Student Solutions of Non-traditional Geometry Tasks

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Abstract **– The article deals with student solutions of non-traditional geometry tasks focused on triangle constructions. The research conducted concerns the solutions proposed by the students of Mathematics Teaching in combination with another subject at the University of Tirana in Tirana, Albania, and Constantine the Philosopher University in Nitra, Slovakia. The second group consisted of technical education students in the field of Software Engineering at the Canadian Institute of Technology in Tirana, Albania. In this comparison, we focused on the success of the solution method used as well as comparison of the solution methods of the individual groups of students in terms of their success.**

Keywords **– non-traditional geometry tasks, algebraic method, Euclidean geometry, university research.**

1. Introduction

Geometry belongs to the oldest parts of mathematics. The beginnings of geometry have been linked to practical needs, such as land surveying, building construction and art. Application of geometric knowledge into practice can be seen from records from ancient civilizations, such as Babylon, Egypt, India, and many other significant cultures of those times. However, the systematic building of the foundations of geometry can only be identified in the times of ancient Greece, where mathematics is closely related to philosophy, i.e. to seek the truth. Therefore, a good bit of perceptible knowledge, mainly from synthetic geometry, bears names identical with the philosophical schools of ancient Greece [3].

Among the first philosophers who contributed to the development of mathematics was Thales of Miletus. He is mentioned in the list of seven sages, which was compiled around 582 BC under the archonship of Damasias. Thales also deserved this place because he travelled about the well-known world and gathered distinguishable knowledge that then influenced his philosophy. He gained his earnestness, among other things, by knowing how to calculate the solar eclipse. This points to his wide-ranging knowledge of geometry, philosophy and astronomy [3].

Another important philosophical school is the Pythagorean School, which blends philosophy with mathematics. In addition to the contribution to geometry, it is also significant by the discovery of irrational numbers that was contrary to their philosophy and thus did not admit their existence.

There are still many philosophical schools in the history of Greek mathematics and philosophy that have contributed to geometry. Finally, we shall mention Euclid of Alexandria. Euclid of Alexandria is known for his work *Elements*, in which he collected and systematically organized all the geometry knowledge. This work is so significant that it has been the only textbook of geometry for several centuries. Because of the significance of the work itself and the contribution of Euclid to geometry by theoretical knowledge, even today flat surface geometry is called Euclidean geometry as well as the most widespread method of ruler-and-compass construction which is called Euclidean construction. A more detailed development of geometry can be found in the literature [3], [10].

Despite the tremendous progress in geometry, some construction tasks have remained unresolved for a long time in geometry history. The most

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prominent of them was the construction of regular n-sided polygons, angle trisection, duplicity of the cube, circular quadrant and many other tasks. The question of the solvability of geometry tasks was approached by Descartes in the Geometry essay. His contribution represented a change in the understanding of the powers of numbers in geometry, as well as the access to the geometric construction in terms of the solution of algebraic equations. Until Descartes' times, understanding the powers was in the matter of increasing the size, which was also related to the names of the powers, for instance, a^2 was read as "square *a*", the third power called *cubus* spoke clearly about the cube, which meant that the volume was involved. Descartes used Euclid's axioms and triangle similarities, so the powers were not seen as the increase of dimensions, in result of which the imagination itself in the construction would fail too. Descartes looked at the powers as modification of proportion arising from the similarity of triangles. Let us note that the very concept of proportion has been known and thoroughly processed since ancient Greece, as evidenced by Plato's *Timaeus* [15]. Another important step for solving geometry tasks is that he considered construction as a specific structure now known as the *algebraic field*. It treats arithmetic consisting of summation, subtraction, multiplication, division, and extraction-root operations, while the latter one was taken as a sort of division. Using this reasoning, he has largely contributed to algebraization of geometry and calculation methods in geometry [6].

Furthermore, we must realize that there had been various challenging tasks in the history that had to be resolved with the required accuracy. These tasks occurred predominantly due to the necessity of technical practice. For this reason, the approximate construction theory for technical purposes has been processed, but normally students do not meet with these methods in the common teaching process.

Approximate solutions are known in numerical mathematics, but geometry in school mathematics keeps its place as an accurate science. Many students in the teaching field of mathematics are familiar with historical tasks which are non-solvable by the Euclidean construction, but their familiarity with such tasks ends herewith. On the other hand, in algebra courses it is not only common to get familiar with higher-degree equations of degree more than five, but students also learn the approximate methods, by which these equations are solved. The teaching of geometry at the school level is focused on these exact constructions, which implies that the tasks are ordered thematically, which also determines the method of solution.

In this article, we focus on the non-traditional geometry tasks that lead students to an approximate construction, because of their complexity, but also because students do not have learned procedures for the type of the tasks considered.

2. Geometric constructions

The solution of a geometry task is essentially the construction of geometric figures. This means that, in conformity with the valid axioms, it is possible to derive or add elements, if necessary, so that a designated figure is constructed [11]. From the historical point of view, Euclidean constructions were thus introduced. In Euclidean constructions, there is a ruler without a scale and a compass that serve as aids. Hence, we use the appropriate combination of the ruler and the compass to solve the task.

Basic Euclidean constructions consist of the following steps:

1. A point is considered to be constructed if its position is given or is intersected by two lines, two circles, or a line and a circle

2. A line is considered to be constructed if at least two of its points are specified

3. A circle $k(S;r)$ is considered to have been constructed if a point S *and a line segment* r *are given*

4. If two different intersecting lines a, b *are given, then we consider their intersection to be constructed*

5. If a circle k *and its secant* m *are given, then we consider their intersections $X \neq Y$ to be constructed*

6. If two circles k_1, k_2 are given, of which we know that they intersect, then we consider their intersections $X \neq Y$ to be constructed [12]

Basic methods of construction tasks

As we described the historical evolution of geometry at the beginning, the ways in which many of the challenging tasks were solved were crystallized into steady-state methods, which have their specific way of solving and thinking [11].

1. A set of points of given properties
2. Geometric projection method
3. Algebraic geometry method
4. Analytic geometry method

The first two methods use geometric thinking and the other two methods are focused on algebraic-computational thinking.

A set of points of given properties

The metric properties of geometric figures are mainly used as follows:

1. Circle: a set of points equally spaced from a specified point (the centre of circle)
2. Line segment axis: a set of points equally spaced from the line segment edges
3. Angle axis: a set of points equally spaced from the angle arms
4. Axis of a segment formed by two parallel lines: a set of points equally spaced from two parallel lines
5. Thales' circle: a set of vertices of all impedance triangles whose hypotenuse is the diameter of the circle
6. A set of points with the same angle

Geometric projection method
To solve tasks using this method, from analyzing the draft we look for symmetry or other geometric projections resulting from the considered solution properties.

Algebraic geometry method
By using known formulas and relationships, we determine an equation whose solution is an unknown parameter that we can then construct geometrically and thus solve the task.

Analytic geometry method
In the draft and analysis, we use the coordinate system, to which the solution is bound. This method belongs to the calculation methods and geometric objects change to algebraic objects, which can facilitate manipulation with them and thus offer a solution.

3. The course of research

In the research, we focused on the students' ability to solve a non-traditional geometry task. In the first step, we created a representative sample of software engineering students and mathematics teaching students, with research being conducted at three universities: the University of Tirana and the Canadian Institute of Technology, both in Tirana, Albania, and Constantine the Philosopher University in Nitra, Slovakia, from April to May 2018. For the students of mathematics teaching, we prepared two tasks. The first task was for everyone focusing on a triangle construction if we know all of its three altitudes. This task is Euclidally solvable, assuming it belongs to non-traditional tasks.

Assignment 1: *Propose a procedure to construct a triangle if you know its altitudes* (solution with altitudes)

Algebraic geometry solution

Working with the algebraic geometry method gives us an expression whose geometrical design is a guide to converting the task into a triangle construction by way of all three sides, or a construction with one known side design and two altitudes. Geometrically, we can construct structures for the sum and the difference of line segments, divide a line segment in a given ratio, and multiply line segments. By using Euclid's postulates and the Pythagorean theorem, we can construct the root of algebraic expressions. When solving construction problems by algebraic geometry method, we use an equation based on known relationships and a subsequent geometric construction of the algebraic expression, which represents the root of the equation formulated. By combining Heron's formula $S = \sqrt{s(s-a)(s-b)(s-c)}$, $s = \frac{a+b+c}{2}$ and the formula to calculate the area of the triangle by means of a side and an altitude to the side $S = \frac{av_a}{2}$, $S = \frac{bv_b}{2}$, $S = \frac{cv_c}{2}$ we get an algebraic expression that defines the procedure for constructing the desired sides:

$$a = \frac{v_a\sqrt{2}}{\sqrt{\left(1+\frac{v_b}{v_a}+\frac{v_c}{v_a}\right)\left(1+\frac{v_b}{v_a}-\frac{v_c}{v_a}\right)\left(1+\frac{v_c}{v_a}-\frac{v_b}{v_a}\right)\left(\frac{v_b}{v_a}+\frac{v_c}{v_a}-1\right)}}$$

The resulting relationship for calculation of the side is directly the procedure for the construction of the triangle at known altitudes, while we can see that the expression consists of operations we can geometrically construct. The side construction is thus only one of many tasks that are designed to construct algebraic expressions, while it is necessary to specify the size of the unit line segment and the height sizes. Although the construction is Euclidean, it is composed of a number of partial structures, which make the construction more demanding. On the other hand, these partial constructions are simple and clearly defined by the rule of the equation.

Construction using a set of points of given properties

By way of the set of points of given properties method, we use a geometric object that meets a part of the properties that characterize the point we search for. As an example, we can specify a circle, which is a geometric object, for which all its points are equally distant from the centre. Similarly, the curves that we can only construct approximately by

Euclidean construction are also worthwhile. These include, for example, a parabola as a set of points having the same distance from a specified line and a selected point.

The task we are addressing is one of the tasks with two unknown points. We start with the situation that we freely choose a fixed point A and the remaining two points are bound by the condition based on the assignment. The altitude in the triangle is commonly interpreted as the distance of the point from the base. We will use this property to determine the altitude v_c by specifying the line p on which the point C is located, and at the same time determining the line on which the points A and B lie. Subsequently, the point C is determined by the properties resulting from the use of two altitudes. In this case, we have a fixed point A and a line q_c on which the points A and B_1 lie. The point A is invariant, and the point B_1 is selected. In this step, we interpret the height to the side as the distance of the line from the selected point. In this case, we create circles $k_a(A; v_a)$ and $k_b(B; v_b)$. Next, we determine the line q_a from the point B_1 as a tangent to the circle k_a and the line q_b from the point B_1 as a tangent to the circle k_b. We get the point C_i as the intersection of the lines q_a and q_b. The point C_i depends on the point B_1. It means that by moving the point B_1 linearly along the line q_c we get a mechanical curve $c(B_1)$. This mechanical curve meets the conditions for the points determined by the parameters v_a and v_b. The searched point C is found as the intersection of the line p and the curve $c(B_1)$. To find the point C, we can choose different methods that are focused on the approximate construction. The first option is the point construction of the curve $c(B_1)$. The second option is to approximate the curve $c(B_1)$ in the vicinity of the predicted position of the point C by means of a circular arc, which is determined by the three points of the curve determined by the point construction of the curve. Another option is to construct the curve using ICT. Here we can use, for example, the "set of points" option in the GeoGebra programme. For this, we just draw one point of the curve $c(B_1)$ and determine the point, on which the point position on the curve depends. Now, that we already know the point C, from this point we trace the tangent q_b to the circle k_a. The point B is defined as the intersection of the line q_A with the line q_c. By connecting the points we find, we get ΔABC.

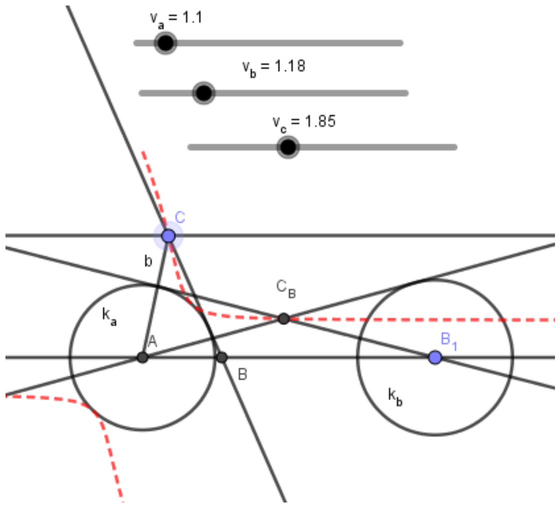

Figure 1: Triangle construction by means of a curve in the GeoGebra 5 programme. We set up the point $\boldsymbol{C_B}$ by which we determined the curve $\boldsymbol{c(B_1)}$. The cross point of the curve $\boldsymbol{c(B_1)}$ with the line $\boldsymbol{p}$ determined the point $\boldsymbol{C}$. From the point $\boldsymbol{C}$ we traced the tangent $\boldsymbol{b}$ to the circle $\boldsymbol{k_a}$. Finally, we determined the point $\boldsymbol{B}$ as the intersection point of the lines $\boldsymbol{q_c}$ and $\boldsymbol{b}$.

The use of the "set of points of given properties" method offers the opportunity to build dynamic thinking in pupils. In this construction, it means creating the perception of how the triangle changes if one parameter is left free and two parameters are fixed. The method has the disadvantage that the searched curve cannot be constructed using a compass and a ruler. On the other hand, it can be used to incorporate solutions to geometric tasks using ICT, or to develop approximation procedures.

Construction using the geometric projection method

The geometric projection method uses the option of an easy-to-build design. By using a suitable geometric projection, the design we get is the object we search for. Congruent and similar projections are therefore used in school geometry. The author [14] solves the task on the basis of homothety. Just like in the algebraic geometry approach, it follows from the formula for the triangle area, which yields the triple ratio $a: b: c = v_c: v_b: v_a$.

Based on this, it creates a triangle $\Delta A'B'C'$which we can easily construct from the data $c' = v_a$, $a' = v_b$, $b' = v_c$. Then, we use homothety the way that we select $A' = A$ and $|AV_a| = v_a$ to be the homothetic centre.

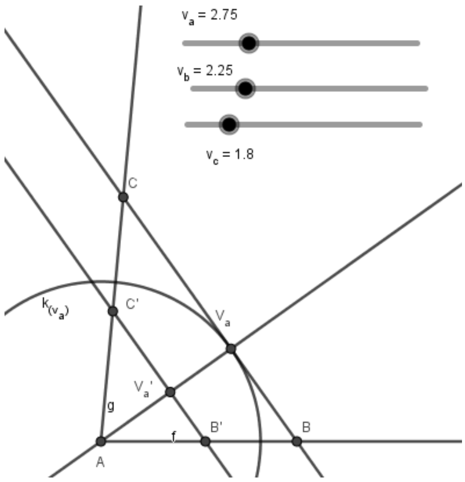

Figure 2: Triangle construction determined by three altitudes using the geometric projection method. In homothety, we determined the point A as the homothetic centre and therefore $\boldsymbol{A'} = \boldsymbol{A}$*. Subsequently, we projected* $\boldsymbol{V_{a'}}$ *into* $\boldsymbol{V_a}$*, by which we were then able to construct the searched* $\Delta \boldsymbol{ABC}$*.*

The second task was to construct a triangle if we know its two sides and the radius of its inscribed circle. This task leads to the formulation of a cubic equation, and therefore its general solution uses non-Euclidean construction methods. As a result, this can also be considered an untraditional task.

Assignment 2: *Propose a procedure for a triangle construction if you know the size of the sides* a, b *and the radius of the inscribed circle* ρ (solution with an inscribed circle)

Algebraic geometry solution method

In the work [2], the solution is focused on formulating a 3rd degree algebraic equation in the following form:

$$c^3 - (a+b)c^3 + (4\rho^2 - (a-b)^2)c + \\ +(a+b)(4\rho^2 + (a-b)^2) = 0$$

Set of points of given properties solution method

In the work [13], the solution is proposed based on a set of points of given properties. In the first step, we start with a line segment $|BC| = a$ and a line m, which is parallel to $|BC|$, with the condition $|mBC| = \rho$ being met. On the line m we place the circle $k(S;\rho)$, $S \in m$. From the points B, C we lead tangent lines $B \in p, C \in q$ to the circle k. Then we get $A' \in p \cap q$. By moving the circle k, we obtain a curve l that satisfies the properties of triangles with a fixed side b and a radius of an inscribed circle ρ. Then, to complete the searched triangle, we determine the side b by using $k_b(C; b)$. Then, $A \in k_b \cap l$. We get 3 solutions, while 2 solutions are in the upper half-plane, and the 3rd solution is in the lower half-plane, while the lines meet the condition of the tangent, but the circle is not inscribed in the developed solution, which corresponds to the negative solution of the algebraic equation.

4. Results and evaluation

The students' solutions have been analyzed according to the chosen method of solution and success. We have defined two methods for solving the task, namely geometric (a set of points of given properties, geometric projection method) and algebraic (algebraic geometry method, analytic geometry). Success has been in the range of 0 (no continuation) to 5 (complete task solving). Subsequently, we dealt with the students' solutions to see how far they came by using the method they applied, and also to know which method they were able to use.

In the first task, we compared whether the technical education students preferred to solve the task using the algebraic method rather than the students of the teaching branch of study. There were 14 technical education students, while the algebraic method was used by three students. So, the ratio of the algebraic solution to the geometric is 3/11. We compared this group with teaching-oriented students, where from 20 students, two students solved the task using the algebraic method. This gives us the ratio of solutions for the teaching students of the algebraic method in regard to the geometric 2/18.

To verify this assumption, we used a statistical test 2-sample test for equality of proportions with continuity correction in the R language. The result of the two-ratio conformity test shows that the ratios of the individual solutions (algebraic vs. geometric) are not statistically significantly different (`X-squared(df)=0.37`, `df=1`, `p=0.541`), while the 95% confidence interval for the difference of these ratios is (`-0,21; 0,54`).

Although the students in technical branches of study use multiple methods of computational mathematics, which we assume that reflects on the method of solution, it turned out that the study itself did not affect the method of solution so strongly.

We can also compare this result with the history of geometric tasks solving, which we have already partially described. We noticed that there was no significant suppression of the method, but only a change of progress was made once in the algebraic (computational) method and once in the geometric method.

We then asked a research question about which method was more successful. For the purpose of testing, we used here testing the `main effects` in the R language:

```
res1<-aov(Score~Group+Solution,data=altitude2)
              Df Sum Sq Mean Sq F value Pr(>F)
Group          1   0.17  0.1681   0.091  0.765
Solution       1   1.03  1.0322   0.560  0.460
Residuals     31  57.18  1.8446
```

5. Conclusion

From the above results, we can see that the influence of the solution method on the success of the solution has not been demonstrated. We have assumed success in favour of the algebraic method, for historical reasons, because many of the tasks have been successfully solved in history because of algebraic methods as seen in constructing regular polygons in the work [9]. Since there is a solution in both the algebraic and geometric methods for the tasks used, the task does not rule out in principle any method. Therefore, the success of the solution depends on how well the method is applied, but also on the preference of the type of thinking as referred to in the article [1], in which the task of algebra determines the preferred style of student thinking based on the method used. The difference that occurs in each type of thought is also found in the work [8], where the author points to intuition and clarity that are essential to geometric thinking and geometric methods. He also points to the benefits of intuition and clarity that move forward in solving, but can also lead us to logical mistakes or inaccuracies. On the other hand, algebraic thinking is more about logical processes that are slower “short-sighted“), meaning that the conclusions are without logical controversy, but this thinking has the problem that the lack of intuition extinguishes the possibility of other logical processes with known information. The statistical analysis of the effectiveness of selected methods in mathematics teaching is dealt with, e.g. in the article [7]. For the procedures in algebraic methods, several studies point to the need to look for the ways of teaching mathematics that lead to divert from learning solution procedures by heart, by e.g. incorporating tasks solved by the substitution method [4], [5].

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