Journal of Applied Fluid Mechanics, Vol. 10, No. 5, pp. 1421-1426, 2017.
Available online at www.jafmonline.net, ISSN 1735-3572, EISSN 1735-3645.
DOI: 10.18869/acadpub.jafm.73.242.27079

Oscillating Motion of an Oldroyd-B Fluid with Fractional Derivatives in a Circular Cylinder

N. Raza[1†], E. U. Haque[2], M. M. Rashidi[3], A. U. Awan[1] and M. Abdullah[4]

[1] *Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore, Pakistan.*
[2] *Department of Mathematics and Statistics, The University of Lahore, Lahore-Pakistan.*
[3] *University of Birmingham, School of Civil Engineering, Birmingham, United Kingdom.*
[4] *Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan.*

†*Corresponding Author Email: raza_nauman@yahoo.com*

(Received August 17, 2016; accepted May 8, 2017)

ABSTRACT

The velocity field and tangential shear stress for unsteady flow of an Oldroyd-B fluid with Caputo fractional derivatives through an infinite long cylinder are evaluated. The fluid in the infinitely long cylinder is initially at rest and at t = 0+, due to shear, the fluid starts to oscillate longitudinally. We have solved the fractional model with the tool of Laplace and finite Hankel transformations. The solutions are in series form and are written in generalized G-function to avoid the entanglement. In limiting cases, the solutions of ordinary Oldroyd-B fluid, Maxwell fluid with fractional as well as ordinary and Newtonian fluid are derived. Finally, behavior of different physical parameters on fluid is illustrated by graphs.

Keywords: Oldroyd-B fluid; Tangential stress; Longitudinal oscillation; Velocity field; Laplace transformation; Hankel transformation.

NOMENCLATURE

A	arbitrary constant	λ_r	retardation time
E	extra shear stress	ζ	shear stress
R	radius of circular cylinder	σ	angular frequency
t	time	ψ	fractional parameter
u	velocity field	φ	similarity parameter
		Θ	direction vector
ρ	density	μ	dynamic viscosity
λ	relaxation time	ν	kinematic viscosity

1. INTRODUCTION

The performance and shape of the movement of non-Newtonian fluids can be narrated well by Navier-Stokes equations. Examples of these non-Newtonian fluids are blood, dough, suspensions, polymer resolution, liquid crystals and bitumen. Due to their multiple applications in industry, engineering and medical, these non-Newtonian fluids achieve much significance. The motion of non-Newtonian fluids between oscillating and rotating cylinders is the most worthy and fascinating phenomenon in engineering.

The phenomenon in which a non-Newtonian fluid is moving in a cylinder is very valuable problem in dynamics. The exact solution of these kind of problems such that rotational oscillation in an infinite rod is firstly presented by Stokes (1886). Casarella and Laura (1969) described the movement of fluid in a cylinder in both longitudinal oscillation and rotation. Rajagopal (1983) and later Rajagopal and Bhatnagar (1995) firstly solved the models for the motion of non-Newtonian fluids. The models of rate type fluids that are viscoelastic have been solved by Rajagopal and Srinivasa (2000).

The extensions to exact solutions of rate type fluids discussed by Fetecau *et al*. (2008 a) and Rubab *et al*. (2009). Some conventional exact solutions to the steady state viscoelastic fluid have been studied by Fetecau *et al*. (2008 b). Exact solutions for the motion due to time-dependent shear to a non-Newtonian fluid have been discussed by Fetecau and Kannan (2005). Rotational motion within annulus of an Oldroyd-B fluid is discussed by Imran *et al.* (2015). Some important attempts to get exact solutions of fractional non-Newtonian fluid models are listed here (Mahmood *et al.* (2009), Tong *et al.* (2005), Sultan and Nazar (2016)).

Recently, the solution of some incompressible fluids which have convection flow in two parallel plates is discussed by Rashidi *et al.* (2013) and solution of Oldroyd-B fluid for magnetohydrodynamic flow is presented by Abbasbandy *et al.* (2013), both are playing key role in dynamics industry. Many papers associated to non-Newtonian fluids have been proclaimed (Awan *et al.* (2010), Keimanesh *et al.* (2011), Galanis and Rashidi (2012), Riaz *et al.* (2016), Jamil *et al.* (2012)). In this paper, exact solutions of velocity field and tangential stress of an Oldroyd-B fluid with fractional derivatives which is oscillating longitudinally are computed. Initially the cylinder is at rest and at t = 0+, the cylinder starts to move. To solve the problem, Laplace and Hankel transformations are used that made the approach to the solutions more attainable.

2. FORMULATION OF THE PROBLEM

The extra-stress E and the velocity U of the movement of fluid are considered as

$$u=u(r,t)e_z,\qquad E=E(r,t), \tag{1}$$

where e_z in the cylindrical coordinate system (r,θ,z) is the unit vector in z-direction. Moreover, when the fluid starts to move, we have

$$u(r,0)=0,\qquad E(r,0)=0. \tag{2}$$

The equations that govern the motion of an Oldroyd-B fluid are given as (Fetecau *et al.* (2008)

$$(1+\lambda\frac{\partial}{\partial t})\zeta(r,t)=\mu(1+\lambda_r\frac{\partial}{\partial t})\frac{\partial u(r,t)}{\partial r} \tag{3}$$

$$\rho\frac{\partial u(r,t)}{\partial t}=\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\zeta(r,t). \tag{4}$$

where $\tau(r,t) = E_{r\theta}(r,t)$ is the only nontrivial shear stress. By eliminating ζ from Eqs. (3) and (4), we have

$$(1+\lambda\partial/\partial t)\partial u(r,t)/\partial t=v(1+\lambda_r\partial/\partial t)\times(\partial/\partial r+1/r)\partial u(r,t)/\partial r \tag{5}$$

The Caputo fractional differential operator (Fetecau *et al.* (2008)) is

$$D_t^{\rho}f(t)=\begin{cases}\frac{1}{(1-\rho)}\frac{d}{dt}\int_0^t\frac{f(x)}{(t-x)^{\rho}}dx, & 0<\rho<1;\\ \frac{df(t)}{dt} & \rho=1,\end{cases} \tag{6}$$

where Γ(·) denotes the Gamma function.

From Eqs. (3) and (5) by using the fractional differential operators D_t^{Φ} and D_t^{Ψ} in place of integer time derivatives, we obtain the equations of motion of an Oldroyd-B fluid with fractional derivatives

$$(1+\lambda^{\Phi}D_t^{\Phi})\zeta(r,t)=\mu(1+\lambda_r^{\Psi}D_t^{\Psi})\frac{\partial u(r,t)}{\partial r}, \tag{7}$$

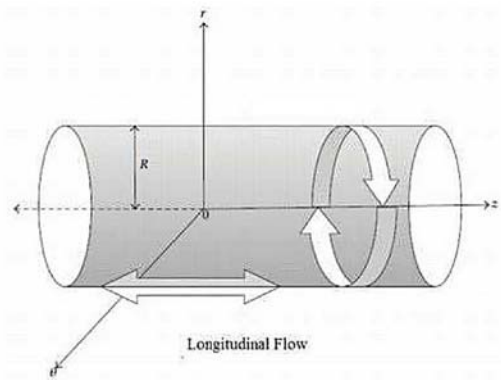

Fig. 1. Oldroyd-B fluid with longitudinal oscillation.

$$(1+\lambda^{\Phi}D_t^{\Phi})\frac{\partial u(r,t)}{\partial t}=v(1+\lambda_r^{\Psi}D_t^{\Psi})\times\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\frac{\partial u(r,t)}{\partial r}. \tag{8}$$

Consider a fractional Oldroyd-B fluid which is initially at rest in an infinitely long cylinder of radius R. In the presence of shear stress, after some time the cylinder starts to oscillate. As a result of applied shear stress, the fluid is gradually moved. The appropriate initial and boundary conditions are

$$u(r,0)=\frac{\partial u(r,t)}{\partial t}\Big|_{t=0}=0,\ \ \zeta(r,0)=0;\ \ r\in[0,R], \tag{9}$$

$$(1+\lambda^{\Phi}D_t^{\Phi})\zeta(r,t)\,|_{r=R}=\mu(1+\lambda_r^{\Psi}D_t^{\Psi})$$

$$\frac{\partial u(r,t)}{\partial r}\Big|_{r=R}=A\sin(\omega t) \tag{10}$$

where ω is angular frequency. Eqs. (7) and (8) involving fractional derivatives are solved by using the tool of Laplace transformation and Hankel transformation.

3. CALCULATION OF THE VELOCITY FIELD

Taking the Laplace transformation of Eqs. (8) and (10), we have

$$(p+\lambda^{\Phi}p^{\Phi+1})\bar{u}(r,p)=v(1+\lambda_r^{\Psi}p^{\Psi})\times\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\frac{\partial\bar{u}(r,p)}{\partial r}, \tag{11}$$

$$\frac{\partial\bar{u}(r,p)}{\partial r}\Big|_{r=R}=\frac{A\omega}{\mu(1+\lambda_r^{\Psi}p^{\Psi})(p^2+\omega^2)}, \tag{12}$$

The finite Hankel transform of u(r, p) is defined as (Fetecau *et al.* (2008))

$$\overline{u_H}(r_n,p)=\int_0^R r\bar{u}(r,p)J_0(rr_n)dr, \tag{13}$$

where $J_0(\cdot)$ is the Bessel function of first type having zeroth order and rn, n = 1,2,3,... are the positive roots of the transcendental equation $J_1(Rr) = 0$. Multiplying Eq. (11) by $rJ_0(rr_n)$, then integrating with respect to 'r' from 0 to R and using the identity

$$\int_0^R r\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]\bar{u}(r,p)J_0(rr_n)dr=RJ_0(Rr_n)\frac{\partial\bar{u}(R,p)}{\partial r}r_n^2\,\overline{u_H}(r_n,p) \tag{14}$$

$$\overline{u_H}(r_n,p)=\frac{AR\omega\nu J_0(Rr_n)}{\mu(p^2+\omega^2)[(p+\lambda^{\Phi}p^{\Phi+1})+\nu r_n^2(1+\lambda_r^{\Psi}p^{\Psi})]} \quad (15)$$

$$\overline{u_H}(r_n,p)=\frac{AR\omega VJ(Rr_n)}{\mu r_n^2}\left[\frac{1}{(1+\lambda_r^{\Psi}p^{\Psi})(p^2+\omega^2)}-\frac{(p+\lambda^{\Phi}p^{\Phi+1})}{(1+\lambda_r^{\Psi}p^{\Psi})(p^2+\omega^2)[(p+\lambda^{\Phi}p^{\Phi+1})+\nu r_n^2(1+\lambda_r^{\Psi}p^{\Psi})]}\right] \quad (16)$$

Taking the inverse Hankel transform, we obtain

$$\bar{u}(r,p)=\frac{Ar^2\omega}{2\mu R}\frac{1}{(1+\lambda_r^{\Psi}p^{\Psi})(p^2+\omega^2)}-\frac{2A\omega}{\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2J_0(Rr_n)}\times\left[\frac{(p+\lambda^{\Phi}p^{\Phi+1})}{(1+\lambda_r^{\Psi}p)(p^2+\omega^2)[(p+\lambda^{\Phi}p^{\Phi+1})+\nu r_n^2(1+\lambda_r^{\Psi}p^{\Psi})]}\right] \quad (17)$$

The suitable form of the last factor of Eq. (17) is

$$\frac{(1+\lambda^{\Phi}p^{\Phi})}{[(p+\lambda^{\Phi}p^{\Phi+1})+Vr_n^2(1+\lambda_r^{\Psi}p^{\Psi})]}=\frac{1}{\lambda^{\Phi}}\sum_{i=0}^{\infty}\sum_{j=0}^{i} c_i^j\lambda_r^j\left(\frac{-Vr_n^2}{\lambda}\right)^i\times\left[\frac{p^{\Psi j-i-1}+\lambda^{\Phi}p^{\Phi+\Psi j-i-1}}{\left(p^{\Phi}+\frac{1}{\lambda^{\Phi}}\right)^{i+1}}\right], \quad (18)$$

where $C_i^j=\frac{i!}{j!(i-j)!}$ is the binomial co-efficient. Introducing Eq. (18) in (17) and applying inverse

Laplace transformation, using convolution theorem and the formula

$$L^{-1}\left[\frac{p^f}{(p^f-d)^g}\right]=G_{e,f,g}(d,t);$$

$$\text{Re(eg-f)}>0,\ \text{Re(p)}>0,\ \left|\frac{d}{p^e}\right|<1, \quad (19)$$

where

$$G_{e,f,g}(d,t)=\sum_{i=0}^{\infty}\frac{d^j\Gamma(g+i)}{\Gamma(g)\Gamma(j+1)}\frac{t^{(g+i)e-f-1}}{\Gamma[(g+j)e-f]}, \quad (20)$$

is the generalized G-function. We find the expression u(r,t) for the velocity field

$$u(r,t)=\frac{Ar^2}{2\mu R\lambda_r^{\Psi}}\left[\sin\omega t*G_{\Psi,0,1}(\tfrac{-1}{\lambda_r^{\Psi}},t)\right]-\frac{2A}{\lambda\mu R\lambda_r^{\Phi}}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2J_0(Rr_n)}\sum_{i=0}^{\infty}\sum_{j=0}^{i}c_i^j(\lambda_r^{\Psi})^j\left(\frac{-Vr_n^2}{\lambda^{\Phi}}\right)^i\times\left[\left\{\sin\omega t*G_{\Psi,0,1}\left(\frac{-1}{\lambda_r^{\Psi}},t\right)\right\}\times\left(G_{\Phi,\Psi j-i-1,i+1}\left(\frac{-1}{\lambda^{\Phi}},t\right)+\lambda^{\Phi}G_{\Phi,\Psi j-i-1,i+1}\left(\frac{-1}{\lambda^{\Phi}},t\right)\right)\right] \quad (21)$$

where $*$ denotes the convolution between two functions.

3.1 Calculation of the Shear Stress

Taking Laplace transform of Eq. (7), we have

$$\bar{\zeta}(r,p)=\frac{\mu(1+\lambda_r^{\Psi}p^{\Psi})}{(1+\lambda^{\Phi}p^{\Phi})}\frac{\partial\bar{u}(R,p)}{\partial r} \quad (22)$$

Differentiating Eq. (17) w.r.t 'r', we obtain

$$\frac{\partial\bar{u}(R,p)}{\partial r}=\frac{Ar\omega}{\mu R}\frac{1}{(1+\lambda_r^{\Psi}p^{\Psi})(p^2+\omega^2)}+\frac{2A\omega}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_nJ_0(Rr_n)}\times\left[\frac{(p+\lambda^{\Phi}p^{\Phi+1})}{(1+\lambda_r^{\Psi}p^{\Psi})(p^2+\omega^2)[(p+\lambda^{\Phi}p^{\Phi+1})+Vr_n^2(1+\lambda_r^{\Psi}p^{\Psi})]}\right] \quad (23)$$

Introducing Eq. (23) into (22), we have

$$\bar{\zeta}(r,p)=\frac{Ar}{R}\frac{\omega}{(1+\lambda^{\Phi}p^{\Phi})(p^2+\omega^2)}+\frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_nJ_0(Rr_n)}\frac{\omega}{(p^2+\omega^2)}\times\left[\frac{\rho}{[(p+\lambda^{\Phi}p^{\Phi+1})+Vr_n^2(1+\lambda_r^{\Psi}p^{\Psi})]}\right] \quad (24)$$

applying inverse Laplace transformation, using convolution theorem and the formula of G-function, we find the expression for shear stress ζ(r,t)

$$\zeta(r,t)=\frac{Ar}{\lambda^{\Phi}R}\left[\sin\omega t*G_{\Phi,0,1}(\tfrac{-1}{\lambda^{\Phi}},t)\right]+\frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_nJ_0(Rr_n)}\times\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i\frac{(\nu r_n^2)^i}{(\lambda^{\Phi})^{i+1}}C_i^j(\lambda_r^{\Phi})^j\times\left[\sin\omega t*G_{\Phi,\Psi j-i-1,i+1}\left(\frac{-1}{\lambda^{\Phi}},t\right)\right] \quad (25)$$

4. LIMITING CASES

The solutions (21) and (25) are in general form. After imposing some appropriate limits on these, we recover solutions for some other fluids.

4.1 Ordinary Oldroyd-B Fluid

Substituting ψ,φ → 1 in Eqs. (21) and (25), we get the expression for the velocity

$$u_{OB}(r,t)=\frac{Ar^2}{2\mu R\lambda_r^{\Psi}}\left[\sin\omega t*G_{1,0,1}\left(\frac{-1}{\lambda_r^{\Psi}},t\right)\right]-\frac{2A}{\lambda^{\Phi}\mu R\lambda_r^{\Psi}}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2J_0(Rr_n)}\times\sum_{i=0}^{\infty}\sum_{j=0}^{i}C_i^j(\lambda_r^{\Psi})^j\left(\frac{-\nu r_n^2}{\lambda^{\Phi}}\right)^i\times\left[\left\{\sin\omega t*G_{1,0,1}\left(\frac{-1}{\lambda_r^{\Psi}},t\right)\right\}*\left(G_{1,j-i-1,i+1}\left(\frac{-1}{\lambda^{\Phi}},t\right)+\lambda^{\Phi}G_{1,j-i-1,i+1}\left(\frac{-1}{\lambda^{\Phi}},t\right)\right)\right] \quad (26)$$

and for the shear stress

$$\zeta_{OB}(r,t)=\frac{Ar}{\lambda^{\Phi}R}\left[\sin\omega t*G_{1,0,1}\left(\frac{-1}{\lambda^{\Phi}},t\right)\right]+\frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_nJ_0(Rr_n)}\times\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-\nu r_n^2)^i}{(\lambda^{\Phi})^{i+1}}C_i^j(\lambda_r^{\Psi})^j\left[\sin\omega t*G_{1,j-i-1,i+1}\left(\frac{-1}{\lambda^{\Phi}},t\right)\right] \quad (27)$$

of an ordinary Oldroyd-B fluid.

4.2 Fractional Maxwell Fluid

Substituting $\lambda_r \rightarrow 0$, ψ → 1 in Eqs. (21) and (25), we get the expression

$$u_{FM}(r,t)=\frac{Ar^2}{2\mu R}(\sin\omega t)-\frac{2A}{\lambda^{\Phi}\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2J_0(Rr_n)}\times\sum_{i=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda^{\Phi}}\right)^i\times\sin\omega t*\left[G_{\Phi,-i-1,i+1}\left(\frac{-1}{\lambda^{\Phi}},t\right)+\lambda\,G_{\Phi,-i-1,i+1}\left(\frac{-1}{\lambda^{\Phi}},t\right)\right] \quad (28)$$

for the velocity field and

$$u_{FM}(r,t)=\frac{Ar}{\lambda^{\Phi}R}\left[\sin\omega t*G_{1,0,1}\left(\frac{-1}{\lambda^{\Phi}},t\right)\right]+\frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_nJ_0(Rr_n)}\times\sum_{i=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda^{\Phi}}\right)^i\left[\sin\omega t*G_{\Phi,-i-1,i+1}\left(\frac{-1}{\lambda^{\Phi}},t\right)\right] \quad (29)$$

for the shear stress of Maxwell fluid with fractional derivatives.

4.3 Ordinary Maxwell Fluid

Substituting $\lambda_r \rightarrow 0$and $\psi,\varphi \rightarrow 1$ for Eqs. (21),(25) (or substituting $\varphi \rightarrow 1$ for Eqs. (28), (29)), we get the expression

$$u_M(r,t) = \frac{Ar^2}{2\mu R}\left(\sin\omega t \right) - \frac{2A}{\lambda^{\Phi}\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)} \times \sum_{i=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda^{\Phi}}\right)^i \times \sin\omega t * \left[G_{1,-i-1,i+1}\left(\frac{-1}{\lambda^{\Phi}},t\right) + \lambda^{\Phi}\, G_{1,-i-1,i+1}\left(\frac{-1}{\lambda^{\Phi}},t\right)\right] \quad (30)$$

for the velocity field and

$$\zeta_M(r,t) = \frac{Ar}{\lambda^{\Phi}R}\left[\sin\omega t * G_{1,0,1}\left(\frac{-1}{\lambda^{\Phi}},t\right)\right] + \frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)} \times \sum_{i=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda^{\Phi}}\right)\left[\sin\omega t * G_{1,-i-1,i+1}\left(\frac{-1}{\lambda^{\Phi}},t\right)\right] \quad (31)$$

for the shear stress of Maxwell fluid with ordinary derivatives.

4.4 Newtonian Fluid

Substituting $\lambda_r,\lambda \rightarrow 0$ and $\psi,\varphi \rightarrow 1$ for Eqs. (21),(25), we get the velocity field expression

$$u_M(r,t) = \frac{Ar^2}{2\mu R}(\sin\omega t) - \frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)}(-\nu r_n^2)\sin\omega t \quad (32)$$

and the shear stress expression

$$\zeta_N(r,t) = \frac{Ar}{R}(\sin\omega t) + \frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)}(-\nu r_n^2)\sin\omega t \quad (33)$$

for a Newtonian fluid.

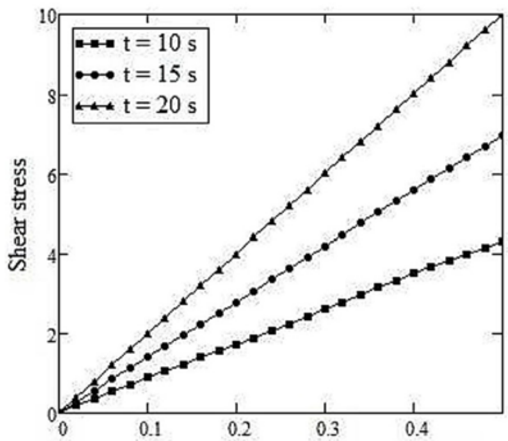

Fig. 2. Shear stress graph for R=0.5, λ_r=3, ν=0.0357541, μ=15, A=90, ω=15, λ=5, ψ=0.5, φ=0.3 and various values of t.

5. NUMERICAL RESULTS AND DISCUS- SION

The velocity field and the tangential stress for un-steady flow of an Oldroyd-B fluid with the Ca-puto derivatives through an infinite long cylinder are evaluated. The fluid in the infinitely long cylinder is initially at rest and at t = 0+, due to shear, the fluid starts to move. The tool of finite Hankel and Laplace transformations is used for determining these solutions. The solutions are presented in generalized G−function. These solutions satisfy all initial and boundary conditions. The solutions of ordinary Oldroyd-B fluid, fractional Maxwell fluid, ordinary Maxwell fluid and Newtonian fluid are achieved in limiting cases. The behavior of solutions is illustrated graphically in the end.

Finally, we have plotted some graphs for velocity and tangential stress of the fluid by using Eqs. (21) and (25) respectively, to see the effect of different physical parameters on our results. These graphs have been plotted against the values of r. Figs. 2 and 3 depict that velocity and shear stress are directly proportional to time. From Figs. 4-7, we conclude that both the tangential stress and the velocity field are decreasing function to λ_r and λ. The impact of the kinematic viscosity ν is analyzed in Figs. 8 and 9, which show that both shear stress and velocity field are directly proportional to the kinematics viscosity.

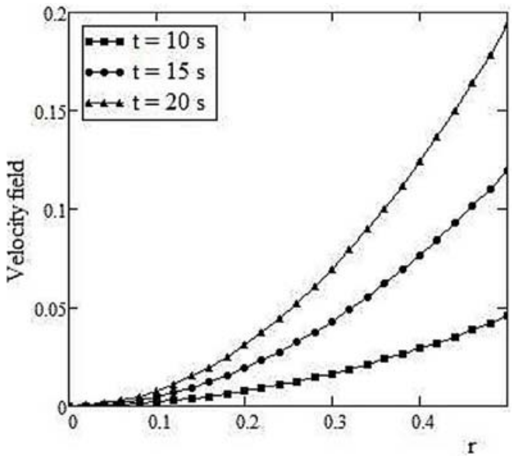

Fig. 3. Velocity field graph for R=0.5, λ_r=3, ν=0.0357541, μ=15, A=90, ω=15, λ=5, ψ=0.5, φ=0.3 and various values of t.

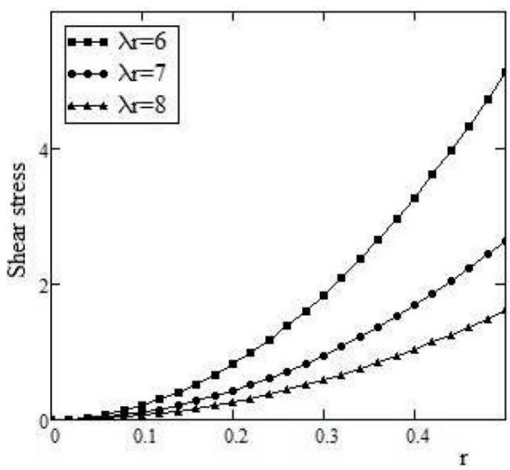

Fig. 4. Shear stress graph for R=0.5, λ_r=3, ν=0.0357541, μ=15, A=90, ω=15, λ=5, ψ=0.5, φ=0.3, t=15 and various values of λ_r.

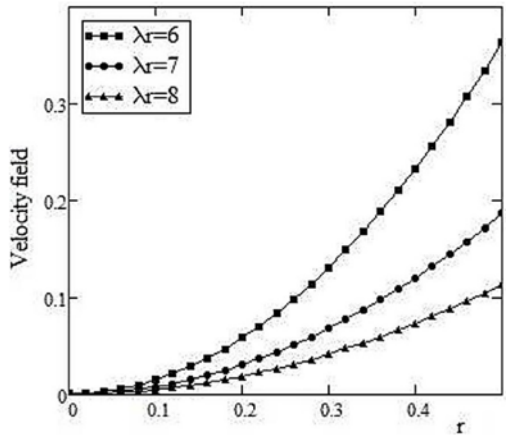

Fig. 5. Velocity field graph for R=0.5, λ_r=3, ν=0.0357541, μ=15, A=90, ω=15, λ=5, ψ=0.5, φ=0.3, t=15 and various values of λ_r.

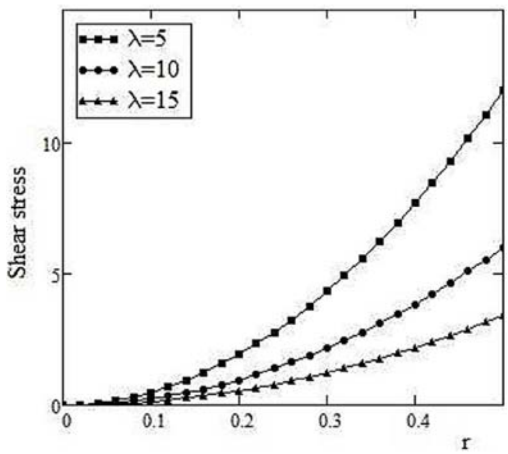

Fig. 6. Shear stress graph for R=0.5, λ_r=3, μ=15, A=90, ω=15, ν=0.0357541, ψ=0.5, φ=0.3, t=15 and various values of λ.

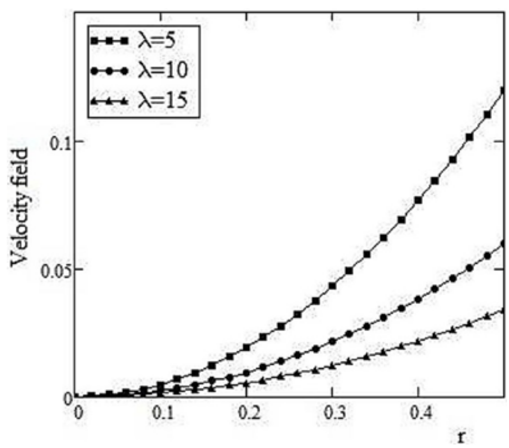

Fig. 7. Velocity field graphs for R=0.5, λ_r=3, μ=15, A=90, ω=15, ν=0.0357541, ψ=0.5, φ=0.3, t=15 and various values of λ.

In fluid dynamics, the cylindrical domain is very useful to solve various engineering problems, that include meteorological geophysical and industrial problems and motion of fluid in oscillating, rotating annulus or flows in pipes. The flows of bio fluids through veins or stenosis arteries are intensively studied in the past decade. The biological fluids are viscoelastic in nature that has complex rheology. These fluids are well described by the non-Newtonian fluid models, including the second grade and Maxwell fluid model.

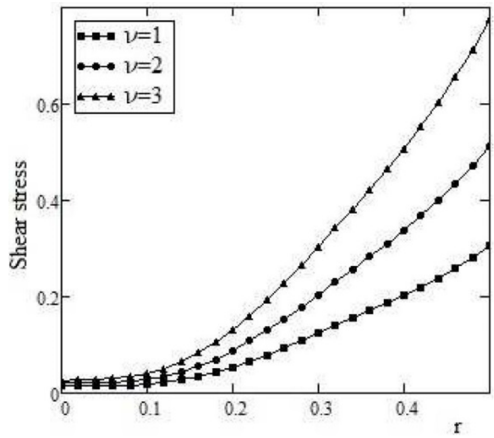

Fig. 8. Shear stress graph for *R*=0:5, λ_r=3, *μ*=15, *A*=90, ω =15, λ=5, y=0:5, φ=0.3, ψ =15 and various values of n.

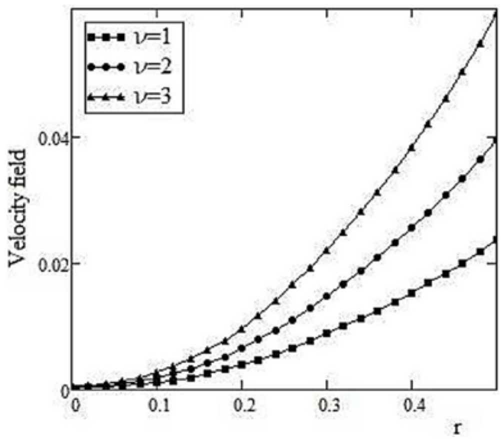

Fig. 9. Velocity field graph for R=0.5, λ_r=3, μ=15, A=90, ω=15, λ=5, ψ=0.5, φ=0.3, t=15 and various values of ν.

6. CONCLUSIONS

In this paper, exact analytical solutions of velocity field and shear stress of an Oldroyd-B fluid with fractional derivatives in cylindrical domain in calculated. To solve the problem, Laplace and Hankel transformations are used that made the approach to the solutions more attainable.

- It is worthy to observe that the fluid layers situated close cylinder surface have a significant motion, while the fluid situated in the central area of the cylinder has a very slow motion.
- The shear stress has the behavior similar with velocity; therefore, it is increasing when the

time increases.

- The physical parameters ν is directly proportional to both velocity function and shear stress.
- The physical parameters λ and λ_r are opposite to both velocity function and shear stress.

ACKNOWLEDGMENTS

The authors are really thankful to the reviewers for their constructive remarks and fruitful suggestions.

REFERENCES

Abbasbandy, S., T. Hayat, A. Alsaedi and M. M. Rashidi (2013). Numerical and analytical solutions for falknerskan flow of MHD Oldroyd-B fluid. *Inter. J. Numer. Meth. Heat and Fluid Flow. 24*, 390–401.

Awan, A. U., C. Fetecau and Q. Rubab (2010). Axial Couette flow of a generalized Oldroyd-B fluid due to a longitudinal time-dependent shear stress. *Quest. Math. 33*, 429–441.

Casarella, M. J. and P. A. Laura (1969). Drag on oscillating rod with longitudinal and torsional motion. *J. Hydronaut. 3*, 883–888.

Fetecau, C. and K. Kannan. (2005). A note on an unsteady flow of an Oldroyd-B fluid. *Int. J. Math. Math. Sci. 19*, 3185–3194.

Fetecau, C., A. Mahmood, C. Fetecau and D. Vieru (2008 a). Some Exact solutions for the helical flow of a generalized Oldroyd-B fluid in a circular cylinder. *Comput. Math. Appl. 56*, 3096–3108.

Fetecau, C., C. Fetecau, M. Khan and D. Vieru (2008 b). Decay of a potential vortex in a generalized Oldroyd-B fluid. *Appl. Math. Comp. 205*, 497–506.

Galanis, N. and M. M. Rashidi (2012). Entropy generation in non-Newtonian fluids due to heat and mass transfer in the entrance region of ducts. *Inter. J. Heat Mass Transf. 48*, 1647–1662.

Imran, M., M. Tahir, M. A. Imran and A. U. Awan (2015). Taylor-Couette flow of an Oldroyd-B fluid in an annulus subject to a time-dependent rotation. *American J. Appl. Math. 3*, 25–31.

Jamil, M., C. Fetecau and M. Rana (2012). Some exact solutions for Oldroyd-B fluid due to time dependent prescribed shear stress. *J. Theoret. Appl. Mech. 50*, 549–562.

Keimanesh, M., M. M. Rashidi, A. J. Chamkha and R. Jafari (2011). Study of a third grade non-Nowtonian fluid flow between two parallel plates using the multi-step differential transform method, *Comput. Math. Appl. 62*, 2871–2891.

Mahmood, A., S. Parveen, A. Ara and N. A. Khan (2009). Exact analytic solutions for the unsteady flow of a non-Newtonian fluid between two cylinders with fractional derivative model. *Commun. Nonlinear Sci. Numer. Simul. 14*, 3309–3319.

Rajagopal, K. R. (1983). Longitudnal and torsional oscillations of a rod in a non-Newtonian fluid. *Acta Mech. 49*, 281–285.

Rajagopal, K. R. and A. R. Srinavasa (2000). A thermodynamic frame work for rate type fluid models. J. Non-Newtonian *Fluid Mech. 88*, 207–227.

Rajagopal, K. R. and R. K. Bhatnagar (1995). Exact solutions for some simple flows of an Oldroyd-B fluid. *Acta Mech. 113*, 233–239.

Rashidi, M. M., T. Hayat, M. Keimanesh and A. A. Hendi (2013). New analytical method for the study of natural convection flow of a non-Newtonian. *Inter. J. Numer. Meth. Heat Fluid Flow. 23*, 436–450.

Riaz, M. B., M. A. Imran and K. Shabbir (2016). Analytic solutions of Oldroyd-B fluid with fractional derivatives in a circular duct that applies a constant couple. *Alex. Eng. J. 55*, 3267–3275.

Rubab, Q., S. M. Husnine and A. Mahmood (2009). Exact solution of generalized Oldroyd-B fluid subject to a time-dependant shear stress in a pipe. *J. Prime Research Math. 5*, 139–148.

Stokes, G. G. (1886). *On the effect of the rotation of cylinders and spheres about their axes in increasing the logarithmic decrement of the arc of vibration.* Cambridge University Press, Cambridge.

Sultan, Q. and M. Nazar (2016). Flow of generalized burgers' fluid between side walls induced by sawtooth pulses stress. *J. Appl. Fluid Mech. 9*, 2195–2204.

Tong, D., R. Wang and H. Yang. (2005). Ex- act solutions for the flow of non- Newtonian fluid with fractional derivative in an annular pipe. Sci. China Ser. G Phys. *Mech. Astro. 48*, 485–495.