r,q) = -\left(1+\lambda_3 q+\lambda_4 q^2\right)\bar{v}(r,q), \tag{29}$$

$$\left(1+\lambda_2\frac{q^2}{\alpha+(1-\alpha)q}+\lambda_1\frac{q}{\alpha+(1-\alpha)q}\right)\bar{\tau}_{rz}(r,q) = \left(1+\lambda_4\frac{q^2}{\beta+(1-\beta)q}+\lambda_3\frac{q}{\beta+(1-\beta)q}\right)\frac{\partial\bar{v}(r,q)}{\partial r}. \tag{30}$$

From Eq. (30) we can write as

$$\bar{\tau}_{rz}(r,q) = \frac{\lambda_4 q^2+(\lambda_3-\beta+1)q+\beta}{\lambda_2 q^2+(\lambda_1-\alpha+1)q+\alpha}\times\frac{(1-\alpha)q+\alpha}{(1-\beta)q+\beta}\frac{\partial\bar{v}(r,q)}{\partial r}. \tag{31}$$

$$F_{\alpha\beta}(q) = \frac{\alpha+(1-\alpha)q}{\beta+(1-\beta)q}\times\frac{\lambda_4 q^2+(\lambda_3-\beta+1)q+\beta}{\lambda_2 q^2+(\lambda_1-\alpha+1)q+\alpha}, \tag{32}$$

Eqs. (31) and (32) gives

$$\bar{\tau}_{rz}(r,q) = F_{\alpha\beta}(q)\frac{\partial\bar{v}(r,q)}{\partial r}. \tag{33}$$

Similarly from Eq. (29) we can write

$$\bar{\mathfrak{R}}(r,q) = -A(q)\bar{v}(r,q) \tag{34}$$

where,

$$A(q) = \frac{\lambda_4 q^2+\lambda_3 q+1}{\lambda_2 q^2+\lambda_1 q+1}. \tag{35}$$

Substituting Eqs. (22), (33) and (34) in Eq. (28)

$$F_{\alpha\beta}(q)\frac{1}{r}\frac{\partial}{\partial r}(r\bar{v}(r,q)) = -P(q) - \frac{K^2 I_0(Kr)}{qI_0(K)} + (q+H_a^2+\frac{1}{K_0}A(q))\bar{v}(r,q), \tag{36}$$

Applying the finite Hankel transform to above equation

$$(F_{\alpha\beta}(q)r_n^2+q+\frac{A(q)}{K_0}+H_a^2)\bar{v}(r_n,q)=\frac{K^2r_nJ_1(r_n)}{q(K^2+r_n^2)}+\frac{P(q)J_1(r_n)}{r_n}, \tag{37}$$

where $\bar{v}(r_n,q)=\int_0^1 r\bar{v}(r,q)J_0(r_nr)dr$, J_0 is the zeroth order Bessel function of first kind and $J_0(r_n)=0$ for all n. With Eqs. (32) and (35) the expression $\bar{v}(r_n,q)$ takes the form

$$\bar{v}(r_n,q)=F(r_n,q)P(q)\frac{J_1(r_n)}{r_n}+\frac{F(r_n,q)}{q}\frac{r_nK^2J_1(r_n)}{K^2+r_n^2}, \tag{38}$$

where,

$$F(r_n,q)=\frac{B_0+B_1q+B_2q^2+B_3q^3+B_4q^4+B_5q^5}{\mathfrak{P}_{0,n}+\mathfrak{P}_{1,n}q+\mathfrak{P}_{2,n}q^2+\mathfrak{P}_{3,n}q^3+\mathfrak{P}_{4,n}q^4+\mathfrak{P}_{5,n}q^5+\mathfrak{P}_{6,n}q^6}, \tag{39}$$

and the functions $\mathfrak{P}_{0,n},\mathfrak{P}_{1,n},\ldots,\mathfrak{P}_{6,n}$ and constants $B_0,B_1,\ldots,B_5$ are given in Appendix A. Let $q_{1,n},q_{2,n},\ldots,q_{6,n}$ be the roots of the polynomial $\mathfrak{P}_{0,n}+\mathfrak{P}_{1,n}q+\mathfrak{P}_{2,n}q^2+\mathfrak{P}_{3,n}q^3+\mathfrak{P}_{4,n}q^4+\mathfrak{P}_{5,n}q^5+\mathfrak{P}_{6,n}q^6$ (These roots can be computed numerically). By simple calculation we can write partial fractions as

$$F(r_n,q)=\frac{B_0+B_1q+B_2q^2+B_3q^3+B_4q^4+B_5q^5}{\mathfrak{P}_{6,n}(q-q_{1,n})(q-q_{2,n})(q-q_{3,n})(q-q_{4,n})(q-q_{5,n})(q-q_{6,n})}=\frac{1}{\mathfrak{P}_{6,n}}\sum_{i=1}^{6}\frac{a_{i,n}}{q-q_{i,n}}, \tag{40}$$

where $a_{i,n}$ ($i=1,2,\ldots,6$) are in Appendix B. The inverse Laplace transform of Eq. (40) is

$$f(r_n,t)=\frac{1}{\mathfrak{P}_{6,n}}\sum_{i=1}^{6}a_{i,n}e^{q_{i,n}t}, \tag{41}$$

Thus the inverse Laplace transform for function $F(r_n,q)/q$ involved in the second term of (38) is

$$d(r_n,t)=\int_0^t f(r_n,\tau)d\tau=\frac{1}{\mathfrak{P}_{6,n}}\sum_{i=1}^{6}\frac{a_{i,n}}{q_{i,n}}(e^{q_{i,n}t}-1). \tag{42}$$

Using Eqs. (41) and (42) we can write the inverse Laplace transform for Eq. (38)

$$v(r_n,t)=f(r_n,t)*p(t)\frac{J_1(r_n)}{r_n}+\frac{K^2r_nJ_1(r_n)}{K^2+r_n^2}d(r_n,t), \tag{43}$$

where $f(r_n,t)*p(t)$ represents the convolution of two functions f and p. If $g(r,t)$, $r\in[0,1]$ has the Hankel transform $g(r_n,t)$, the inverse Hankel is $g(r,t)=2\sum_{n=1}^{\infty}J_0(rr_n)/J_1^2(r_n)\times g(r_n,t)$. The required solution for velocity component can be obtained by applying the inverse Hankel transform to Eq. (43) and is given by

$$v(r,t)=2\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_nJ_1(r_n)}\int_0^t f(r_n,\tau)p(t-\tau)d\tau+2K^2\sum_{n=1}^{\infty}\frac{r_nJ_0(rr_n)}{J_1(r_n)(K^2+r_n^2)}d(r_n,t). \tag{44}$$

4. Thermal Transport in Electro-magneto-hydrodynamics Flow

In this section we will derive solution for the time-fractional temperature distribution due to EMHD flow of a generalized burger fluid. The energy equation which describes the mathematical model for the temperature distribution Θ is [30]

$$\frac{\gamma_0}{\rho C_p r}\frac{\partial}{\partial r}(r\frac{\partial\Theta(r,t)}{\partial t})+\frac{\sigma}{\rho C_p}(B^2v^2(r,t)+E^2)+\frac{\nu}{C_p}(\frac{\partial v(r,t)}{\partial r})^2=\frac{\partial\Theta(r,t)}{\partial t} \tag{45}$$

where γ_0 represents the thermal conductivity of the fluid and C_p represents the specific heat capacity of the fluid at constant pressure. Let Θ_w denotes the constant wall temperature through microchannel walls due to combined effects of viscous and Joule heating [31], then the appropriate initial and boundary conditions are

$$\Theta(r,0)=\Theta_w,\quad 0\le r\le R, \tag{46}$$

$$\Theta(R,t)=\Theta_w,\quad 0<t. \tag{47}$$

Using the dimensionless parameters defined in Eq. (13) along with the parameters $\Theta^*=(\Theta-\Theta_w)/(\Theta_1-\Theta_w)$, $\Theta_1=\sigma R^2E_z^2/k$ Eqs. (45)-(47) become (after dropping * notation)

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\Theta}{\partial r})+S_z+E_cH_a^2u^2+E_c(\frac{\partial u}{\partial r})^2=Pr\frac{\partial\Theta}{\partial t} \tag{48}$$

$$\Theta(r,0)=0,\quad 0\le r\le 1\quad,\quad \Theta(1,t)=0,\quad 0<t, \tag{49}$$

where the constants E_c, H_a, S_z and P_r are the Eckert number, the Hartmann number, the dimensionless volumetric heat generation due to Joule heating, and the Prandtl number respectively. These constants are defined as

$$E_c=\frac{(\epsilon\psi_wE_z)^2}{\mu\gamma_0(\Theta_1-\Theta_w)},H_a=B_0R\sqrt{\frac{\sigma}{\mu}},S_z=\frac{\Theta_1}{(\Theta_1-\Theta_w)},Pr=\frac{C_p\mu}{\gamma_0}. \tag{50}$$

The time fractional form of temperature distribution can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\Theta}{\partial r})+\mathcal{D}(r,t)=Pr^{CF}\partial_t^{\gamma}\Theta(r,t), \tag{51a}$$

where

$$\mathcal{D}(r,t)=S_z+E_cH_a^2v^2(r,t)+E_c(\frac{\partial v(r,t)}{\partial r})^2, \tag{51b}$$

in which γ is the fractional parameter and the velocity $v(r,t)$ is given by Eq. (44). The Laplace transformed form of Eqs. (50)-(51) along with Eq. (49) is given by

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\overline{\Theta}}{\partial r})+\overline{\mathcal{D}}=\frac{Prq\overline{\Theta}}{(1-\gamma)q+\gamma}, \tag{52}$$

$$\overline{\Theta}(1,q)=0 \tag{53}$$

where $\mathcal{L}\{\Theta(r,q)\}=\overline{\Theta}(r,q)$ and $\mathcal{L}\{\mathcal{D}(r,q)\}=\overline{\mathcal{D}}(r,q)$. Now we apply the finite Hankel transform to Eq. (52) along with Eq. (53), we get

$$\overline{\Theta}(r_n,q)=\frac{(1-\gamma)q+\gamma}{[(1-\gamma)r_n^2+Pr]q+\gamma r_n^2}\overline{\mathcal{D}_1}(r_n,q) \tag{54}$$

with

$$\overline{\mathcal{D}_1}(r_n,q)=\int_0^1 rJ_0(rr_n)\overline{\mathcal{D}}(r,q)dr \tag{55}$$

Denoting by

$$\overline{\mathcal{D}_2}(r_n,q)=\frac{(1-\gamma)q+\gamma}{[1-\gamma)r_n^2+Pr]q+\gamma r_n^2}, \tag{56}$$

we can rewrite it

$$\overline{\mathcal{D}_2}(r_n,q)=\gamma_n+\beta_n\frac{1}{q+\delta_n}, \tag{57}$$

where

$$\gamma_n=\frac{(1-\gamma)}{(1-\gamma)r_n^2+Pr},\quad \beta_n=\frac{\gamma Pr}{[(1-\gamma)r_n^2+Pr]^2},\quad \delta_n=\frac{\gamma r_n^2}{(1-\gamma)r_n^2+Pr}. \tag{58}$$

From Eqs. (55) and (57),

$$\mathcal{D}_1(r_n,t)=\mathcal{L}^{-1}\{\overline{\mathcal{D}_1}(r_n,q)\}=\int_0^1 r\mathcal{D}(r,t)J_0(rr_n)dr, \tag{59}$$

$$\mathcal{D}_2(r_n,t)=\mathcal{L}^{-1}\{\overline{\mathcal{D}_2}(r_n,q)\}=\gamma_n\delta(t)+\beta_n e^{-\delta_n t}, \tag{60}$$

where $\delta(t)$ represents the Dirac delta distribution. By using convolution theorem and Eqs. (59) and (60), the Laplace transformed form of temperature distribution can be written as,

$$\Theta(r_n,t)=\gamma_n\mathcal{D}_1(r_n,t)+\beta_n e^{-\delta_n t}\int_0^t e^{\delta_n\tau}\mathcal{D}_1(r_n,\tau)d\tau. \tag{61}$$

The thermal transport $\Theta(r,t)$ in the electro-magneto-hydrodynamics flow of time fractional form of generalized Burgers' fluid can be obtained by applying the finite inverse Hankel transform to Eq. (61)

$$\Theta(r,t)=2\sum_{n=1}^{\infty}\frac{\gamma_n\mathcal{D}_1(r_n,t)}{J_1^2(r_n)}J_0(rr_n)+2\sum_{n=1}^{\infty}\frac{\beta_n e^{-\delta_n t}}{J_1^2(r_n)}J_0(rr_n)\int_0^t e^{\delta_n\tau}\mathcal{D}_1(r_n,\tau)d\tau \tag{62}$$

4.1 Special case $\gamma=1$

We know that

$$\frac{\partial\Theta(r,t)}{\partial t}=\lim_{\gamma\to 1}{}^{CF}\partial^{\gamma}\Theta(r,t). \tag{63}$$

Applying the limit $\gamma\to 1$ to Eq. (62) we obtain the thermal transport $\Theta(r,t)$ in the EMHD flow of ordinary generalized Burgers' fluid as,

$$\Theta(r,t)=\frac{2}{Pr}\sum_{n=1}^{\infty}\frac{e^{-\frac{r_n^2}{Pr}t}}{J_1^2(r_n)}J_0(rr_n)\int_0^t e^{\frac{r_n^2}{Pr}\tau}\mathcal{D}_1(r_n,\tau)d\tau, \tag{64}$$

with $\mathcal{D}_1(r_n,t)$ is given by Eq. (59), $\mathcal{D}(r,t)$ given by Eq. (51) but with velocity $v(r,t)$ given by Eq. (44) corresponding to ordinary fluid.

5. Numerical Results and Discussion

The axial unsteady electro-magneto-hydrodynamic flows of fractional generalized Burgers' fluids within a circular tube with isothermal surface have been studied. The mathematical model is described by the fractional constitutive equation with time-fractional Caputo-Fabrizio derivatives with exponential kernel. In the considered model, the histories of the velocity gradient influence the shear stress. Also, the thermal memory effects are considered in fractional energy equation.

The fluid motion is generated by the combined influence of external electric and magnetic fields and the pressure gradient in the axial direction. Exact analytical solutions for the fluid velocity and thermal transport have been determined by using the Laplace transform with respect to time variable *t* and the finite Hankel transform of order zero with respect to the radial coordinate *r*.

By employing the Mathcad software, numerical calculations were carried out in order to obtain numerical values of velocity $v(r,t)$ and temperature $T(r,t)$. In the analytical expressions (44) for velocity and (62) for the temperature needs to know the positive roots of the equation $J_0(x)=0$. These roots have been determined with the subroutine "root(*f*(*x*),*x*,*a*,*b*)" from Mathcad. In our numerical calculations, we used $n = 200$ terms in the series solutions (44) and (62). By numerical simulations, we have observed that the influence of terms corresponding to $n > 200$ is insignificant.

For the numerical computations presented in this paper, we have used the oscillating pressure gradient $p(t)=0.2+0.1\cos(3\pi t/2)+0.05\sin(\pi t)$, the Hartman number Ha = 0.1, Prandtl number Pr = 7 and the thermal fractional parameter $\gamma=0.6$.

The influence of permeability K_0 of porous medium on the fluid velocity has been analyzed in graphs from Figs. 1- 4. In these figures we have compared the ordinary fluids corresponding to fractional parameters (a) $\alpha=\beta=1$ with the fractional ones corresponding to the fractional parameters (b) $\alpha,\beta\in(0,1)$. We have considered following cases: generalized Burgers' fluids with $\lambda_1=10, \lambda_2=1.9, \lambda_3=0.9, \lambda_4=1.4$, Burgers' fluids for which $\lambda_1=10, \lambda_2=1.9, \lambda_3=0.9, \lambda_4=0$, Oldroyd-B fluids characterized by $\lambda_1=10, \lambda_2=0, \lambda_3=0.9, \lambda_4=0$ and Maxwell fluids with $\lambda_1=10, \lambda_2=0, \lambda_3=0, \lambda_4=0$.

It is observed from Figs. 1-4 that the values of fluid velocity are increasing with the permeability of porous medium. This fact is due to the Darcy's resistance which decreases for increasing values of the parameter K_0. Also, it is important to note that fractional fluids flow slower than the ordinary fluids. This behavior is produced by the stronger damping of the velocity gradient in the case of fractional fluids due to the kernel of the Caputo-Fabrizio derivatives (See Eq. (33)). Graphs from Figs. 1-4 reveals that the Burgers' fluids flow faster that the generalized Burgers fluids and Oldroyd-B fluids, whereas the velocity of Maxwell fluids has the highest values. The different fluid behaviors are due to the different way in which the gradient of the velocity is damped (the mathematical expressions of the damping kernels differ from fluid to fluid).

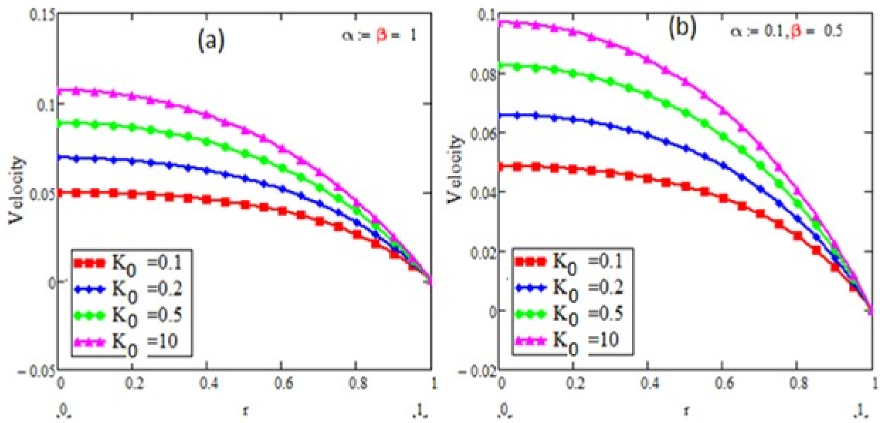


Fig. 1. Velocity profile $v(r,t)$ of Burgers' fluid ($\lambda_4 = 1.4, \lambda_3 = 0.9, \lambda_2 = 1.9, \lambda_1 = 10$) for $t = 1$, $K = 0.1$, $H_a = 0.1$ and various values of K_0 (a) Generalized Burgers' fluid; (b) Fractional Generalized Burgers' fluid.

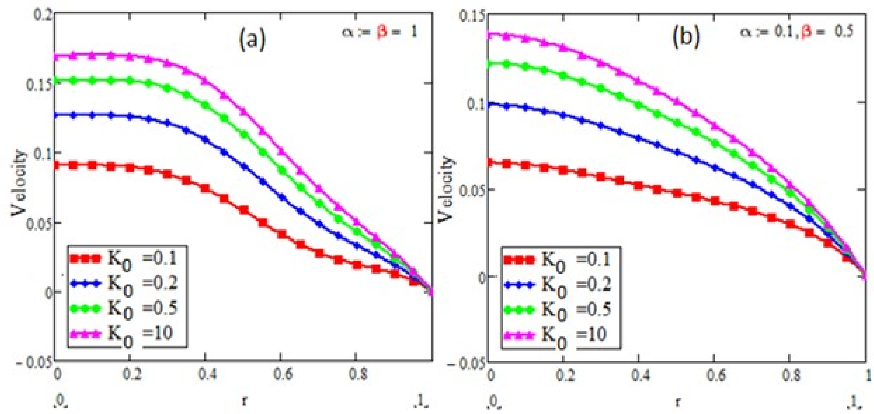


Fig. 2. Velocity profile $v(r,t)$ of Burgers' fluid ($\lambda_4 = 0, \lambda_3 = 0.9, \lambda_2 = 1.9, \lambda_1 = 10$) for $t = 1$, $K = 0.1$, $H_a = 0.1$ and various values of K_0 (a) Burgers' fluid; (b) Fractional Burgers' fluid.

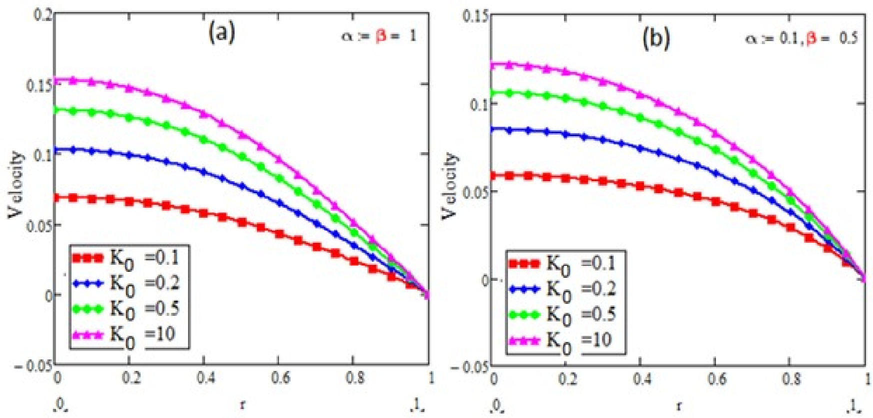


Fig. 3. Velocity profile $v(r,t)$ of Oldroyd-B fluid ($\lambda_4 = 0, \lambda_3 = 0.9, \lambda_2 = 0, \lambda_1 = 10$) for $t = 1$, $K = 0.1$, $H_a = 0.1$ and various values of K_0 (a) Oldroyd-B fluid; (b) Fractional Oldroyd-B fluid.

Fig. 5 has been plotted to highlight the influence of the fractional parameter α on the fluid motion when the fractional parameter $\beta = 0.7$. It can seen from Fig. 5 that for small values of the time *t*, the values of velocity are increasing with the fractional parameter α while, for large values of the time t velocity values are decreasing with the fractional parameter α.

Fig. 6 has been plotted to highlight the influence of the fractional parameter β on the fluid motion when the fractional parameter $\alpha = 0.4$. One can see from Fig. 6 that for small values of the time *t*, the values of velocity are decreasing with the fractional parameter β while, for large values of the time t velocity values are increasing with the fractional parameter β. The behaviors highlighted in Figs 5 and 6 are due to the time-evolution of the weight function of the velocity gradient in the constitutive equation, namely $h_{\alpha\beta}(t) = L^{-1}\{F_{\alpha\beta}(q)\}$ with graphs given in Fig. 9.

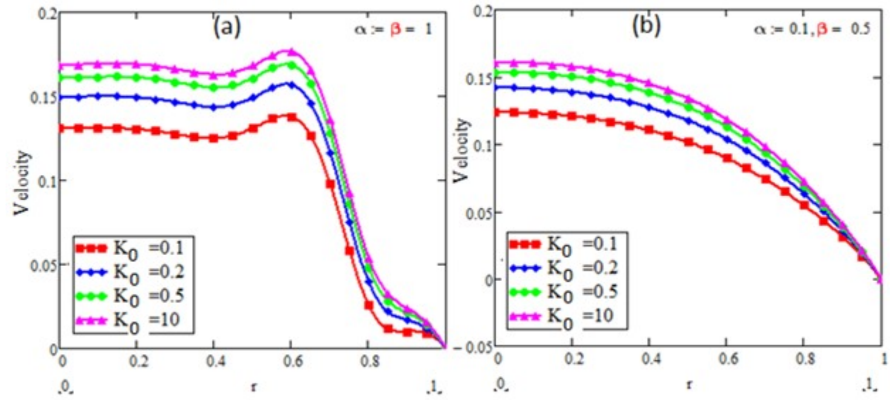


Fig. 4. Velocity profile $v(r,t)$ of Maxwell fluid ($\lambda_4 = 0, \lambda_3 = 0, \lambda_2 = 0, \lambda_1 = 10$) for $t = 1$, $K = 0.1$, $H_a = 0.1$ and various values of K_0 (a) Maxwell fluid; (b) Fractional Maxwell fluid.

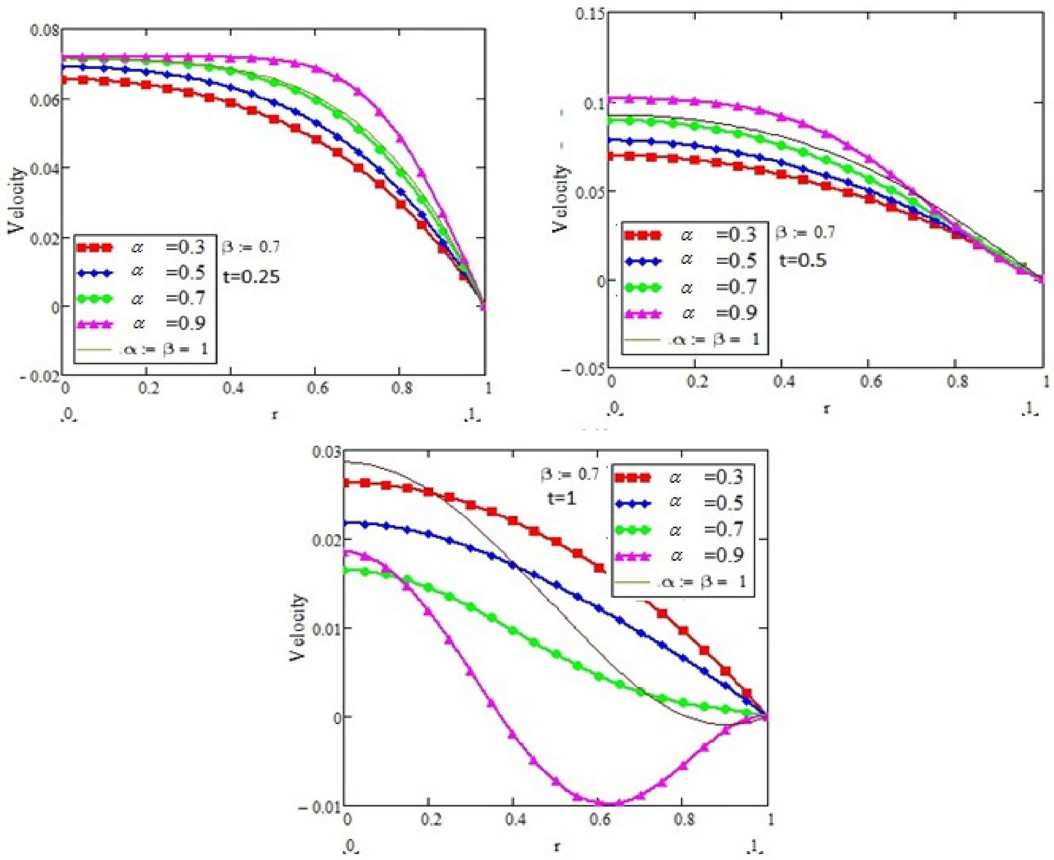
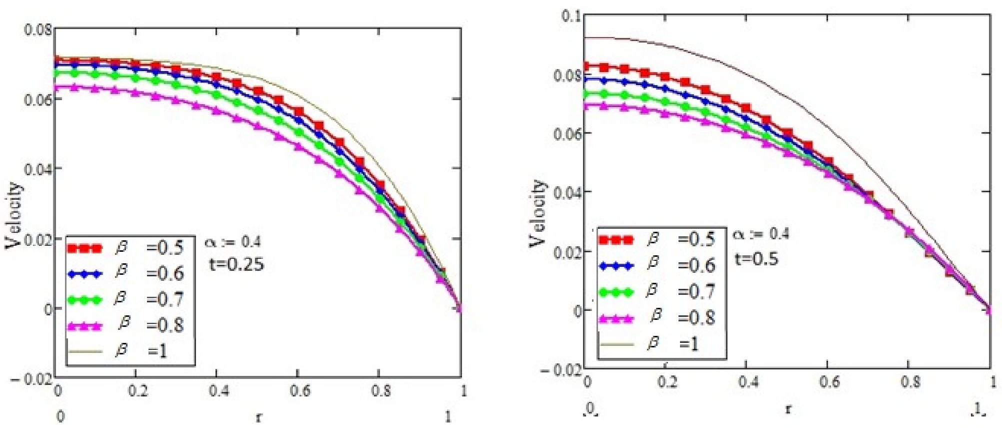


Fig. 5. Velocity profile $v(r,t)$, versus r, of generalized Burgers' fluid ($\lambda_4 = 0.2, \lambda_3 = 0.1, \lambda_2 = 0.8, \lambda_1 = 0.15$) for $K = 0.1$, $K_0 = 0.4$, $H_a = 0.1$ and different values of α with β fixed.



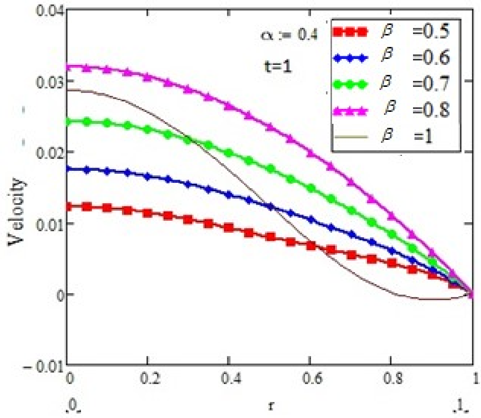


Fig. 6. Velocity profile $v(r,t)$, versus r, of generalized Burgers' fluid ($\lambda_4 = 0.2, \lambda_3 = 0.1, \lambda_2 = 0.8, \lambda_1 = 0.15$) for $K = 0.1, K_0 = 0.4, H_a = 0.1$ and different values of β with $\alpha \le \beta$.

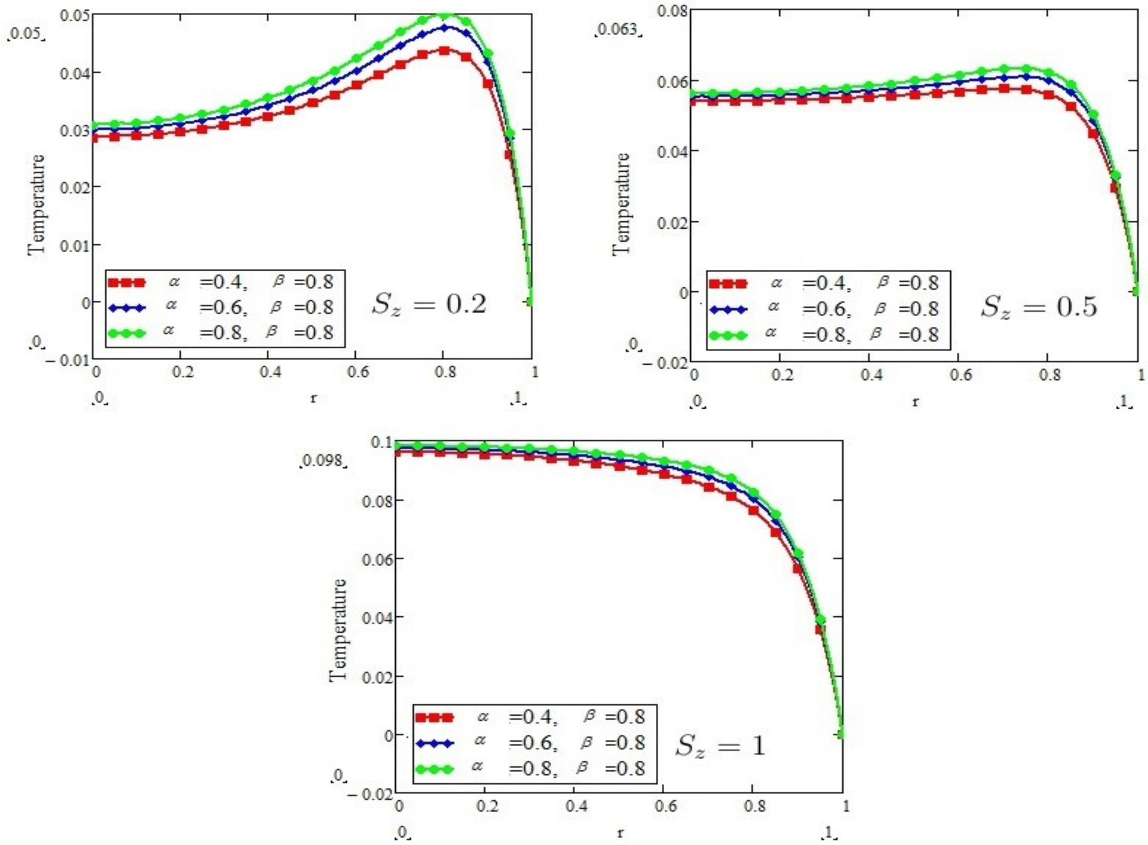
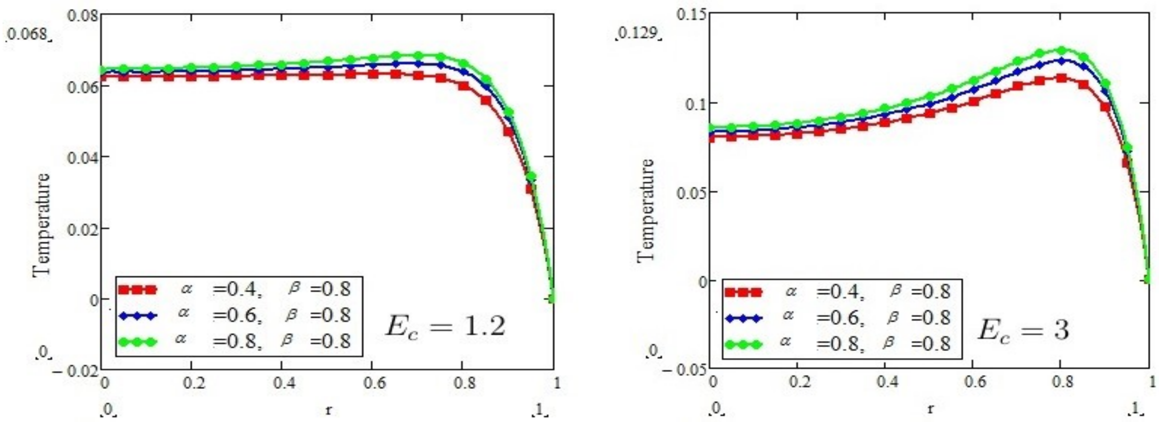


Fig. 7. Profile of temperature $\Theta(r,t)$ versus r of generalized Burgers' fluid ($\lambda_4 = 0.4, \lambda_3 = 0.1, \lambda_2 = 0.3, \lambda_1 = 0.15$) for $H_a = 0.1, K = 4.5, K_0 = 0.4, E_c = 1.2, \gamma = 0.6$ and different values of α with $\alpha \le \beta$.



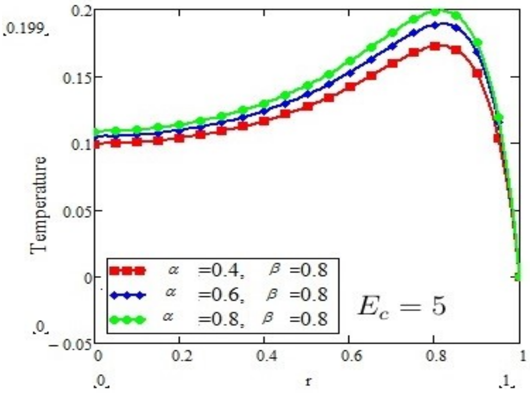


Fig. 8. Profile of temperature $\Theta(r,t)$ versus r of generalized Burgers' fluid ($\lambda_4 = 0.4, \lambda_3 = 0.1, \lambda_2 = 0.3, \lambda_1 = 0.15$) for $H_a = 0.1, K = 4.5, K_0 = 0.4,\ S_z = 0.6, \gamma = 0.6$ and different values of α with $\alpha \le \beta$.

Figs. 7 is prepared to demonstrate the effects of S_z, the Joule heating coefficient, on the temperature profile together with the fractional parameters. The temperature profile increases with the increase in the values of S_z . Fig. 8 represents the effects of Eckert number E_c along with the fractional parameters on the thermal transport. We used higher values of Eckert number E_c . An increase in fluid thermal transport is observed with the increase in the values of E_c .

In refs. [32, 33], authors used Eckert number to investigate the EMHD thermofluidic temperature profile in a channel flow. They have used for the Eckert number the values $E_c = 1, 1.2, 1.4$. For the higher values of Eckert number, a remarkable increase in thermal transport profile was observed in refs. [32, 33], therefore a good resemblance of the results from this paper it is found with results given in [32, 33].

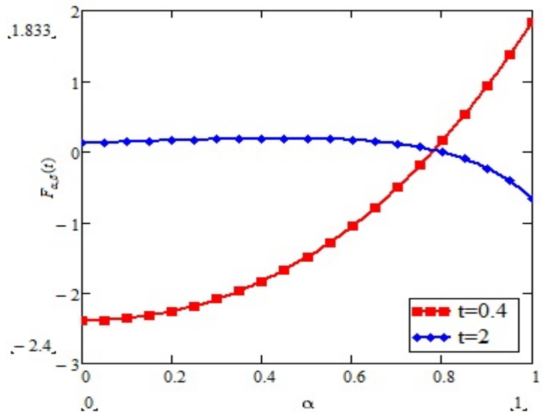


Fig. 9. Profile of function $F_{\alpha,\beta}(t)$ versus α for $\lambda_4 = 0.4, \lambda_3 = 0.1,\ \lambda_2 = 0.3, \lambda_1 = 0.15, \beta = 0.7$ and different values of time t .

6. Concluding remarks

The exact analytical solution of time fractional model EMHD fluid flow through a cylindrical micro channel is derived for both velocity and thermal transport profiles with the help of Laplace and finite Hankel transforms. The following is a summary of main findings in the paper.

- As anticipated, the drag force falls as the permeability of porous medium rises and this increases the velocity profile for all kinds of fluids (a generalized Burgers' fluids, a Burgers' fluid, a Maxwell fluid and an Oldroyd-B fluid). This is resemblance with the fact that velocity profile reduces with permeability. Furthermore an opposite effect is observed with the increase in the magnitude of applied magnetic fields.
- A comparative study has been carried out numerically to distinguish the distinct behavior of ordinary model solutions and fractional model solutions.
- It is observed that we can control the fluid velocity and thermal transport performance with the variation of fractional parameters.
- The ordinary fluid moves slower/faster than the fractional fluid for certain values of time t.
- The Joule heating parameter can be used to control the fluid thermal transport.
- For large values of the Eckert number, increase in fluid temperature is observed.

Fractional order EMHD flow through capillary and its thermal behavior have important applications in Biochip technology. Thus, our results are also important in the control of liquid samples of nano-volumes present in microfluidic devices. These liquid samples have applications in medical diagnosis and biological analysis.

Conflict of Interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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Appendix A

$$B_0 = -K_0\alpha\beta \tag{A.1}$$

$$B_1 = K_0(-\alpha - \beta + 2\alpha\beta - \beta\lambda_1 - \alpha\beta\lambda_1) \tag{A.2}$$

$$B_2 = K_0(-1 + \alpha + \beta - \alpha\beta - \lambda_1 - \alpha\lambda_1 + 2\alpha\beta\lambda_1 - \beta\lambda_1^2 - \beta\lambda_2 - \alpha\beta\lambda_2), \tag{A.3}$$

$$B_3 = K_0(-\lambda_1 + \alpha\lambda_1 + \beta\lambda_1 - \alpha\beta\lambda_1 - \lambda_1^2 + \beta\lambda_1^2 - \lambda_2 - \alpha\lambda_2 + 2\alpha\beta\lambda_2 - 2\beta\lambda_1\lambda_2), \tag{A.4}$$

$$B_4 = K_0(-\lambda_2 + \alpha\lambda_2 + \beta\lambda_2 - \alpha\beta\lambda_2 - 2\lambda_1\lambda_2 + 2\beta\lambda_1\lambda_2 - \beta\lambda_2^2), \tag{A.5}$$

$$B_5 = K_0 q^5(-\lambda_2^2 + \beta\lambda_2^2) \tag{A.6}$$

$$\mathfrak{P}_{0,n} = -\alpha\beta - Ha^2K_0\alpha\beta - K_0r_n^2\alpha\beta, \tag{A.7}$$

$$\begin{gathered}\mathfrak{P}_{1,n} = -\alpha - Ha^2K_0\alpha - K_0r_n^2\alpha - \beta - Ha^2K_0\beta - K_0r_n^2\beta + 2\alpha\beta - K_0\alpha\beta + 2Ha^2K_0\alpha\beta + \\ 2K_0r_n^2\alpha\beta - \beta\lambda_1 - Ha^2K_0\beta\lambda_1 - Ha^2K_0\alpha\beta\lambda_1 - K_0r_n^2\alpha\beta\lambda_1 - K_0r_n^2\alpha\lambda_3 - \alpha\beta\lambda_3\end{gathered} \tag{A.8}$$

$$\begin{gathered}\mathfrak{P}_{2,n} = -1 - Ha^2K_0 - K_0r_n^2 + \alpha - K_0\alpha + Ha^2K_0\alpha + K_0r_n^2\alpha + \beta - K_0\beta + Ha^2K_0\beta + \\ K_0r_n^2\beta - \alpha\beta + 2K_0\alpha\beta - Ha^2K_0\alpha\beta - K_0r_n^2\alpha\beta - \lambda_1 - Ha^2K_0\lambda_1 - Ha^2K_0\alpha\lambda_1 - K_0r_n^2\alpha\lambda_1 + \\ \beta\lambda_1 - K_0\beta\lambda_1 - K_0r_n^2\beta\lambda_1 - K_0\alpha\beta\lambda_1 + 2Ha^2K_0\alpha\beta\lambda_1 + 2K_0r_n^2\alpha\beta\lambda_1 - Ha^2K_0\beta\lambda_1^2 - \beta\lambda_2 - \\ Ha^2K_0\beta\lambda_2 - Ha^2K_0\alpha\beta\lambda_2 - K_0r_n^2\alpha\beta\lambda_2 - K_0r_n^2\lambda_3 - \alpha\lambda_3 + K_0r_n^2\alpha\lambda_3 - \beta\lambda_3 + 2\alpha\beta\lambda_3 - \\ K_0r_n^2\alpha\lambda_1\lambda_3 - \beta\lambda_1\lambda_3 - K_0r_n^2\alpha\lambda_4 - \alpha\beta\lambda_4\end{gathered} \tag{A.9}$$

$$\begin{gathered}\mathfrak{P}_{3,n} = -K_0 + K_0\alpha + K_0\beta - K_0\alpha\beta - K_0\lambda_1 - Ha^2K_0\lambda_1 - K_0r_n^2\lambda_1 - K_0\alpha\lambda_1 + Ha^2K_0\alpha\lambda_1 + \\ K_0r_n^2\alpha\lambda_1 + Ha^2K_0\beta\lambda_1 + K_0r_n^2\beta\lambda_1 + 2K_0\alpha\beta\lambda_1 - Ha^2K_0\alpha\beta\lambda_1 - K_0r_n^2\alpha\beta\lambda_1 - Ha^2K_0\lambda_1^2 - \\ K_0\beta\lambda_1^2 + Ha^2K_0\beta\lambda_1^2 - \lambda_2 - Ha^2K_0\lambda_2 - Ha^2K_0\alpha\lambda_2 - K_0r_n^2\alpha\lambda_2 + \beta\lambda_2 - K_0\beta\lambda_2 - K_0r_n^2\beta\lambda_2 - \\ K_0\alpha\beta\lambda_2 + 2Ha^2K_0\alpha\beta\lambda_2 + 2K_0r_n^2\alpha\beta\lambda_2 - 2Ha^2K_0\beta\lambda_1\lambda_2 - \lambda_3 + \alpha\lambda_3 + \beta\lambda_3 - \alpha\beta\lambda_3 - \\ \lambda_1\lambda_3 - K_0r_n^2\lambda_1\lambda_3 + K_0r_n^2\alpha\lambda_1\lambda_3 + \beta\lambda_1\lambda_3 - K_0r_n^2\alpha\lambda_2\lambda_3 - \beta\lambda_2\lambda_3 - K_0r_n^2\lambda_4 - \alpha\lambda_4 + \\ K_0r_n^2\alpha\lambda_4 - \beta\lambda_4 + 2\alpha\beta\lambda_4 - K_0r_n^2\alpha\lambda_1\lambda_4 - \beta\lambda_1\lambda_4\end{gathered} \tag{A.10}$$

$$\mathfrak{P}_{4,n} = -K_0\lambda_1 + K_0\alpha\lambda_1 + K_0\beta\lambda_1 - K_0\alpha\beta\lambda_1 - K_0\lambda_1^2 + K_0\beta\lambda_1^2 - K_0\lambda_2 - Ha^2K_0\lambda_2 - K_0r_n^2\lambda_2 - \tag{A.11}$$

$$K_0\alpha\lambda_2 + Ha^2K_0\alpha\lambda_2 + K_0r_n^2\alpha\lambda_2 + Ha^2K_0\beta\lambda_2 + K_0r_n^2\beta\lambda_2 + 2K_0\alpha\beta\lambda_2 - Ha^2K_0\alpha\beta\lambda_2 - K_0r_n^2\alpha\beta\lambda_2 - 2Ha^2K_0\lambda_1\lambda_2 - 2K_0\beta\lambda_1\lambda_2 + 2Ha^2K_0\beta\lambda_1\lambda_2 - Ha^2K_0\beta\lambda_2^2 - \lambda_2\lambda_3 - K_0r_n^2\lambda_2\lambda_3 + K_0r_n^2\alpha\lambda_2\lambda_3 + \beta\lambda_2\lambda_3 - \lambda_4 + \alpha\lambda_4 + \beta\lambda_4 - \alpha\beta\lambda_4 - \lambda_1\lambda_4 - K_0r_n^2\lambda_1\lambda_4 + K_0r_n^2\alpha\lambda_1\lambda_4 + \beta\lambda_1\lambda_4 - K_0r_n^2\alpha\lambda_2\lambda_4 - \beta\lambda_2\lambda_4$$

$$\mathfrak{P}_{5,n} = -K_0\lambda_2 + K_0\alpha\lambda_2 + K_0\beta\lambda_2 - K_0\alpha\beta\lambda_2 - 2K_0\lambda_1\lambda_2 + 2K_0\beta\lambda_1\lambda_2 - Ha^2K_0\lambda_2^2 - K_0\beta\lambda_2^2 + Ha^2K_0\beta\lambda_2^2 - \lambda_2\lambda_4 - K_0r_n^2\lambda_2\lambda_4 + K_0r_n^2\alpha\lambda_2\lambda_4 + \beta\lambda_2\lambda_4 \tag{A.12}$$

$$\mathfrak{P}_{6,n} = -K_0\lambda_2^2 + K_0\beta\lambda_2^2 \tag{A.13}$$

$$a_{i,n} = \frac{1}{\mathfrak{P}_{6,n}}\prod_{j=1,j\neq i}^{6}\frac{B_0 + B_1q_{i,n} + B_2q_{i,n}^2 + B_3q_{i,n}^3 + B_4q_{i,n}^4 + B_5q_{i,n}^5}{(q_{i,n} - q_{j,n})}\$,\ \$i = 1,2,3,4,5,6. \tag{A.14}$$