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# Exact Solution of Unsteady Tank Drainage for Ellis Fluid

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## ABSTRACT

In this work, we investigate the the problem of an unsteady tank drainage while considering an isothermal and incompressible Ellis fluid. Exact solution is gotten for a resulting non-linear PDE (partial differential equation)-subject to proper boundary conditions-. The special cases such as Newtonian, Power law, and as well as Bingham solution are retrieved from this suggested model of Ellis fluid. Expressions for velocity profile, shear stress on the pipe, volume flux, average velocity, and the relationship between the time vary with the depth of a tank and the time required for complete drainage are obtained. Impacts of different developing parameters on velocity profile $v_z$ and depth $H(t)$ are illustrated graphically. The analogy of the Ellis, power law, Newtonian, and Bingham Plastic fluids for the relation of depth with respect to time, unfold that the tank can be empty faster for Ellis fluid as compared to its special cases.

**Keywords:** Tank drainage, Ellis fluid; Exact solution.

## NOMENCLATURE

| | |
|---|---|
| $\mathbf{A_1}$ | first Rivlin–Ericksen tensr |
| $d$ | diameter of the pipe |
| $\frac{D}{Dt}$ | material derivative |
| $\mathbf{f}$ | body force |
| $g$ | gravitational acceleration |
| $H(t)$ | depth of fluid in the tank at any time |
| $H_0$ | initial depth of the fluid |
| $L$ | length of the pipe |
| $n$ | power law index |
| $p$ | dynamic pressure |
| $p_1$ | pressure inlet |
| $p_2$ | pressure outlet |
| $Q$ | flow rate |
| $R$ | radius of the pipe |
| $R_T$ | radius of the tank |
| $r_1$ | yield radius in circular pipe flow |
| $r$ | Radial coordinate |
| $\mathbf{S}$ | extra stress tensor |
| $S_0$ | yield stress |
| $S_{\frac{1}{2}}$ | material constant |
| $t$ | time |
| $\mathbf{V}$ | velocity vector |
| $\bar{V}$ | average velocity |
| $z$ | axial coordinate |

**Greek letters**

| | |
|---|---|
| $\alpha$ | material constant |
| $\eta$ | scalar quantity |
| $\eta_0$ | material constant |
| $\rho$ | constant density |

## 1. INTRODUCTION

In recent years, non-Newtonian fluids have remained the focus of modern research due to their numerous biological, industrial and technological applications such as tooth paste, drilling mud, greases, paints, blood, polymer melts, clay coatings etc. Non-Newtonian fluid is an expansive class of fluids; so, unlike Navier-Stokes equation for Newtonian fluids, there is not a single model that can delineate all the properties of non-Newtonian fluids (Chhabra & Richardson, 1999). Hence, several constitutive equations (e.g., Power law fluid model, second order fluid model, third order fluid model, Sisco fluid model, eyring fluid model, and Phan-Thien-Tanner fluid model) have been proposed to predict the physical structure and behavior of various types of non-Newtonian fluids (Memon *et al.*, 2014; Dunn & Rajagopal, 1995).

For such models, the exact solutions are rare to be obtained for the equations of motion specially for non-Newtonian fluids, because of the nonlinear nature of those equations (Farooq, Rahim, Islam, &

Siddiqui, 2013; Farooq, Rahim, Islam, & Arif, 2014); a few number of exact solutions are found in the existing review of literature. Specially when the cylindrical coordinates are used these types of solutions come even to be rare, for the reason that of the nonlinearity in the higher order viscosity part and in inertial part (Farooq *et al*., 2014). Numerical results of the differential equations, no problem how correct they are, still have not been the exact solutions for the reason that of the parameters involved in equation, which should have to be given values for each result.

There are certain factors and reasons for which the exact solutions are considered to be important (Wang, 1991; Wang, 1989; Benharbit & Siddiqui, 1992; Siddiqui & Kaloni, 1986; Rajagopal & Gupta, 1981; Wang, 1966), such as;

(*i*). these (exact solutions) signify basic fluid-dynamic flows. Consequently, there is possibility, for the fundamental episodes defined by the Navier-Stokes equations, to be examined more thoroughly due to the uniform rationality of exact solutions.

(*ii*). these solutions (Exact) help in scrutinizing phase for empirical, numerical, experimental, and asymptotic methods. Even though, the integration of the equations of motion is completely made feasible through computer techniques, whereas the accuracy of the outcomes is possible to be established by comparing them with an exact solution.

A detailed and exceptional review of exact solutions of the Navier-Stokes equation is provided by Wang (Wang, 1991).

Study of tank drainage flow has received significant attention due to the focus upon the practical applications of these flows in the contemporary sciences. Since the formulation of these types of flows, there have been many research attempts for their analysis. The Newtonian fluid has been used for tank drainage flow by (Papanastasiou, 1994) and power law fluid by (Bird, Stewart and Lightfoot, 1960) to investigate and solve the problem exactly.

The theory describing the efflux time of a tank has been derived by Crosby (Crosby, 1961)and by Bird, Stewart, and Lightfoot (Bird, Stewart and Lightfoot, 1960), and further extended to systems with the installed fittings by Hanesian (Hanesian, 1984). It is a founding fact that, when the tank is drained by a hole, Torricelli's equation is used to describe the discharge velocity and flow rate which is given in (De Nevers, 2005; Bird, Stewart and Lightfoot, 1960).These problems are revisited in (Joye & Barrett, 2003). Under conditions of turbulent flow in the exit pipe, relationship between the efflux time and height of the liquid to the bottom of the exit pipe is calculated by (Wilkes, 2006), further the Mechanics of the slow draining for a large tank under gravity is briefly explained in (Van Dongen & Roche, 1999). Unsteady draining flows from a rectangular tank (two dimensional and two layered) is given by (Forbes & Hocking, 2007), and for circular tanks a three dimensional draining flow for two-fluid system is studied by (Forbes & Hocking, 2010). Efflux Time and comparison of a cylindrical tank with differential form is given in (Subbarao, 2011; Devi, Singh, Reddy, Dharwal & Subbarao, 2011; Reddy & Subbarao, 2011; Devi, Padma & Subbarao, 2011). Slow draining of large spherical tank under the action of gravity is studied by (Subbarao, Rao, Raju & Prasad, 2012), in which mathematical and experimental values have been compared, and found to be in good agreement with the model. Usage of polymer solutions for drag reduction in gravity driven flow systems is given in (Subbarao, Madhavi, Naidu & King , 2013; Subbarao, Yadav & King, 2013), and exact solution of tank drainage for Newtonian fluid with slip condition have been solved by Memon, Siddiqui, & Shah, 2017.

In this manuscript, we studied tank drainage problem of Ellis fluid (Bird, Stewart and Lightfoot, 2002; Afanasiev, Münch, & Wagner, 2007; Ali, Abbasi, & Ahmad, 2015), resulting non-linear partial differential equations-subject to boundary conditions-are acquired analytically with exact solutions and also we have retrieved the special cases, such as Newtonian, Power law and Bingham plastic fluid. For the very high yield stress $\left(S_{\frac{1}{2}} \rightarrow \infty\right)$ solution of the problem retrieve for Newtonian case (Papanastasiou, 1994), on replacement$\left(S_{\frac{1}{2}} = \eta_0,\ \alpha = \frac{1}{n}\right)$, we get the solutions for Power law case (Bird, Stewart and Lightfoot, 2002) and when we substitute ($\alpha = 0$) we get the solution for Bingham plastic fluid. Subsequently, expressions for velocity profile, shear stress on pipe, flow rate, average velocity, depth of fluid in the tank, and the time required for complete drainage are obtained. As per the best of our insight, the solution of the problem has not been accounted for in the existing literature.

In section 2 of this paper, the governing equations of Ellis fluid model are specified. Section 3 provides formulation and solutions of the tank drainage problem, such as velocity profile, shear stress on the pipe, flow rate, average velocity, shear stress on the pipe, relationship between the variation in time and in the depth of fluid of the tank, and also between the time required for complete drainage. Section 4 deals with its special cases of Ellis fluid model . The results and discussion are included in section 5, while section 6 concludes the study.

## 2. BASIC EQUATIONS

Essential governing equations for incompressible Ellis fluid flow, disregarding thermal effects are:

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{f} + \nabla \cdot \mathbf{S}, \tag{2}$$

The symbol $\mathbf{V}$ represents velocity vector, $\rho$ denotes the constant density, $p$ be the dynamic pressure, $\mathbf{f}$ is the body force and $\mathbf{S}$ the extra stress tensor. The operator $\frac{D}{Dt}$ denotes the material derivative. The extra stress tensor describing an Ellis (Bird, Stewart and

Lightfoot, 2002; Afanasiev, Münch, & Wagner, 2007; Ali, Abbasi, & Ahmad, 2015) is made by:

$$\mathbf{S} = \eta \mathbf{A}_1, \tag{3}$$

Where the parameter $\eta$ is defined as:

$$\frac{1}{\eta} = \frac{1}{\eta_0}\left[1 + \left|\sqrt{\frac{\frac{1}{2}(\mathbf{S}:\mathbf{S})}{S_{\frac{1}{2}}}}\right|^{\alpha-1}\right], \tag{4}$$

$$A_1 = \nabla V + (\nabla V)^T \tag{5}$$

Here $A_1$ is first Rivilin- Ericksen tensor, $\eta$ is a scalar quantity and Eq. (4) contains three constants $\eta_0, \alpha, S_{\frac{1}{2}}$ which may be determined experimentally for each fluid and also the symbol represent in Eq. (4) is

$$S:S = \text{trace}\left(S^2\right). \tag{6}$$

Special cases are essential for Ellis fluid model, at low shear stress $\left(S_{\frac{1}{2}} \to \infty\right)$, this model controls to a Newtonian fluid model, and appropriately at high shear rates the proposed model converts into the Power law fluid model.

## 3. TANK DRAINAGE

Let suppose a cylindrical tank is of the radius $R_T$ containing an isothermal, incompressible Ellis fluid and the pipe of the diameter $d$ is attached at the center of the bottom of the tank . The initial depth of the fluid is taken to be $H_0$. The fluid in the tank is drained down through by a a pipe having length $L$ and radius $R$. Additionally more, the depth of fluid in the tank at any time $t$ is assumed to be $H(t)$ in the tank, flow of the fluid in the pipe is due to hydrostatic pressure of the fluid in the tank and gravity.

our plane is to determine velocity profile, shear stress on the pipe, flow rate, average velocity, relationship between the variation in time and in the depth of fluid of the tank, and also with the time required for complete drainage. Here we have use cylindrical coordinates ($r$, $\theta$ $z$) with *r-axis* (normal to the pipe), and *z-axis* along the center of the pipe in vertical direction. As the flow is individual in the *z*-direction, and the $\theta$ and $r$ components of velocity vector $\mathbf{V}$ are equal to zero,

$$\mathbf{V}, = [v_r, v_\theta, v_z] = [0, 0, v_z(r,t)], \qquad \mathbf{S} = S(r,t) \tag{7}$$

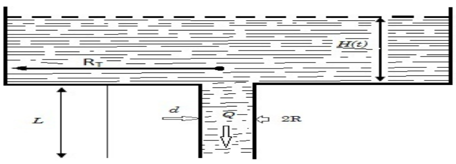


**Fig. 1. Tank drainage flow down by mean of circular pipe (Papanastasiou, 1994).**

By means of Eq. (7), the equation of continuity (1) is indistinguishably fulfilled; and we have neglect the convected part of the acceleration, the momentum Eq. (2) diminishes towards

$r-$*component of momentum* :

$$\frac{\partial p}{\partial r} = \frac{1}{r}\frac{\partial (rS_{rr})}{\partial r} - \frac{S_{\theta\theta}}{r}, \tag{8}$$

$\theta-$*component of momentum*:

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = \frac{1}{r^2}\frac{\partial (r^2 S_{r\theta})}{\partial r}, \tag{9}$$

$z-$*component of momentum* :

$$\rho\frac{\partial v_z}{\partial t} + \frac{\partial p}{\partial z} - \rho g. = \frac{1}{r}\frac{\partial (rS_{rz})}{\partial r} \tag{10}$$

The flow in the pipe of radius $R$ is due to both hydrostatic pressure and gravity. At the entrance and exit in the pipe, the pressures can be described as:

$$at \quad z = 0, \quad p = p_1 = \rho g H(t), \tag{11}$$

$$at \quad z = L, \quad p = p_2 = 0, \tag{12}$$

so that,

$$\frac{\partial p}{\partial z} = -\frac{\rho g H(t)}{L} \tag{13}$$

The velocity in the pipe flow almost constant with time $t$, due to slow draining, so that we may neglect the time derivative $\frac{\partial v_z}{\partial t}$. The equation for momentum for this combined flow is:

$$\frac{1}{r}\frac{\partial (rS_{rz})}{\partial r} = -\frac{\rho g H(t)}{L} - \rho g. \tag{14}$$

The associated boundary condition's at drainage pipe

$$at \quad r = 0, \quad S_{rz} = 0, \quad \text{(centerline)} \tag{15}$$

$$at \quad r = R, \quad v_z = 0 \quad \text{(no-slip)} \tag{16}$$

According to the (centerline) boundary condition (15) "shear stresses" are zero in the middle of the pipe as a result of the maximisation of the velocity of the fluid. On the other hand, as per 'No-Slip' the boundary condition (16) the velocity of the fluid particles is zero at the walls of the pipe.

By intigrating Eq. (14) with respect to $r$, keeping $t$ as a constant, we obtaint

$$S_{rz} = -\frac{\rho g r}{2L}\left(H(t) + L\right) + f_1(t). \tag{17}$$

Here $f_1(t)$ is arbitarary function of $t$,by using (centerline) boundary condition (15) , we get

$$S_{rz} = -\frac{\rho g r}{2L}\left(H(t) + L\right). \tag{18}$$

Now by using Eq. (18) in the constitutive equations, the arrangement of Eqs. (3) and (4) is,

$$\frac{\partial v_z}{\partial r} = \frac{1}{\eta_0}\left[1 + \left|\frac{S_{rz}}{S_{\frac{1}{2}}}\right|^{\alpha-1}\right] S_{rz} \tag{19}$$

We consider $|S_{rz}|$ as positive because of drainage flow problem, therefore for relaxing to the absolute condition and by using Eq. (18) in Eq. (19), we get

$$\frac{\partial v_z}{\partial r} = -\frac{1}{\eta_0}\left[\begin{array}{l}\frac{\rho g r}{2L}\left(H(t)+L\right)+\\ S_{\frac{1}{2}}^{1-\alpha}\left\{\frac{\rho g r}{2L}\left(H(t)+L\right)\right\}^{\alpha}\end{array}\right] \quad (20)$$

on integrating (20) with respect to *r*, we have:

$$v_z = -\frac{\rho g}{4L\eta_0} r^2\left\{H(t)+L\right\} - \frac{S_{\frac{1}{2}}^{1-\alpha}}{\eta_0(\alpha+1)}\left\{\frac{\rho g}{2L}\left(H(t)+L\right)\right\}^{\alpha} r^{\alpha+1} + f_2(t) \quad (21)$$

Where $f_2(t)$ is an arbitarary function of *t,* after using no slip consition, we get

$$f_2(t) = \frac{\rho g}{4L\eta_0} R^2\left\{H(t)+L\right\} + \frac{S_{\frac{1}{2}}^{1-\alpha}}{\eta_0(\alpha+1)}\left\{\frac{\rho g}{2L}\left(H(t)+L\right)\right\}^{\alpha} R^{\alpha+1} \quad (22)$$

Hence Eq. (21) reduces to

$$v_z = -\frac{\rho g}{4L\eta_0}\left(r^2-R^2\right)\left\{H(t)+L\right\} - \frac{S_{\frac{1}{2}}^{1-\alpha}}{\eta_0(\alpha+1)}\left\{\frac{\rho g}{2L}\left(H(t)+L\right)\right\}^{\alpha}\left(r^{\alpha+1}-R^{\alpha+1}\right) \quad (23)$$

## 3.1 Flow Rate, Average Velocity, Shear Stress on the Pipe and the Relation of Time with Depth of the Fluid for Complete Drainage

The “flow rate *Q*” per unit width is specified through the formula

$$Q = \int_0^{2\pi}\int_0^{R} r v_z(r,t)\,dr\,d\theta = \int_0^{R} 2\pi r v_z(r,t)\,dr. \quad (24)$$

Using velocity profile (23) in Eq. (24), the flow rate can be calculated

$$Q = \frac{\rho g\pi R^4}{8L\eta_0}\left(H(t)+L\right) + \frac{S_{\frac{1}{2}}^{1-\alpha}R^{\alpha+3}\pi}{\eta_0(\alpha+3)}\left\{\frac{\rho g(H(t)+L)}{2L}\right\}^{\alpha}. \quad (25)$$

We determine the average velocity, $\overline{V}$ by using the following formula (26)

$$\overline{V} = \frac{Q}{\pi R^2}. \quad (26)$$

So by the use of Eq. (25) in Eq. (26), average velocity of the fluid can be

$$\overline{V} = \frac{\rho g R^2}{8L\eta_0}\left(H(t)+L\right) + \frac{S_{\frac{1}{2}}^{1-\alpha}R^{\alpha+1}}{\eta_0(\alpha+3)}\left\{\frac{\rho g(H(t)+L)}{2L}\right\}^{\alpha}. \quad (27)$$

Shear stress on the pipe is given by

$$S_{rz}\big|_{r=R} = -\left(\frac{\rho g R}{2L}\right)\left[H(t)+L\right]. \quad (28)$$

Mass balance over the entire tank is

$$\frac{d}{dt}\left[\pi R_T^2 H(t)\right] = -Q(t). \quad (29)$$

Substituting flow rate from Eq. (25) into Eq. (29), and rewite the differential equation as following

$$\frac{dH(t)}{\frac{\rho g R^4}{8L\eta_0 R_T^2}\left(H(t)+L\right) + \frac{S_{\frac{1}{2}}^{1-\alpha}R^{\alpha+3}}{\eta_0 R_T^2(\alpha+3)}\left\{\frac{\rho g(H(t)+L)}{2L}\right\}^{\alpha}} = -dt \quad (30)$$

Intigrate to Eq. (30) on both sides, we get

$$\frac{-8LR_T^2\eta_0}{\rho g R^4(1-\alpha)}\ln\left(\begin{array}{l}-\frac{\rho g R^4}{8LR_T^2\eta_0}\left(H(t)+L\right)^{1-\alpha} - \\ \frac{S_{\frac{1}{2}}^{1-\alpha}R^{\alpha+3}\left(\frac{\rho g}{2L}\right)^{\alpha}}{\eta_0(\alpha+3)R_T^2}\end{array}\right) = t + C_0. \quad (31)$$

By means of initial condition *H(t) = H₀* at *t = 0* for tank drainage, we acquire that

$$\frac{-8LR_T^2\eta_0}{\rho g R^4(1-\alpha)}\ln\left(\begin{array}{l}-\frac{\rho g R^4}{8LR_T^2\eta_0}\left(H_0+L\right)^{1-\alpha} - \\ \frac{S_{\frac{1}{2}}^{1-\alpha}R^{\alpha+3}\left(\frac{\rho g}{2L}\right)^{\alpha}}{\eta_0(\alpha+3)R_T^2}\end{array}\right) = C_0 \quad (32)$$

The fluid elevation in the tank then descends slowly agreeing to

$$H(t) = \left\{\begin{array}{l}\frac{-8LR_T^2\eta_0 e^{-\frac{\rho g R^4}{8LR_T^2\eta_0}(1-\alpha)t}}{\rho g R^4} \\ \left(\begin{array}{l}\frac{-\rho g R^4}{8LR_T^2\eta_0}\left(H_0+L\right) \\ -\frac{S_{\frac{1}{2}}^{1-\alpha}R^{\alpha+3}\left(\frac{\rho g}{2L}\right)^{\alpha}}{\eta_0(\alpha+3)R_T^2}\end{array}\right) \\ -\frac{4S_{\frac{1}{2}}^{1-\alpha}R^{\alpha-1}}{(\alpha+3)}\left(\frac{\rho g}{2L}\right)^{(\alpha-1)}\end{array}\right\}^{\frac{1}{1-\alpha}} - L \quad (33)$$

and the time required for complete drainage for Ellis fluid is obtained by taking *H(t) = 0,* in

$$\frac{-8LR_T^2\eta_0}{\rho g R^4(1-\alpha)}\ln\left(\frac{-\frac{\rho g R^4}{8LR_T^2\eta_0}\left(H(t)+L\right)^{1-\alpha} - \frac{S_{\frac{1}{2}}^{1-\alpha}R^{\alpha+3}\left(\frac{\rho g}{2L}\right)^{\alpha}}{\eta_0(\alpha+3)R_T^2}}{-\frac{\rho g R^4}{8LR_T^2\eta_0}\left(H_0+L\right)^{1-\alpha} - \frac{S_{\frac{1}{2}}^{1-\alpha}R^{\alpha+3}\left(\frac{\rho g}{2L}\right)^{\alpha}}{\eta_0(\alpha+3)R_T^2}}\right) = t. \quad (34)$$

# 4. SPECIAL CASES: NEWTONIAN, POWER LAW, BINGHAM PLASTIC FLUIDS

## 4.1 Newtonian Fluid

When yield stress is very high, mathematically for $\left(S_{\frac{1}{2}} \to \infty\right)$, Ellis solutions reduces to the

Newtonian fluid (Papanastasiou, 1994), that is

$$v_z = \frac{\rho g}{4L\eta_0}\left(R^2 - r^2\right)\{H(t)+L\} \tag{35}$$

$$Q = \frac{\rho g \pi R^4}{8L\eta_0}(H(t)+L). \tag{36}$$

$$\bar{V} = \frac{\rho g \pi R^2}{8L\eta_0}(H(t)+L) \tag{37}$$

$$H(t) = \{H_0 + L\} e^{-\frac{\rho g R^4}{8LR_T^2\eta_0}t} - L \tag{38}$$

$$t = -\frac{8LR_T^2\eta_0}{\rho g R^4}\ln\left(\frac{H(t)+L}{H_0+L}\right) \tag{39}$$

, which provides a reasonable estimate for the Tank drainage for Newtonian fluid.

### 4.2 Power-law Fluid

For substitution $\left(S_{\frac{1}{2}} = \eta_0,\ \alpha = \frac{1}{n}\right)$ Ellis solution gives the result for Power law fluid (Bird, Stewart and Lightfoot, 2002),

$$v_z = -\frac{n}{n+1}\left\{\frac{\rho g}{2\eta_0 L}(H(t)+L)\right\}^{\frac{1}{n}}\left(r^{\frac{1}{n}+1} - R^{\frac{1}{n}+1}\right) \tag{40}$$

$$Q = \frac{n\pi R^{\frac{1}{n}+3}}{1+3n}\left\{\frac{\rho g}{2L\eta_0}(H(t)+L)\right\}^{\frac{1}{n}}. \tag{41}$$

$$\bar{V} = \frac{nR^{\frac{1}{n}+1}}{1+3n}\left\{\frac{\rho g}{2L\eta_0}(H(t)+L)\right\}^{\frac{1}{n}}. \tag{42}$$

By using Eq. (41) in (29), we will get

$$H(t) = \left\{(H_0+L)^{\frac{n-1}{n}} - \frac{(n-1)R^{\frac{1}{n}+3}}{(1+3n)R_T^2}\left(\frac{\rho g}{2\eta_0 L}\right)^{\frac{1}{n}} t\right\}^{\frac{n}{n-1}} - L \tag{43}$$

, and time required for complete drainage from a tank for Power law fluid is by taking $H(t) = 0$ in

$$t = -\frac{(1+3n)R_T^2}{(n-1)R^{\frac{1}{n}+3}}\left(\frac{\rho g}{2\eta_0 L}\right)^{\frac{-1}{n}}\left\{(H(t)+L)^{\frac{n-1}{n}} - (H_0+L)^{\frac{n-1}{n}}\right\}. \tag{44}$$

### 4.3 Bingham Plastic Fluid

This fluid is defined by the model:

$$S_{rz} = S_0 + \eta_0\frac{\partial v_z}{\partial r}, \qquad for \qquad S_{rz} > S_0 \tag{45}$$

$$\frac{\partial v_z}{\partial r} = 0, \qquad for \qquad S_{rz} \le S_0 \tag{46}$$

Thus, for stresses larger than $S_0$ Eq. (19) is simplified to (45) by setting $\left(S_{\frac{1}{2}} = S_0,\ \alpha = 0\right)$ while for $S_{rz} \le S_0$ (46) is obtained from (3) for the reason highly viscous and also yield stress is very high. From Eq. (18), which upon substitution of $S_0$ for $S_{rz}$ yields the distance between the radius of the pipe $r_1$ beyond which Eq. (46) holds:

$$r_1 = -\frac{2LS_0}{\rho g(H(t)+L)} \tag{47}$$

$$v_{z_1} = \frac{\rho g}{4\eta_0 L}(H(t)+L)\left[(R^2 - r^2) + 2r_1(r-R)\right],$$

$$for \qquad 0 < r < r_1 \tag{48}$$

$$v_{z_2} = 0.\ \ for\ r_1 \le r \le R \tag{49}$$

The volume flow rate is given by

$$Q = 2\pi\left(\int_0^{r_1} r v_{z_1}(r)dr + \int_{r_1}^{R} r v_{z_2}(r)dr\right). \tag{50}$$

Through using equation number (47) and (48), in equation number (49), we will get total flow rate

$$Q = \frac{\rho g r_1^2 \pi}{24\eta_0 L}\{H(t)+L\}.$$

$$\left[3(2R^2 - r_1^2) + 4(2r_1 - 3R)\right] \tag{51}$$

By using Eq. (50) in Eq. (26), the average film velocity will be,

$$\bar{V} = \frac{\rho g r_1^2}{24\eta_0 LR^2}\{H(t)+L\}.$$

$$\left[3(2R^2 - r_1^2) + 4(2r_1 - 3R)\right] \tag{52}$$

Making the use of Eq. (50) in (29) we will get

$$H(t) = \{H_0 + L\}e^{-\frac{\rho g r_1^2}{24LR_T^2\eta_0}\left[3(2R^2-r_1^2)+4(2r_1-3R)\right]t} - L \tag{53}$$

, and the time required for complete drainage from a tank for Bingham plastic fluid is by taking $H(t) = 0$ in

$$t = \frac{-24LR_T^2\eta_0}{\rho g r_1^2\{3(2R^2 - r_1^2) + 4(2r_1 - 3R)\}}\ln\left(\frac{H(t)+L}{H_0+L}\right) \tag{54}$$

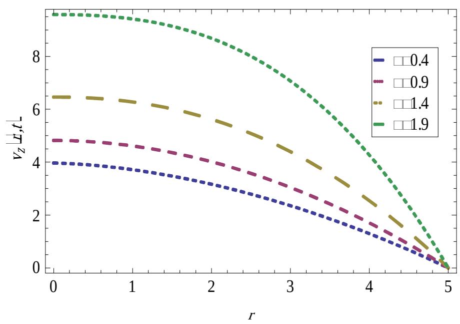


**Fig. 2. Effect of $\alpha$ on velocity profile for Ellis fluid, when** $\eta_0 = 100\ poise, \rho = 1.38\ g/cm^3$, $R = 5cm,\ S_{\frac{1}{2}} = 21.554 dyn/cm^2$, $L = 10cm,\ H(t) = 20cm$

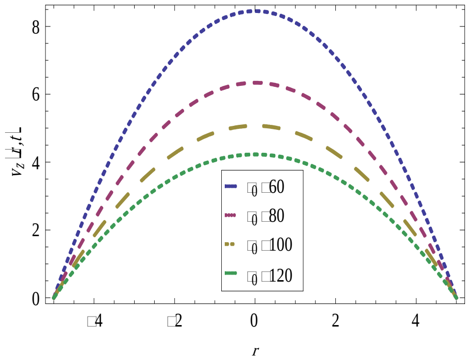


**Fig. 3. Effect of $\eta_0$ on velocity profile for Ellis fluid, when** $\alpha=1\,poise, \rho=1.38\,g/cm^3$,

$R=5cm,\ S_{\frac{1}{2}}=21.554dyn/cm^2$,

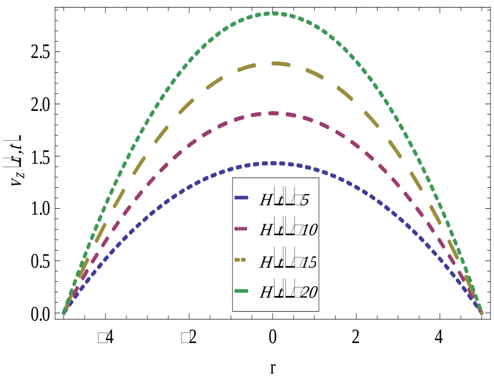


**Fig. 4. Difference of *H(t)* on the velocity profile for Ellis fluid, when** $\alpha=1, \rho=0.78\,g/cm^3$,

$\eta_0=100\,poise,\ S_{\frac{1}{2}}=21.554dyn/cm^2$,

$L=10cm,\ R=5cm$.

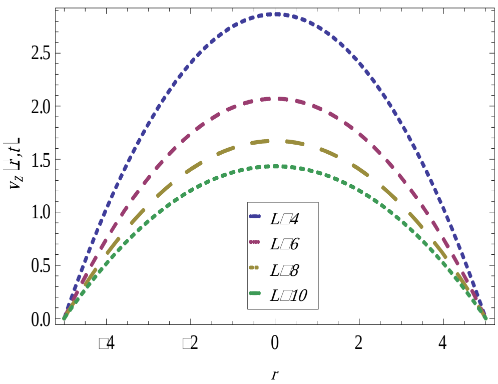


**Fig. 5. Difference of *L* on the velocity profile for Ellis fluid, when** $\alpha=1, \rho=0.78\,g/cm^3$,

$\eta_0=200\,poise,\ S_{\frac{1}{2}}=21.554dyn/cm^2$,

$H(t)=20cm,\ R=5cm$.

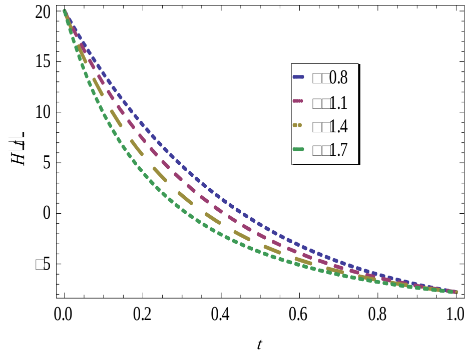


**Fig. 6. Effect of $\alpha$ on depth for Ellis fluid, when** $H_0=20cm, \rho=1.38\,g/cm^3, R_T=25cm$

$\eta_0=0.13\,poise,\ S_{\frac{1}{2}}=21.554dyn/cm^2$,

$L=10cm,\ R=5cm$

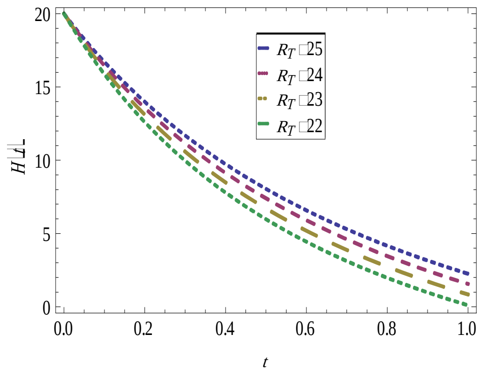


**Fig. 7. Effect of $R_T$ on depth for Ellis fluid, when** $H_0=20, \rho=1.38\,g/cm^3, \alpha=1.9$

$\eta_0=0.6\,poise,\ S_{\frac{1}{2}}=21.554dyn/cm^2$,

$L=10cm,\ R=5cm$.

**Table 1 Relation of depth with respect to time for Ellis fluid and its special cases when**

$H_0=20cm, \rho=0.78\,g/cm^3$,

$\alpha=1.7, \eta_0=11.5\,poise,\ S_{\frac{1}{2}}=100.554dyn/cm^2$,

$L=10cm,\ R=5cm,\ R_T=25cm, r_1=2.4cm, n=1.9$

| t | Ellis | Newtonian | Power Law | Bingham Plastic |
|---|---|---|---|---|
| 0 | 20 | 20 | 20 | 20 |
| 2 | 19.2279 | 19.5056 | 19.7364 | 19.8596 |
| 4 | 18.4805 | 19.0193 | 19.474 | 19.7198 |
| 6 | 17.7568 | 18.5411 | 19.2129 | 19.5806 |
| 8 | 17.0559 | 18.0707 | 18.953 | 19.4422 |
| 10 | 16.3769 | 17.6081 | 18.6943 | 19.3043 |

## 5. RESULTS AND DISCUSSION

In this work, we examined unsteady tank drainage problem by means of an isothermal, incompressible Ellis fluid, through which exact results for the non-linear differential equation were obtained. The variation of velocity profile $v_z$ and depth *H(t)* has been investigated on different parameters. The effects of the Ellis index $\alpha$, length of pipe *L,* viscosity $\eta_0$, depth of the tank *H(t)* on velocity profile are detected through Figs. 2-5, and effect of the $\alpha$ ,radius of the tank $R_T$, and the depth *H(t)* is shown in the Figs. 6 and 7. In the Fig. 2, it is observed that the magnitude of velocity increases with the increase in Ellis index $\alpha$. This explains that the magnitude of velocity decrease with the decrease of fluid viscosity. The effect of $\eta_0$ and *H(t)*on velocity profile $v_z$ is shown in the Figs. 3-5. In these figures, it can be noticed that as $\eta_0$ and *H(t)* increase, the magnitude of the velocity distribution also increases; and vice versa. The velocity distribution, viscosity, radius of the pipe, and the depth of the tank are found to be interlinked with each other. The length of the pipe is also found to be inversely proportional to the velocity profile, which can be seen in Fig. 5. The effects of Ellis index $\alpha$ and radius of the tank $R_T$ on height of the fluid in tank *H(t),* are shown in the Figs. 6 and 7. An increase in $\alpha$ causesdecrease in the depth of fluid $H(t)$; and the depth increases when radius of tank is increased. A comparison of depth of Ellis fluid with its special cases is presented in Table 1- with fixed parameters-, which are mentioned in caption of the table. The results presented in Table 1 are computed numerically. Tabulated data showing depth at various times indicates that the depth of Ellis fluid is lower than the depth of its special cases. This also explains the reason that velocity profile of Ellis fluid is higher than its special cases.

## 6. CONCLUSION

Considering equation for unsteady tank drainage flow for incompressible and isothermal Ellis fluid, we have obtained exact solutions for the proposed model. Also we have exactly retrieved the special cases for Ellis model, such as Newtonian, Power law, and Bingham plastic. A relationship between (33), (34), and the variation in time and the depth is derived. It is, noted that as the fluid becomes thicker, the velocity of the fluid increases; and it is also important to note that Ellis fluid drains quickly as compared to its special cases.

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