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An Analytical Solution for the Laminar Forced Convection in a Pipe with Temperature-Dependent Heat Generation

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ABSTRACT

An analytical solution is presented for the case of laminar forced convection in a pipe with heat generation linearly dependent on the local temperature of the fluid. The flow is fully developed and the boundary conditions of the third kind. Within the general analysis presented, some particular cases are identified and discussed. A detailed analysis of the thermal entrance is given. It is shown that in the fully developed region the temperature distribution does not depend on the axial coordinate. An analytical expression of the fully developed Nusselt number is given. Finally, the practical significance of the problem is discussed.

Keywords: Forced convection; Internal heat generation; Ohmic heating; Third kind boundary condition.

NOMENCLATURE

A	constant
c_p	specific heat at constant pressure
D	diameter
f	Auxiliary function, Eq. (23)
g	Auxiliary function, Eq. (24)
${}_1F_1$	confluent hypergeometric function
h	heat transfer coefficient
I_α	modified Bessel function of the first kind of order α
J_α	Bessel function of the first kind of order α
k	thermal conductivity
Nu_x	local Nusselt number, Eq. (31)
pe	Poiseuille function, Eq. (23)
Pe	Péclet number, $Pe=WD/\alpha$
q'''	wall heat flux
q_w	dimensionless wall heat flux, Eq. (30)
S_o	dimensionless heat generation parameter
S_1^2	dimensionless heat generation parameter

T'	temperature
T	dimensionless temperature
W	average axial velocity
r'	radial coordinate
r	dimensionless radial coordinate, $r=r'/D/2$
x'	axial coordinate
x^*	dimensionless axial coordinate, $x^*=x'/(PeD/2)$
α	thermal diffusivity
β'_o	temperature heat generating rate
λ	constant of separation, Eqs. (19) and (20)
μ	auxiliary position, $\mu=\lambda^{1/2}$

Subscripts

a	environment
b	bulk
fd	fully developed
o	inlet, origin

1. INTRODUCTION

Laminar forced convection with internal heat generation is of relevance to many fields of science and technology such as chemical process equipment, ohmic heating of foods and molten salt nuclear reactors. Excellent reviews of theoretical and experimental work on forced convection in circular ducts are given by Shah and London (1978), Shah and Batti (1987) and Ebadian and Dong (1998). These reviews make evident that, although the enormous research on the Graetz-like problems has addressed all the special effects (axial conduction, heat generation by viscous dissipation, effects of pressure work, etc), all the relevant geometries, and all the significant boundary conditions (of first, second and third kinds) in detail, not many studies have been undertaken to examine the influence of a heat generating source on the laminar forced convection in circular ducts. Related studies are now reviewed, in approximately historical order.

An analytical solution for the thermal entrance region in circular ducts with uniform wall temperature and constant internal heat generation was first proposed by Topper (1955). Sparrow and Siegel (1958) extended Topper's (1955) analysis to the case of arbitrary internal heat generation and wall heat flux. The particular case of adiabatic wall and uniform internal heat generation was experimentally investigated by Inman (1962), who found a very good agreement with the relevant predicted data in Sparrow and Siegel (1958).

Fully developed laminar Nusselt number for non-circular ducts with uniform heat flux boundary conditions and constant internal heat generation was addressed by Tao (1961) by means of an analytical approach based on the properties of complex analytical functions. Tyagi (1966) extended the Tao's (1961) analysis by including viscous dissipation.

The thermal entrance region for a power-law fluid in a circular duct with boundary condition of uniform wall temperature and internal heat generation variable linearly with the temperature, was analysed by Foraboschi and Di Federico (1964).

A similar problem, but for newtonian fluids, boundary conditions of the third kind and uniform internal heat generation, was analysed by Hsu (1971). Mori *et al*. (1977) addressed forced convection in a circular duct in laminar flow with temperature dependent internal heat generation. Obtained the solution for uniform wall temperature boundary condition, they were able to solve the case of arbitrary temperature distribution at the wall-fluid interface and the conjugated problem with constant temperature imposed at the outer surface of the conductive wall.

In the work reported here, we have extended the analysis of Hsu (1971) to the case of boundary conditions of the third kind and temperature dependent internal heat generation. Within the general analysis presented, the works of Foraboschi and Di Federico (1964), for Newtonian fluids, and Mori *et al.* (1977) are identified as particular cases.

The present approach has significant advantages in comparison to the methods used in the studies quoted above. In our analysis the eigenfunctions are calculated analytically in terms of Poiseuille functions. Instead of the hand calculations used in Foraboschi and Di Federico (1964) or the numerical procedure used in Hsu (1971) and in Mori *et al.* (1977), our analytical solution permits high quality results and fast calculations.

The present study, addressing forced convection with internal linear temperature-dependent heat generation for boundary conditions of the third kind, is discussed in detail. The effects of the dimensionless parameters on the heat transfer quantities and temperature field are analysed and discussed. Also, the cases of uniform wall temperature and adiabatic wall for variable and constant internal heat generation are detailed.

The interest in forced convection with internal temperature-dependent heat generation is primarily related to a practical problem: the sterilization of liquid foods by continuous ohmic heating (also joule or volumetric heating). In one of the most diffused configuration the product flows in pipes from one electrode to the other and the flow is parallel to the electric potential. The local internal heat generation is due to the product of the square of the axial gradient of electric potential per the thermal conductivity of the fluid. Typically, this last parameter in liquid foods, like fruit puree or fruit juices, increases with temperature. For an experimental evidence of this dependence see for instance Icier and Ilicali (2005) and Darvishi *et al.* (2011). Since this dependence gives rise to a local internal generation decreasing with the temperature, as shown by Pesso and Piva (2009) we can say that the essential of an ohmic sterilizer is the temperature dependency of the heat generating term.

2. GOVERNING EQUATIONS AND SOLUTION STRATEGIES

The problem considered is schematically shown in Fig. 1. A constant properties newtonian fluid flows steadily in a circular tube with internal heat generation variable linearly with the temperature. The velocity profile is fully established, while the temperature profile at the inlet of the thermal developing section is uniform. At the wall, a convective boundary condition with uniform heat transfer coefficient and ambient temperature is considered. Viscous dissipation and axial heat conduction in the fluid are neglected. The problem is stated mathematically as follows:

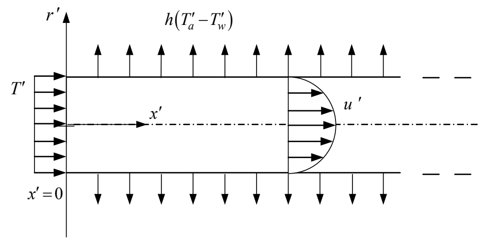

Fig. 1. Geometry and boundary conditions.

Conservation of energy:

$$\rho c_p 2W\left(1-\left(\frac{r'}{D/2}\right)^2\right)\frac{\partial T'}{\partial x'} = k\frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial T'}{\partial r'}\right) + q_o'''\left(1+\beta_o'\left(T'-T_o'\right)\right) \quad (1)$$

Boundary conditions:

$$T'\left(0,r'\right) = T_o' \qquad \text{for } x'=0 \quad (2)$$

$$\left.\frac{\partial T'}{\partial r'}\right|_{r'=0} = 0 \qquad \text{for } 0 < x' < +\infty \quad (3)$$

$$-k\left.\frac{\partial T'}{\partial r'}\right|_{D/2} = h\left(T'\big|_{D/2} - T_a'\right) \qquad \text{for } 0 < x' < +\infty \quad (4)$$

Equations (1-4) are then expressed in dimensionless form, according to the following dimensionless variables:

$$x^* = x'/(Pe\,D/2)\ ,\ r = r'/(D/2)$$
$$T = (T' - T'_a)/\Delta T'_{ref}$$

Different choices for $\Delta T'_{ref}$ are possible: the most often used are $\Delta T'_{ref} = T'_a - T'_o$ and $\Delta T'_{ref} = q'''_o D^2/4k$. In this work the first choice leads to a more convenient expression for the treatment of the dimensionless inlet boundary condition and thence it is used.

Equation (1) and related boundary conditions, Eqs. (2-4), become:

$$\left(1-r^2\right)\frac{\partial T}{\partial x^*} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + S_o + S_1^2 T \qquad (5)$$

$$T(0,r) = -1 \qquad \text{for } x^* = 0 \qquad (6)$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0 \qquad \text{for } 0 < x^* < +\infty \qquad (7)$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=1} + \frac{1}{2R_w}\left.T\right|_{r=1} = 0 \qquad \text{for } 0 < x^* < +\infty \qquad (8)$$

The dimensionless parameters R_w, the dimensionless environmental thermal resistance, S_o and S_1^2, descen ding from the heat generating term, are given by:

$$R_w = \frac{k}{hD}$$

$$S_o = \frac{q'''_o\left(1+\beta'_o\left(T'_a - T'_o\right)\right)D^2}{4k\left(T'_a - T'_o\right)}\ ,\ S_1^2 = \frac{q'''_o \beta'_o D^2}{4k}$$

The solution of Eqs. (5-8) can be decomposed as:

$$T(x^*,r) = T_1(r) + T_2(x^*,r) \qquad (9)$$

When substituting Eq. (9) in Eqs. (5-8), it follows that the function $T_1(r)$ is the solution of the following ordinary differential equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_1}{dr}\right) + S_o + S_1^2 T_1 = 0 \qquad (10)$$

with the boundary conditions:

$$\left.\frac{dT_1}{dr}\right|_{r=0} = 0 \qquad \text{for } 0 < x^* < +\infty \qquad (11)$$

$$\left.\frac{dT_1}{dr}\right|_{r=1} + \frac{1}{2R_w}\left.T_1\right|_{r=1} = 0 \qquad \text{for } 0 < x^* < +\infty \qquad (12)$$

The solution of Eqs. (10-12) is given by:

$$T_1(r) = \frac{S_o}{S_1^2}\frac{J_0(S_1 r)}{J_0(S_1) - 2R_w S_1 J_1(S_1)} - \frac{S_o}{S_1^2} \qquad (13)$$

Equation (13) is not valid for values of S_1^2 which makes the denominator of Eq.(13) equal to zero.

The function $T_2(x^*,r)$ is then the solution of the following partial differential equation:

$$\left(1-r^2\right)\frac{\partial T_2}{\partial x^*} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) + S_1^2 T_2 \qquad (14)$$

with the boundary conditions:

$$T_2(0,r) = -1 - T_1(r) \qquad \text{for } x^* = 0 \qquad (15)$$

$$\left.\frac{\partial T_2}{\partial r}\right|_{r=0} = 0 \qquad \text{for } 0 < x^* < +\infty \qquad (16)$$

$$\left.\frac{\partial T_2}{\partial r}\right|_{r=1} + \frac{1}{2R_w}\left.T_2\right|_{r=1} = 0 \qquad \text{for } 0 < x^* < +\infty \qquad (17)$$

Following the method of separation of variables, the function $T_2(x^*,r)$ can be expressed as:

$$T_2(x^*,r) = \sum_{i=1}^{\infty} A_i\, g_i(x^*) f_i(r) \qquad (18)$$

The functions $g_i(x^*)$ and $f_i(r)$ are solution of:

$$\frac{dg_i}{dx^*} = -\lambda_i g_i \qquad \text{for } 0 < x^* < +\infty \qquad (19)$$

$$\frac{d^2 f_i}{dr^2} + \frac{1}{r}\frac{df_i}{dr} + S_1^2 f_i = -\lambda_i f_i\left(1-r^2\right)$$

for $0 < r < 1$ (20)

Equation (20) is completed by the following boundary conditions:

$$\left.\frac{df_i}{dr}\right|_{r=0} = 0 \qquad (21)$$

$$\left.\frac{df_i}{dr}\right|_{r=1} + \frac{1}{2R_w}\left.f_i\right|_{r=1} = 0 \qquad (22)$$

Introducing the following positions

$$\mu_i = \lambda_i^{1/2} \quad , \quad \alpha_i = \frac{1}{2}\left(1 - \frac{\mu_i}{2} - \frac{S_1^2}{2\mu_i}\right)$$

a generic solution of Eq.(20) is given by that Poiseuille function bounded at the centerline of the duct, so that the boundary condition of axial symmetry, Eq.(21), is always satisfied (Piva, 1996):

$$f_i(r) = pe(r,\mu_i) = \exp\left[-\mu_i r^2/2\right]{}_1F_1\left[\alpha_i, 1, \mu_i r^2\right] \qquad (23)$$

The admissible values of the constant of separation λ_i are the root of the transcendental equation Eq.(22). The search of these roots can be easily executed by means of a numerical procedure.

A general solution of Eq.(19) can be expressed in terms of the exponential function:

$$g_i(x^*) = \exp(-\lambda_i x^*) \qquad \text{for } 0 < x^* < +\infty \qquad (24)$$

The constants A_i that satisfy the boundary condition Eq. (15) are determined by means of the following procedure. From Eqs. (15, 18, 24) it follows:

$$\sum_{i=1}^{\infty} A_i f_i(r) = -1 - T_1(r) \quad (25)$$

It can be easily proved that the functions $f_i(r)$, with the boundary conditions, Eqs.(21) and (22), guarantee:

$$\int_{r=0}^{r=1} \left(1-r^2\right) f_i(r) f_j(r) r dr \begin{cases} = 0, & if\ i \neq j \\ \neq 0, & if\ i = j \end{cases} \quad (26)$$

By multiplying both sides of Eq.(25) by $(1-r^2) f_j(r) r$ and integrating from $r = 0$ to $r = 1$, on the basis of the property given by Eq. (26), it follows:

$$A_j = \frac{\int_{r=0}^{r=1} \left(-1 - T_1(r)\right) f_j(r) \left(1-r^2\right) r dr}{\int_{r=0}^{r=1} f_j^2(r) \left(1-r^2\right) r dr} \quad (27)$$

From Eqs. (9, 13, 18, 24) it follows:

$$T\left(x^*, r\right) = \sum_{i=1}^{\infty} A_i \exp\left(-\lambda_i x^*\right) f_i(r) + \frac{S_o}{S_1^2} \frac{J_0(S_1 r)}{J_0(S_1) - 2R_w S_1 J_1(S_1)} - \frac{S_o}{S_1^2} \quad (28)$$

The dimensionless bulk temperature, wall heat flux (positive when entering in the system) and the local Nusselt number are given by:

$$T_b\left(x^*\right) = 4 \int_{r=0}^{r=1} \left(1-r^2\right) T r dr \quad (29)$$

$$q_w(x^*) = \frac{q''_w}{2k\left(T'_a - T'_o\right)/D} = \left.\frac{\partial T}{\partial r}\right|_{r=1} \quad (30)$$

$$Nu_x = 2 \frac{\left.\partial T/\partial r\right|_{r=1}}{T_w - T_b} \quad (31)$$

From Eqs. (20, 28, 29) it follows:

$$T_b\left(x^*\right) = -4 \sum_{i=1}^{\infty} A_i \exp\left(-\lambda_i x^*\right) \cdot \left[\frac{1}{\lambda_i} \left.\frac{df_i}{dr}\right|_{r=1} + \frac{S_1^2}{\lambda_i} \int_{r=0}^{r=1} f_i(r) r dr \right] + \frac{S_o}{S_1^2} \frac{1}{J_0(S_1) - 2R_w S_1 J_1(S_1)} \frac{8 J_2(S_1)}{S_1^2} - \frac{S_o}{S_1^2} \quad (32)$$

From Eq. (28), the dimensionless heat flux, Eq. (30), becomes:

$$q_w\left(x^*\right) = \sum_{i=1}^{\infty} A_i \exp\left(-\lambda_i x^*\right) \left.\frac{df_i}{dr}\right|_{r=1} - \frac{S_o}{S_1^2} \frac{S_1 J_1(S_1)}{J_0(S_1) - 2R_w S_1 J_1(S_1)} \quad (33)$$

In the present case, the condition of full development (Shah and London, 1978) is achieved for $x^* \to \infty$. In this region, the dimensionless temperature distribu tion, Eq. (28), is independent on x^*. By calculating the limit of Eq. (28) *for* $x^* \to \infty$, it follows:

$$T_{fd}(r) = \frac{S_o}{S_1^2} \frac{J_0(S_1 r)}{J_0(S_1) - 2R_w S_1 J_1(S_1)} - \frac{S_o}{S_1^2} \quad (34)$$

The fully developed Nusselt number is obtained by substituting Eqs. (28, 32, 33) in Eq. (31) and by calculating the limit for $x^* \to \infty$:

$$Nu_{fd} = 2 \frac{-S_1 J_1(S_1)}{J_0(S_1) - 8 J_2(S_1)/S_1^2} \quad (35)$$

3. PARTICULAR CASES

Within the general analysis presented in Par.2, some particular cases can be identified, as follows.

3.1 Uniform Wall Temperature

In the literature, the case of uniform wall temperature boundary condition has been analysed by Foraboschi and Di Federico (1964). In the present analysis, this case is achieved for R_w=0. From Eq. (28) it follows:

$$T\left(x^*, r\right) = \sum_{i=1}^{\infty} A_i \exp\left(-\lambda_i x^*\right) f_i(r) + \frac{S_o}{S_1^2} \frac{J_0(S_1 r)}{J_0(S_1)} - \frac{S_o}{S_1^2} \quad (36)$$

The eigenvalues λ_i are the roots of the following transcendental equation:

$$\left. f_i \right|_{r=1} = 0 \quad (37)$$

The fully developed Nusselt number is still given by Eq. (35).

$$T_b\left(x^*\right) = -4 \sum_{i=1}^{\infty} A_i \exp\left(-\lambda_i x^*\right) \left[\frac{1}{\lambda_i} \left.\frac{df_i}{dr}\right|_{r=1} + \frac{S_1^2}{\lambda_i} \int_{r=0}^{r=1} f_i(r) r dr \right] + \frac{S_o}{S_1^2} \frac{1}{J_0(S_1)} \frac{8 J_2(S_1)}{S_1^2} - \frac{S_o}{S_1^2} \quad (38)$$

$$q_w\left(x^*\right) = \sum_{i=1}^{\infty} A_i \exp\left(-\lambda_i x^*\right) \left.\frac{df_i}{dr}\right|_{r=1} - \frac{S_o}{S_1^2} \frac{S_1 J_1(S_1)}{J_0(S_1)} \quad (39)$$

Furthermore, in the present analysis the functions $f_i(r)$ are expressed as Poiseuille functions, Eq. (23), while in Foraboschi and Di Federico (1964) they were calculated by means of the Frobenius method.

3.2 Adiabatic Wall

A particular case of the wall boundary condition of Eq. (8), is that of an adiabatic wall, achieved for $R_w \to \infty$. From Eq. (28) it follows:

$$T\left(x^*, r\right) = \sum_{i=1}^{\infty} A_i \exp\left(-\lambda_i x^*\right) f_i(r) - \frac{S_o}{S_1^2} \quad (40)$$

The eigenvalues λ_i are the roots of the following transcendental equation:

$$\left.\frac{df_i}{dr}\right|_{r=1} = 0 \quad (41)$$

Equation (26) is still satisfied. Then the constants A_i are given by Eq. (27).

The dimensionless bulk temperature, Eq.(32), becomes:

$$T_b\left(x^*\right)=-4\sum_{i=1}^{\infty}A_i\exp\left(-\lambda_i x^*\right)\frac{S_1^2}{\lambda_i}\int_{r=0}^{r=1}f_i\left(r\right)rdr-\frac{S_o}{S_1^2} \tag{42}$$

3.3 Constant Internal Heat Generation with Boundary Conditions of The Third Kind

In the literature, using boundary conditions of the third kind, the case of uniform internal heat generation has been analysed by Hsu (1971). The fully developed Nusselt number for this case has also been calculated by Mori et al. (1977) as a sub-case of their general analysis. This dealt with the conjugate heat transfer in a circular duct in laminar flow with temperature dependent internal heat generation and constant temperature imposed on the outer surface of the conductive wall.

In the present analysis, the case of uniform heat generation is achieved for S_1^2 =0. In this case Eq.(13), is no longer valid. Following the method proposed by Hsu (1971), the dimensionless temperature distribu tion in the thermal entrance region can be expressed as:

$$T\left(x^*,r\right)=\sum_{i=1}^{\infty}A_i\exp\left(-\lambda_i x^*\right)f_i\left(r\right)+ R_w S_o+\frac{S_o}{4}\left(1-r^2\right) \tag{43}$$

The functions $f_i(r)$ of Eq. (43) are those given by Eq.(23).

The eigenvalues λ_i are the roots of Eq. (22). Equation (26) is still satisfied. Then the constants A_i are given by Eq. (27).

The dimensionless bulk temperature and wall heat flux distributions become respectively:

$$T_b\left(x^*\right)=-4\sum_{i=1}^{\infty}\frac{A_i}{\lambda_i}\exp\left(-\lambda_i x^*\right)\left.\frac{df_i}{dr}\right|_{r=1}+ R_w S_o+\frac{1}{6}S_o \tag{44}$$

$$q_w\left(x^*\right)=\sum_{i=1}^{\infty}A_i\exp\left(-\lambda_i x^*\right)\left.\frac{df_i}{dr}\right|_{r=1}-\frac{S_o}{2} \tag{45}$$

The fully developed Nusselt number, Eq. (35), in fully agreement with that calculated by Mori et al. (1977), becomes:

$$Nu_{fd}=6 \tag{46}$$

In the present analysis the functions $f_i(r)$, Eq. (23), are calculated analytically, while in Mori et al. (1977) they were obtained numerically. In Hsu (1971) the axial length of the tube analysed was not long enough to attain the condition of full development; the fully developed Nusselt number, Eq. (46), is not exactly achieved in the diagram of Nu versus x^* presented by Hsu (1971).

3.4 Constant Internal Heat Generation with Adiabatic Wall

The final particular case in the literature, is that of uniform heat generation with adiabatic wall which was analysed by Sparrow and Siegel (1958). For the present analysis this case is achieved for $R_w\to\infty$ and S_1^2 =0.

$$T\left(x^*,r\right)=\sum_{i=1}^{\infty}A_i\exp\left(-\lambda_i x^*\right)f_i\left(r\right)+ 2S_o x^*-1+\frac{S_o}{4}\left(r^2-\frac{1}{2}r^4-\frac{1}{4}\right) \tag{47}$$

The functions $f_i(r)$ in Eq. (47) are given by Eq. (23). The eigenvalues λ_i are the roots of the following transcendental equation:

$$\left.\frac{df_i}{dr}\right|_{r=1}=0 \tag{48}$$

Equation (26) is still satisfied. Then the constant A_i are given by:

$$A_i=\frac{\int_{r=0}^{r=1}\left(-\frac{S_o}{4}\left(r^2-\frac{1}{2}r^4-\frac{1}{4}\right)\right)f_i\left(r\right)\left(1-r^2\right)rdr}{\int_{r=0}^{r=1}\left(1-r^2\right)f_i^2\left(r\right)rdr} \tag{49}$$

The dimensionless bulk temperature is given by:

$$T_b\left(x^*\right)=2S_o x^*-1 \tag{50}$$

As before, for the present analysis the functions $f_i(r)$ are calculated analytically, while in Sparrow and Siegel (1958) they were obtained numerically.

4. RESULTS AND DISCUSSION

In the present analysis only negative values of the dimensionless parameter S_1^2 are considered. This is what relates to the situation of ohmic heating of food in collinear heaters, as shown by the application of the present analytical solution for a thermo-fluid analysis of laminar flow in a cylindrical collinear ohmic sterilizer (Pesso and Piva 2009). In this equipment, the temperature change of the thermal properties of the fluid can be significant. Their full effect can be taken into account, for instance, by means of a numerical procedure. However, as discussed in more detail in Pesso and Piva (2009), the essential thermo-fluid feature of the ohmic sterilizer is the temperature dependency of the heat generating term. This is due to the marked temperature change of the electrical conductivity of liquid foods. Therefore, for this application the present analytical approach becomes competitive, due to its inherent simplicity and quickness of calculation.

In the present context, negative values of S_1^2 introduce a stabilizing effect on the temperature field, since the internal heat generation decreases as the temperature increases. Every negative value of S_1^2 is allowed (see Appendix I for further details).

The influence of the dimensionless parameters S_o, S_1^2 and R_w on both the heat transfer quantities and the radial distributions of the dimensionless temperature, is analysed.

Although the local Nusselt number is traditionally the main focus in a presentation of heat transfer results, there are two main reasons for not making it so in the present analysis. As observed by Faghri and Sparrow (1980) in a conjugate heat transfer problem, in the present case the local Nusselt number, Eq. (31), in-volves three unknowns, $\partial T/\partial r|_{r=1}$, T_w and T_b. S_o, for a given value of Nu_x, a relationship among these three unknowns is established, but there is no way of obtaining their actual values. Furthermore, at certain specific axial locations, depending on R_w, S_o, and S_1^2, the wall and the bulk temperature reach the same value, leading to a singularity in the Nu_x distribution. In such a situation, as observed by Shah and London (1978), the concept of heat transfer coefficient loses its significance. In view of these deficiencies, it is necessary to show the results for q_w, T_b and T_w.

The roots of Eqs. (23), (37) and (41) have been calculated numerically, giving the eigenvalues λ_i as a function of R_w and S_1^2. The present calculations have been validated by comparison against certain published data (Sparrow & Siegel 1958, Hsu 1971, Barr & Wiginton 1977a, Barr & Wiginton 1977b), as summarized in Tab. 1. The eigenvalues for R_w=0 and S_1^2=0 and those for R_w=0.25 and S_1^2=0 are in excellent agreement with those provided by Barr and Wiginton (1977a) and Hsu (1971), respectively. The eigenvalues for another particular case - $R_w \rightarrow \infty$ and S_1^2=0 - also agree excellently with those of Barr and Wiginton (1977b) and Sparrow and Siegel (1958).

For all cases documented in the present paper, twenty eigenvalues λ_i have been usedinthe calculation. This assures an accurate dimensionless temperature distribu-tion for $x^* \geqslant 5 \cdot 10^{-3}$.

The values of λ_i for R_w=0.25 and S_1^2=0,-0.1,-1 and -10 have been summarized in Tab 2. It can be observed that λ_i increases as S_1^2 decreases. Moreover, the difference between the values of λ_i corresponding to two different values of S_1^2 is less marked as the index *i* increases.

In Fig. 2 the first eigenvalue λ_1 as a function of S_1^2 is shown. It can be observed that when S_1^2<-10, λ_1 becomes independent of R_w. Furthermore, since λ_1 increases as S_1^2 decreases, the extent of the thermal entrance region decreases as S_1^2 decreases.

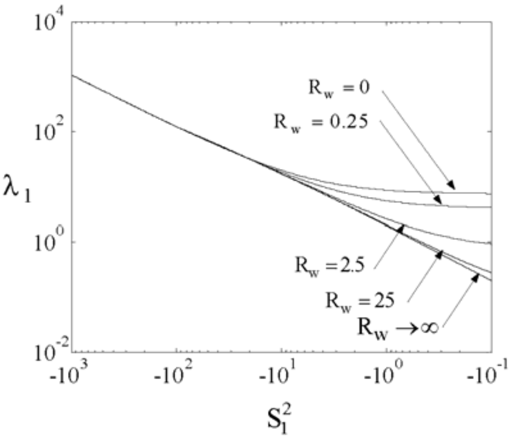

Fig. 2. First eigenvalues as a function of S_1^2.

The dimensionless temperature profiles for R_w=0.25, S_o= -10 and +10, and S_1^2=0 are depicted in Fig. 3 for selected values of x^*. These distributions are completely different for the parameter S_o having positive or negative values.

At the beginning of the duct, the radial dimensionless temperature distribution is nearly flat. For S_o=10 the presence of a maximum in the radial profile of the dimensionless temperature can be observed. This denotes that heat is radially transferred in two opposite directions.

The presence of the maximum is less marked for S_o= -10. Downstream, for x^*=1 and 10, the temperature distribution attains the fully developed profile and becomes monotonic, with a marked difference between its highest and lowest values.

In the fully developed region, the dimensionless temperature gradient at the wall is negative for S_o=10, and positive for S_o=-10. In the applications, positive and negative value of S_o are achieved when $(T'_a-T'_o)$>0 or <0, respectively. Thus, on the basis of the present rules of adimensionalization, in the fully developed region the physical temperature gradient at the wall is always negative, irrespective of whether S_o is > or <0.

The influence of S_1^2 on the dimensionless temperature distributions is depicted in Fig. 4 for the fully develo- ped region. For fixed values of R_w and S_o, the dimensionless temperature distribution becomes more flat as S_1^2 decreases. Furthermore, this effect is more evident for high values of R_w. Since the effect of the internal heating decreases as S_1^2 decreases, the fully developed temperature distribution tends to zero as S_1^2 decreases.

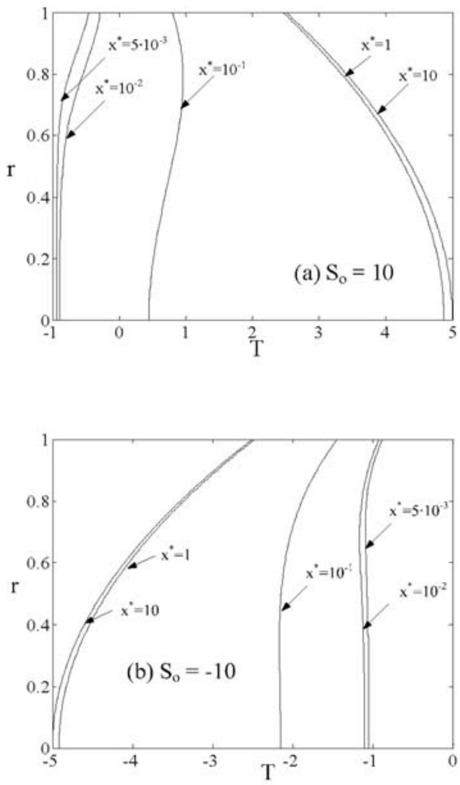

Fig. 3. Radial dimensionless temperature distribution at selected axial stations, R_w=0.25, S_1^2=0: (a) S_o=10; (b) S_o=-10.

In Fig. 5 the dimensionless distributions of bulk and wall temperature and wall heat flux are represented for R_w=0, 0.25 and 25, S_o=-10,10 and S_1^2=0. By choosing to represent the cases with S_o of order ten the effect of the heat generation is made strongly significant. In the fully developed region all the above distributions are constant with x^*. This means that in the fully developed region the whole heat generated in the fluid is transferred to the environment due to the external convection. Therefore, for fixed values of R_w, S_o and S_1^2 (with $S_1^2 \leq 0$), a limiting value of the fluid temperature exists. At a specific axial location, T_w becomes equal to T_b while q_w is finite, giving rise to a singularity in the Nu_x distribution, Eq. (31).

In Fig. 6 the influence of the parameter S_1^2 on the dimensionless bulk temperature and wall heat flux distributions is shown. All the cases depicted in this figure refer to S_o=-10. In the practical applications, a negative value of S_o is the most frequent case, because $T'_o > T'_a$.

In Fig. 6 the T_w distribution has not been plotted. This distribution can be obtained on the basis of the q_w distribution, by means of Eq. (8). From Fig. 6, it can be observed that the T_b and q_w distributions are considerably influenced by the parameter S_1^2, whose effect is more marked in the fully developed region. In this region, as observed in the discussion of Fig. 5, the distributions become constant with x^*. For fixed values of R_w and S_o, the magnitudes of the dimension less bulk temperature and wall heat flux increase as S_1^2 increases up to 0.

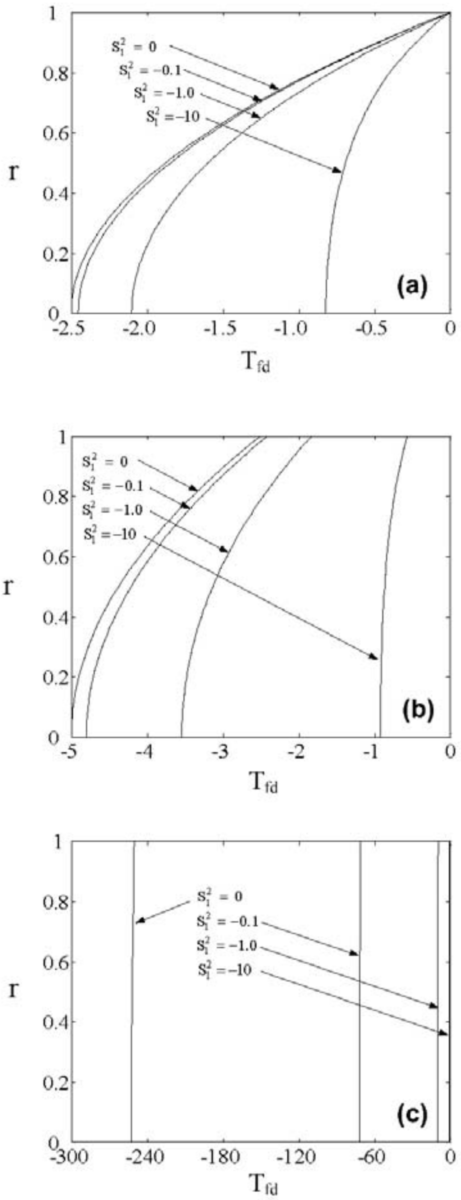

Fig. 4. Influence of S_1^2 on the radial dimensionless temperature in the fully developed region, S_o=-10; (a): R_w=0; (b): R_w=0.25; (c): R_w = 25.

In Fig. 7 the influence of S_1^2 on the fully developed Nusselt number is shown. Since in Eq. (35) the first root of the denominator corresponds to a positive value of the parameter S_1^2 ($S_1^2 \approx$29.61), the distribution Nu_{fd}(S_1^2) never becomes singular in the range of analysis. In Fig. 7 it can be also observed that Nu_{fd} de creases as S_1^2 increases up to 0. This is because, when S_1^2 increases, the increase of $q_{w,fd}$ is less marked than that of the difference $(T_w-T_b)_{fd}$.

In Fig. 8 a comparison between the dimensionless

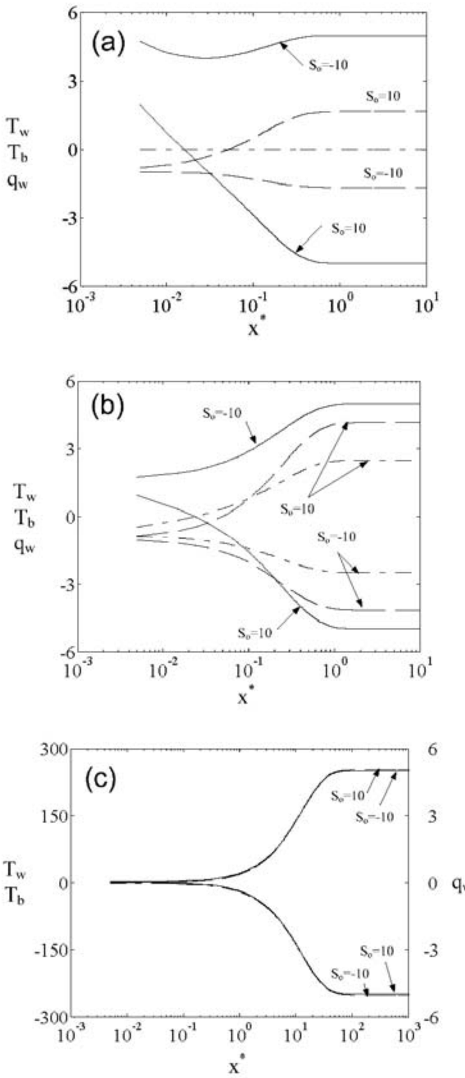

Fig. 5. Dimensionless T_w, T_b and q_w distributions: (a): R_w=0 ; (b): R_w=0.25; (c): R_w=25; ("— -": T_w; "— —": T_b; "—": q_w).

bulk temperature distributions of the adiabatic tube ($R_w \to \infty$) and of a tube with a high dimensionless environment thermal resistance (R_w=25 and $2.5 \cdot 10^5$) is depicted for the case of constant internal heat generation. The heat generating dimensionless parameters have been chosen to give S_o=10 and S_1^2=0. In the fully developed region, the T_b distributions along x^* corresponding to a finite value of R_w always detach from the T_b distribution corresponding to $R_w \to \infty$. Therefore, when R_w is high, the idealization of "adiabatic wall" can be valid only for a finite axial length from the inlet section, as shown in Fig. 8. For the case of constant internal heat generation, Inman (1962) found excellent agreement between his experimental data for a fiberglass insulated tube and the analytical results of Sparrow and Siegel (1958) for the adiabatic wall case. On the basis of Fig. 8, this agreement is probably due to the test section length of the experimental setup ($x^*_{max} \approx 0.1$), which was not long enough to attain the fully developed condition.

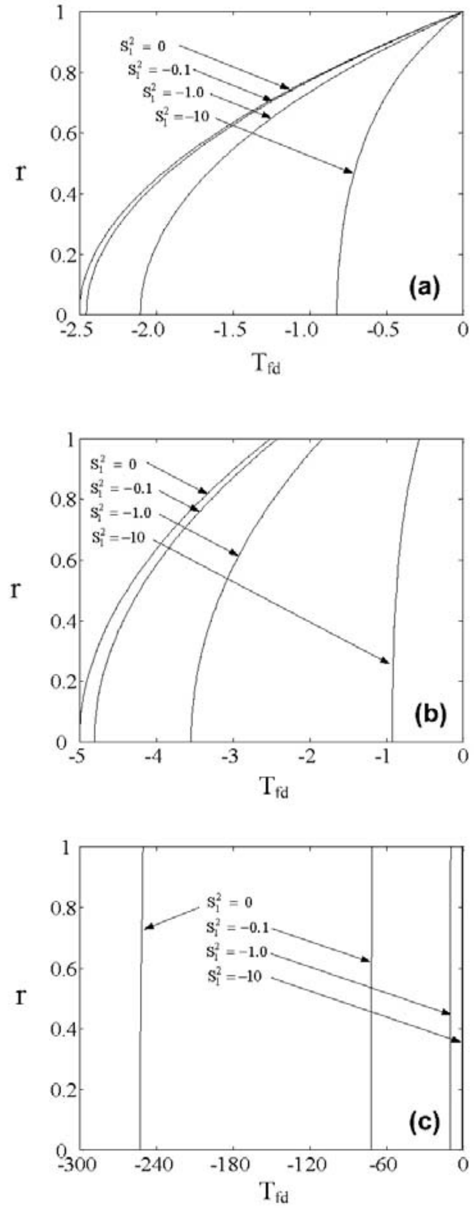

Fig. 6. Influence of S_1^2 on the dimensionless T_b and q_w distribution, S_o=-10: (a): R_w=0; (b): R_w=0.25; (c): R_w=25; ("— —": T_b; "—": q_w).

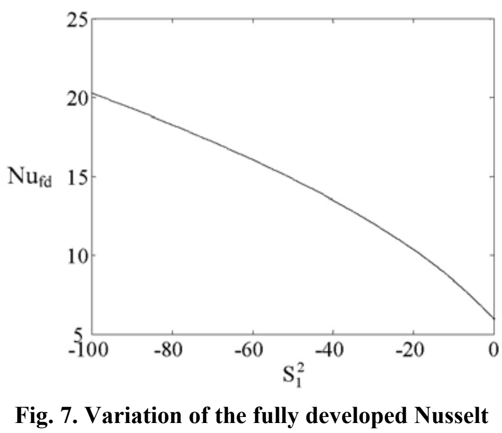

Fig. 7. Variation of the fully developed Nusselt number with S_1^2.

Table 1 Comparison of the present against the literature results for S_1^2=0 (uniform heat generation)

N	R_w=0		R_w=0.25		$R_w\to\infty$		
	Present	Barr (1971a)	Present	Hsu (1971)	Present	Barr (1971b)	Sparrow
1	7.313587	7.313586	4.000000	4.00000	25.679612	25.67961	25.6796
2	44.609461	44.609451	32.99264983	32.99261697	83.86176	83.86176	83.8618
3	113.921031	113.921021	93.0271811	93.0271824	174.16674	174.16673	174.167
4	215.240543	215.240540	184.6970279	184.6962541	296.53630	296.53613	296.536
5	348.564115	348.563965	308.1710673	308.2974106	450.94719	450.94702	450.947

Table 2 Calculated eigenvalues for R_w=0.25 as a function of S_1^2

N	S_1^2= 0	S_1^2= -0.1	S_1^2= -1.0	S_1^2= -10
1	4.0000000	4.1521725	5.5089197	18.105621
2	32.992650	33.177743	34.840864	51.060880
3	93.027181	93.219717	94.952734	112.22680
4	184.697028	184.892447	186.65197	204.28750
5	308.171067	308.367925	310.14046	327.92958
6	463.521362	463.719050	465.49900	483.36472
7	650.785602	650.983816	652.76843	670.67695
8	869.986106	870.184679	871.97244	889.90695
9	1121.13726	1121.33609	1123.1261	1141.07739
10	1404.24891	1404.44793	1406.2396	1424.20227
11	1719.32812	1719.52728	1721.3202	1739.29092
12	2066.38015	2066.57942	2068.3733	2086.34989
13	2445.40901	2445.60837	2447.4030	2465.38400
14	2856.41786	2856.61730	2858.4126	2876.39690
15	3299.40922	3299.60872	3301.4045	3319.39144
16	3774.38515	3774.58470	3776.3809	3794.36991
17	4281.34733	4281.54692	4283.3435	4301.33416
18	4820.29718	4820.49681	4822.2937	4840.28571
19	5391.23591	5391.43557	5393.2327	5411.22584
20	5994.16453	5994.36422	5996.1616	6014.15564

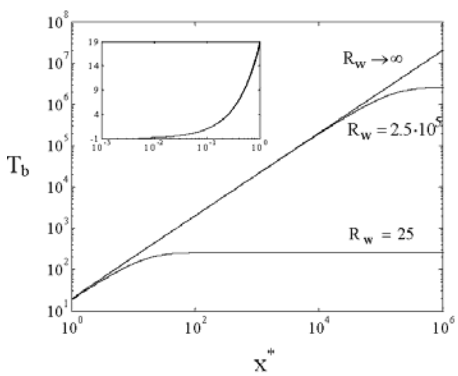

Fig. 8. Dimensionless bulk temperature distributions for R_w=25, 2.5·10^5 and ∞; S_o=10, S_1^2 =0.

5. CONCLUDING REMARKS

Steady laminar forced convection in circular pipes with negative temperature heat generating rate shows some features of considerable interest:

a. In the thermal entrance region, the temperature distribution depends mainly on S_o and R_w. The effect of the parameter S_1^2 is more marked in the fully developed region.
b. In the fully developed region, the quantities T_w, T_b and q''_w are constant with respect to the axial co ordinate. In the practical applications, this means that in this region the whole heat generated is transferred to the environment without further heating of the fluid.
c. The fully developed Nusselt number depends only on S_1^2 and it can be calculated by means of an exact analytical equation.
d. For S_1^2 =0 (constant internal heat generation), the fully developed conditions corresponding to the adiabatic case are never achieved for $R_w\to\infty$. Otherwise, for S_1^2 <0 (negative temperature heat generating rate), the dimensionless temperature distribution corresponding to the adiabatic case is always achieved for $R_w\to\infty$.
e. The negative temperature heat generating rate, which for example is pertinent to the practical applications dealing with ohmic heating of liquid foods in a collinear heater, has a

stabilizing effect on the temperature distribution.

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Appendix I

In this Appendix it is shown that the denominator of Eq.(13):

$$J_0\left(S_1\right) - 2\,R_w S_1 J_1\left(S_1\right) = 0 \tag{I.1}$$

has no roots for S_1^2 <0.

For S_1^2 <0, the argument of the Bessel functions in Eq. (I.1) is purely imaginary. Denoting with $I_\alpha(x)$ the modified Bessel function of the first kind of order α, Eq. (I.1) can be rewritten as:

$$I_0\left(\left|S_1\right|\right) + 2\,R_w \left|S_1\right| I_1\left(\left|S_1\right|\right) = 0 \tag{I.2}$$

Since $|S_1|$ is a real positive quantity, the functions $I_0(|S_1|)$ and $I_1(|S_1|)$ are real and positive (Abramowitz and Stegun 1972). Therefore Eq. (I.2), and then Eq. (I.1), have no roots.