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THE INTERSECTION GRAPH OF ANNIHILATOR SUBMODULES OF A MODULE

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Abstract. Let R be a commutative ring and M be a Noetherian R-module. The intersection graph of annihilator submodules of M, denoted by $GA(M)$ is an undirected simple graph whose vertices are the classes of elements of $Z_R(M) \setminus \mathrm{Ann}_R(M)$, for $a, b \in R$ two distinct classes $[a]$ and $[b]$ are adjacent if and only if $\mathrm{Ann}_M(a) \cap \mathrm{Ann}_M(b) \neq 0$. In this paper, we study diameter and girth of $GA(M)$ and characterize all modules that the intersection graph of annihilator submodules are connected. We prove that $GA(M)$ is complete if and only if $Z_R(M)$ is an ideal of R. Also, we show that if M is a finitely generated R-module with $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$ and $|m - \mathrm{Ass}_R(M)| = 1$ and $GA(M)$ is a star graph, then $r(\mathrm{Ann}_R(M))$ is not a prime ideal of R and $|V(GA(M))| = |\operatorname{Min}\mathrm{Ass}_R(M)| + 1$.

Keywords: prime submodule, annihilator submodule, intersection annihilator graph.

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1. INTRODUCTION

Let R be a commutative ring and M be an R-module. The intersection graph of ideals of a ring introduced and studied in [7] and then in [1] the intersection graph of submodules of a module was defined. The intersection graph of submodules of a module, denoted by $G(M)$, is a graph whose vertices are in one to one correspondence with all non-trivial submodules of M and two distinct vertices are adjacent if and only if the corresponding submodules of M have non-zero intersection. Also, the complement of the intersection graph of submodules of a module studied in [2], for more works on the intersection graph of modules, see [10, 15].

The zero-divisor graph of R, denoted by $\Gamma(R)$, is a graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$ and two distinct vertices a and b are adjacent if and only if $ab = 0$, see [4, 6]. The compressed zero-divisor graph of R, $\Gamma_E(R)$, that is constructed from equivalence classes of zero-divisors, rather than individual zero-divisors themselves was introduced in [11] and studied in some literatures, for examples [3, 8, 13]. This graph has some advantages over the zero-divisor graph. For examples, in many cases, the compressed zero-divisor graph of R is finite when the zero-divisor graph is infinite and another important aspect of the compressed zero-divisor graph is the connection to the associated primes of R.

In this paper, with inspire by the above ideas, we introduce the intersection graph of annihilator submodules of M. Let $a, b \in R$, we say that $a \sim b$ if and only if $\mathrm{Ann}_M(a) = \mathrm{Ann}_M(b)$. As noted in [11], $\sim$ is an equivalence relation. If $[a]$ denotes the class of a, then $[a] = \mathrm{Ann}_R(M)$ whenever $a \in \mathrm{Ann}_R(M)$ and $[a] = R \setminus Z_R(M)$ whenever $a \in R \setminus Z_R(M)$; the other equivalence classes form a partition of $Z_R(M)$. The intersection graph of annihilator submodules of M, denoted by $GA(M)$, is an undirected simple graph whose vertices are the classes of elements of $Z_R(M) \setminus \mathrm{Ann}_R(M)$, for $a, b \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ two distinct classes $[a]$ and $[b]$ are adjacent if and only if $\mathrm{Ann}_M(a) \cap \mathrm{Ann}_M(b) \neq 0$. Let M be a Noetherian R-module. In section two, we study connectivity, diameter and the girth of $GA(M)$. We show that $GA(M)$ is a disconnected graph if and only if $m - \mathrm{Ass}_R(M) = \{P_1, P_2\}$ and $P_1 \cap P_2 = 0$. In section three, we show that $GA(M)$ is a complete graph if and only if $Z_R(M)$ is a prime ideal of R and we show that if $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$ and $|m - \mathrm{Ass}_R(M)| = 1$ and $GA(M)$ is a star graph with at least three vertices, then $r(\mathrm{Ann}_R(M))$ is not a prime ideal of R and

$$|V(GA(M))| = |\operatorname{Min}\mathrm{Ass}_R(M)| + 1.$$

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For each pair of vertices $u, v \in V(G)$, if u is adjacent to v, then we write $u - v$. The degree of a vertex u, denoted by $\deg(u)$, is the number of edges incident to u, and u is called end-vertex if $\deg(u) = 1$. A graph with no edge is called null graph. The complement graph of G, denoted by $\overline{G}$, is a graph with the same vertices such that two vertices of G are adjacent if and only if they are not adjacent in $\overline{G}$. Recall that G is connected if there is a path between any two distinct vertices of G. For vertices x and y of G, let $\mathrm{d}(x, y)$ be the length of a shortest path from x to y ($\mathrm{d}(x, x) = 0$ and $\mathrm{d}(x, y) = \infty$ if there is no such path). The diameter of G is

$$\mathrm{diam}(G) = \sup\{\mathrm{d}(x, y) \mid x \text{ and } y \text{ are vertices of } G\}.$$

The girth of G, denoted by $\mathrm{gr}(G)$, is the length of the shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G contains no cycles). A graph G is complete if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K_n. A complete bipartite graph is a graph G which may be partitioned into two disjoint non-empty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. If one of the vertex set is a singleton, then we call G a star graph. A clique of G is a complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the clique number.

Throughout, R denotes a commutative ring with non-zero identity, $Z(R)$ its set of zero-divisors and for ideal I of R,

$$r(I) = \{r \in R : \text{ there exists } n \in \mathbb{N} \text{ with } r^n \in I\}$$

denotes the radical of I. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the ring of integers and the ring of integers modulo n, respectively. Let M be an R-module and

$$\mathrm{Ass}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} = \mathrm{Ann}_R(m) \text{ for some } 0 \neq m \in M\}.$$

A proper submodule P of M is said to be prime submodule whenever for $r \in R$ and $m \in M$, $rm \in P$ implies that $m \in P$ or $r \in \mathrm{Ann}_R(M/P)$. Let $\mathrm{Spec}_R(M)$ denote the set of prime submodules of M and

$$m - \mathrm{Ass}_R(M) = \{P \in \mathrm{Spec}_R(M) : P = \mathrm{Ann}_M(a) \text{ for some } 0 \neq a \in R\},$$

where for $a \in R$, $\mathrm{Ann}_M(a) = \{m \in M : am = 0\}$. For notations and terminologies not given in this article, the reader is referred to [12].

2. CONNECTIVITY, DIAMETER AND GIRTH OF $GA(M)$

Recall that R is a commutative ring and M is an R-module with property that its zero submodule is not a prime submodule. In this section, the annihilator submodules of M and the intersection graph of annihilator submodules of M are studied.

Theorem 2.1. *Let M be an R-module and $a, b \in R$. Then the following statements are true:*

- (i) [5, *Theorem 2*] *If $a \notin r(\mathrm{Ann}_R(M))$and $\mathrm{Ann}_M(a)$ is a prime submodule of M, then it is a minimal prime submodule of M.*
- (ii) [5, *Theorem 6*(ii)] *If $a, b \notin r(\mathrm{Ann}_R(M))$ and $\mathrm{Ann}_M(a)$, $\mathrm{Ann}_M(b)$ are distinct prime submodules of M, then $abM = 0$.*
- (iii) [5, *Theorem 5*(ii)] *If $a \in r(\mathrm{Ann}_R(M))$, then $\mathrm{Ann}_M(a)$ is an essential submodule of M.*

Lemma 2.2. *Let M be a Noetherian R-module. Then*

$$r(\mathrm{Ann}_R(M))M \subseteq \cap_{P \in m-\mathrm{Ass}_R(M)} P.$$

Proof. By hypotheses and [9, Proposition 3.2], $m - \mathrm{Ass}_R(M) \neq \emptyset$. Suppose that $a \in r(\mathrm{Ann}_R(M))$ and $P \in m - \mathrm{Ass}_R(M)$ we have to show that $aM \subseteq P$. By assumption there is a positive integer t such that $a^t M = 0$. Thus $a^t M \subseteq P$. By hypotheses P is a prime submodule of M, thus $\mathrm{Ann}_R(M/P)$ is a prime ideal of R so by $a^t \in \mathrm{Ann}_R(M/P)$ it follows that $a \in \mathrm{Ann}_R(M/P)$. Hence, $aM \subseteq P$, we are done. □

Theorem 2.3. *Let M be a Noetherian R-module with $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$ and $a, b \in R \setminus \mathrm{Ann}_R(M)$. If $P_1 = \mathrm{Ann}_M(a)$ and $P_2 = \mathrm{Ann}_M(b)$ are prime submodules of M such that $P_1 \cap P_2 = 0$, then $|m - \mathrm{Ass}_R(M)| = 2$.*

Proof. Let $P = \mathrm{Ann}_M(c)$ is a prime submodule of M. Then $c \notin \mathrm{Ann}_R(M)$. By hypotheses, $P_1 \cap P_2 = 0$. Thus $P_1 \cap P_2 \subseteq P$. If $P_1 \nsubseteq P$ and $P_2 \nsubseteq P$, then there exist $m_1 \in P_1 \setminus P$ and $m_2 \in P_2 \setminus P$ such that $am_1 = bm_2 = 0 \in P$ which implies that $a, b \in \operatorname{Ann}_R(M/\operatorname{Ann}_M(c)) = \operatorname{Ann}_R(cM)$. So that $cM \subseteq \operatorname{Ann}_M(a) \cap \operatorname{Ann}_M(b) = P_1 \cap P_2$. Hence, $cM = 0$ and so $c \in \operatorname{Ann}_R(M)$, contrary to the assumption. Therefore, either $P_1 \subseteq P$ or $P_2 \subseteq P$ and so either $P_1 = P$ or $P_2 = P$ by Theorem 2.1(i). Thus $|m - \operatorname{Ass}_R(M)| = 2$. □

Corollary 2.4. *Let M be an R-module with $r(\operatorname{Ann}_R(M)) = \operatorname{Ann}_R(M)$ and $a, b \in R \setminus \operatorname{Ann}_R(M)$. If $P_1 = \operatorname{Ann}_M(a)$ and $P_2 = \operatorname{Ann}_M(b)$ are prime submodules of M such that $P_1 \cap P_2 = 0$, then $aM + bM \cong aM \oplus bM$.*

Proof. Let $a, b \in R \setminus \operatorname{Ann}_R(M)$, $P_1 = \operatorname{Ann}_M(a)$ and $P_2 = \operatorname{Ann}_M(b)$ are prime submodules of M such that $P_1 \cap P_2 = 0$. Then by Theorem 2.1(ii) it follows that $abM = 0$. Hence $aM \cap bM \subseteq \operatorname{Ann}_M(b) \cap \operatorname{Ann}_M(a) = 0$. Thus $aM + bM \cong aM \oplus bM$. □

Corollary 2.5. *Let M be an R-module with $r(\operatorname{Ann}_R(M)) = \operatorname{Ann}_R(M)$ and $a \in R \setminus \operatorname{Ann}_R(M)$. If $\operatorname{Ann}_M(a)$ is a prime submodule of M, then $Z_R(aM)$ is a prime ideal of R.*

Proof. First, we show that $Z_R(aM) = \operatorname{Ann}_R(aM)$. Suppose that $c \in R$. We show that either $\operatorname{Ann}_{aM}(c) = 0$ or $\operatorname{Ann}_{aM}(c) = aM$. Assume that $\operatorname{Ann}_{aM}(c) \neq 0$ and $0 \neq am \in \operatorname{Ann}_{aM}(c)$. Thus $cam = 0$. Hence, $cm \in \operatorname{Ann}_M(a)$ and $\operatorname{Ann}_M(a)$ is a prime submodule of M, it follows that $caM = 0$. Thus $\operatorname{Ann}_{aM}(c) = aM$. Hence, $Z_R(aM) = \operatorname{Ann}_R(aM) = \operatorname{Ann}_R(M/\operatorname{Ann}_M(a))$ is a prime ideal of R, as claimed. □

Let R be a commutative ring and M be an R-module. Assume $Z^*_R(M)$ denotes the set of non-zero zero-divisors of M. For $a, b \in R$, we say that $a \sim b$ if and only if $\operatorname{Ann}_M(a) = \operatorname{Ann}_M(b)$. As noted in [11], $\sim$ is an equivalence relation. If $[a]$ denotes the class of a, then $[a] = \operatorname{Ann}_R(M)$ whenever $a \in \operatorname{Ann}_R(M)$ and $[a] = R \setminus Z_R(M)$ for all $a \in R \setminus Z_R(M)$; the other equivalence classes form a partition of $Z_R(M)$. The intersection graph of annihilator submodules of M, denoted by $GA(M)$, is an undirected simple graph whose vertices are the classes of elements in $Z^*_R(M) \setminus \operatorname{Ann}_R(M)$, and two distinct classes $[a]$ and $[b]$ are adjacent if and only if $\operatorname{Ann}_M(a) \cap \operatorname{Ann}_M(b) \neq 0$.

Lemma 2.6. *Let M be an R-module. If $r(\operatorname{Ann}_R(M)) \neq \operatorname{Ann}_R(M)$, then $GA(M)$ is a connected graph.*

Proof. Let $a \in r(\operatorname{Ann}_R(M)) \setminus \operatorname{Ann}_R(M)$. Then by Theorem 2.1(iii), $\operatorname{Ann}_M(a)$ is an essential submodule of M. So that $[a]$ is a universal vertex in $GA(M)$, which show that $GA(M)$ is connected. □

Theorem 2.7. *Let M be a Noetherian R-module. Then $GA(M)$ is a disconnected graph if and only if $m - \operatorname{Ass}_R(M) = \{P_1, P_2\}$ and $P_1 \cap P_2 = 0$.*

Proof. ($\Leftarrow$) Suppose that P_1 and P_2 are two prime annihilator submodules of M such that $P_1 \cap P_2 = 0$. Thus by Lemma 2.2, $r(\operatorname{Ann}_R(M))M \subseteq P_1 \cap P_2 = 0$ and so $r(\operatorname{Ann}_R(M)) = \operatorname{Ann}_R(M)$. Assume that $a, b \in R \setminus \operatorname{Ann}_R(M)$ and $P_1 = \operatorname{Ann}_M(a)$ and $P_2 = \operatorname{Ann}_M(b)$. It is our claim that, there is no path between two vertices $[a]$ and $[b]$. Set $X = \{\operatorname{Ann}_M(c) \ : \ c \notin \operatorname{Ann}_R(M)\}$. If P_1 is not a maximal element of X, then there exists a maximal element P in X different form P_2 such that $P_1 \subseteq P$. By [9, Proposition 3.2], P is a prime submodule of M so that $|m - \operatorname{Ass}_R(M)| \geq 3$, contrary to hypotheses. Therefore, P_1 and P_2 are maximal elements of X. Thus for each $[c] \in V(GA(M))$ we have either $\mathrm{Ann}_M(c) \subseteq P_1$ or $\mathrm{Ann}_M(c) \subseteq P_2$. On the contrary, suppose that $[a] - [c] - [d] - [b]$ is a path between $[a]$ and $[b]$, where $\mathrm{Ann}_M(c) \subseteq P_1$ and $\mathrm{Ann}_M(d) \subseteq P_2$. Then $0 \neq \mathrm{Ann}_M(c) \cap \mathrm{Ann}_M(d) \subseteq P_1 \cap P_2$ that is a contradiction. In any other cases we have the same contradiction. Hence, there is no path between $[a]$ and $[b]$, as claimed. Therefore, $GA(M)$ is a disconnected graph.

($\Rightarrow$) By hypotheses and Lemma 2.6 we can assume that $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$. If $|m - \mathrm{Ass}_R(M)| = 1$, then $GA(M)$ has a universal vertex such as $[a]$, where $a \in R \setminus \mathrm{Ann}_R(M)$ and $\mathrm{Ann}_M(a) = P \in m - \mathrm{Ass}_R(M)$. So that $|m - \mathrm{Ass}_R(M)| \geq 2$. Assume that $P' \cap P'' \neq 0$, for each prime annihilator submodules P' and P'' of M and we look for a contradiction. Let $[c]$ and $[d]$ be two arbitrary vertices of $GA(M)$. Then there exist two prime annihilator submodules $P_1 = \mathrm{Ann}_M(a)$ and $P_2 = \mathrm{Ann}_M(b)$ of M such that $\mathrm{Ann}_M(c) \subseteq P_1$ and $\mathrm{Ann}_M(d) \subseteq P_2$. Consequently, $[c] - [a] - [b] - [d]$ is a path between two vertices $[c]$ and $[d]$. Hence, $GA(M)$ is a connected graph contrary to assumption. So that there exist two annihilator prime submodules P' and P'' of M such that $P' \cap P'' = 0$. Now, the assertion follows from Theorem 2.3. □

Corollary 2.8. *Let M be a Noetherian R-module. Then the following statements are equivalent:*

(i) *$GA(M)$ is a connected graph,*
(ii) *Either $m - \mathrm{Ass}_R(M) = \{P_1, P_2\}$ and $P_1 \cap P_2 \neq 0$ or $|m - \mathrm{Ass}_R(M)| \neq 2$,*
(iii) *Any two distinct elements of $m - \mathrm{Ass}_R(M)$ are adjacent in $GA(M)$.*

The independence number of a graph G is the maximum size of an independent vertex set and is denoted by $\alpha(G)$.

Corollary 2.9. *Let M be a Noetherian R-module. If $GA(M)$ is a connected graph, then*

$$\alpha(GA(M)) \leq |V(GA(M))| - |m - \mathrm{Ass}_R(M)| + 1.$$

Proof. By Corollary 2.8, any two distinct elements of $m - \mathrm{Ass}_R(M)$ are connected by an edge in $GA(M)$, so that $|m - \mathrm{Ass}_R(M)| \leq \omega(GA(M))$. On the other hand, by [14, proposition 5.1.7]

$$\omega(GA(M)) \leq |V(GA(M))| - \alpha(GA(M)) + 1.$$

Thus the result follows. □

Remark 2.10. Let $G(M)$ be a disconnected graph. Then by [1, Theorem 2.1], $M = P_1 \oplus P_2$ and $m - \mathrm{Ass}_R(M) = \{P_1, P_2\}$, where P_1, P_2 are prime submodules of M and $P_1 \cap P_2 = 0$, so that $GA(M)$ is a disconnected graph by Theorem 2.7, whenever $|V(GA(M))| \geq 2$.

The converse is not true in general. Consider $M = \mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$ as a $\mathbb{Z}$-module. It is easy to see that $GA(M)$ is a null graph with two vertices, but $G(M)$ is a connected graph with more than two vertices.

Theorem 2.11. *Let M be a Noetherian R-module and $GA(M)$ be a connected graph. Then* $\mathrm{diam}(GA(M)) \leq 2$.

Proof. Let $[a]$ and $[b]$ be two arbitrary vertices of $GA(M)$. If $\mathrm{Ann}_M(a)\cap\mathrm{Ann}_M(b)\neq 0$, then $d([a],[b])=1$. Let $\mathrm{Ann}_M(a)\cap\mathrm{Ann}_M(b)=0$. If $r(\mathrm{Ann}_R(M))\neq\mathrm{Ann}_R(M)$, then by Theorem 2.1(iii), there is a universal vertex in $GA(M)$ denoted by $[x]$ such that $x\in r(\mathrm{Ann}_R(M))\setminus\mathrm{Ann}_R(M)$. Hence $GA(M)$ has the path $[a]-[x]-[b]$ as a subgraph and so $d([a],[b])=2$.

Suppose that $r(\mathrm{Ann}_R(M))=\mathrm{Ann}_R(M)$. If $ab\notin\mathrm{Ann}_R(M)$, then $[ab]\in V(GA(M))$ and $\mathrm{Ann}_M(a)\subset\mathrm{Ann}_M(ab)$. In this case, if $\mathrm{Ann}_M(ab)=\mathrm{Ann}_M(a)$, then we must have $\mathrm{Ann}_M(b)\subseteq\mathrm{Ann}_M(a)$ which contradicts the fact that $\mathrm{Ann}_M(a)\cap\mathrm{Ann}_M(b)=0$. So $\mathrm{Ann}_M(a)\subset\mathrm{Ann}_M(ab)$ and a similar argument shows that $\mathrm{Ann}_M(b)\subset\mathrm{Ann}_M(ab)$. Hence, $GA(M)$ has the path $[a]-[ab]-[b]$ as a subgraph and so $d([a],[b])=2$.

In the sequel, if $ab\in\mathrm{Ann}_R(M)$, then there exist two vertices $[x],[y]$ of $GA(M)$ such that $\mathrm{Ann}_M(a)\subseteq\mathrm{Ann}_M(x)$ and $\mathrm{Ann}_M(b)\subseteq\mathrm{Ann}_M(y)$ where $\mathrm{Ann}_M(x)$, $\mathrm{Ann}_M(y)\in m-\mathrm{Ass}_R(M)$ and $x,y\notin r(\mathrm{Ann}_R(M))$. Thus by Corollary 2.8, $GA(M)$ has the path $[a]-[x]-[y]-[b]$ as a subgraph. Suppose that $0\neq m\in\mathrm{Ann}_M(x)\cap\mathrm{Ann}_M(y)$. If either $m\in\mathrm{Ann}_M(a)$ or $m\in\mathrm{Ann}_M(b)$, then $GA(M)$ has the path $[a]-[y]-[b]$ or $[a]-[x]-[b]$ as a subgraph, respectively. Now, let $m\notin\mathrm{Ann}_M(a)\cup\mathrm{Ann}_M(b)$. By $abM=0$ it follows that $bM\subset\mathrm{Ann}_M(a)$. Hence $bm\in\mathrm{Ann}_M(a)\cap\mathrm{Ann}_M(y)$ which implies that $GA(M)$ has the path $[a]-[y]-[b]$ as a subgraph. Hence, $d([a],[b])=2$ and therefore $\mathrm{diam}(GA(M))\leq 2$. □

Theorem 2.12. *Let M be a Noetherian R-module and $GA(M)$ be a disconnected graph. Then $\overline{GA(M)}$ is a connected graph and* $\mathrm{diam}(\overline{GA(M)})\leq 2$.

Proof. By Theorem 2.7, there exist two prime annihilator submodules $P_1=\mathrm{Ann}_M(a)$ and $P_2=\mathrm{Ann}_M(b)$ of M such that $P_1\cap P_2=0$. Thus $[a]$ and $[b]$ are adjacent in $\overline{GA(M)}$. Also, P_1 and P_2 are the only maximal elements of $X=\{\mathrm{Ann}_M(d): d\notin\mathrm{Ann}_R(M)\}$ so for every $[c]\in V(GA(M))\setminus\{[a],[b]\}$, we have either $\mathrm{Ann}_M(c)\subset\mathrm{Ann}_M(a)$ or $\mathrm{Ann}_M(c)\subset\mathrm{Ann}_M(b)$. Assume that $[c]$ and $[d]$ are two arbitrary vertices of $\overline{GA(M)}$. If either $\mathrm{Ann}_M(c),\mathrm{Ann}_M(d)\subset P_1$ or $\mathrm{Ann}_M(c),\mathrm{Ann}_M(d)\subset P_2$, then $\overline{GA(M)}$ has the path $[c]-[b]-[d]$ or $[c]-[a]-[d]$ as a subgraph, respectively. Also, if $\mathrm{Ann}_M(c)\subset P_1$ and $\mathrm{Ann}_M(d)\subset P_2$, then $\overline{GA(M)}$ has a cycle $[c]-[b]-[a]-[d]-[c]$ as a subgraph. Hence, $\overline{GA(M)}$ is a connected graph and $\mathrm{diam}(\overline{GA(M)})\leq 2$. □

Theorem 2.13. *Let M be a Noetherian R-module and let $GA(M)$ be a connected graph. Then the following statements are true:*

(i) *If either $|m-\mathrm{Ass}_R(M)|=1$ or $|m-\mathrm{Ass}_R(M)|=2$ and $V(GA(M))\leq 3$, then* $\mathrm{gr}(GA(M))\in\{3,\infty\}$.
(ii) *If either $|m-\mathrm{Ass}_R(M)|\geq 3$ or $|m-\mathrm{Ass}_R(M)|=2$ and $V(GA(M))\geq 4$, then* $\mathrm{gr}(GA(M))=3$.

Proof. If $r(\mathrm{Ann}_R(M))\neq\mathrm{Ann}_R(M)$, then it follows from Theorem 2.1(iii) that $GA(M)$ has a universal vertex such as $[a]$, where $a\in r(\mathrm{Ann}_R(M))\setminus\mathrm{Ann}_R(M)$. Thus $\mathrm{gr}(GA(M))\in\{3,\infty\}$. So in the following we can assume that $r(\mathrm{Ann}_R(M))=\mathrm{Ann}_R(M)$. If $m-\mathrm{Ass}_R(M)=\{P\}$, then $P=\mathrm{Ann}_M(a)$ for some $a\in R\setminus\mathrm{Ann}_R(M)$. So $[a]$ is a universal vertex of $GA(M)$ and the result follows. For $|m-\mathrm{Ass}_R(M)|\geq 3$ the result follows from Corollary 2.8.

Now, suppose that $m-\mathrm{Ass}_R(M) = \{P_1, P_2\}$, where $P_1 = \mathrm{Ann}_M(a), P_2 = \mathrm{Ann}_M(b)$ and $a, b \notin Ann_R(M)$. If $|V(GA(M))| < 4$, then by Corollary 2.8 the proof is obvious. Thus we can assume that $|V(GA(M))| \geq 4$. Let $|V(GA(M))| > 4$. Then there exist two vertices $[x], [y]$ which are adjacent to $[a]$. If $0 = \mathrm{Ann}_M(x) \cap \mathrm{Ann}_M(y)$, then $\mathrm{Ann}_M(x) \cap \mathrm{Ann}_M(y) \subseteq P_2$ so by an argument similar to that of Theorem 2.3 one can show that either $\mathrm{Ann}_M(x) \subset P_2$ or $\mathrm{Ann}_M(y) \subset P_2$. Thus $GA(M)$ has the cycle $[x]-[b]-[a]-[x]$ or $[y]-[b]-[a]-[y]$ as a subgraph, respectively and $\mathrm{gr}(GA(M)) = 3$. If $0 \neq \mathrm{Ann}_M(x) \cap \mathrm{Ann}_M(y)$, then $[a]-[x]-[y]-[a]$ is a cycle in $GA(M)$. Thus the result follows. Finally, assume that $|V(GA(M))| = 4$ and $[x], [y] \in V(GA(M)) \setminus \{[a], [b]\}$. If both of $[x]$ and $[y]$ are adjacent to either $[a]$ or $[b]$, then the proof is similar to the previous procedure. Otherwise, without loss of generality, let $[x]-[a]-[b]-[y]$. Then $\mathrm{Ann}_M(x) \subset \mathrm{Ann}_M(a)$ and $\mathrm{Ann}_M(y) \subset \mathrm{Ann}_M(b)$. Since $\mathrm{Ann}_M(a) \cap \mathrm{Ann}_M(b) \neq 0$, then there exist $0 \neq m \in M$ such that $m \in \mathrm{Ann}_M(a) \cap \mathrm{Ann}_M(b)$. If $m \in \mathrm{Ann}_M(x)$ or $m \in \mathrm{Ann}_M(y)$, then $GA(M)$ has the cycle $[x]-[a]-[b]-[x]$ or $[y]-[b]-[a]-[y]$ as a subgraph, respectively. Now, let $m \notin \mathrm{Ann}_M(x) \cup \mathrm{Ann}_M(y)$. If $xyM = 0$, then we have $yM \subseteq \mathrm{Ann}_M(x)$ and so $ym \in \mathrm{Ann}_M(x)$. It follows that $ym \in \mathrm{Ann}_M(x) \cap \mathrm{Ann}_M(b)$ which implies that $GA(M)$ has the cycle $[x]-[b]-[a]-[x]$ as a subgraph. If $xyM \neq 0$, then $xy \neq 0$ and we have the following two cases, since $|V(GA(M))| = 4$,

Case 1. If $\mathrm{Ann}_M(xy) = \mathrm{Ann}_M(a)$, then $\mathrm{Ann}_M(y) \subseteq \mathrm{Ann}_M(a)$. Thus $[y]-[a]-[b]-[y]$ is a cycle in $GA(M)$. Similarly, if $\mathrm{Ann}_M(xy) = \mathrm{Ann}_M(b)$, then $[x]-[b]-[a]-[x]$ is a cycle in $GA(M)$.

Case 2. If $\mathrm{Ann}_M(xy) = \mathrm{Ann}_M(x)$, then $\mathrm{Ann}_M(y) \subseteq \mathrm{Ann}_M(a)$. Thus $[y]-[a]-[b]-[y]$ is a cycle in $GA(M)$. Similarly, for $\mathrm{Ann}_M(xy) = \mathrm{Ann}_M(y)$, we have $[x]-[b]-[a]-[x]$ is a cycle in $GA(M)$.

Therefore, $gr(GA(M)) = 3$ and the proof is completed. □

Corollary 2.14. *Let M be a Noetherian R-module and $|m-\mathrm{Ass}_R(M)| = 2$. If $GA(M)$ is a connected triangle-free graph with at least three vertices, then $GA(M) \cong K_{1,2}$.*

Proof. Let $\mathrm{Ann}_M(a), \mathrm{Ann}_M(b) \in m - \mathrm{Ass}_R(M)$. Then by hypotheses $a, b \notin r(\mathrm{Ann}_R(M))$ and by the proof of Theorem 2.13, we have $|V(GA(M))| = 3$. Assume that $[c] \in V(GA(M)) \setminus \{[a], [b]\}$. Then it follows that either $\mathrm{Ann}_M(c) \subset \mathrm{Ann}_M(a)$ or $\mathrm{Ann}_M(c) \subset \mathrm{Ann}_M(b)$ since $GA(M)$ is triangle-free. Thus $GA(M)$ is either $[b]-[a]-[c]$ or $[a]-[b]-[c]$, we are done. □

3. MODULES WHOSE $GA(M)$ IS COMPLETE OR STAR

In this section, R is a commutative ring and M is an R-module. The subset

$$T(M) = \left\{m \in M \ : \ rm = 0 \text{ for some } 0 \neq r \in R\right\}$$

of M is called the torsion subset of M.

Theorem 3.1. *Let M be a Noetherian R-module. Then $GA(M)$ is a complete graph if and only if $Z_R(M)$ is a prime ideal of R.*

Proof. First, let $GA(M)$ is a complete graph and $a, b \in Z_R(M)$. If $\mathrm{Ann}_M(a) = \mathrm{Ann}_M(b)$, then $a+b \in Z_R(M)$ and the proof is completed. Otherwise, there is an edge between $[a]$ and $[b]$, since $GA(M)$ is a complete graph, thus $\mathrm{Ann}_M(a) \cap \mathrm{Ann}_M(b) \neq 0$. Hence, there exists $0 \neq m \in M$ such that $am = bm = 0$. Therefore, $(a+b)m = 0$ and so $a+b \in Z_R(M)$ which complete the proof.

Conversely, suppose that $[a], [b]$ are two disjoint arbitrary vertices of $GA(M)$ and suppose that $Z_R(M)$ is an ideal of R. Let $I = \mathrm{Ann}_R(M)$. It is easy to see that R/I is a Noetherian ring and M is Noetherian R/I-module. By [12, Corollary 9.36], $Z_{R/I}(M) = \cup_{\mathfrak{P} \in Ass_{R/I}(M)} \mathfrak{P}$. It follows from The Prime Avoidance Theorem, [12, Theorem 3.61], that $Z_{R/I}(M) = \mathrm{Ann}_{R/I}(m)$, for some $0 \neq m \in M$. We have $a+I, b+I \in Z^*_{R/I}(M)$ so that $am = bm = 0$. Thus $m \in \mathrm{Ann}_M(a) \cap \mathrm{Ann}_M(b)$. Hence, $GA(M)$ is a complete graph. □

Theorem 3.2. *Let M be a Noetherian R-module such that $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$ and $|m - \mathrm{Ass}_R(M)| = 1$. If $GA(M)$ is a star graph with at least three vertices, then $r(\mathrm{Ann}_R(M))$ is not a prime ideal of R and*

$$|V(GA(M))| = |\operatorname{Min}\operatorname{Ass}_R(M)| + 1.$$

Proof. Suppose that $GA(M)$ is a star graph and $a \in R \backslash \mathrm{Ann}_R(M)$ and $P = \mathrm{Ann}_M(a) \in m - \mathrm{Ass}_R(M)$. By [9, Proposition 3.2], each maximal element of $X = \{\mathrm{Ann}_M(d) : d \notin \mathrm{Ann}_R(M)\}$ is a prime submodule of M which implies that $\mathrm{Ann}_M(b) \subseteq P$, for any $b \in Z_R(M) \setminus \mathrm{Ann}_R(M)$. Thus $[a]$ is the only universal vertex of $GA(M)$. Let $[b], [c]$ be two end-vertices of $GA(M)$. Then by Theorem 2.2(iii), $b, c \notin r(\mathrm{Ann}_R(M))$. We show that $bc \in r(\mathrm{Ann}_R(M))$ and so $r(\mathrm{Ann}_R(M))$ is not a prime ideal of R. If $bc \in \mathrm{Ann}_R(M)$ we are done. Now, assume that $bc \in Z_R(M) \setminus \mathrm{Ann}_R(M)$. If $\mathrm{Ann}_M(b) = \mathrm{Ann}_M(bc)$, then two vertices $[b]$ and $[c]$ are adjacent in $GA(M)$ which contradicts the fact that $GA(M)$ is a star graph. Thus $\mathrm{Ann}_M(b) \neq \mathrm{Ann}_M(bc)$, a similar argument shows that $\mathrm{Ann}_M(c) \neq \mathrm{Ann}_M(bc)$. Hence, $[b] - [bc] - [c]$ is a path in $GA(M)$ which implies that $\mathrm{Ann}_M(a) = \mathrm{Ann}_M(bc)$. Therefore, the previous paragraph shows $\mathrm{Ann}_M(d) \subseteq \mathrm{Ann}_M(bc)$, for any $d \in Z_R(M) \setminus \mathrm{Ann}_R(M)$. Now, it is easy to see that for all $\mathfrak{p} \in \mathrm{Ass}_R(M)$ we have $\mathrm{Ann}_R(M) \subset \mathfrak{p}$ and $bc \in \mathfrak{p}$. Thus $bc \in r(\mathrm{Ann}_R(M))$.

Let $[b]$ be an end-vertex of $GA(M)$. We show that $[b]$ lies in a minimal element of $\mathrm{Ass}_R(M)$. By hypotheses $b \in Z_R(M) \setminus r(\mathrm{Ann}_R(M))$ and an argument similar to that of Theorem 3.1 shows that $Z_R(M) = \cup_{\mathfrak{p} \in Ass_R(M)} \mathfrak{p}$. So that $b \in \mathfrak{p}$ for some $\mathfrak{p} \in Ass_R(M)$. Suppose that $\mathfrak{p} = \mathrm{Ann}_R(m)$, where $m \in M$. If $\mathfrak{p}$ is a minimal element of $\mathrm{Ass}_R(M)$ we are done. Otherwise, there exists a minimal element $\mathfrak{q}$ of $\mathrm{Ass}_R(M)$ such that $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{q} = \mathrm{Ann}_R(m')$, for some $m' \in M$. Again, if $b \in \mathfrak{q}$ we are done. Assume that $b \notin \mathfrak{q}$. By the previous paragraph $r(\mathrm{Ann}_R(M)) \subset \mathfrak{q}$. Assume that $c \in \mathfrak{q} \setminus r(\mathrm{Ann}_R(M))$. Thus $c \neq b$. Now, $\mathrm{Ann}_M(b) \neq \mathrm{Ann}_M(c)$ since $cm' = 0$ but $bm' \neq 0$ and $m \in \mathrm{Ann}_M(b) \cap \mathrm{Ann}_M(c)$. This is a contradiction with the fact that $GA(M)$ is a star graph. Thus every end-vertices of $GA(M)$ lies in a minimal element of $\mathrm{Ass}_R(M)$. If $b, c \in \mathfrak{p} = \mathrm{Ann}_R(m)$, for two end-vertices $[b], [c]$ of $GA(M)$, then $m \in \mathrm{Ann}_M(b) \cap \mathrm{Ann}_M(c)$ which is a contradiction. Hence, any end-vertex is located in a unique minimal associated prime ideal of R. Therefore,

$$|V(GA(M))| \leq |\operatorname{Min}\operatorname{Ass}_R(M)| + 1.$$

By hypotheses and the previous paragraph $|\operatorname{Min}\operatorname{Ass}_R(M)| \geq 2$. Let $\mathfrak{p} \in \operatorname{Min}\operatorname{Ass}_R(M)$. Then $\mathfrak{p} \nsubseteq r(\operatorname{Ann}_R(M))$. Assume that $b \in \mathfrak{p} \setminus r(\operatorname{Ann}_R(M))$ thus $[b]$ is a end-vertices of $GA(M)$ and so

$$|V(GA(M))| \geq |\operatorname{Min}\operatorname{Ass}_R(M))| + 1. \qquad \square$$

A graph G is planar if it has a drawing without crossings. In a graph G subdivision of an edge with endpoints u and v is a operation of replacing the edge $u - v$ with the path $u - w - v$, through a new vertex w. The subdivision of a graph G is a graph obtained from G by subdividing some of the edges of G.

Theorem 3.3. *Let R_1, R_2, R_3 be commutative rings and $R = R_1 \times R_2 \times R_3$ and let $M = M_1 \times M_2 \times M_3$, where M_i is an R_i-module, for $i = 1, 2, 3$. Then $GA(M)$ is a planar graph if and only if $Z_{R_i}(M_i) = \operatorname{Ann}_{R_i}(M_i)$, for $i = 1, 2, 3$.*

Proof. First, we assume that $Z_{R_i}(M_i) = \operatorname{Ann}_{R_i}(M_i)$, for all $1 \leq i \leq 3$. Then

$$\begin{aligned} V(GA(M)) = \{&x_1 = [(0,0,1)], x_2 = [(0,1,0)], x_3 = [(1,0,0)], \\ &x_4 = [(1,1,0)], x_5 = [(1,0,1)], x_6 = [(0,1,1)]\}. \end{aligned}$$

Here, it is easy to check that the graph $GA(M)$ is isomorphic to the Figure 1 which contains no subdivision of K_5 and $K_{3,3}$. Thus by [14, Theorem 6.2.2], $GA(M)$ is a planar graph.

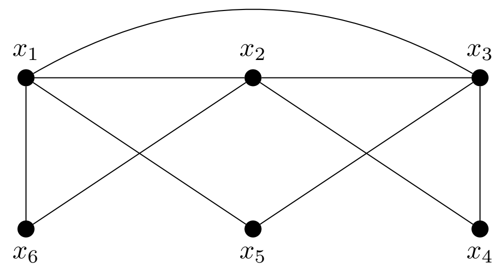

Fig. 1. A graph with no subdivision of K_5 and $K_{3,3}$

Conversely, suppose that the graph $GA(M)$ is planar. On the contrary and with no loss of generality, let $Z_{R_1}(M_1) \neq \operatorname{Ann}_{R_1}(M_1)$. Then there exists $r \in R_1 \setminus \{0, 1\}$ such that $0 \neq \operatorname{Ann}_{M_1}(r) \subset M$. It is easy to check that the vertex set

$$\begin{aligned} \mathcal{F} = \{&x_1 = [(1,1,0)], x_2 = [(1,0,0)], x_3 = [(0,0,1)], x_4 = [(1,0,1)], \\ &x_5 = [(r,0,1)], x_6 = [(r,1,0)], x_7 = [(r,0,0)], x_8 = [(0,1,0)]\} \end{aligned}$$

induces a subgraph of $GA(M)$ which contains a subdivision of $K_{3,3}$, see Figure 2 for more details. By [14, Theorem 6.2.2], $GA(M)$ is not planar which is a contradiction. So $Z_{R_i}(M_i) = \operatorname{Ann}_{R_i}(M_i)$, for all $1 \leq i \leq 3$.

$\square$

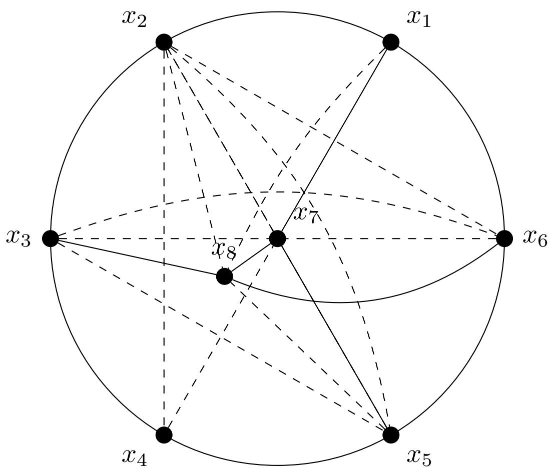

Fig. 2. An induced subgraph of $GA(M)$ whose bold lines form a subdivision of $K_{3,3}$

Theorem 3.4. *Let $R_1, \ldots, R_n$ $(n \geq 4)$, be commutative rings and $R = R_1 \times \ldots \times R_n$ and let $M = M_1 \times \ldots \times M_n$, where M_i is an R_i-module, for all $1 \leq i \leq n$. Then $GA(M)$ is not a planar graph.*

Proof. It is sufficient to show that $GA(M)$ is not a planar graph for $n = 4$. Consider the vertex set

$$\mathcal{F} = \{x_1 = [(1,1,0,0)], x_2 = [(0,0,1,0)], x_3 = [(1,1,1,0)], x_4 = [(0,1,0,0)], x_5 = [(1,0,0,0)]\}.$$

It is easy to see that $\mathrm{Ann}_M(x_i) \neq \mathrm{Ann}_M(x_j)$ and $\mathrm{Ann}_M(x_i) \cap \mathrm{Ann}_M(x_j) \neq 0$, for all i, j with $i \neq j$ and $1 \leq i, j \leq 5$. Thus the set $\mathcal{F}$ induces a complete subgraph of $GA(M)$. Hence, by [14, Theorem 6.2.2] the result follows. □

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